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Distribution of roots of random real generalized polynomials

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Abstract. The average density of zeros for monic generalized polynomials, $P_n(z) = \phi(z) + \sum_{k=1}^{n} c_k f_k(z)$, with real holomorphic ϕ, f_k and real Gaussian coefficients is expressed in terms of correlation functions of the values of the polynomial and its derivative. We obtain compact expressions for both the regular component (generated by the complex roots) and the singular one (real roots) of the average density of roots. The density of the regular component goes to zero in the vicinity of the real axis like $|\mathrm{Im}\, z|$. We present the low and high disorder asymptotic behaviors. Then we particularize to the large n limit of the average density of complex roots of monic algebraic polynomials of the form $P_n(z) = z^n + \sum_{k=1}^{n} c_k z^{n-k}$ with real independent, identically distributed Gaussian coefficients having zero mean and dispersion $\delta = \frac{1}{\sqrt{n\lambda}}$. The average density tends to a simple, *universal* function of $\xi = 2n\log|z|$ and λ in the domain $\xi \coth\frac{\xi}{2} \ll n|\sin\arg(z)|$ where nearly all the roots are located for large n.

I. INTRODUCTION

Let P_n be a (monic) algebraic polynomial of degree n,

$$P_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n. \tag{1.1}$$

The roots of P_n are (algebraic) functions of its coefficients $a_1, a_2, \cdots, a_n$. If the coefficients are random variables, then the roots of the polynomial will also be random variables. The study of the distribution of the roots of random polynomials began with the investigation by Bloch and Polya [1], continued by Littlewood and Offord [2,3], Kac [4] and many others (for reviews see [5,6]) of the number and distribution law of the real roots of random polynomials with real coefficients.

Most investigations of the distribution of zeros for random algebraic polynomials have dealt either with the initial problem – the real zeros of real polynomials – or with the complex zeros of complex polynomials. In the latter case, one is helped by the fact that it is possible to transform the joint distribution function of the coefficients into the joint distribution function of the roots explicitly using the relations between the coefficients of an algebraic polynomial and its roots.[1] The average distribution of roots for homogeneous algebraic polynomials,

$$P_{n-1}(z) = \sum_{j=0}^{n-1} c_j z^j,$$

with real normal Gaussian coefficients c_j, was recently investigated by Shepp and Vanderbei[2] [10]. They obtain a generalized Kac-Rice formula for the density of complex roots and explore its large n limit. The fraction of the expected number of roots contained in an

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[1]Which follow by comparing the polynomial $P_n(z) = \sum_{k=0}^{n} a_k z^{n-k}$ with its roots expansion $P_n(z) = a_0 \prod_{j=1}^{n}(z - z_j)$. The Jacobi determinant for this transformation was computed by Girschick and Hammersley [7–9].

[2]We thank an anonymous referee for bringing this paper to our attention

angular sector $S(\theta_1,\theta_2)$ which does not intersect the real axis[3] tends to $|\theta_2-\theta_1|/2\pi$. Most roots are concentrated in a small annulus (with width $\sim n^{-1}$) near the unit circle. $\mathcal{N}(R)$, the expected number of roots in a ball of radius R, satisfies

$$\lim_{n\to\infty}\frac{1}{n}\mathcal{N}(\mathrm{e}^{\frac{s}{2n}})=\frac{1}{1-\mathrm{e}^{-s}}-\frac{1}{s}. \tag{1.2}$$

In a recent preprint by Ibragimov and Zeitouni this result is generalized to coefficient distributions belonging to the domain of attraction of a α-stable law [12].

In this paper we study the distribution of the roots for more general random polynomials with *real* coefficients. The initial motivation for the present work was given by some interesting properties of the roots of the Szegö orthogonal polynomials [13] related to the Wiener transfer function [14]. Szegö polynomials are orthogonal polynomials in the non negative measure on the unit circle of the complex plane $\mathrm{d}\mu(\theta)$ generated by a non-negative Töplitz form $T(k-l)\,;k,l=0,1,\ldots$. Here

$$T(k)=\int_{-\pi}^{\pi}\mathrm{d}\mu(\theta)\mathrm{e}^{-\mathrm{i}k\theta} \tag{1.3}$$

are the measure's moments. In the case of the Wiener transfer function the moments are equal to the autocorrelation function of a discrete finite real signal sample ($X(p)=0;\ p<0$ or $p\geq N$):

$$T(k)=F_N(k)=\sum_{p=0}^{N-1}X(p)X(p+k) \tag{1.4}$$

Let the signal be the sum of a useful signal, consisting of several harmonic components with frequencies ω_ν, and noise. Then there is strong numerical and some analytic evidence that for signal-to-noise ratios that are not too low, the roots of the Szegö polynomials of order n with $1\ll n\ll N$ break into two groups. The first one converges rapidly to the unit circle at the points $\mathrm{e}^{\mathrm{i}\omega_\nu}$, [15–17]. The other group, which also converges to the unit circle, but

[3]The analogous result for complex coefficients was obtained by Shparo and Shur [11].

more slowly, is nearly equispaced – resembling an one-dimensional crystal. It has universal statistical properties even when the harmonic components of the signal are absent [18,19].

