ELLIPTIC EQUATIONS, PRINCIPAL EIGENVALUE AND DEPENDENCE ON THE DOMAIN

Jose M. Arrieta*
Department of Mathematics
University of Texas at Austin
Austin, TX 78712. U.S.A.

Abstract

We consider a general second order uniformly elliptic differential operator L and also the set Θ of all open sets (not neccessarily smooth) in the unit ball of $I\!R^n$. We define a metric d in this set (up to an equivalence relation $\sim$) that makes the space $(\Theta/\sim, d)$ a complete metric space. We show that the principal eigenvalue and eigenfunction of L are continuous with the metric d. Similar results are obtained for the solutions of the equation $Lv = f$.

1 Introduction.

The main motivation of this paper comes from the work of Beresticki, Nirenberg and Varadhan in [6]. In this paper, for a general bounded domain (not neccessarily smooth) the relation between the sign of the principal eigenvalue and a refined version of the maximum principle is obtained. Also, the existence of a principal eigenfunction is established along with a general existence theorem of bounded solutions of an elliptic equation in a general bounded domain.

To be more especific let us denote by L an elliptic operator of the form

$$L = a^{ij}\partial_{ij} + b^i\partial_i + c \tag{1.1}$$

where there exist positive numbers c_0 and C_0 with

$$c_0|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq C_0|\xi|^2 \tag{1.2}$$

Also,

$$\begin{array}{l} a^{ij} \in C(I\!R^n) \\ b^i, c \in L^\infty(I\!R^n) \\ (\sum |b^i|^2)^{\frac{1}{2}}, |c| \leq b \end{array} \tag{1.3}$$

for some $b \geq 0$.

Denote by M the operator $M = a^{ij}\partial_{ij} + b^i\partial_i$, that is $L = M + c$.

It is well known that, in general, if the domain Ω is not smooth, there is not an existence theorem for the Dirichlet problem

*E-mail: arrieta@math.utexas.edu

$$
\begin{aligned}
Lv &= \ f, \qquad \Omega,\\
v &= \ 0, \qquad \partial\Omega.
\end{aligned}
$$

where $v = 0$ in $\partial\Omega$ means that $\lim_{x\to\partial\Omega} v(x) = 0$. It turns out that the Dirichlet condition $\lim_{x\to\partial\Omega} v(x) = 0$ is a rather strong condition.

To avoid this difficulty one has to relax the way in which the boundary condition is satisfied. Following the ideas in [6], this can be done in the following way: for the bounded domain Ω let us define a family of smooth domains H_j, $j \geq 1$, so that $\bar{H}_j \subset \Omega$, $\bar{H}_j \subset H_{j+1}$ and $\bigcup_{j\geq 1} H_j = \Omega$. Denote by u_j the solution of,

$$
\begin{aligned}
Mu_j &= \ -1, \qquad H_j,\\
u_j &= \ 0, \qquad \partial H_j.
\end{aligned}
$$

(this solution exists since H_j is smooth). It can be proved that u_j is a nondecreasing sequence which converge to a function u^Ω weakly in $W^{2,p}(J)$ and strongly in $C^1(J)$, for any compact set $J \subset \Omega$. Moreover, u^Ω is a strong solution of the problem $Mu = -1$ in Ω. It is not difficult to prove that the function u^Ω is independent of the domains H_j and only depends in Ω and the operator M.

The Dirichlet condition $v = 0$ in Ω is substituted by the weaker Dirichlet condition $v \stackrel{u^\Omega}{=} 0$, where

$$
v \stackrel{u^\Omega}{=} 0 \text{ means } v(x_j) \to 0, \text{ for any sequence } \{x_j\} \subset \Omega, \text{ with } x_j \to \partial\Omega \text{ and } u^\Omega(x_j) \to 0.
$$

We wil also use the notation $x_j \stackrel{u^\Omega}{\to} \partial\Omega$ to denote that $x_j \to \partial\Omega$ and $u^\Omega(x_j) \to 0$.

With this notion the following Refined Maximum Principle can be stated

Definition 1.1 *The operator L satisfies the RMP in Ω if $Lw \geq 0$ in Ω, w bounded above and $\limsup w(x_j) \leq 0$ if $x_j \stackrel{u^\Omega}{\to} \partial\Omega$ imply $w \leq 0$ in Ω.*

The principal eigenvalue of the operator L in a domain Ω is defined as

$$
\lambda(L, \Omega) = \sup\{\mu : \text{ there exists } \phi > 0 \text{ in } \Omega, (L+\mu)\phi \leq 0 \text{ in } \Omega\}
$$

The relation between the maximum principal and the sign of the principal eigenvalue is expressed in the following

Theorem 1.2 *The refined maximum principle holds for L in Ω if and only if $\lambda(\Omega) > 0$.*

With respect to the solutions of the equation $Lv = f$ we have

Theorem 1.3 *Assume $\lambda(L, \Omega) > 0$. Then, given $f \in L^n(\Omega)$, there is a unique solution v, with $|v|$ bounded, of*

$$
\begin{aligned}
Lv &= \ f, \quad \Omega,\\
v &\stackrel{u^\Omega}{=} \ 0, \quad \partial\Omega.
\end{aligned}
$$

Furthermore, for a constant A depending only on Ω, c_0, C_0, b and λ

$$
\|v\|_\infty \leq A\|f\|_n
$$

For a proof of these results the reader is referred to [6].

One of the main ingredients for the proof of these results is the following theorem by Alexandroff, Bakelman and Pucci.

Theorem (ABP). *If $c \leq 0$ and w satisfies $Lw \geq f$ in Ω and $\limsup_{x \to \partial\Omega} w(x) \leq 0$, then*

$$\sup_{\Omega} w \leq B\|f\|_{L^n}$$

where the constant B depends only on n, c_0, b and the diameter of Ω.

Also in [6] a refined version of this theorem is given.

Theorem (Refined ABP). *Assume $\lambda(L,\Omega) > 0$. Suppose v is a function, bounded above, satisfying*

$$Lv \geq f \quad in \quad \Omega, \quad f \leq 0, \quad f \in L^n(\Omega),$$

$$\limsup v(x_j) \leq \beta \quad if \quad x_j \overset{u^{\Omega}}{\longrightarrow} \partial\Omega, \quad \beta \geq 0.$$

Then

$$\sup v \leq A(\|f\|_{L^n} + \beta(\sup c^+)|\Omega|^{1/n}) + \beta$$

where A is the constant from Theorem 1.3.

For a proof of Theorem (ABP) see [10] and for a proof of the Refined ABP see [6].

In this paper we study the behavior of the solutions of the equation $Lv = f$, $v \overset{u^{\Omega}}{=} 0$ and the principal eigenvalue and eigenfunction of the operator L under a perturbation of the domain.

We will need to start by giving a meaning to the phrase "perturbation of the domain". If Ω is smooth the natural way to talk about a perturbed domain is to ask that in the neighborhood of each point $x \in \partial\Omega$ the boundary of Ω and the boundary of Ω_ϵ (the perturbed domain) are close in certain sense, for instance in $C^{k,\alpha}$. This can be done because for a smooth domain the boundary looks locally like the graph of certain smooth function. In general domains this cannot be done and new ideas are needed.

We propose the following way. For a domain Ω and an operator M we can construct the function u^{Ω}. Extending to zero outside of Ω we have a function defined in $I\!R^n$. We will say that Ω_ϵ converge to Ω if the function u^{Ω_ϵ} converge to the function u^{Ω} in certain norm, for instance in $L^{\infty}(I\!R^n)$.

We will show that under this concept of convergence of domains, the principal eigenvalue and eigenfunction, and the solutions of the equation $Lv = f$ in Ω, $v \overset{u^{\Omega}}{=} 0$ behave continuously.

When the operator L is the Laplace operator there are results concerning the dependence of the eigenvalues and eigenfunctions of L under different types of boundary conditions. Courant and Hilbert in [9] already established the continuous dependence of the eigenvalues of $\triangle$, with Dirichlet, Neumann or mixed boundary conditions, under certain type of perturbations of the domain. They already pointed out that the behavior with Neumann boundary conditions is more singular than under Dirichlet's. Also, for a general $2m^{th}$-order selfadjoint elliptic operator with Dirichlet boundary conditions, Babuska and Viborny in [4] proved the continuous dependence of the eigenvalues under rather general perturbations of the domain.

For other results related to the different behavior of the operator L when the domain is perturbed (continuity results or singular behavior) under different kind of boundary condition (especially Neumann) the reader is referred to the bibliography at the end of the article.

Most of the results in the above articles exploit the variational techniques associated with an operator that can be written in divergence form (as it is the case of a selfadjoint operator). In our case the operator L cannot be written in divergence form (since the coefficients a^{ij} lack enough regularity) and therefore the variational techniques cannot be used.

We describe now the contents of the paper.

Although the function u is constructed for bounded domains, it can be easily defined for bounded open sets (not neccessarily connected). If Ω is a bounded open set then $\Omega = \bigcup_{j=1}^{N} C^j$ where $n \in I\!N \cup \{\infty\}$ and C^j are the connected components of Ω. Then we can construct for each C^j the function u^{C^j} and we can define $u^{\Omega} = u^{C^j}$ in C^j for $j \geq 1$ and $u^{\Omega} = 0$ in $I\!R^n \setminus \Omega$.

In Section 2 we consider the set $\Theta = \{\Omega \subset B_1 : \Omega \text{ open}\}$, where B_1 is the unit ball in $I\!R^n$ and define the functional

$$\begin{aligned} d : \Theta \times \Theta &\longrightarrow I\!R^+ \\ (\Omega_1, \Omega_2) &\longrightarrow d(\Omega_1, \Omega_2) \equiv \|u^{\Omega_1} - u^{\Omega_2}\|_{L^\infty(B_1)} \end{aligned} \tag{1.4}$$

We would like to say that $d(\cdot, \cdot)$ is a metric in Θ but this is not true since we can have two different open sets Ω_1, Ω_2 with $u^{\Omega_1} = u^{\Omega_2}$ a.e. in $I\!R^n$ (for instance $\Omega_1 = B(p, r)$, the ball of radius r centered at a point p and $\Omega_2 = B(p, r) \setminus \{p\}$). To overcome this difficulty we will define an equivalence relation in Θ with the property that if $\Omega_1 \sim \Omega_2$ then $u^{\Omega_1} = u^{\Omega_2}$ a.e. in $I\!R^n$. Moreover, we will show that if $\Omega_1 \sim \Omega_2$ then $\lambda(\Omega_1) = \lambda(\Omega_2)$, $\varphi(\Omega_1) = \varphi(\Omega_2)$ a.e. in $I\!R^n$, where $(\lambda(\Omega), \varphi(\Omega))$ are the principal eigenpairs of L in Ω, and the bounded solutions of $Lv = f$ in Ω_1 and Ω_2 will coincide. This implies that if we define the set $\tilde{\Theta} = \Theta / \sim$ then for each equivalence class in $\tilde{\Theta}$ we have a well defined principal eigenvalue and eigenfunction and a well defined set of solutions of $Lv = f$. Moreover we will see that the space $(\tilde{\Theta}, d)$ is a complete metric space. We also include in Section 2 some examples of sequence of domains that converge in this metric.

