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Charged fields, Higgs Phenomenon and Confinement. Lesson from soluble models *

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Abstract. Higgs phenomenon and confinement are discussed on the basis of a general framework for gauge and Poincaré transformations on field algebras arising in gauge theories, in positive gauges. The resulting phenomena and structures are explicitly controlled for a class of soluble models.

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1. Motivations

The rôle of gauge theories has evolved, in the last 30 years, from that of a special case, Quantum Electrodynamics (QED), to that of a common tool for the description of all the fundamental interactions. The fact that QED, and in general gauge theories, are usually described in terms of field algebras which do not completely fit into the Wightman framework [1], poses therefore a general problem, and requires the identification and investigation of characteristic structures.

In fact, locality and covariance of gauge fields hold only in "local gauges", where positivity fails; on the other hand, the status of gauge fields in positive gauges is problematic, especially when there is confinement (i.e. absence of charged sectors with finite energy), or screening ("Higgs phenomenon").

The space–time symmetries present problems already in the QED case, where the Lorentz group is spontaneously broken in the charged sectors [2] [3], with substantial implications on the transformation properties of charged fields. The absence of charged sectors with well–defined (unitarily implemented) time evolution in Quantum Cromodynamics (QCD) is not a very distant phenomenon from the breaking of the Lorentz group in QED, the main difference being produced by the smallness of the coupling constant in QED, which makes the problem relevant only at very large distances.

We sketch below a general framework for these features and problems in terms of characteristic structures of gauge field algebras in positive gauges. As we will see,

i) unbroken gauge groups cannot commute with those Poincaré transformations which do not leave the corresponding charged sectors invariant; the gauge group does not therefore commute with the Lorentz transformations in QED, and with the time translations in QCD.

ii) In the case of a (partially) broken gauge group, the algebra of observables is strongly dense in the field subalgebra which is pointwise invariant under the residual unbroken gauge subgroup, and there are no charged states corresponding to the broken charges.

iii) In case ii), Poincaré automorphisms commuting with the gauge group can be defined only on an extension of the field algebra with non–trivial centre; the appearance of central variables in the time evolution of the field algebra is at the basis of non–trivial mass spectra associated to the breaking of the gauge group. These phenomena can be seen as the relativistic counterpart of general features of non–relativistic systems with Coulomb interactions, where variables at infinity appear in the time evolution of local variables as a consequence of the removal of infrared or volume cutoffs.

Before introducing *field* algebras, we recall the point of view and the results of general Algebraic Local Quantum Theory, which always apply to the *observable* algebras.

2. Observables and fields

The *observable* algebra of a gauge QFT can be assumed to be given by a net $\mathcal{A}(\mathcal{O})$ of local (Von Neumann) algebras, defined in the vacuum representation. The whole physical content of the theory can in principle be extracted from the observable algebra, in terms of the classification of its representations, under suitable criteria of "physical relevance". We do not consider for the moment the problem of which properties distinguish the observable nets which arise in gauge theories from those which arise in standard Quantum Field Theory (QFT), and therefore we consider any local net $\mathcal{A}(\mathcal{O})$, with the standard assumptions [4]. The following possibilities arise for relevant representations:

1. Representations labelled by "localizable" charges, obtained from the vacuum representation through morphisms which are localizable in finite space–time regions, in the sense that they leave invariant all observables in the causal complement of a double cone. This has been thoroughly investigated by Doplicher, Haag and Roberts (DHR). Space–time covariance of the charged sectors follows, together with the classification of statistics and the construction of a field algebra and a compact "gauge group" which classifies the representations of $\mathcal{A}$ [5].

2. Representations stable under space–time translations, defined by (particle) states with energy–momentum spectrum in an isolated iperboloid with positive mass. By the results of ref. [6] they can all be obtained by morphisms which are either localized in the sense of DHR, or can be localized in *any* space–like cone, i.e. the morphism can be chosen so that all observables localized in the causal complement of any given spacelike cone are left invariant.

3. Representations stable under space–time translations, with relativistic spectral condition, labelled by charges which obey a Gauss' law (which forbids [3] eigenstates of the mass operator). The localization region of the corresponding morphisms includes at least a spacelike cone [7].

QED, being characterized by a charge which obeys a Gauss' law, is a candidate for case 3. However, the generalization of the DHR analysis to this case, and the full characterization of the properties of the charged sectors (including particle statistics) seems to require further information which, in our opinion, can be obtained by studying the properties of the charged fields in the (standard) positive gauge formulations.

QCD seems to be characterized by the absence of charged sectors stable under space–time translations and with positive energy. It is not clear how such a theory can be characterized as a gauge theory along the lines of 1–3. Perhaps, again some insight can be obtained by studying the algebra of charged fields in positive gauges, which should define sectors not covariant under (space)–time translations.

The Higgs–Kibble model, and more generally gauge theories exhibiting the Higgs phenomenon, do not have sectors labelled by a Gauss' charge, because of screening. The spontaneous breaking of gauge symmetry implies that charged fields in positive gauges cannot give rise to charged sectors, (see Proposition 3 below). Again, even if for different reasons, it is not clear how such a kind of theories can be characterized by a gauge group, and, more generally, whether they can be recognized in terms of properties of the observable algebra. Here too, the study of the field algebra in the standard approach should give relevant hints.