If the useful signal is absent, the Szegö polynomial of order n is well approximated by (1.1) with $a_k = N^{-\frac{1}{2}} c_k$ with approximately Gaussian coefficients c_k if $1 \ll n \ll N$ [18]. In the presence of a useful signal, the polynomials have a more complicated form, which can be obtained by substituting more general polynomials for the monomials z^k in (1.1). Thus, we are led to considering *generalized random polynomials.*

Let

$$\phi(z),\ f_k(z),\ k = 1, 2, \ldots, n\ \ ,\ n \in \mathbb{N} \tag{1.5}$$

be holomorphic and linearly independent functions of the complex variable z in a domain of the complex plane, $z = x + \mathrm{i}y \in D \subset \mathbb{C}$. Let $\mathbf{c}(\omega)$ be a random n-dimensional vector with components $c_k(\omega),\ k = 1, 2, \ldots, n\ ,\ n \in \mathbb{N}$. Here $\omega \in \Omega$ where Ω is a probability space.

We define random generalized *monic* polynomials of degree n by

$$P_n(z) = \phi(z) + \sum_{1 \le k \le n} c_k f_k(z). \tag{1.6}$$

Homogeneous polynomials (of degree $n-1$) correspond to setting $\phi \equiv 0$. Alternatively, a monic polynomial of degree n can be regarded as a homogeneous one, having a singular distribution for the coefficient c_{n+1} ($\delta(c_{n+1} - 1)$). Setting $\phi(z) = z^n, f_k(z) = z^{n-k},\ k = 1, 2, \ldots, n\ , n \in \mathbb{N}$ we obtain the random monic algebraic polynomials, (1.1). Trigonometric, hyperbolic and other types of random polynomials [5] can be obtained by suitably defining ϕ and f_k. In the following we will often omit the qualifier *generalized* and call the objects defined by Eq.(1.6) simply polynomials.

Let us note that any non-random affine transformation of the random vector of the coefficients,

$$c_k \to a_k + \sum \mathsf{K}_{km} \tilde{c}_m \tag{1.7}$$

where $\mathbf{a}$ is a constant vector and K a non singular matrix, transforms the polynomial P_n into one of the same form (1.6) but with the basis set (1.5) replaced by

$$\tilde{\phi} = \phi + \sum_m a_m f_m; , \quad \tilde{f}_k = \sum_m \mathsf{K}_{mk} f_m. \tag{1.8}$$

Thus, by redefining the basis set, we can always consider that the mathematical expectation of the polynomial is equal to its deterministic part, ϕ.

Let $F(z;\omega)$ be a random holomorphic function of $z \in D \subset \mathbb{C}$, *i.e.* a family of functions $F(z;\omega)$ indexed by $\omega \in \Omega$ which are almost surely holomorphic for z in the domain D. Let

$$z_r(\omega) = x_r(\omega) + iy_r(\omega), \ \ r = 1, \ldots \tag{1.9}$$

be the solutions of the equation

$$F(z;\omega) = 0. \tag{1.10}$$

The zeros of F are random variables. In each compact subdomain $D_1 \subset D$, there may be only a finite number $N(D_1;\omega)$, of zeros for each realization $\omega \in \Omega$, since the accumulation points of zeros cannot lie inside the domain of holomorphy.

The density of zeros of the random function F is the random distribution on $\mathbb{R}^2$:

$$\begin{aligned}\rho(x, y;\omega) &= \sum_r \delta(x - x_r(\omega))\delta(y - y_r(\omega)) \\ &= \sum_r \delta^{(2)}(z - z_r(\omega)),\end{aligned} \tag{1.11}$$

where δ and $\delta^{(2)}$ are, respectively, the Dirac distribution on $\mathbb{R}$ and on $\mathbb{R}^2$. Using the definition of the Dirac δ we may rewrite (1.11) as

$$\rho(x, y;\omega) = \left|\dot{F}(z;\omega)\right|^2 \delta^{(2)}(F(z;\omega)), \tag{1.12}$$

where the Cauchy-Riemann conditions were used to calculate the Jacobian. The compact notation

$$\dot{F} = \frac{\mathrm{d}F}{\mathrm{d}z} \tag{1.13}$$

for the derivative of F will be used throughout this paper.