In Section 3 we show that the principal eigenvalue and eigenfunction are continuous in the space $(\tilde{\Theta}, d(\cdot, \cdot))$ (Proposition 3.4). As a byproduct we will obtain that the constant A from Theorem 1.3 and Theorem (Refined ABP) can be choosen independent of the domain in a neighborhood (in the sense of the metric $d(\cdot, \cdot)$) of each bounded domain with a positive principal eigenvalue (Corollary 3.5)

In Section 4 we show that if v_k are the bounded solutions of $Lv_k = f_k$ in the domains $\Omega_k \in \Theta$ for $k \geq 0$ where $d(\Omega_k, \Omega_0) \stackrel{k \to \infty}{\longrightarrow} 0$, $\lambda(\Omega_0) > 0$ and $f_k \stackrel{k \to \infty}{\longrightarrow} f_0$ in L^n then $v_k \stackrel{k \to \infty}{\longrightarrow} v_0$ in certain norms (Theorem 4.1).

Throughout this paper we will use repeatedly a convergence result that we sumarize in the following

Proposition 1.4 *Let Ω be a bounded domain and let $v_k \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, be a strong solution of $Lv_k + \mu_k v_k = f_k$ in Ω, where $f_k \in L^p(\Omega)$, $\|f_k\|_{L^p(\Omega)} \leq C$, $\|v_k\|_{L^p} \leq C$ for certain constant C independent of k. Then, there exist subsequences $v_{k'}$, $f_{k'}$ and $\mu_{k'}$, functions $f \in L^p(\Omega)$, $v \in W^{2,p}_{loc}(\Omega)$ and a number μ, such that $\mu_{k'} \stackrel{k' \to \infty}{\longrightarrow} \mu$ $\int_\Omega f_{k'} \chi \stackrel{k' \to \infty}{\longrightarrow} \int_\Omega f \chi$ for any $\chi \in C_0^\infty(\Omega)$, $v_{k'} \stackrel{k \to \infty}{\longrightarrow} v$ w-$W^{2,p}(J)$ for each $J \subset\subset \Omega$ and v is a strong solution of $Lv + \mu v = f$ in Ω.*

If $p = \infty$ then $v \in W^{2,q}_{loc}(\Omega)$ for all $q > 0$ and $v_{k'} \stackrel{k \to \infty}{\longrightarrow} v$ w-$W^{2,q}(J)$ and s-$W^{1,\infty}(J)$ for each $J \subset\subset \Omega$.

Proof. The basic ingredient for the proof are interior estimates, see Theorem 9.11 in [10]. Fom this theorem we have that for any $\Omega' \subset\subset \Omega$, $\|v_k\|_{W^{2,p}(\Omega')} \le C(\|v_k\|_{L^p(\Omega)} + \|f_k\|_{L^p(\Omega)}) \le \tilde{C}$ for certain $\tilde{C}$ independent of k. This uniform bound in k implies that we can get weakly convergent subsequences in Ω'. In particular for each of the domains H_j we can get a convergent subsequence. Using a standard diagonalization procedure we can get a convergent subsequence $v_{k'}$ in w-$W^{2,p}(J)$ for any $J \subset\subset \Omega$. With a similar argumet for f_k and from the boundedness of μ_k we get the existence of subsequences $v_{k'}$, $f_{k'}$ and $\mu_{k'}$, functions $f \in L^p(\Omega)$, $v \in W^{2,p}_{loc}(\Omega)$ and a number μ, such that $\mu_{k'} \overset{k'\to\infty}{\longrightarrow} \mu$ $\int_\Omega f_{k'}\chi \overset{k'\to\infty}{\longrightarrow} \int_\Omega f\chi$ for any $\chi \in C_0^\infty(\Omega)$, $v_{k'} \overset{k\to\infty}{\longrightarrow} v$ w-$W^{2,p}(J)$ for each $J \subset\subset \Omega$.

Using the weak expression of the equation $Lv_{k'} + \mu_{k'}v_{k'} = f_{k'}$, letting $k' \to \infty$ and using the weak convergence of $v_{k'}$ and $f_{k'}$ we get that v is a strong solution of $Lv + \mu v = f$.

The case $p = \infty$ is obtained from the above argument plus some Sobolev inclusions. This proves the proposition.

Notation. Throughout this paper the letter u will always denote the funtion associated to the open set Ω as described above. When we want to stress the dependence of u on the domain we will write u^Ω. If several opens are involved, Ω, Ω^*, Ω_k, Ω^η etc, we will use the symbols u, u^*, u_k, u^η etc. Similarly λ and φ will always denote the principal eigenvalue and eigenfunction in Ω. If we need to stress the dependence on Ω or in the operator L we will write $\lambda(\Omega)$, $\varphi(\Omega)$ or $\lambda(L,\Omega)$, $\varphi(L,\Omega)$. Also we will abreviate $\lambda(\Omega^*)$ by λ^*, etc..

L will always denote the elliptic operator defined by (1.1) satisfying (1.2) and (1.3). Also, M will be the operator $M = a^{ij}\partial_{ij} + b^i\partial_i$, so that $L = M + c$. The constants c_0, C_0 and b will always denote the constants from (1.2) and (1.3).

All the open sets in this paper are bounded, and when a family of open sets are considered they will always be uniformly bounded. Hence, without loss of generality, we can assume that all open sets are contained in B_1 the unit ball in $I\!R^n$.

2 The space of open sets.

This section is divided in three subsections.

In the first one we define an operation, denoted by $*$, in the set $\Theta = \{\Omega \subset B_1 : \quad \Omega \text{ open}\}$, that is

$$\begin{aligned} * : \Theta &\longrightarrow \Theta \\ \Omega &\longrightarrow \Omega^*, \end{aligned}$$

with the property that $\Omega \subset \Omega^* \subset \bar{\Omega}$ and $u^\Omega = u^{\Omega^*}$ a.e.. Moreover Ω^* is the biggest among all open sets U with $u^\Omega = u^U$ a.e. $x \in I\!R^n$. Also we will see that $\lambda(\Omega) = \lambda(\Omega^*)$, $\varphi(\Omega) = \varphi(\Omega^*)$ and the bounded solutions of the equations $Lv = f$, $v \overset{u}{=} 0$ in Ω and $Lv^* = f$, $v^* \overset{u^*}{=} 0$ in Ω^* are the same.

In the second subsection we use the operation $*$ to define and equivalence relation $\sim$ in the set Θ by

$$\Omega_1 \sim \Omega_2 \quad \text{if} \quad \Omega_1^* = \Omega_2^*$$

Denoting $\tilde{\Theta} = \Theta/\sim$ we define a metric $d(\cdot,\cdot)$ in $\tilde{\Theta}$ that makes the space $(\tilde{\Theta}, d)$ a complete metric space.

In the third one we give some examples of sequence of open sets in Θ that exhibit the convergence induced by the metric $d(\cdot,\cdot)$.

2.1. Construction of Ω^*.

Let us start with certain results.

Lemma 2.1 *Let Ω be a bounded open set, $x_0 \in \partial\Omega$ and $\{x_k\} \subset \Omega$ a sequence satisfying $x_k \stackrel{k\to\infty}{\longrightarrow} x_0$ and $u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Let $\tilde{\Omega}$ be another bounded open set with the property that there exists a positive number ρ so that $B(x_0,\rho)\cap\Omega = B(x_0,\rho)\cap\tilde{\Omega}$. Then $\tilde{u}(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$.*

Proof. Let $\tilde{H}_j$ be the sets used for the construction of $\tilde{u}$. Choose a constant $C>0$ such that $\|u\|_{L^\infty} \leq C$ and $\|\tilde{u}\|_{L^\infty} \leq C$. Denote by $V_j = \tilde{H}_j \cap B(x_0,\frac{\rho}{2})$. Obviously $V_j \subset\subset \Omega\cap\tilde{\Omega}$. Fix $\epsilon > 0$ small enough so that $M\epsilon|x-x_0|^2 \leq \frac{1}{2}$ in $\Omega\cap B(x_0,\frac{\rho}{2})$. Denote by $h = u + \epsilon|x-x_0|^2$ so that $Mh \leq -\frac{1}{2}$ in $\Omega\cap B(x_0,\frac{\rho}{2})$. Moreover $h \geq \epsilon\frac{\rho^2}{4}$ in $\partial V_j \cap \{|x-x_0| = \frac{\rho}{2}\}$. If we denote by $\delta = \epsilon\frac{\rho^2}{4}$ and define $\xi = \frac{C}{\delta}h - \tilde{u}_j$ in V_j we trivially have that $\xi \geq 0$ in ∂V_j and $M\xi \leq \frac{-C}{2\delta}+1 \leq 0$ in V_j. From the maximum principle we get $\xi \geq 0$ in V_j which implies $\tilde{u}_j \leq \frac{C}{\delta}h$ in V_j. In other words, $\tilde{u}_j \leq \frac{C}{\delta}(u+\epsilon|x-x_0|^2)$ in V_j. Letting $j\to\infty$ we obtain $\tilde{u}(x) \leq \frac{C}{\delta}(u(x)+\epsilon|x-x_0|^2)$ for $x \in \Omega\cap B(x,\frac{\rho}{2})$. In particular $\tilde{u}(x_k) \leq \frac{C}{\delta}(u(x_k)+\epsilon|x_k-x_0|^2) \to 0$ when $k\to\infty$. This proves the Lemma.

Definition 2.2 *Let Ω and $\tilde{\Omega}$ be open sets and K an arbitrary set. We say that $\Omega \stackrel{K}{=} \tilde{\Omega}$ if for every $x \in K$ there exists a $r(x) > 0$ such that $\Omega\cap B(x,r(x)) = \tilde{\Omega}\cap B(x,r(x))$.*

We can prove now the following,

Lemma 2.3 *Let Ω be a bounded open set. Define the set*

$$\Gamma_\Omega = \{x \in \partial\Omega : s.t.\ there\ exists\ x_k \in \Omega,\ with\ x_k \stackrel{k\to\infty}{\longrightarrow} x,\ and\ u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0\}$$

Let $\tilde{\Omega}$ be another bounded open set with $\Omega \stackrel{\Gamma_\Omega}{=} \tilde{\Omega}$. Let $\{x_k\} \subset \Omega$ be a sequence with $u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Then there exists a k_0 such that $x_k \in \tilde{\Omega}$ for $k \geq k_0$ and $\tilde{u}(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$.

Proof. Let us see first that there exists a k_0 so that $x_k \in \tilde{\Omega}$ for $k \geq k_0$. If this is not the case we can take a convergent subsequence, that we will call it x_k again such that $x_k \notin \tilde{\Omega}$ for all k and $x_k \stackrel{k\to\infty}{\longrightarrow} \bar{x}$ for certain $\bar{x}$. Since $u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$ and $x_k \in \Omega$ we have that $\bar{x} \in \partial\Omega$. Hence from Definition 2.2 we obtain that $\bar{x} \in \Gamma_\Omega$. Therefore there exists $r(\bar{x}) > 0$ with $\Omega\cap B(\bar{x},r(\bar{x})) = \tilde{\Omega}\cap B(\bar{x},r(\bar{x}))$. This contradicts the fact that $x_k \notin \tilde{\Omega}$.

Let us see now that $\tilde{u}(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Similarly, if this is not the case we can take a convergent subsequence that we will call it again x_k, a positive number α and a $\bar{x}$ so that $x_k \stackrel{k\to\infty}{\longrightarrow} \bar{x}$, and $\tilde{u}(x_k) \geq \alpha$. As above, $\bar{x} \in \Gamma_\Omega$. But from Lemma 2.1 we get that $\tilde{u}(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$, in contradiction with the existence of $\alpha > 0$ satisfying $\tilde{u}(x_k) > \alpha$. This proves the Lemma.