The difficulty in the characterization of the phenomena exhibited by gauge theories on the basis of the classification of the representations of the observable algebra can in general be traced back in the idea that the "gauge group" must be identified in terms of the "particle representations" of $\mathcal{A}$; these exist only in the "QED case", and even in this case do not have the localization properties which appear as most natural from the point of view of the local structure of the observables.

In conclusion, in order to i) investigate a possible algebraic characterization of gauge QFT, ii) discuss possible algebraic characterizations of confinement and of screening, iii) get information on the charged sectors of QED, we propose to study the general properties of the algebra of fields in positive gauges. This strategy is very similar to that used by DHR in their first paper [8] for theories with localizable charges, namely to abstract general structural properties from concrete field algebras, and use them as a basis for a general algebraic approach.

3. Charged field algebra as an extension of the observable algebra

The charged fields in gauge theories are characterized by being charged with respect to a Gauss' charge, and therefore they cannot be local with respect to the observables in the abelian case, and cannot be relatively local in the non–abelian case. Furthermore, the experience with QED suggests that charged fields cannot in general be covariant under the Lorentz group. Locality and covariance are obtained in renormalizable gauges, but at the price of giving up positivity; the physical interpretation is then obtained through a subsidiary condition on the states, and the solutions of such a condition require, for the charged sectors, a non–local construction (see ref. [9]. This is at the basis of the absence of locality and covariance of the states and of the morphisms which may define them in terms of representations of the observable algebra.

In a spirit similar to that of ref.[8], we consider a C^* algebra $\mathcal{F}$, as the *algebra of fields* in a positive gauge. The *gauge group* $\mathcal{G}$ is defined as the group of $*$ automorphisms β_g, $g \in \mathcal{G}$, of $\mathcal{F}$, which leave an *observable subalgebra* $\mathcal{A} \subset \mathcal{F}$ pointwise invariant. Clearly, given $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{A}$ can be defined as the subalgebra of $\mathcal{F}$ which is pointwise invariant under $\mathcal{G}$, or, alternatively, one may consider $\mathcal{A}$ as a given *observable* algebra, $\mathcal{F}$ an extension of $\mathcal{A}$, and $\mathcal{G}$ *defined* by the above relation. For the observable algebra $\mathcal{A}$ the standard general assumptions can be made, i.e.:

i) *$\mathcal{A}$ is the C^* completion of a local net*

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

defined for all double cones $\mathcal{O}$ in Minkowski space, $\mathcal{A} = \overline{\cup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{\|\ \|}$, and the Poincaré group is assumed to act as a group of automorphisms $\alpha_{a,\Lambda}$ of $\mathcal{A}$.

A vacuum state is assumed to exist as a pure state on $\mathcal{A}$, with unique (pure) extension to $\mathcal{F}$:

ii) *there exists a pure state ω_0 on $\mathcal{F}$, such that its restriction to $\mathcal{A}$ is pure and Poincaré invariant.*

The first issue is the space–time covariance properties of the field algebra and the relation between the space–time translations and the gauge group. In the standard approach to gauge QFTs in positive gauges, the gauge group is believed to be a “global” one, the local gauge group having been broken by fixing the gauge, and the folklore seems to take for granted that such (“residual”) gauge group commutes with the space–time translations. It is worthwhile to see whether this property can indeed be assumed and what is its origin in the present framework. Assume therefore that

iii) *a subgroup $\mathcal{P}_0$ of the Poincaré group defines a group of automorphisms α_p, $p \in \mathcal{P}_0$ of $\mathcal{F}$, which extend the Poincaré automorphisms defined on $\mathcal{A}$.*

We will also use later the assumptions

iv) *As a state on $\mathcal{F}$, ω_0 is left invariant by the automorphisms α_p:* $\omega_0(B) = \omega_0(\alpha_p(B))$, $\forall B \in \mathcal{F}$, $p \in \mathcal{P}_0$.

v) *in the GNS representation* π_0 *defined by* ω_0 *and* $\mathcal{F}$, ω_0 *is the only state invariant under all* α_p, $p \in \mathcal{P}_0$.

Since ω_0 is pure, $\mathcal{F}$ is irreducible in π_0; all the gauge automorphisms β_g which leave ω_0 invariant have unitary implementers in π_0, which commute with $\mathcal{A}$, and it is reasonable to assume that all unitary operators in π_0 which commute with $\mathcal{A}$ define automorphisms of $\mathcal{F}$; in fact, by irreducibility of $\mathcal{F}$, this can always be achieved, by enlarging $\mathcal{F}$, if necessary, with strong limits in π_0.