Now we can define the operation $*$.

Definition 2.4 *For any bounded open set we will denote $\Omega^* = \bar{\Omega} \setminus \Gamma_\Omega$*

In order to study the different properties of Γ_Ω and Ω^* we still need some aditional results.

Lemma 2.5 *Let Ω be a bounded open set. Let $x_0 \in \partial\Omega$ such that there exist positive numbers α, r with $u(x) > \alpha$ in $B(x_0, r) \cap \Omega$. Then if $\tilde{\Omega} = \Omega \cup B(x_0, \frac{r}{2})$ we have $\tilde{u} = u$ in Ω. In particular u admits a C^1 extension to a neighborhood of x_0.*

Proof. If $x \in \Gamma_\Omega$ it is easy to see that we have $|x - x_0| \geq r$. This implies that $B(x, \frac{r}{2}) \cap \Omega = B(x, \frac{r}{2}) \cap \tilde{\Omega}$ for all $x \in \Gamma_\Omega$. In other words $\Omega \overset{\Gamma_\Omega}{=} \tilde{\Omega}$.

From Lemma 2.3 we obtain that if $x_k \xrightarrow{u} \partial\Omega$ then $\tilde{u}(x_k) \overset{k\to\infty}{\longrightarrow} 0$. In other words $\tilde{u} \overset{u}{=} 0$ in Ω. Hence the function $\tilde{u}$ satisfies

$$\begin{aligned} M\tilde{u} = &\ -1, \qquad \Omega, \\ \tilde{u} \overset{u}{=} &\ 0. \end{aligned}$$

By uniqueness of solutions, we get $\tilde{u} = u$ in Ω. The Lemma is proved.

Lemma 2.6 *Let B be domain and Ω a bounded open set with $B \cap \Omega \neq \emptyset$. Assume there exists an $\alpha > 0$ such that $u(x) > \alpha$ for $x \in B \cap \Omega$. Then $B \subset \bar{\Omega}$.*

Proof. Assume it is not true, then there exist two points $a \in B \setminus \bar{\Omega}$ and $b \in B \cap \Omega$. By connectedness there exists an arc $\gamma \subset B$ connecting a and b. By compactness of γ and openness of B there exists a positive number r so that $B(z, r) \subset B$ for all $z \in \gamma$, $B(a, r) \subset B \setminus \bar{\Omega}$ and $B(b, r) \subset B \cap \Omega$. Again by connectedness there exists a point $z_0 \in \gamma$ so that $B(z_0, r) \subset B \setminus \bar{\Omega}$ but $\partial B(z_0, r) \cap \partial\Omega \neq \emptyset$. Hence, if $y \in \partial B(z_0, r) \cap \partial\Omega$ then y is a barrier point since it admits an exterior sphere condition. Therefore u admits a continuous extension to y with $u(y) = 0$. This is in contradiction with the assumption that $u > \alpha$ in B.

Proposition 2.7 *We have the following,*
(i). Γ_Ω is closed.
(ii). Ω^* is open.
(iii). $\Omega \subset \Omega^* \subset \bar{\Omega}$.
(iv). $\partial\Omega^* = \Gamma_\Omega$
(v). $\partial\Omega \setminus \partial\Omega^* \subset \Omega^*$.

Proof. A diagonalization procedure proves (i) inmediately.

To prove (ii), we observe first that $\Omega \subset \Omega^*$. Hence it is enough to see that if $x \in \Omega^* \setminus \Omega$ then there is a neighborhood of x contained in Ω^*. If $x \in \Omega^* \setminus \Omega$ then $x \in \partial\Omega \setminus \Gamma_\Omega$. In particular there exist α and ρ positive so that $u(y) > \alpha$ in $B(x, \rho) \cap \Omega$. Now, from Lemma 2.6 we get that $B(x, \rho) \subset \bar{\Omega}$ and obviously $B(x, \rho) \cap \Gamma_\Omega = \emptyset$. This implies that $B(x, \rho) \subset \Omega^*$ and Ω^* is open.

The proofs of (iii), (iv) and (v) are trivial.

We can prove now the following important result,

Proposition 2.8 *Let Ω be a bounded open set and construct the open set Ω^* as above. Denote by u and u^* the functions associated with Ω and Ω^*, respectively. Then*
$u = u^*$ in Ω.

Proof Since $\Omega \subset \Omega^*$ we get $u \leq u^*$ in Ω.

Let $K \subset\subset \Omega \subset \Omega^*$. Let $\{H_j^*\}$ be the domains used for the definition of u^* and let j_0 be so that $K \subset\subset H_{j_0}^*$. Since $H_j^* \subset\subset \Omega^*$, we have that for all $x \in \partial\Omega^* = \Gamma_\Omega$ there exists an $r(x)$ with $B(x, r(x)) \cap H_j^* = \emptyset$. This implies that $\Omega \overset{\Gamma_\Omega}{=} \Omega \cup H_j^*$ and from Lemma 2.5 we get that $u^{\Omega \cup H_j^*} = u$ in Ω. But since $u^{\Omega \cup H_j^*} \geq 0$ in ∂H_j^*, using the maximum principle we get $u^{\Omega \cup H_j^*} \geq u_j^*$ in H_j^* or equivalently $u \geq u_j^*$ in H_j^*. Hence if $j \geq j_0$ we get $u \geq u_j^*$ in K and letting $j \to \infty$ we get $u \geq u^*$ in K. Therefore $u = u^*$ in Ω. This completes the proof.

Proposition 2.9 *Let Ω be a bounded open set and let $\tilde{\Omega}$ another open set satisfying $\Omega \subset \tilde{\Omega} \subset \bar{\Omega}$, and $u = \tilde{u}$ in Ω. Then $\Omega \subset \Omega^*$. (In other words, Ω^* is the biggest of all open sets $\tilde{\Omega}$ containing Ω and contained in $\bar{\Omega}$ whith the property that $u = \tilde{u}$ in Ω).*

Proof. If $x \in \tilde{\Omega}$ then $\tilde{u}(x) > 0$. Hence there exist r and α positive numbers so that $\tilde{u}(y) > \alpha$ in $B(x, r) \subset \tilde{\Omega}$. Therefore $u(y) > \alpha$ in $B(x, r) \cap \Omega$. This means that $x \notin \Gamma_\Omega$. Hence $x \in \tilde{\Omega} \setminus \Gamma_\Omega \subset \Omega^*$.

We also have more properties,

Proposition 2.10 *Let Ω, Ω_1, Ω_2, Ω_i be bounded open sets. Then,*
(i). If $\Omega_1 \subset \Omega_2$ then $\Omega_1^* \subset \Omega_2^*$.
(ii). $(\Omega^*)^* = \Omega^*$.
(iii). $\Omega_1 \cap \Omega_2 = \emptyset$ if and only if $\Omega_1^* \cap \Omega_2^* = \emptyset$.
(iv). If Ω_1 and Ω_2 are disjoint, then $(\Omega_1 \cup \Omega_2)^* = \Omega_1^* \cup \Omega_2^*$.
(v). Ω is connected if and only if Ω^* is connected.
(vi). If $\Omega = \bigcup_{i=1}^\infty \Omega_i$ where Ω_i are pairwise disjoint opens then $\Omega^* = \bigcup_{i=1}^\infty \Omega_i^*$

Proof. We will denote by u, u_1, u_2, u_i the canonical functions associated to the open sets Ω, Ω_1, Ω_2, Ω_i.
(i). Obviously $u_1 \leq u_2$. If $x \in \Omega_1^*$, there exist $r > 0$ and $\alpha > 0$ with $B(x, r) \subset \bar{\Omega}_1$ and $u_1(B(x, r) \cap \Omega_1) > \alpha$ which implies $u_2(B(x, r) \cap \Omega_1) > \alpha$. Let us see that $u_2(B(x, r) \cap \Omega_2) \geq \alpha$. If this is not the case then there exists $y \in B(x, r) \cap \Omega_2$ with $u_2(y) < \alpha$ and by continuity $u_2(B(y, \rho)) < \alpha$ for certain $\rho > 0$. Choosing ρ small enough so that $B(y, \rho) \subset B(x, r)$ we get that $B(y, \rho) \cap \Omega_1 = \emptyset$. This is in contradiction with the fact that $B(x, r) \subset \bar{\Omega}_1$.
(ii). Since $\Omega \subset \Omega^*$ we get from (i) that $\Omega^* \subset (\Omega^*)^*$. Moreover $\Omega \subset \Omega^* \subset (\Omega^*)^* \subset \text{clos}(\Omega^*) \subset \bar{\Omega}$. Moreover, from Proposition 2.8 we get that $u = u^* = u^{**}$ in Ω. Hence from Proposition 2.9, taking $\tilde{\Omega} = (\Omega^*)^*$, we get $(\Omega^*)^* \subset \Omega^*$.
(iii). This is straightforward.
(iv). Using (i) it is easy to prove that $\Omega_1^* \cup \Omega_2^* \subset (\Omega_1 \cup \Omega_2)^*$. Since Ω_1 and Ω_2 are disjoint we know that $u_{1,2}$ (the function associated to $\Omega_1 \cup \Omega_2$) coincides with u_1 in Ω_1 and with Ω_2 in Ω_2. This implies that $\Gamma_{\Omega_1 \cup \Omega_2} = \Gamma_{\Omega_1} \cup \Gamma_{\Omega_2}$. But, by definition $(\Omega_1 \cup \Omega_2)^* = clos(\Omega_1 \cup \Omega_2) \setminus \Gamma_{\Omega_1 \cup \Omega_2} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \setminus (\Gamma_{\Omega_1} \cup \Gamma_{\Omega_2}) \subset (\bar{\Omega}_1 \setminus \Gamma_{\Omega_1}) \cup (\bar{\Omega}_2 \setminus \Gamma_{\Omega_2}) = \Omega_1^* \cup \Omega_2^*$
(v). Since $\Omega \subset \Omega^* \subset \bar{\Omega}$ we obtain that if Ω is connected then Ω^* is also connected. If Ω is disconnected then $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 nonempty disjoint open sets. From (iv), $\Omega^* = \Omega_1^* \cup \Omega_2^*$ and from (iii) Ω_1^* and Ω_2^* are disjoint. Then Ω^* is not connected.
(vi). Let us assume first that Ω_i are connected for all $i \in N$. If we denote by $\hat{\Omega}_i = \Omega \setminus \Omega_i$ so that $\hat{\Omega}_i$ is open, we get from (iv) that $\Omega^* = \Omega_i^* \cup \hat{\Omega}_i^*$ and from (iii) $\Omega_i^* \cap \hat{\Omega}_i^* = \emptyset$. This implies

that Ω_i^* is an open connected component of Ω^*. Moreover, since $\Omega_i \subset \Omega$ we get $\Omega_i^* \subset \Omega^*$ for all $i \in N$ so that $\bigcup_{i=1}^{\infty} \Omega_i^* \subset \Omega^*$. Hence if this inclusion is strict it means that there exists another open connected component C of Ω^* and therefore $C \cap (\bigcup_{i=1}^{\infty} \Omega_i^*) = \emptyset$. But since C is open and $C \subset \Omega^*$ we obtain $C \cap \Omega \neq \emptyset$ so that there exists $i \in N$ with $C \cap \Omega_i \neq \emptyset$. This implies that $C \cap \Omega_i^* \neq \emptyset$ for certain i, in contradiction with $C \cap (\bigcup_{i=1}^{\infty} \Omega_i^*) = \emptyset$. For the general case ($\Omega_i$ not neccesarily connected) we know that each open set Ω_i can be decomposed into a countable union of connected components, so that $\Omega_i = \bigcup_{k=1}^{N_i} \Omega_i^k$ where Ω_i^k is connected, and $N_i \in N \cup \{\infty\}$. Hence, $\Omega_i^* = \bigcup_{k=1}^{N_i} (\Omega_i^k)^*$ and $\Omega^* = (\bigcup_{i=1}^{\infty} \bigcup_{k=1}^{N_i} \Omega_i^k)^* = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{N_i} (\Omega_i^k)^* = \bigcup_{i=1}^{\infty} \Omega_i^*$. This concludes the proof of the Proposition.

Remark 2.11 It is important to mention that the conclusions of (iv) and (vi) are not true in general if Ω_1 and Ω_2 are not disjoint. To see this just consider the domains in $I\!R^2$ given by $\Omega_1 = \{(r, \theta) : 0 < r < 1, -\frac{3}{4}\pi < \theta < \frac{3}{4}\pi\}$ and $\Omega_2 = \{(r, \theta) : 0 < r < 1, \frac{\pi}{4} < \theta < \frac{7}{4}\pi\}$. Obviously $\Omega_1 = \Omega_1^*$ and $\Omega_2 = \Omega_2^*$ but $\Omega_1 \cup \Omega_2 = B(0,1) \setminus \{(0,0)\}$ and $(B(0,1) \setminus \{(0,0)\})^* = B(0,1)$.

One of the main results of the section is the following

Theorem 2.12 *Let Ω be a bounded open set and denote by $\{\Omega_i\}_{i \geq 1}$ the connected components of Ω, so that $\Omega = \bigcup_{i \geq 1} \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Let L be the operator $L = M + c(x)$ and denote by $\{\lambda_i, \varphi_i\}_{i \geq 1}$, $\{\lambda_i^*, \varphi_i^*\}_{i \geq 1}$ the principal eigenpairs of L in Ω_i and Ω_i^*, respectively. Then*
(i). $|\Omega^* \setminus \Omega| = 0$ and therefore $u^{\Omega} = u^{\Omega^*}$ a.e. in $I\!R^n$.
(ii). For all $i \geq 1$, $\lambda_i = \lambda_i^*$ and if we normalize φ_i and φ_i^* by requiring that $\varphi_i(p_i) = \varphi_i^*(p_i)$ for certain $p_i \in \Omega_i$ then $\varphi_i = \varphi_i^*$ a.e. in $I\!R^n$.
(iii). If $\mathcal{U}$ is an open set such that $u^{\Omega} = u^{\mathcal{U}}$ a.e. in $I\!R^n$, then $\mathcal{U}^* = \Omega^*$.

Proof. From Proposition 2.10 we have that $\Omega^* = \bigcup_{i \geq 1} \Omega_i^*$. This implies that $u^{\Omega^*}_{|\Omega_i^*} = u^{\Omega_i^*}$

Fix $i \geq 1$. Let us see first that $\lambda_i = \lambda_i^*$ and that $\varphi_i = \varphi_i^*$ in Ω_i. Since Ω_i is connected, we know from Proposition 2.10 that Ω_i^* is also connected.

By definition,

$$\lambda_i = \sup\{\mu : \text{ there exists } \varphi > 0 \text{ in } \Omega_i, (L + \mu)\varphi \leq 0 \text{ in } \Omega_i\}$$

Since $\Omega_i \subset \Omega_i^*$ we have by monotonicity $\lambda_i \geq \lambda_i^*$. Moreover, $(L + \lambda_i)\varphi_i^* = (L + \lambda_i^*)\varphi_i^* + (\lambda_i - \lambda_i^*)\varphi_i^* \geq 0$ in Ω_i. Also, there exists a positive constant c such that $\varphi_i^* \leq c u^{\Omega_i^*}$ in Ω_i^* and with Proposition 2.9 we get that $\varphi_i^* \leq c u^{\Omega_i}$ in Ω_i. Noting that φ_i^* is bounded we obtain that φ_i^* is a constant multiple of the principal eigenfunction in Ω_i (see [6] Corollary 2.2). Hence $\varphi_i^* = c\varphi_i$ and with the normalization condition we get $\varphi_i^* = \varphi_i$ in Ω_i, and therefore $\lambda_i = \lambda_i^*$.

From this result and Theorem 2.4 of [6] we get that $\Omega_i^* \setminus \Omega_i$ has zero Lebesgue measure and therefore $\Omega^* \setminus \Omega$ has also zero Lebesgue measure. Hence $\varphi_i^* = \varphi_i$ a.e.. Moreover, from Proposition 2.8 we obtain that $u^{\Omega_i} = u^{\Omega_i^*}$ a.e. and therefore $u^{\Omega} = u^{\Omega^*}$ a.e.. This proves (i) and (ii).

Let us prove (iii). If $u^{\mathcal{U}} = u^{\Omega}$ a.e. then it is easy to see that $\bar{\mathcal{U}}^* = \bar{\Omega}^*$ and therefore $\text{clos}(\mathcal{U}^* \cap \Omega^*) = \bar{\mathcal{U}}^* = \bar{\Omega}^*$. Hence $\mathcal{U}^* \cap \Omega^* \subset \mathcal{U}^* \subset \text{clos}(\mathcal{U}^* \cap \Omega^*)$. From Proposition 2.9 we obtain $\mathcal{U}^* \subset (\mathcal{U}^* \cap \Omega^*)^*$ But obviously $(\mathcal{U}^* \cap \Omega^*)^* \subset (\mathcal{U}^*)^* = \mathcal{U}^*$. This implies that $(\mathcal{U}^* \cap \Omega^*)^* = \mathcal{U}^*$. A similar argument will show that $(\mathcal{U}^* \cap \Omega^*)^* = \Omega^*$. Therefore $\mathcal{U}^* = \Omega^*$. This concludes the proof of the theorem.

With respect to the solutions of the equation $Lv = f$ we have the following,

Theorem 2.13 *Let Ω be a bounded open set and let $f \in L^n(\Omega) = L^n(\Omega^*)$. Define,*

$$C(f,\Omega) = \{v : L^\infty - \textit{solution of } Lv = f, \textit{ in } \Omega, \, v \overset{u}{=} 0\}$$

$$C(f,\Omega^*) = \{v^* : L^\infty - \textit{solution of } Lv^* = f, \textit{ in } \Omega^*, \, v^* \overset{u^*}{=} 0\}$$

Then $C(f,\Omega) = C(f,\Omega^*)$.

Proof. Since $\Omega \subset \Omega^*$ and $u = u^*$ in Ω, it is easily seen that $C(f,\Omega^*) \subset C(f,\Omega)$. Now, let $v \in C(f,\Omega)$. Choose a constant N big enough so that $\lambda(\Omega, L-N)$, the principal eigenvalue of $L-N$ in Ω, is positive. Denote by $\hat{f}(x) = f(x) - Nv(x) \in L^\infty(\Omega)$. Then v is the unique bounded solution of

$$\begin{aligned} Lv - Nv = {} & \tilde{f}, \qquad \Omega, \\ v \overset{u}{=} {} & 0. \end{aligned} \tag{2.1}$$

But since $\lambda(\Omega^*, L-N) = \lambda(\Omega, L-N) > 0$, we have the existence and uniqueness of a solution of,

$$\begin{aligned} Lv^* - Nv^* = {} & \tilde{f}, \qquad \Omega^*, \\ v^* \overset{u^*}{=} {} & 0. \end{aligned}$$

Again, since $\Omega \subset \Omega^*$ and $u = u^*$ in Ω, we have that v^* is also a solution of (2.1). This implies that $v = v^*$, which proves the proposition.

2.2. The metric space $(\tilde{\Theta}, d)$. As we said earlier, we introduce in the set Θ the equivalence relation $\sim$ defined by: $\Omega_1 \sim \Omega_2$ if $\Omega_1^* = \Omega_2^*$. With some abuse of notation we will denote by Ω^* the equivalence class generated by Ω. In $\tilde{\Theta} \equiv \Theta/\sim$ we define a metric d by $d(\Omega_1^*, \Omega_2^*) = \|u_1^* - u_2^*\|_{L^\infty}$.

Let us start with a lemma.

Lemma 2.14 *Let Ω be a bounded open set. For $\eta > 0$ define $\Omega^\eta = \{x \in \Omega : u^\Omega(x) > \eta\}$. Then $u^{\Omega^\eta} = u^\Omega - \eta$ in Ω^η.*

Proof. Notice that in general $\bar{\Omega}^\eta$ is not contained in Ω. Let H_j, u_j be the domains and functions used for the construction of u^Ω. Consider $H_j^\eta = \{x \in H_j : u_j(x) > \eta\}$. Since H_j is smooth we have that $\bar{H}_j^\eta \subset H_j$ and $\partial H_j^\eta = \{x \in \bar{H}_j^\eta : u_j(x) = \eta\}$. Moreover $\cup_{j\geq 1} H_j^\eta = \Omega^\eta$. Since $M(u_j - \eta) = -1$ in H_j^η and $u_j - \eta = 0$ in ∂H_j^η we have from the maximum principle that $0 \leq u_j - \eta \leq u^{\Omega^\eta}$ in H_j^η. Letting $j \to \infty$ we get that $0 \leq u^\Omega - \eta \leq u^{\Omega^\eta}$ and $M(u^\Omega - \eta) = -1$. By uniqueness of solutions we get that $u^\Omega - \eta = u^{\Omega^\eta}$. This proves the lemma.

We have the following result,

Proposition 2.15 *The space $(\tilde{\Theta}, d(\cdot,\cdot))$ is a complete metric space.*

Proof. To check that $d(\cdot,\cdot)$ is actually a metric is easy. Notice that if $d(\Omega_1^*, \Omega_2^*) = 0$ then $u_1^* = u_2^*$ a.e.. Hence, from Theorem 2.12, $\Omega_1^* = \Omega_2^*$.

Now we check for completeness. Let $\{\Omega_k^*\}$ be a cauchy sequence in $\tilde{\Theta}$. Then $\{u_k^*\}$ is a cauchy sequence in $L^\infty(B_1)$ and therefore there exists a function $v \in L^\infty(B_1)$ so that $u_k^* \overset{k\to\infty}{\longrightarrow} v$ in

$L^\infty(B_1)$. Let us see that there is an open set Ω so that $v = u^\Omega$. Our candidate for Ω will be the following:

$$\Omega = \{x \in B_1 : \text{s.t. there exists } \rho = \rho(x),\ \alpha = \alpha(x) \text{ with } v(y) > \alpha, \text{ a.e. } y \in B(x,r)\}$$

Obviously Ω is open and contained in B_1. Since $u_k^* \overset{k\to\infty}{\longrightarrow} v$ in L^∞ we have that there exists a $k_0 = k_0(x)$ such that if $k \geq k_0$ then $u_k^*(y) \geq \alpha/2$ for a.e. $y \in B(x,\rho)$. This implies that $B(x,\rho) \subset \Omega_k^*$ for $k \geq k_0$. Also since u_k^* are strong solutions of $Mu_k^* = -1$ in Ω_k^*, from Proposition 1.4 we get that v is a strong solution of $Mv = -1$ in Ω. We only need to check now the boundary condition.