The representation π_0 of $\mathcal{F}$ will be our primary object of interest; but an important rôle is played in the following by *gauge invariant* strong topologies on $\mathcal{F}$, defined by *gauge invariant* representations π of $\mathcal{F}$, i.e. by representations which are stable under the action of the gauge group, $\pi \circ \beta_g = \pi$, $\forall g \in \mathcal{G}$; the gauge automorphisms are automatically continuos with respect to any gauge invariant strong topology, as a consequence of the invariance under ${\beta_g}^*$ of the folium of states associated to π. Assuming iv) and v), the representation π_0 will turn out to be gauge invariant if and only if ω_0 is invariant under β_g; in any case, a gauge invariant representation is obtained by taking the direct sum over the gauge group of the GNS representations defined by the states $\omega_g \equiv \omega_0 \circ \beta_g$. This representation will be denoted by π_0^{inv}.

The following Propositions show the implications of the assumption that a group of automorphisms of $\mathcal{F}$, in particular α_p, $p \in \mathcal{P}_0$, have "gauge invariant generators".

Proposition 1. *Let* π *be a gauge invariant representation of* $\mathcal{F}$, *and* γ *an automorphism of* $\mathcal{F}$.
i) *if* γ $\mathcal{F}$ *is the strong limit in* π *of automorphisms* γ_L *which commute with* β_g, *then* $[\gamma, \beta_g] = 0$.
ii) *in particular, if* $\alpha_p(A)$, $p \in \mathcal{P}_0$, *is the strong limit of* $U_L(p) A U_L(p)^*$, $U_L \in \mathcal{A}$, *then*

$$[\alpha_p, \beta_g] = 0 \quad \forall g \in \mathcal{G} \tag{1}$$

iii) *if* γ $\mathcal{F}$ *is implemented in* π *by (unitary) operators* U *in the Von Neumann algebra generated by* $\mathcal{A}$ *in* π, *then*

$$[\gamma, \beta_g] = 0 \quad \forall g \in \mathcal{G}$$

Proof. i): The stability of π under β_g implies that β_g is strongly continuous, and therefore, $\forall A \in \mathcal{F}$,

$$\beta_g \gamma(A) = \beta_g \, s - \lim \gamma_L(A) = s - \lim \beta_g \gamma_L(A) = s - \lim \gamma_L \beta_g(A) = \gamma \beta_g(A)$$

ii): All β_g are strongly continuous and have therefore a unique strongly continuous extension to the Von Neumann algebra generated by $\mathcal{F}$ in π, which leaves the Von Neumann algebra generated by $\mathcal{A}$ pointwise invariant.

We conclude that if Poincaré transformations of $\mathcal{F}$ can be constructed from observable local implementers, or are implemented by strong limits of observables operators, then they commute with the gauge group. The delicate points are here:
a) The use of a gauge invariant strong topology, which is essential for the argument; we will see below the non–trivial implication of this fact for broken gauge groups.
b) The Poincaré group is not always implemented by observable operators in gauge theories, nor do local implemeters always converge; this cannot in fact be the case for the Lorentz boosts in QED, since they do not leave the charged sectors stable [2],[3],[7],[9], and for the time translations in confined models, if confinement corresponds to the instability of charged sectors under time translations, as also suggested by the models discussed below.

¿From Proposition 1 it also follows:

Proposition 2. *If $[\alpha_p, \beta_g] \neq 0$ for some $p \in \mathcal{P}_0$ and $g \in \mathcal{G}$, then, in any representation π, either*
1*) β_g is broken, or*
2*) α_p is not implemented by operators in the strong closure of the observable algebra.*

In the case of time translations α_t, Proposition 2 says that if $[\alpha_t, \beta_g] \neq 0$ for some $g \in \mathcal{G}$, then one has for β_g either the Higgs phenomenon or the confinement; a non–zero mass spectrum is in fact associated in general to the breaking of β_g, as a consequence of the lack of commutativity with α_t, see ref. [10], and the second alternative exclude the existence of energy as a (non local) observable. The alternative 2) can be replaced, for the group of space–time translations α_x, by
2′) *α_x is not implemented by a unitary group satisfying the relativistic spectral condition.*
In fact, by Borchers' theorem [11] the implementers could then be chosen in the strong closure of $\mathcal{A}$.

A converse of Proposition 2, i.e. the fact that if a Poincaré transformation commutes with the gauge group, then it is implemented by observables, requires to consider the possibility of broken gauge transformations, and will be given below (Propositions 5 and 6).

Now we discuss the possibility that the gauge group is broken in the vacuum representation π_0 of $\mathcal{F}$. This point has sharp implications on the relation between $\mathcal{A}$

and $\mathcal{F}$, and it is convenient to remark first that *in a gauge invariant representation* π $\mathcal{A}$ *cannot be strongly dense in* $\mathcal{F}$, if the gauge group is non–trivial. This follows immediately from the strong continuity of β_g, which forbids the existence of a strongly dense pointwise invariant subalgebra.

Moreover, for a GNS representation over a pure state ω invariant under the gauge group, in particular for π_0 whenever it is stable under gauge transformations, the GNS subspace generated by $\mathcal{A}$ is never dense, if the gauge group is not trivial: in fact, the invariance of ω implies the existence of unitary implementers of the gauge group, which reduce to the identity on the GNS subspace generated by $\mathcal{A}$.

Given a representation π of $\mathcal{F}$, the unbroken subgroup $\mathcal{G}^0_\pi$ of the gauge group is given by the gauge automorphisms β_g which leave π invariant, $\pi \circ \beta_g = \pi$. For the vacuum representation π_0, if iv), v) hold, and β_g commutes α_p, $\forall g \in \mathcal{G}$, $p \in \mathcal{P}_0$, the unbroken subgroup $\mathcal{G}_0$ is given by the gauge automorphisms β_g satisfying ${\beta_g}^*(\omega_0) = \omega_0$. We call $\mathcal{F}^0_\pi$ ($\mathcal{F}_0$ when $\pi = \pi_0$) the subalgebra of $\mathcal{F}$ pointwise invariant under $\mathcal{G}^0_\pi$. From the definition of the gauge group it follows that for a gauge invariant representation $\mathcal{F}^0_\pi = \mathcal{A}$, whereas, for representation with gauge group broken to the identity, $\mathcal{F}^0_\pi = \mathcal{F}$.

Proposition 3. *In the GNS representation π_0 of $\mathcal{F}$ defined by ω_0, the observable algebra $\mathcal{A}$ is strongly dense in $\mathcal{F}_0$. The GNS subspace generated by $\mathcal{A}$ is therefore dense in the GNS subspace generated by $\mathcal{F}_0$. The strong closure $\overline{\mathcal{A}} = \overline{\mathcal{F}_0}$ coincides with the Von Neumann algebra of the operators, in the representation space of π_0, which are invariant under (the unique continuous extension of) β_g, $g \in \mathcal{G}_0$.*

Proof. Assume that the strong closure of $\mathcal{A}$ is contained properly in the strong closure of $\mathcal{F}_0$; the commutant of $\mathcal{F}_0$ is then contained properly in the commutant of $\mathcal{A}$, i.e. there exists an operator in the representation space $\mathcal{H}_\pi$ which commutes with $\mathcal{A}$ but not with $\mathcal{F}_0$; by taking the hermitean (or antihermitean) part, and using the spectral theorem, a unitary operator is constructed with the same properties. This defines an automorphism of $\mathcal{F}$ which leaves $\mathcal{A}$ pointwise invariant, and therefore a gauge automorphism of $\mathcal{F}$ implemented in π_0, which does not act trivially on $\mathcal{F}_0$, contrary to its definition. By the same argument one proves the last statement.

It follows from Proposition 3 that, if the gauge group is broken to the identity, then $\mathcal{F}^0_\pi = \mathcal{F}$, and the representation space of π_0 coincides with that of the GNS representation of the observables over ω_0, i.e. all the states are obtained by applying observables to the vacuum.

The Poincaré automorphisms of $\mathcal{A}$ are then implementeted by unitary operators $U(a, \Lambda)$ in this representation, which belong, by irreducibility, to the strong closure of $\mathcal{A}$. If the extension of the Poincaré automorphisms to $\mathcal{F}$ is done in π_0 by $U(a,\Lambda)BU^*(a,\Lambda)$, $B \in \mathcal{F}$, it does not in general commute with the gauge group; in fact, even if $U(a,\Lambda)$ are the strong limits in π_0 of $U_L(a,\Lambda) \in \mathcal{A}$, such limits are taken in a strong topology which is not gauge invariant, and the gauge automorphisms are not continuous with respect to it.

A gauge invariant extension requires the use of a gauge invariant strong topology, given by a representation stable under β_g. In the representation π_0^{inv}, obtained as a direct sum of the GNS representations of $\mathcal{F}$ over $\omega_0 \circ \beta_g$, the strong convergence of $\pi_0(U_L(a,\Lambda))$ implies the strong convergence of $\pi_0^{inv}(U_L(a,\Lambda))$, by definition of π_0^{inv} and invariance of U_L under β_g. Their limit is invariant under the unique strongly continuous extension of β_g, and defines an extension of the Poincaré group to the strong closure of $\mathcal{F}$ in π_0^{inv}, which commutes with the gauge group.

It is immediate to see that the strong closure of the field algebra in the representation π_0^{inv} has a centre, $\mathcal{Z}_F$, which is abelian, because π_0^{inv} is a direct sum if irreducible representations of $\mathcal{F}$, and has a spectrum isomorphic to the gauge group (with the discrete topology). The Poincaré automorphisms do not in general leave $\mathcal{F}$ stable, and it follows from their construction that they leave invariant the algebra generated by $\mathcal{F}$ and $\mathcal{Z}_F$, which may be taken as a new field algebra, on which the Poincaré automorphisms always exist and commute with the gauge group. We have therefore proven

Proposition 4. *If the gauge group is broken in π_0 to the identity, then the Poincaré automorphisms extend to automorphisms of the algebra generated by $\mathcal{F}$ and the centre of the Von Neumann algebra generated by $\pi_0^{inv}(\mathcal{F})$, and commute with the gauge group.*