Let η be a small positive parameter and define the opens $\Omega_k^\eta = \{x \in \Omega_k : u_k^*(x) > \eta\}$. Choose $k_0 = k_0(\eta)$ such that if $k \geq k_0$ then $\|u_k - v\|_{L^\infty} \leq \frac{\eta}{2}$. It is easy to see that $\Omega_k^\eta \subset \Omega$ for $k \geq k_0$ which implies that $u^{\Omega_k^\eta} \leq u^\Omega$. From Lemma2.14 we have that $u_k - \eta \leq u^\Omega$. Letting $k \to \infty$ and since η is arbitrary we get that $v \leq u^\Omega$. This implies that v is a strong solution of $Mv = -1$ in Ω and satisfies $0 \leq v \leq u^\Omega$. By uniqueness we have that $v = u^\Omega$. This concludes the proof.

2.3. Examples.

In this subsection we will show some examples were the convergence of domains is fullfilled. We will need the following estimate from [10],

Theorem (Estimates at the Boundary). *Let $v \in W^{2,n}(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lv \geq f$ in Ω, $v \leq 0$ in $B \cap \partial\Omega$ where $B = B_{2R}(y)$ is a ball in $I\!R^n$ and $f \in L^n$. Then for any $p > 0$ we have,*

$$\sup_{\Omega\cap B_R(y)} v \leq C\Big(\Big(\frac{1}{|B|}\int_{B\cap\Omega} |v^+|^p\Big)^{\frac{1}{p}} + \frac{R}{c_0}\|f\|_{L^n(B\cap\Omega)}\Big) \tag{2.2}$$

where $C = C(n, c_0, C_0, R^2, p)$.

Remark 2.16 We want to apply the previous theorem to the function u^Ω associated to a bounded domain $\Omega \subset B_1$ not neccessarily smooth. Notice that $u = u^\Omega$ satisfies $Mu = -1$ but in general u is not continuous at the boundary. If we approximate u by the functions u_j defined in $H_j \subset\subset \Omega$ we know that $u_j \leq u$ and H_j is smooth so that $u_j \in W^{2,n}(H_j) \cap C^0(\bar{H}_j)$. Moreover, although in general u_j does not converge to u in L^∞, we will always have that if A is a bounded open, then $\sup_{A\cap H_j} u_j \overset{j\to\infty}{\longrightarrow} sup_{A\cap\Omega} u$. Applying (2.2) to u_j we get

$$\sup_{H_j\cap B_R(y)} u_j \leq C\Big(\Big(\frac{1}{|B|}\int_{B\cap H_j} |u_j|^p\Big)^{\frac{1}{p}} + \frac{R}{c_0}\|1\|_{L^n(B\cap H_j)}\Big) \leq$$

$$C\Big(\Big(\frac{1}{|B|}\int_{B\cap\Omega} |u|^p\Big)^{\frac{1}{p}} + \frac{R}{c_0}\|1\|_{L^n(B\cap\Omega)}\Big) \leq C\Big(\|u^{B_1}\|_\infty\Big(\frac{|B\cap\Omega|}{|B|}\Big)^{\frac{1}{p}} + \frac{R}{c_0}|B\cap\Omega|^{\frac{1}{n}}\Big)$$

¿From this,by choosing $p = n$, we easily obtain

$$\sup_{\Omega\cap B_R(y)} u \leq C|B_{2R}(y) \cap \Omega|^{\frac{1}{n}} \tag{2.3}$$

where $C = C(n, c_0, C_0, R,)$.

Example 2.17 *Let Ω_k be a sequence of open sets such that $\Omega_k \subset B_1$ and $|\Omega_k| \stackrel{k\to\infty}{\longrightarrow} 0$. Then $u_k \stackrel{k\to\infty}{\longrightarrow} 0$ in $L^\infty(B_1)$.*

This is a straightforward application of (2.3).

Example 2.18 *Let $\Omega \subset B_1$ be a bounded open set which satisfies an exterior cone condition at each point of its boundary. Let $\Omega_k \subset\subset B_1$ be a sequence of open sets satisfying,*
(i). If $J \subset\subset \Omega$, there exists a $k_0 = k(J)$ such that $J \subset\subset \Omega_k^*$ for all $k \geq k_0$.
(ii). $|\Omega_k \setminus \Omega| \stackrel{k\to\infty}{\longrightarrow} 0$.
Then $u_k \stackrel{k\to\infty}{\longrightarrow} u$ in $L^\infty(B_1)$.

Remark 2.19 *Notice that many cases studied in the literature are particular cases of Example 2.18. For instance, the case of a smooth perturbation of a domain, the case of a dumbbell domain or even the case studied in [4].*

From (2.3) and (ii) we immediately get that

$$\text{if } \; K \subset\subset I\!R^n \setminus \bar{\Omega} \text{ then } \|u_k\|_{L^\infty(K)} \stackrel{k\to\infty}{\longrightarrow} 0 \tag{2.4}$$

From Proposition 1.4 and from (i), we also have that there exists a function ϕ so that $u_k \stackrel{k\to\infty}{\longrightarrow} \phi$ w-$W^{2,n}(J)$ and s-$W^{1,\infty}(J)$ for each $J \subset\subset \Omega$. Moreover ϕ is a strong solution of $M\phi = -1$ in Ω. We need to show that ϕ satisfies a Dirichlet condition and that $u_k \stackrel{k\to\infty}{\longrightarrow} \phi$ in $L^\infty(B_1)$.

We will show the following claim,

Claim. *For each point $x \in \partial\Omega$ and for each $\epsilon > 0$ there exists $\eta > 0$ and k_0 such that $u_k \leq \epsilon$ in $B(x, \eta)$ for all $k \geq k_0$.*

Assuming the claim is proved, by compactness of $\partial\Omega$ we have that for each $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of $\partial\Omega$ and a k_0 such that $u_k(x) \leq \epsilon$ in $\mathcal{U}$ for $k \geq k_0$. This implies that $\phi = 0$ in $\partial\Omega$ and therefore by uniqueness we have $\phi = u$. Also we deduce that $u_k \stackrel{k\to\infty}{\longrightarrow} u$ in L^∞.

Proof of the Claim. We define the cone with vertex at the point $x \in I\!R^n$ with axis $w \in I\!R^n$, $|w| = 1$ of height $\rho > 0$ and with opening $\alpha \in (0, \pi)$ by

$$C(x, w, \rho, \alpha) = \{y \in I\!R^n : |y - x| < \rho, \; (y - x) \cdot w > |y - x| \cos(\alpha)\}$$

Also we define $D(x, w, \rho, \alpha) = B(x, \rho) \setminus \bar{C}(x, w, \rho, \alpha)$

In [18] the following property is proved: for fixed α and ρ, there exists a function $f(r, \theta)$ for $0 < r < \rho$ and $\theta \in (\alpha, \pi]$ such that if

$$h_{x,w}(y) = f(|y - x|, \cos^{-1}(\frac{(y - x) \cdot w}{|y - x|}))$$

then $h_{x,w}(0) = 0$ and

$$h_{x,w} > 0, \; h_{x,w} \in C^2, M h_{x,w} \leq -1 \qquad \text{in } D(x, w, \rho, \alpha)$$

Notice that the function f is independent of x and w.

Consider a point $x \in \partial\Omega$. Since Ω has the exterior cone property, we have that there exists $w \in I\!R^n$, $|w| = 1$, $\rho > 0$ and $\alpha \in (0, \pi/2)$ such that $C(x, w, 2\rho, 2\alpha) \subset I\!R^n \setminus \Omega$. Consider the function $f(r, \theta)$ constructed above for ρ and α. Choose $\eta \in (0, \rho)$ small enough so that $f(r, \theta) \leq \epsilon$ in $0 < r < 2\eta$, $\theta \in (\alpha, \pi]$. Define $x_\eta = x + \eta w$ and consider the cones $C(x_\eta, w, \rho, \alpha) \subset C(x_\eta, w, \rho, 2\alpha) \subset\subset C(x, w, 2\rho, 2\alpha) \subset I\!R^n \backslash \Omega$. From (2.4) we get that $u_k \stackrel{k\to\infty}{\longrightarrow} 0$ in $L^\infty(C(x_\eta, w, \rho, 2\alpha))$. Let δ be a small parameter and choose $k_0 = k_0(\delta)$ so that $u_k \leq \delta$ in $C(x_\eta, w, \rho, 2\alpha)$ for $k \geq k_0$.

Denote by $s = \min\{f(r, \theta) : r = \rho,\ \theta \in [2\alpha, \pi]\} > 0$. If $m = \max\{\|u^{B_1}\|_\infty / s, 1\}$ and we define the function $\tilde{h} = m h_{x_\eta, w} + \delta$, we have that $\tilde{h} \geq u_k$ in $\partial D(x_\eta, w, \rho, \alpha)$ for $k \geq k_0$. For a fixed $k \geq k_0$ consider the domains H_j^k and the functions $(u_k)_j$, $j \geq 1$, used to define u_k. We know that $(u_k)_j \leq u_k$ in H_j^k for all $j \geq 1$. In particular $\tilde{h} \geq (u_k)_j$ in $\partial D(x_\eta, w, \rho, \alpha)$ and since $(u_k)_j$ satisfies a strong Dirichlet condition in ∂H_j^k we have that $\tilde{h} \geq (u_k)_j$ in $\partial(H_j^k \cap D(x_\eta, w, \rho, \alpha))$. We also have that $M(\tilde{h} - (u_k)_j) \leq 0$ in $H_j^k \cap D(x_\eta, w, \rho, \alpha)$. The maximum principle implies now that $\tilde{h} \geq (u_k)_j$ in $H_j^k \cap D(x_\eta, w, \rho, \alpha)$. Letting $j \to \infty$ we obtain that $\tilde{h} \geq u_k$ in $D(x_\eta, w, \rho, \alpha) \cap \Omega_k$ for $k \geq k_0$.

In $B(x_\eta, 2\eta)$ we have that $\tilde{h} \leq m\delta + \delta$. This implies that $u_k \leq m\delta + \delta$ in $B(x, \eta) \subset B(x_\eta, 2\eta)$ for $k \geq k_0$. Choosing $\delta = \epsilon/(m+1)$, we prove the claim.

3 Eigenvalue Behavior.

In this Section we want to study the dependence of the principal eigenvalue and eigenfunction on the domain Ω.

Let us start with a simple proposition.

Proposition 3.1 *For all bounded domain Ω we have*

$$\varphi(x) \leq (|\lambda| + b)\|\varphi\|_{L^\infty(\Omega)} u(x)$$

where $\{\lambda, \varphi\}$ is the principal eigenpair of the operator L in Ω.

Proof. Obviously $\{\lambda, \varphi\}$ satisfies

$$\begin{aligned} L\varphi + \lambda\varphi = &\ 0, \qquad \Omega, \\ \varphi \stackrel{u}{=} &\ 0. \end{aligned}$$

Easy computations show that

$$\begin{aligned} M(\varphi - (b + |\lambda|)\|\varphi\|_{L^\infty(\Omega)} u(x) \geq &\ 0, \qquad \Omega, \\ \varphi - (b + |\lambda|)\|\varphi\|_{L^\infty(\Omega)} u(x) \stackrel{u}{=} &\ 0. \end{aligned}$$

The refined maximum principle proves the Proposition.