The centre can be essential for the gauge invariance of the Poincaré automorphisms, as we will see in the models below. The appearance of central variables in the dynamics of $\mathcal{F}$ allows for an evasion of the Goldstone theorem, and is at the basis of the (Higgs) phenomenon of mass generation accompanying the spontaneous breaking of the gauge group (*spontaneous* indicating the commutation between the gauge group and the time translations) [10]. The point is that in general central variables appear if the Poncaré automorphisms and the gauge group are formulated so that they commute; in the ordinary Goldstone theorem such central variables are excluded by the assumption that a symmetry is generated by a local current. If the action of local implementers converges strongly on $\mathcal{F}$ in π_0, then a very similar argument to that given above shows that they converge strongly in π_0^{inv}, and the limit may then involve central variables *as a consequence of the non local character of the charged fields.* The same structures are present in non relativistic models with long range (Coulomb) or mean field interactions [10],[12], a prototype being Haag's treatment of the BCS model [13]. We may also observe that Proposition 3 applies to any symmetry group, with $\mathcal{A}$ playing the rôle of the neutral subalgebra, but a mass gap is produced only if central variables appear in the dynamics of charged fields, and therefore only if the latter are sufficiently non–local with respect to the observables.

We can now discuss a converse of Proposition 2, and its implications on confinement.

Proposition 5. *If* $[\alpha_p, \beta_g] = 0$ *for all* $p \in \mathcal{P}_0$ *and for all* β_g *which are not broken in* π_0*, then the automorphisms* α_p *are implemented in* π_0 *by unitary operators belonging to the strong closure of* $\pi_0(\mathcal{A})$.

Proof. ω_0 is invariant under β_g, as a consequence of v), and under α_p, because of iv); there exist therefore implementers $U(p)$ and $V(g)$, which by construction leave invariant ψ_0, the representative vector of ω_0; hence

$$U(p)V(g)B\psi_0 = \alpha_p\beta_g(B)\psi_0 = \beta_g\alpha_p(B)\psi_0 V(g)U(p)B\psi_0 \quad \forall B \in \mathcal{F}$$

and therefore U and V commute, by the ciclicity of ψ_0. Thus, by Prop.3, $U(p) \in \overline{\pi_0}(\mathcal{A})$.

By applying the construction in the proof of Proposition 4, which only uses the fact that in π_0 α_p is implemented by strong limits of observables, the assumptions of Proposition 5 imply that all α_p, $p \in \mathcal{P}_0$, extend to automorphisms of the algebra generated by $\mathcal{F}$ and $\mathcal{Z}_F$ in π_0^{inv}, which commute with β_g for *all* $g \in \mathcal{G}$. Since the form of the automorphisms which commute with the *broken* gauge transformations is determined by the construction given in the proof of Proposition 4, a version of Proposition 5 also applies to a field algebra with Poincaré transformations invariant under all the gauge group, and implies the existence of implemeters invariant under all the gauge group:

Proposition 6. *If* $[\alpha_p, \beta_g] = 0$ *for all* $g \in \mathcal{G}$*, then all the automorphisms* α_p *are implemented in* π_0^{inv} *by gauge invariant operators, i.e. operators in the strong closure of* $\pi_0^{inv}(\mathcal{A})$.

It follows from Propositions 5 and 6 that the existence of Poincaré automorphisms commuting with the gauge group always leads to implementers which leave invariant the Hilbert sectors defined (in π_0 or in π_0^{inv}) by the representations of the observable algebra $\mathcal{A}$; Poincaré automorphisms are therefore in this case never broken in the observable sectors. Since the Lorentz boosts are broken in QED, and since the breaking of time translations is typical of confined models (see below), we conclude that

Poincaré automorphism, if they exist, cannot commute with the gauge group in these cases; the lack of commutativity between gauge transformations and time translations may in fact characterize confinement [14], since it is equivalent (Prop.2 and Prop.5) to the non–existence of the energy as an observable, in the charged sectors. However, such characterization does not cover the case of time translations implemented by an (observable) energy unbounded from below, a mechanism which seems to occur in QED(2+1), if the "photons" do not acquire a mass [15].

4. Models

The general structures outlined above can be seen and explicitly controlled in soluble models. We discuss in the following the Stückelberg–Kibble and the Schwinger model; the first is a prototype of the Higgs phenomenon, the second of confinement. As we shall see, however, confinement takes place also in the S–K model, for low space dimensions, and this phenomenon is explicitely seen to depend in a very direct way upon the general alternative discussed above for the field algebra and the gauge group.

The S–K model is defined by a linearization of the (abelian) Higgs–Kibble model, corresponding to fixing the modulus of the Higgs field and treating the phase as a scalar field; the Lagrangean density is

$$\mathcal{L}_{SK} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(\partial_\mu \chi - eA_\mu)^2$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and χ a scalar (Higgs) field. It will be considered for space–time dimensions $d+1$, $d = 3,\ 2,\ 1$.

The Schwinger model is given, in bosonized form, by the Lagrangean density

$$\mathcal{L}_S = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(\partial_\mu \varphi)^2 + e\,\varepsilon^{\mu\nu}\partial_\mu\varphi A_\nu$$

with φ the (pseudo–) scalar field in terms of which the fermion field is expressed, in two space–time dimensions.

Observable algebras

We will first discuss the models in terms of observable algebras. These are defined by the correlation functions of gauge invariant fields on the vacuum, which are by definition independent of the gauge fixing. The rôle of the gauge fixing is that of giving (non local) relations between the fields which appear in the Lagrangean and the observables, allowing for the construction of the corresponding field algebras. The construction follows therefore a different logic, compared to that of ref. [16], where the field algebra at a fixed time was first defined in terms of canonical variables, and then the time evolution was constructed, meeting problem and features very close to those of non–relativistic Coulomb systems [17].

To obtain the observable algebras, we start from the equations of motion for the observable fields given by the above Lagrangeans and assume local commutativity for the corresponding quantum fields; it is then easy to characterize (all) the Poincaré invariant correlation functions of such fields satisfying the relativistic spectral condition, and obtain the complete algebraic structure of the observable fields, which will correspond to a canonical (Weyl) algebra.

For both models, the equation of motion for $F_{\mu\nu}$ are the Maxwell equations,

$$\partial_\mu F^{\mu\nu} - j^\nu = 0 \tag{2}$$

with j^ν given, in the S–K model, by

$$j^\nu = e\,(\partial^\nu \chi - eA^\nu) \tag{3}$$

and, in the Schwinger model, by

$$j^\nu = e\,\varepsilon^{\mu\nu}\partial_\mu \varphi \tag{4}$$

For the S–K model, from eq.(4) it follows

$$e^2\,F^{\mu\nu} = -(\partial^\mu j^\nu - \partial^\nu j^\mu) \tag{5}$$

and, from eqs. (2) and (5),

$$\Box j^\mu + e^2 j^\mu = 0 \qquad\qquad \partial_\mu j^\mu = 0 \tag{6}$$

The current fields j^μ are therefore, in the S–K model, free massive fields of mass $m^2 = e^2$. Their two point function on a Poincaré invariant state satisfying the relativistic spectral condition is determined (up to a constant factor) by eq.(6),

$$< j^\mu(x) j^\nu(y) >= J^{\mu\nu}(x-y)$$

$$\tilde{J}^{\mu\nu}(k) = (g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2})\,\delta(k^2 - m^2)\,\theta(k^0) \tag{7}$$

By local commutativity and the the Jost–Schroer theorem, all the truncated correlation funtions of j^μ vanish, and the commutator $[j^\mu(x), j^\nu(y)]$ is a c–number valued distribution, determined by eq.(7). It follows that, in the Hilbert space $\mathcal{H}$ given by the Wightman reconstruction, the exponentials of the smeared fields $W(f) \equiv \exp(j^\mu(f_\mu))$, f_μ real in the Schwartz space $\mathcal{S}(\mathbb{R}^{d+1})$, generate a Weyl algebra $\mathcal{A}$, defined by the symplectic form

$$< f, g >= \int d^4k \, (g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}) \, \delta(k^2 - m^2) \, \varepsilon(k^0) \, \tilde{f}_\mu(-k) \, \tilde{g}_\nu(k) \tag{8}$$

The Poincaré invariant state on the Wightman fields j^μ defines a state ω_0 on $\mathcal{A}$, given by

$$\omega_0(W(f)) = e^{-[f,f]/4} \tag{9}$$

$$[f, f] = \int d^4k \, (g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}) \, \delta(k^2 - m^2) \, \tilde{f}_\mu(-k) \, \tilde{g}_\nu(k) \tag{10}$$

and $\mathcal{H}$ is the GNS space of $\mathcal{A}$ on the state ω_0. We have therefore obtained the observable algebra and its vacuum representation.

For the Schwinger model, the equation of motion for φ is

$$\partial_\mu(-\partial^\mu \varphi + e \, \varepsilon^{\mu\nu} A_\nu) = 0 \tag{11}$$

Eq. (11) can also be written

$$\Box \varphi - e F_{01} = 0 \tag{12}$$

Equations (2) and (4) give the relations

$$\partial_\mu(e\varphi - F_{01}) = 0 \qquad\qquad \mu = 0, 1 \tag{13}$$

and therefore

$$e \, \varphi = F_{01} + \sigma \tag{14}$$

with σ a field invariant under space–time translations. Eqs.(12) and (14) imply that F_{01} is a free massive fields, of mass $m^2 = e^2$. It is important to remark that the (Wick) exponentials of φ are observable, since they correspond to the bilinears of the fermion field, so that the observable algebra must include (as unbounded operators affiliated to the local Von Neumann algebras) F_{01} *and* φ.

The one point function of F_{01} on a state invariant under the proper Poincaré group vanishes by eq.(12), and the two point function is that of a massive field. Assuming as before local commutativity, which implies the vanishing of all higher order truncated correlation functions, the observable algebra must then be identified as the algebra $\mathcal{A}_S$ generated by the Weyl exponentials of the massive free field $F_{01}(f)$, $f \in \mathcal{S}(\mathbb{R}^2)$, and by the variable $\exp i\alpha\sigma$, which, by local commutativity and space–time invariance, is in the centre of $\mathcal{A}_S$.