A simple corollary can be obtained easily,

Corollary 3.2 *If Ω_k is a sequence of domains such that $u_k \stackrel{k\to\infty}{\longrightarrow} 0$ in L^∞ then the principal eigenvalue λ_k satisfies $\lambda_k \stackrel{k\to\infty}{\longrightarrow} +\infty$.*

We are going to need the following results.

Lemma 3.3 *Let Ω be a bounded domain with $\lambda(\Omega) > 0$.*
(i). There exists a constant $D = D(c_0, C_0, b, \Omega, \lambda(\Omega))$ such that if $\hat{\Omega}$ is a bounded domain $\hat{\Omega} \subset \Omega$ and $\hat{v}$ is the strong solution of

$$\begin{aligned} L\hat{v} = & \ \hat{f} \quad \hat{\Omega} \\ \hat{v} \overset{\hat{u}}{=} & \ 0 \end{aligned}$$

for $\hat{f} \in L^n(\hat{\Omega})$, then $\|\hat{v}\|_{L^\infty(\hat{\Omega})} \le D\|\hat{f}\|_{L^n(\hat{\Omega})}$.
(ii). Let $\hat{\Omega}$ be a bounded domain $\hat{\Omega} \subset \Omega$. Suppose $\hat{v}$ is a function, bounded above, satisfying

$$L\hat{v} \ge \hat{f} \quad in \quad \hat{\Omega}, \quad \hat{f} \le 0, \quad \hat{f} \in L^n(\hat{\Omega}),$$

$$\limsup \hat{v}(x_j) \le \beta \quad if \quad x_j \overset{u^{\hat{\Omega}}}{\longrightarrow} \partial\hat{\Omega}, \quad \beta \ge 0.$$

Then

$$\sup \hat{v} \le D(\|\hat{f}\|_{L^n} + \beta(\sup c^+)|\hat{\Omega}|^{1/n}) + \beta$$

where $D = D(c_0, C_0, b, \Omega, \lambda(\Omega))$ is the constant from (i).

Proof. (i). For the sake of notation we will denote by $G(g, U)$ the solution of

$$\begin{aligned} Lv = & \ g \quad U \\ v \overset{u^U}{=} & \ 0 \end{aligned}$$

If f is the extension to zero outside of $\hat{\Omega}$ and we denote by $\hat{f}^+$, f^+ the positive part of $\hat{f}$, f and similarly for $\hat{f}^-$, f^-, we will easily get from the refined maximum principle, that $G(f^+, \Omega) \le G(\hat{f}^+, \hat{\Omega}) \le 0$ and $G(f^-, \Omega) \le G(\hat{f}^-, \hat{\Omega}) \le 0$.

This implies that $\|G(\hat{f}^+, \hat{\Omega})\|_{L^\infty(\hat{\Omega})} \le \|G(f^+, \Omega)\|_{L^\infty(\Omega)} \le A\|f^+\|_{L^n(\Omega)} = A\|\hat{f}^+\|_{L^n(\hat{\Omega})}$ and $\|G(\hat{f}^-, \hat{\Omega})\|_{L^\infty(\hat{\Omega})} \le \|G(f^-, \Omega)\|_{L^\infty(\Omega)} \le A\|f^-\|_{L^n(\Omega)} = A\|\hat{f}^-\|_{L^n(\hat{\Omega})}$ where A is the constant from Theorem 1.3. Since $\hat{v} = G(\hat{f}^+, \hat{\Omega}) - G(\hat{f}^-, \hat{\Omega})$ we get that $\|\hat{v}\|_{L^\infty(\hat{\Omega})} \le A(\|\hat{f}^+\|_{L^n(\hat{\Omega})} + \|\hat{f}^-\|_{L^n(\hat{\Omega})}) \le 2A\|\hat{f}\|_{L^n(\hat{\Omega})}$. Taking $D = 2A$ we prove (i).
(ii). This part is straightforward from the Refined ABP theorem.

Now we come to one of the main results of the paper where a continuity result for the principal eigenvalue and eigenfunction is established.

Proposition 3.4 *Let $\{\Omega_k\}_{k\ge 0}$, be a sequence of bounded domains with $\Omega_k \subset B_1$ for all k. Denote by $\{\lambda_k, \varphi_k\}$ the principal eigenpairs with the normalization condition that $\|\varphi\|_{L^\infty} = 1$. Then, if $d(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0$, we have*

$$\begin{aligned} \lambda_k & \overset{k\to\infty}{\longrightarrow} \lambda_0, \\ \varphi_k & \overset{k\to\infty}{\longrightarrow} \varphi_0, \ w - W^{2,p}(J), \ p > 1, \quad s - W^{1,\infty}(J) \ \mathit{for\ all}\ J \subset\subset \Omega_0^* \\ \|\varphi_k - \varphi_0\|_{L^\infty(B_1)} & \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Proof. Let $J \subset\subset \Omega_0^*$. Then there exists and $\alpha > 0$ such that $u_0^* \geq \alpha$ in J. Hence there exists a neighborhood of J in Ω_0^*, U^J such that $u_0^* \geq \frac{\alpha}{2}$ in $\bar{U}^J$. Since $\|u_k^* - u_0^*\|_\infty \stackrel{k\to\infty}{\longrightarrow} 0$ we get the existence of a k_1 such that $u_k^* \geq \frac{\alpha}{4}$ in U^J for $k \geq k_1$. From the construction of Ω_k^* we get that $J \subset \Omega_k^*$ for $k \geq k_1$ and $u_k^* \geq \frac{\alpha}{4}$ in J.

Since $J \subset \Omega_k^*$ we get that the λ_k is a bounded sequence and we can get a subsequence, that we denote by λ_k again, such that $\lambda_k \stackrel{k\to\infty}{\longrightarrow} \mu$. Also, from Proposition 1.4 we get a function $\phi \in W_{loc}^{2,p} \cap W_{loc}^{1,\infty}$ so that $\varphi_k \stackrel{k\to\infty}{\longrightarrow} \phi$ w-$W^{2,p}(J)$, s-$W^{1,\infty}(J)$ for all $J \subset\subset \Omega_0^*$. The function ϕ satisfies the equation $L\phi + \mu\phi = 0$ in Ω_0^*, and obviously $\phi \geq 0$. From Proposition 3.1 we have that $\varphi_k \leq Cu_k$ for some constant$C$ independent of k. Going to the limit we obtain $\phi \leq Cu_0^*$. The only thing we need to prove is that ϕ is not identically zero in Ω_0^*.

If $\phi \equiv 0$ then $\varphi_k \stackrel{k\to\infty}{\longrightarrow} 0$ in $L^\infty(J)$ for all $J \subset\subset \Omega_0^*$. Let $J_\delta \subset\subset \Omega_0^*$ so that $|\Omega_0^* \setminus J_\delta| \leq \delta$. We have $\|\varphi_k\|^n_{L^n(\Omega_k^*)} = \|\varphi_k\|^n_{L^n(\Omega_k^*\setminus J_\delta)} + \|\varphi_k\|^n_{L^n(J_\delta)} \leq \|\varphi_k\|^n_{L^n(\Omega^*\setminus J_\delta)} + \|\varphi_k\|^n_{L^n(\Omega_k^*\setminus\Omega_0^*)} + \|\varphi_k\|^n_{L^n(J_\delta)} \leq |\Omega_k^* \setminus J_\delta| + C^n\|u_k\|^n_{L^\infty(\Omega_k^*\setminus\Omega_0^*)} + \|\varphi_k\|^n_{L^n(J_\delta)} \leq \delta + C^n\|u_k - u_0\|^n_{L^\infty(\Omega_k^*)} + \|\varphi_k\|^n_{L^n(J_\delta)}$. This last inequaity implies that $\limsup k \to \infty \|\varphi_k\|^n_{L^n(\Omega_k^*)} \leq \delta$. Since δ is arbitrary we havet that $\lim_{k\to\infty} \|\varphi_k\|^n_{L^n(\Omega_k^*)} = 0$.

Therefore, the function φ_k satisfies $\|\varphi_k\|_{L^\infty} = 1$ and $\lim_{k\to\infty} \|\varphi_k\|^n_{L^n(\Omega_k^*)} = 0$. Let us see that this is impossible. Notice that if we denote by $f_k = -(c(x) + \lambda_k)\varphi_k(x)$ we have that $f_k \in L^n(\Omega_k)$ and $\|f_k\|_n \leq C\|\varphi_k\|_n$ for some constant C independent of k. Moreover, φ_k is the unique solution of the problem,

$$\begin{aligned} M\varphi_k = &\quad f_k, \quad \Omega_k \\ \varphi_k \stackrel{u_k}{=} &\quad 0. \end{aligned}$$

from Lemma 3.3 of BNV, we get that there exists a constant B independent of k so that $\|\varphi_k\|_{L^\infty(\Omega_k)} \leq B\|f_k\|_{L^n(\Omega_k)} \leq \tilde{B}\|\varphi_k\|_{L^n(\Omega_k)}$ where $\tilde{B}$ is another constant independent of k. This inequality is in contradiction with the fact that $\|\varphi_k\|_\infty = 1$ and $\lim_{k\to 0} \|\varphi_k\|_{L^n(\Omega_k^*)} = 0$. Therefore ϕ is not idetically zero and $\phi \geq 0$. Using a Harnack type inequality we get that $\phi > 0$ in Ω and this implies that $\phi = \varphi$ and $\mu = \lambda$.

To prove the last part of the proposition let us start by proving that $\|\varphi_k - \varphi_0\|_{L^n} \stackrel{k\to\infty}{\longrightarrow} 0$. As before, choose a set $J_\delta \subset\subset \Omega_0^*$ such that $|\Omega_0^* \setminus J_\delta| \leq \delta$. Then

$$\|\varphi_k - \varphi_0\|^n_{L^n(B_1)} = \|\varphi_k\|^n_{L^n(\Omega_k^*\setminus\Omega_0^*)} + \|\varphi_0\|^n_{L^n(\Omega_0^*\setminus\Omega_k^*)} + \|\varphi_k - \varphi_0\|^n_{L^n(\Omega_0^*\cap\Omega_k^*)}$$

Let k be big enough to garantee that $J_\delta \subset \Omega_0^* \cap \Omega_k^*$. With the previous equality, we obtain

$$\|\varphi_k - \varphi_0\|^n_{L^n(B_1)} \leq C^n\|u_k - u_0\|^n_{L^n} + \|\varphi_0\|^n_{L^n(\Omega_0^*\setminus J_\delta)} + \|\varphi_k - \varphi_0\|^n_{L^n(\Omega_0^*\cap\Omega_k^*\setminus J_\delta)} + \|\varphi_k - \varphi_0\|^n_{L^n(K_\delta)}$$

and this inequality implies that $\limsup_{k\to 0} \|\varphi_k - \varphi_0\|^n_{L^n(B_1)} \leq C\delta$ for some constant C independent of δ. Since δ is arbitrary we get that $\|\varphi_k - \varphi_0\|^n_{L^n(B_1)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$.