The appearence of central variables in the observable algebra is related to chiral symmetry, which is here well defined as an automorphism of $\mathcal{A}$, commuting with the (proper) Poincaré group:

$$\beta^\lambda \, e^{i\varphi(f)} = e^{i\lambda\tilde{f}(0)} \, e^{i\varphi(f)} \qquad \beta^\lambda \, e^{iF_{01}(f)} = e^{iF_{01}(f)} \tag{15}$$

The presence of the central variable σ is essential for the validity of eq.(15), i.e. for the existence of chiral automorphisms commuting with the (proper) Poincaré group. In fact, if a factorial, in particular irreducible, representation π of $\mathcal{A}$ is considered, then $\pi(\sigma)$ is a number, and any automorphism of $\pi(\mathcal{A})$ which shifts φ must also shift the massive field F_{01}, and cannot therefore commute with the space–time translations.

One recovers in this way the alternative, typical for symmetries in systems with long range forces, between
i) a symmetric algebraic dynamics (which naturally arises as the thermodynamic limit of a symmetric finite volume dynamics), which involves central variables [10]
ii) the use of a simple algebra, with the consequence of a non–symmetric dynamics; this is obtained as the infinite volume limit of finite volume dynamics generated by Hamiltonians with non–symmetric boundary terms [18].

Moreover, the mass spectrum of the Schwinger model can be seen as the spectrum associated to the spontaneous breaking of the chiral transformations [16].

Field algebras

The field algebras defined by the lagrangean variables in the Coulomb gauge can now be constructed as extended Weyl algebras. For the details, see refs. [14],[16]. We start from the Coulomb gauge relation

$$-\triangle A^0 = j^0 \tag{16}$$

in order to construct A^0. Once the variable A^0 has been constructed, the Higgs field χ and the A^i fields in the S–K model follow immediately from eq.(3), which gives $\partial^0\chi$ in terms of A^0, and A^i in terms of j^i and $\partial^i\chi$.

We look therefore for an operator valued distribution solution of eq.(16), defined in a Hilbert space $\mathcal{H}$, with a cyclic vector ψ_0 invariant under space–time translations.

The one point function of A^0 is then constant, and we will fix it to 0 for the moment; the two point function $W(x-y)$ is of positive type, and its Fourier transform is therefore a measure, $\tilde{W}(k)$, satisfying

$$|\mathbf{k}|^4 \, \tilde{W}(k) = J^{00}(k) \tag{17}$$

The solution of eq.(17) is unique up to $\delta(\mathbf{k})a(k^0)$, and this term is excluded if in $\mathcal{H}$ there is only one vector invariant under space translations. Moreover, the solution exists if and only if

$$\frac{\tilde{J}^{00}(k)}{|\mathbf{k}|^4} = \frac{1}{m^2 \mathbf{k}^2} \, \delta(k^2 - m^2) \, \theta(k^0) \tag{18}$$

is a measure (see eqs.(9),(10)). This is true for the S–K model in space dimensions $d = 3$, but not for $d = 2, 1$, nor for the Schwinger model.

1. Stückelberg–Kibble model in $3+1$ dimensions.

The solution of eq.(16) is in this case uniquely determined by the one and two point functions, assuming that all the higher order truncated correlation functions vanish. In order to express the solution in terms of the observable Weyl algebra $\mathcal{A}$, it is enough to notice that the form $[g, g]$, which defines the state ω_0 on $\mathcal{A}$, remains finite on $\triangle^{-1}\mathcal{S}$, defined in Fourier space (on the support of $\delta(k^2 - m^2)$) by $\{\tilde{f}/\mathbf{k^2}\}$, $f \in \mathcal{S}(\mathbb{R}^4)$; the Weyl operators can then be extended by strong continuity to $\triangle^{-1}\mathcal{S}$, since sequences $W(f_n)$ of Weyl operators converge strongly, in a GNS representation over a quasi–free state ω, if and only if f_n converge strongly in the scalar product $[f, f]$ which defines ω (eq.(9)). The solution of eq.(16) exists therefore in the strong closure of the observable algebra, in the vacuum representation.

Given A^0, the Higgs field χ is determined by eq.(3), namely $e\,\partial^0 \chi = -\partial_0^2 A^0$, and, as already discussed for A^0, we may construct χ as

$$\chi = -\frac{1}{e} \, \partial_0 A^0 \tag{19}$$

We have therefore constructed the field algebra, in the Coulomb gauge, as the Weyl algebra over the extended space $\triangle^{-1}\mathcal{S}$; moreover, since this algebra is regularly represented by ω_0, we may include in the field algebra all the bounded functions of the fields.