To show the convergence in L^∞ we proceed as follows. For a positive η define the open set $\Omega_0^\eta = \{x \in \Omega_0^* : \quad u_0^*(x) > \eta\}$. Since $u_k \stackrel{k\to\infty}{\longrightarrow} u_0$ in L^∞ we have that there exists a $k_1 = k_1(\eta)$ such that $\Omega_0^\eta \subset \Omega_k^*$ for $k \geq k_1$. Choose $k_2 \geq k_1$ so that $\|u_k - u_0\|_{L^\infty} \leq \eta$. Hence for $k \geq k_2$, we have

$$\|\varphi_k - \varphi_0\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \leq \|\varphi_0\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \leq C\|u_0^*\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \leq C\eta$$

Similarly,

$$\begin{aligned}\|\varphi_k - \varphi_0\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} &\le \|\varphi_k\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} \le C\|u_k^*\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \le \\ & C\|u_k^* - u_0^*\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} + C\|u_0^*\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \le 2C\eta\end{aligned}$$

We only need to prove that $\|\varphi_k - \varphi_0\|_{L^\infty(\Omega_0^\eta)}$ is small as $\eta \to 0$. In Ω_0^η we have that $L(\varphi_k - \varphi_0) = -\lambda_k\varphi_k + \lambda_0\varphi_0 = -\lambda_k(\varphi_k - \varphi_0) + (\lambda_0 - \lambda_k)\varphi_0$. Calling $f_k = -\lambda_k(\varphi_k - \varphi_0) + (\lambda_0 - \lambda_k)\varphi_0$ we have that $\|f_k\|_{L^n} \overset{k\to\infty}{\longrightarrow} 0$. Moreover,

$$\varphi_k - \varphi_0 \le \varphi_k \le Cu_k = C(u_k - u_0) + C(u - \eta) + C\eta \le C\|u_k - u\|_{L^\infty} + Cu^{\Omega_0^\eta} + C\eta$$

Hence if $k \ge k_2$ we get

$$\limsup_{x_j \overset{\Omega_0^\eta}{\longrightarrow} \partial\Omega_0^\eta} (\varphi_k(x_j) - \varphi_0(x_j)) \le 2C\eta$$

Now from Lemma 3.3, we have

$$\sup\{\varphi_k(x) - \varphi_0(x) : \quad x \in \Omega_0^\eta\} \le D(\|f_k\|_{L^n} + C\eta) + C\eta$$

We can give a similar argument for $\varphi_0 - \varphi_k$ and we will obtain that

$$\|\varphi_k - \varphi_0\|_{L^\infty(\Omega_0^\eta)} \le D\|f_k\|_{L^n} + C\eta$$

Hence $\limsup_{k\to 0} \|\varphi_k - \varphi_0\|_{L^\infty(\Omega_0^\eta)} \le C\eta$. This implies the last conclusion of the Proposition.

An important corollary, that will be used in the next section, can be obtained now.

Corollary 3.5 *Let $\Omega_0 \in \Theta$ be a bounded domain with $\lambda(L, \Omega_0) > 0$. There exists an $\epsilon = \epsilon(L, \Omega_0)$ such that if*

$$\mathcal{D}(\Omega_0, \epsilon) = \{\Omega \in \Theta : \ \Omega \textit{ is a domain, and } \ d(\Omega_0^*, \Omega^*) < \epsilon\}$$

then
(i). $\lambda(L, \Omega) > 0$ for all $\Omega \in \mathcal{D}(\Omega_0, \epsilon)$.
(ii). For any $\Omega \in \mathcal{D}(\Omega_0, \epsilon)$ there is a function w defined in Ω satisfying $Lw \le 0$ and $1 \le w \le C$ where $C = C(c_0, C_0, b, \Omega_0, \lambda)$.
(iii). There exists a constant $D = D(c_0, C_0, b, \Omega_0, \lambda)$ such that if $\Omega \in \mathcal{D}(\Omega_0, \epsilon)$ and v is the strong solution of

$$\begin{aligned} Lv =& \ f, \quad \Omega \\ v \overset{u}{=}& \ 0, \quad \partial\Omega. \end{aligned}$$

for certain $f \in L^n(\Omega)$, then

$$\|v\|_{L^\infty(\Omega)} \le D\|f\|_{L^n(\Omega)}. \tag{3.1}$$

(iv). If $\Omega \in \mathcal{D}(\Omega_0, \epsilon)$ and h is a function in Ω, bounded from above and satisfying

$$Lh \ge f \quad in \quad \Omega, \quad f \le 0, \quad f \in L^n(\Omega),$$

$$\limsup h(x_j) \le \beta \quad if \quad x_j \stackrel{u^\Omega}{\longrightarrow} \partial\Omega, \quad \beta \ge 0.$$

Then

$$\sup h \le D(\|f\|_{L^n} + \beta(\sup c^+)|\Omega|^{1/n}) + \beta \tag{3.2}$$

where D is the constant from part (iii).

Proof.
(i). This is clear from Proposition 3.4.
(ii). This result is related to Proposition 6.1 from [6]. We will follow the proof given in [6] with some modifications. Let us consider a parameter η small enough and consider the open set $\Omega_0^\eta = \{x \in \Omega_0 : u_0(x) > \eta\} \subset \Omega_0$. Choose $0 < \epsilon_1 \le \eta/2$ so that $\|u - u_0\|_\infty \le \eta/2$ and $\lambda \ge \lambda_0/2$ for all $\Omega \in \mathcal{D}(\Omega_0, \epsilon_1)$. This obviously implies that $u > \eta/2$ in the set Ω_0^η for all $\Omega \in , \mathcal{D}(\Omega_0, \epsilon_1)$. Hence, $\Omega_0^\eta \subset \Omega^*$ for all $\Omega \in , \mathcal{D}(\Omega_0, \epsilon_1)$. Let w^η be the unique bounded solution of

$$Mw^\eta = \begin{cases} -2b & \Omega \setminus \Omega_0^\eta \\ 0 & \Omega_0^\eta \end{cases}$$

$$w^\eta \stackrel{u}{=} \ 0$$

By the refined maximum principle we have $|w^\eta(x)| \le 2bu(x)$. This implies that if $x \in \Omega \setminus \Omega_0^\eta$ then $|w^\eta(x)| \le 2b(u(x) - u_0(x)) + 2bu_0(x) \le 2b\eta/2 + 2b\eta = 3b\eta$. Also, $\|w^\eta\|_\infty \le 2b\|u\|_\infty \le 2b\|u^{B_1}\|_\infty$ which is independent of η and Ω.

Therefore, in $\Omega \setminus \Omega_0^\eta$ we have $L(1 + w^\eta) = Mw^\eta + c(1 + w^\eta) \le -2b + b + 3b^2\eta$. Choosing $\eta \le 1/3b$ we have that $L(1 + w^\eta) \le 0$ in $\Omega \setminus \Omega_0^\eta$.

Let us see now that there exists a $\delta = \delta(\eta, \Omega_0) > 0$ such that $\inf\{\varphi_0(x),\, x \in \Omega_0^\eta\} \ge \delta$, where φ_0 is the principal eigenfunction of L in Ω_0 with the normalization condition $\|\varphi_0\|_\infty = 1$. If this is not the case then there will exist a sequence $\{x_k\} \subset \Omega_0^\eta$, and therefore $u_0(x_k) > \eta$, so that $\varphi_0(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$. But this is in contradiction with the fact that $u_0 \stackrel{\varphi_0}{=} 0$ (see Theorem 5.1 of [6]).

From Proposition 3.4 we can choose a $0 < \epsilon_2 \le \epsilon_1$ such that $\|\varphi - \varphi_0\|_{L^\infty} \le \delta/2$ for all $\Omega \in \mathcal{D}(\Omega_0, \epsilon_2)$. This implies that $\varphi > \delta/2$ in Ω_0^η.

Hence, if we denote by $w = 1 + w^\eta + \alpha\varphi$ for certain $\alpha > 0$, to be fixed later, we get

$$Lw = L(1 + w^\eta) - \alpha\lambda\varphi \le \begin{cases} 0 & \Omega \setminus \Omega_0^\eta \\ b(1 + 2b\|u^{B_1}\|_\infty) - \alpha\frac{\lambda_0\delta}{4} & \Omega_0^\eta. \end{cases}$$

Choosing $\alpha = 4b(1 + 2b\|u^{B_1}\|_\infty)/\lambda_0\delta$ we have that $Lw \le 0$ and $1 \le w \le 1 + 2b\|u^{B_1}\|_\infty + 4b(1 + 2b\|u^{B_1}\|_\infty)/\lambda\delta \equiv C$. This proves part (ii) of the Corollary.
(iii) and (iv). Once part (ii) is proved we can follow the proof given in [6] (see Proposition 6.1, 6.2 and the proofs of Theorem 1.2 and 1.3 given in Section 6 of [6]). If C is the constant from (ii) we can see that $D = (1 + \frac{2}{\lambda_0}Cb)\|u^{B_1}\|_\infty bB$ where B is the constant from the Theorem (ABP), that we know it is independent of the domain. This proves the Corollary.

Remark 3.6 If $\Omega \subset B_1$ is a general open set we can write $\Omega = \bigcup_{i\ge1} C^i$ where C^i are the connected components of Ω. If we denote by λ^i the principal eigenvalue of the operator L in C^i, we can define a generalized principal eigenvalue of L in Ω by,

$$e(\Omega) = \min\{\lambda^i,\ i \ge 1\}$$

Moreover, since $|\Omega| < \infty$ this minimum will be attained only at a finite number of points. Without loss of generality we can assume that there exists an $m \geq 1$ such that $\lambda^i = e(\Omega)$ $1 \leq i \leq m$ and $\lambda^i > e(\Omega)$ for $i > m$.

If we consider the eigenfunctions φ^i, extended to zero outside C^i, with some normalization condition, for example $\|\varphi^i\|_\infty = 1$, we can define a generalized eigenfunction in Ω as the set of functions $E(\Omega)$ where

$$E(\Omega) \equiv \{\sum_{i=1}^{m} \delta_i \varphi^i : \delta^i = 0 \text{ or } 1, \text{ and } \sum_{i=1}^{m} \delta_i > 0\}$$

With this definitions and with the same ideas of Proposition 3.4 we can obtain the following result:

Proposition 3.7 *Let $\{\Omega_k\}_{k \geq 0}$, be a sequence of bounded open sets with $\Omega_k \subset B_1$ for all k. Denote by $\{e(\Omega_k), E(\Omega_k)\}$ the generalized principal eigenpairs. Then, if $d(\Omega_k, \Omega_0) \overset{k\to\infty}{\longrightarrow} 0$, we have $e(\Omega_k) \overset{k\to\infty}{\longrightarrow} e(\Omega_0)$. Moreover for any sequence of functions $\chi_k \in E(\Omega_k)$, $k \geq 1$, there exists a subsequence $\chi_{k'}$ and a function $\chi_0 \in E(\Omega_0)$ depending on the subsequence such that*

$$\begin{aligned} \chi_{k'} &\overset{k'\to\infty}{\longrightarrow} \chi_0, \; w - W^{2,p}(J), \quad s - W^{1,\infty}(J) \text{ for all } J \subset\subset \Omega_0^* \\ \|\chi_k' - \chi_0\|_{L^\infty(B_1)} &\overset{k'\to\infty}{\longrightarrow} 0. \end{aligned}$$

4 The equation $Lv = f$.

In this section we want to study the dependence on the domain of the solutions of the equation $Lv = f$ in Ω, $v \overset{u}{=} 0$ in $\partial\Omega$.