The observable algebra is *strongly dense* in the field algebra, in the vacuum representation π_0, and there are *no charged states. The gauge group* consists of the automorphisms $\gamma^{\lambda\mu}$ defined by

$$\gamma^{\lambda\mu} \, (e^{iA^0(f)}) = e^{i\lambda \; Re \tilde{f}(m,\mathbf{0})} \; e^{i\mu \; Im \tilde{f}(-m,\mathbf{0})} \; e^{iA^0(f)} \tag{20}$$

corresponding to

$$A^0(\mathbf{x},0) \mapsto A^0(\mathbf{x},0) + \lambda \qquad\qquad \chi(\mathbf{x},0) \mapsto \chi(\mathbf{x},0) + \mu$$

and *it is broken* in π_0 to the identity (See Proposition 3).

It follows imediately from eq.(20) that the space–time translations, defined (by construction) on the field algebra by the unitary group which implements the space–time translations for the observable algebra in the vacuum representation, do not commute with $\gamma^{\lambda\mu}$. A representation π_0^{inv} of the field algebra can be immediately constructed as the direct sum of the GNS representations over the states $\gamma^{\lambda\mu\,*}\omega_0$, and space time translations commuting with all $\gamma^{\lambda\mu}$ can be easily constructed on the algebra generated by the Weyl algebra over $\triangle^{-1}\mathcal{S}$ and the centre of its strong closure in π_0^{inv}; they have gauge invariant implementers, given in each irreducible representation by the action of the gauge automorphisms (which are strongly continuos in π_0^{inv}) on the implementers in π_0. (See Prop.4).

A field algebra with a non–trivial centre, and the same structure for the space–time translation automorphisms, is also obtained if the time evolution is constructed [16] as a strong limit of infrared cut–off dynamics defined by Hamiltonians invariant under the gauge group, with a strong topology invariant under the gauge automorphisms. The mass spectrum of the model is associated to the spontaneous breaking of the automorphisms $\gamma^{\lambda\mu}$, through a generalized Goldstone theorem [10].

2. Stückelberg–Kibble model in $2+1$ and $1+1$ dimensions; Schwinger model.

For the S–K model in 2+1 and 1+1 dimensions, and for the Schwinger model, the above construction does not apply, since the quadratic form which defines the vacuum state on the observables is divergent on $\triangle^{-1}\mathcal{S}$, and cannot in fact be extended to a positive form on $\triangle^{-1}\mathcal{S}$ which still majorizes the extension of the symplectic form to $< \triangle^{-1}f, g >$, $f, g \in \mathcal{S}$, a necessary condition for the positivity of the resulting state.

The construction of the field algebras for these models can be done by an extension of the observable Weyl algebra to an algebra defined in an abstract way as the Weyl algebra over a space of C^∞ functions (linearly bounded in the space variables) $\triangle^{-1}\mathcal{S}$, with a symplectic form $< f, g >$ invariant under space–time translations and extending the symplectic form on $\mathcal{S}\times\mathcal{S}$ [14].

As a result of an analysis of the relation between the symplectic form $< f, g >$, and the quadratic form $[g, g]$, $f \in \triangle^{-1}\mathcal{S}$, $g \in \mathcal{S}$, the extension of the vacuum state to the Weyl algebra of the fields is found to be unique and given by

$$\omega_0(\exp i\alpha A^0(f)) = 0 \qquad \text{if} \quad \tilde{f}(m,0) \neq 0 \tag{21}$$

The vacuum representation of the field algebra is therefore *non regular*, i.e. not strongly continuous in the parameters of the Weyl groups; the variables A^0, and χ for the S–K model, do not exist as field variables, but only in the exponentiated (Weyl) form $\exp i\alpha A^0(f)\,,\ \exp i\beta\chi(f)\,,\ f \in \mathcal{S}\,.$

It follows from eq.(21) that the automorphisms $\gamma^{\lambda\mu}$ are unbroken in the GNS representation of the field algebra defined by the unique extension of the vacuum state. The application to the vacuum of the charged field variables gives therefore rise to charged states, orthogonal to the vacuum representation of the observables, and the representations of the observable algebra obtained by the GNS construction over such states are easily seen to be inequivalent to the vacuum representation. The expectation value of the electric field gives rise to a non–trivial Gauss charge in the charged sectors.

The space translation automorphisms are well defined on the (Coulomb gauge) field algebra and are implemented by strongly continuous unitary groups; the (space) momentum is therefore well defined also in the charged sectors. The time evolution automorphisms, which exist on the field algebra as a consequence of the invariance under time translations of the extended symplectic form, leave the (unique extension of) the vacuum state invariant; they are therefore unitarily implemented and define a time evolution of the charged states, which give rise to representations of the observable algebras which are inequivalent for different times. The same result is obtained, by invariance of the vacuum under time translations, if one considers the states obtained from a charged state by applying time translations automorphisms to the *observable* algebra. The implementers of the time translations are therefore not strongly continuous, and have no generator, i.e. the Hamiltonian does not exist in the charged sectors. (See Propositions 2 and 5).

As in the general analysis given above, the reason is that the gauge automorphisms $\gamma^{\lambda\mu}$ do not commute with the time evolution automorphisms, and therefore the time evolution of a charged state gives rise to states with different values of the (unbroken) charges, with an electric flux at infinity which oscillates in time.

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