We have the following result,

Theorem 4.1 *Let $\{\Omega_k\}_{k \geq 0}$ be a sequence of domains with $\Omega_k \subset B_1$ for all $k \geq 0$ and $\lambda(L, \Omega_0) > 0$. Let $f_k \in L^n(B_1)$, $k \geq 0$, be a sequence of functions. Denote by v_k, $k \geq 0$, the bounded solution of*

$$\begin{aligned} Lv_k = &\; f_k, \quad \Omega_k \\ v_k \overset{u_k}{=} &\; 0, \quad \partial\Omega_k. \end{aligned}$$

Then if $d(\Omega_k^*, \Omega_0^*) \overset{k\to\infty}{\longrightarrow} 0$ and $\|f_k - f_0\|_{L^n} \overset{k\to\infty}{\longrightarrow} 0$ we have

$$v_k \overset{k\to\infty}{\longrightarrow} v_0, \quad w - W^{2,p}(J), \quad s - W^{1,\infty}(J) \text{ for all } J \subset\subset \Omega_0^*, \quad s - L^\infty(B_1)$$

Proof. From Corollary 3.5 we can assume that $\lambda(L, \Omega_k) > 0$ for all k. Moreover, since $d(\Omega_k^*, \Omega_0^*) \overset{k\to\infty}{\longrightarrow} 0$, there exists a constant D independent of k such that $\|v_k\|_{L^\infty} \leq D\|f_k\|_{L^n}$. Using Proposition 1.4 we can get a subsequence of v_k, that we denote by v_k again, and a function $\phi \in W^{2,p}_{loc} \cap W^{1,\infty}_{loc}$ such that $v_k \overset{k\to\infty}{\longrightarrow} \phi$ w-$W^{2,p}(J)$, s-$W^{1,\infty}(J)$ for all $J \subset\subset \Omega_0^*$. Moreover the function ϕ is an strong and bounded solution of $L\phi = f$. We just need to show that $\phi \overset{u_0}{=} 0$ and that $\|v_k - \phi\|_{L^\infty} \overset{k\to\infty}{\longrightarrow} 0$.

Let us consider two different cases.

(a). Let us assume first that the functions $f_k = f \in L^\infty$. In this case, the functions v_k satisfy $Mv_k = f - cv_k$ and from the refined maximum principle we get that

$$|v_k(x)| \le \|f\|_{L^\infty}(1 + bD)u_k(x) \tag{4.1}$$

Letting $k \to \infty$ in (2.1) and taking into account that $u_k \overset{k\to\infty}{\longrightarrow} u_0$ in L^∞ we get that $|\phi(x)| \le Cu_0(x)$ which shows that $\phi = v_0$.

The proof that $v_k \overset{k\to\infty}{\longrightarrow} v_0$ in L^∞ follows the same ideas of the proof of Proposition 3.4.

Let us start by proving that $\|v_k - v_0\|_{L^n} \overset{\epsilon\to 0}{\longrightarrow} 0$. As before, choose a set $J_\delta \subset\subset \Omega_0^*$ such that $|\Omega_0^* \setminus J_\delta| \le \delta$. Then

$$\|v_k - v_0\|^n_{L^n(B_1)} = \|v_k\|^n_{L^n(\Omega_k^*\setminus\Omega_0^*)} + \|v_0\|^n_{L^n(\Omega_0^*\setminus\Omega_k^*)} + \|v_k - v_0\|^n_{L^n(\Omega_0^*\cap\Omega_k^*)}$$

Let k be big enough to garantee that $J_\delta \subset \Omega_0^* \cap \Omega_k^*$. With the previous equality, we obtain

$$\|v_k - v_0\|^n_{L^n(B_1)} \le C^n\|u_k - u_0\|^n_{L^n} + \|v_0\|^n_{L^n(\Omega_0^*\setminus J_\delta)} + \|v_k - v_0\|^n_{L^n(\Omega_0^*\cap\Omega_k^*\setminus J_\delta)} + \|v_k - v_0\|^n_{L^n(J_\delta)}$$

and this inequality implies that $\limsup_{k\to 0}\|v_k - v_0\|^n_{L^n(B_1)} \le C\delta$ for some constant C independent of δ. Since δ is arbitrary we get that $\|v_k - v_0\|^n_{L^n(B_1)} \overset{\epsilon\to 0}{\longrightarrow} 0$.

To show the convergence in L^∞ we proceed as follows. For a positive η define the open set $\Omega_0^\eta = \{x \in \Omega_0^* : \quad u_0^*(x) > \eta\}$. Since $u_k \overset{k\to\infty}{\longrightarrow} u_0$ in L^∞ we have that there exists a $k_1 = k_1(\eta)$ such that $\Omega_0^\eta \subset \Omega_k^*$ for $k \ge k_1$. Choose $k_2 \ge k_1$ so that $\|u_k - u_0\|_{L^\infty} \le \eta$. Hence for $k \ge k_2$, we have

$$\begin{aligned}
\|v_k - v_0\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} &\le \|v_k\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} + \|v_0\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} \le \\
&C\|u_k^*\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} + C\|u_0^*\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} \le \\
&C\|u_k^* - u_0^*\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} + 2C\|u_0^*\|_{L^\infty(\Omega_k^*\setminus\Omega_0^\eta)} \le 3C\eta
\end{aligned}$$

Similarly $\|v_k - v_0\|_{L^\infty(\Omega_0^*\setminus\Omega_0^\eta)} \le 3C\eta$

We only need to prove that $\|v_k - v_0\|_{L^\infty(\Omega_0^\eta)}$ is small as $\eta \to 0$. In Ω_0^η we have that $L(v_k - v_0) = 0$. Moreover,

$$|v_k - v_0| \le C(u_k + u_0) = C(u_k - u_0) + 2C(u_0 - \eta) + 2C\eta \le C\|u_k - u_0\|_{L^\infty} + 2Cu^{\Omega_0^\eta} + 2C\eta$$

Hence if $k \ge k_2$ we get

$$\limsup_{x_j \overset{\Omega_0^\eta}{\longrightarrow} \partial\Omega_0^\eta} |v_k(x_j) - v_0(x_j)| \le 3C\eta$$

Now from Lemma 3.3, we have

$$\sup\{|v_k(x) - v_0(x)| : \quad x \in \Omega_0^\eta\} \le D(\|f_k\|_{L^n} + C\eta) + C\eta$$

Hence $\limsup_{k\to 0}\|v_k - v_0\|_{L^\infty(\Omega_0^\eta)} \le C\eta$.

(b). For the general case, let $\epsilon > 0$ be small enough and choose a function $f_0^\epsilon \in L^\infty(B^1)$ such that $\|f_0 - f_0^\epsilon\|_{L^n} \le \epsilon/2$. We can choose a $k_1 = k_1(\epsilon)$ such that if $k \ge k_1$ we have

$\|f_k - f_0^\epsilon\|_{L^n} \leq \epsilon$. Denote by v_k^ϵ the bounded solution of $Lv_k^\epsilon = f_0^\epsilon$ in Ω_k, $v_k^\epsilon \overset{u_k}{=} 0$ in $\partial\Omega_k$, for all $k \geq 0$. From Corollary 3.5 we have that $\|v_k^\epsilon - v_k\|_\infty \leq D\|f_0^\epsilon - f_k\|_{L^n} \leq D\epsilon$ and $\|v_0 - v_0^\epsilon\|_{L^\infty} \leq D\epsilon$. Also, from (a), there exists a $k_2 = k_2(\epsilon) \geq k_1$ such that $\|v_k^\epsilon - v_0^\epsilon\|_{L^\infty} \leq D\epsilon$ for $k \geq k_2$. This implies that $\|v_k - v_0\|_{L^\infty} \leq 3D\epsilon$ for $k \geq k_2$. Since ϵ is arbitrary, this shows that $v_0 = \phi$ and the convergence of v_k to v_0 in L^∞. This proves the Proposition.

References

[1] J.M. Arrieta, "Neumann Eigenvalue Problems on Exterior Perturbations of the Domain", *J. Diff. Eq.* 118, No. 1, (1995) 54-103.

[2] J.M. Arrieta, "Rates of Eigenvalues on a Dumbbell Domain. Simple Eigenvalue Case", *Transactions of the A.M.S.* Vol. 347, N. 9, (1995) 3503-3531

[3] J.M. Arrieta, J.K. Hale and Q. Han, "Eigenvalue Problems for Nonsmoothly Perturbed Domains", *J. Diff. Eq.* 91, No.1, (1991) 24-52.

[4] I. Babuska and R. Vyborny, "Continuous Dependence of Eigenvalues on the Domains," *Czech. Math. J.,* 15(1965) 169-178.

[5] J.T. Beale, "Scattering Frequencies of Resonators," *Comm. Pure Appl. Math.,* 26 (1973) 549-563.

[6] H. Berestycki, L. Nirenberg and S.R.S. Varadhan, "The Principal Eigenvalue and Maximum Principle for Second-Order Elliptic Operators in General Domains", *Comm. Pure Appl. Math.,* 47, (1994) 47-92.

[7] R. Brown, P.D. Hislop and A. Martinez, "Eigenvalues and Resonances for Domains with Tubes: Neumann Boundary Conditions", Vol. 115, No. 2 (1995) 458-476.

[8] I. Chavel and E.A. Feldman, "Spectra of Domains in Compact Manifolds," *J. Funct. Anal.,* 30 (1978) 198-222.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol I,* Wiley-Interscience, New York 1953.

[10] D. Gilbarg and N. Trudinger, *" Elliptic Partial Differential Equations of Second Order"* Springer-Verlag, New York/Berlin, 1983.

[11] J. K. Hale and G. Raugel, "Reaction-Diffusion Equations on Thin Domains", *J. Math. Pures Appl.,* 71 (1992), 33-95.

[12] J. K. Hale and J. M. Vegas, "A Nonlinear Parabolic Equation with Varying Domain", *Arch. Rational Mech. Anal.* 86 (1984), 99-123.

[13] R. Hempel, L. Seco and B. Simon, "The Essential Spectrum of Neumann Laplacians on some Bounded Singular Domains", *J. Funct. Anal.,* 102 (1991), 448-483.

[14] P.D. Hislop and A. Martinez,"Scattering Resonances of a Helmholtz Resonator", *Indiana U. Math. J.* 40, (1991) 767-788.

[15] S. Jimbo, "Perturbation Formula of Eigenvalues in a Singularly Perturbed Domain", *J. Math. Soc. Japan* 45 (2), (1993) 339-356.

[16] S. Jimbo and Y. Morita, "Remarks on the Behavior of Certain Eigenvalues on a Singularly Perturbed Domain with Several Thin Channels", *Comm. Part. Diff. Eq.*, 17 (3&4), (1992), 523-552.

[17] M. Lobo-Hidalgo and E. Sanchez-Palencia, "Sur Certaines Propriétés Spectrales des Perturbations du Domaine dans les Problémes aux Limites," *Comm. Part. Diff. Eq.*, 4 (1979) 1085-1098.

[18] K. Miller, "Barriers on Cones for Uniformly Elliptic Operators" *Ann. Mat. Pura Appl.* 76, 1967, 161-177.

[19] S. Ozawa, "Singular Variations of Domains and Eigenvalues of the Laplacian," *Duke Math. J.,* 48 (1981) 769-778.

[20] S. Ozawa, "Spectra of Domains with Small Spherical Neumann Boundary", *J. Fac. Sci. Univ. Tokyo* 30 (1983), 259-277.

[21] J. Rauch and M. Taylor, "Potential and Scattering Theory on Wildly Perturbed Domains," *J. Funct. Anal.,* 18 (1975) 27-59.

[22] J.M. Vegas, "A Functional-Analytic Framework for the Study of Elliptic Equations on Variable Domains" *Proc. Roy. Soc. Edinburgh. Sect. A* 116, no. 3-4, (1990) 367-380.