The expected (average) density of zeros of F is obtained by averaging over the realizations:

$$\mathcal{D}(x,y) = \mathbb{E}\left\{\rho(x,y;\circ)\right\}. \tag{1.14}$$

In a similar way, one may define the two-point correlation function of the zeros by

$$\mathcal{D}_2(x_1,y_1;x_2,y_2) = \mathbb{E}\left\{\rho(x_1,y_1;\circ)\rho(x_2,y_2;\circ)\right\} \tag{1.15}$$

and higher, m-point, correlation functions for the roots.

Substituting (1.12) into (1.14) and introducing the joint distribution function of the values of the function and its derivative at the point $z = x + \mathrm{i}y$

$$\mathcal{P}(\alpha,\tilde{\alpha};z) = \mathbb{E}\left\{\delta^{(2)}(\alpha - F(z;\circ))\delta^{(2)}(\tilde{\alpha} - \dot{F}(z;\circ))\right\}, \tag{1.16}$$

we see that the average density of roots at the point $z = x + \mathrm{i}y$ is given by the Kac-Rice [4,5,20] formula

$$\begin{aligned}\mathcal{D}(x,y) &= \mathbb{E}\{|\dot{F}(z;\circ)|^2\delta^{(2)}(F(z;\circ))\} \\ &= \int |\tilde{\alpha}|^2\mathcal{P}(0,\tilde{\alpha};z)\mathrm{d}^{(2)}\tilde{\alpha}.\end{aligned} \tag{1.17}$$

in terms of $\mathcal{P}(\alpha,\tilde{\alpha};z)$. Similar formulas, involving only the joint distribution function of the values of the function and its derivative at the selected points may be written for the m-point correlation functions of the roots such as (1.15). We will study the average distribution of roots, Eq.(1.14), in the case when $F(z,\omega)$ is a generalized random polynomial, (1.6), with Gaussian coefficients.

If the components of $\mathbf{c}(\omega)$ in (1.6) are Gaussian (real or complex) random variables, the joint distribution function of the values of the polynomial and its derivative at some point z (1.16) will also be also a Gaussian distribution which is determined by the correlation functions (covariance matrix) of the values of the polynomial and its derivative at that point. This means that we may compute explicitly the integral in Eq.(1.17) for the average density of roots in the Gaussian case and the corresponding expressions for the m-point correlation functions of the roots.

In Section III we will obtain the average density of roots, Eq.(1.14), for *real* Gaussian generalized monic polynomials, when the basis functions in (1.5) are of *real* type

$$[\phi(z)]^* = \phi(z^*), \quad [f_k(z)]^* = f_k(z^*). \tag{1.18}$$

Here * denotes complex conjugation. The coefficients c_k are independent, identically distributed (*(iid)*) real Gaussian random variables with zero expectation value. The assumption of a joint Gaussian distribution is important, while the restriction to the *iid* case is inconsequential. Indeed, by a suitable linear mapping of type (1.7) any finite Gaussian distribution may be mapped onto the standard one (*iid* with zero average and unit dispersion). By the above remark this leads to a redefinition (1.8) of the basis set.

We will obtain a general formula giving the expected density of complex roots at points with $\mathrm{Im}(z) \neq 0$. Due to the reality condition, (1.18), there is also a singular component of the expected density of roots located on the real axis. We will obtain a generalization of the Kac-Rice formula for it. For homogeneous polynomials with non-central Gaussian distributed coefficients we recover a result by Edelman and Kostlan [6], who also pointed out that taking a singular correlation matrix limit for that case will yield the density of real roots in the monic case. We will show that the density of complex roots tends to zero, like $|\mathrm{Im}(z)|$, in the vicinity of the real axis. In the high randomness limit (large dispersion δ of the Gaussian distribution) the expected density of roots for monic polynomials approaches, obviously, that for the homogeneous ones ($\phi \equiv 0$). In the weak randomness limit, when the dispersion δ goes to zero, the roots concentrate near the zeros of the deterministic part ϕ. Near simple zeros of ϕ the distribution tends to a Gaussian one. Near zeros of ϕ with multiplicity $k > 1$, it has generically $2k$ maxima at a distance $\sim \delta^{\frac{1}{k}}$ from the zero's position. An interesting situation arises when the multiplicity of the zero is large.

Since only the joint distribution function of the polynomial and its derivative is assumed to be Gaussian, the results of Section III could also be relevant to cases when the coefficients are non-Gaussian but one may prove a limit theorem for the joint distribution function.

In Section IV we will apply the general results of Section III to the density of roots of monic algebraic random polynomials, (1.1), with *iid* coefficients having a real Gaussian distribution with zero average and dispersion $\delta = N^{-\frac{1}{2}}$ in the large n and N limit. In this case the deterministic part has a real zero of multiplicity n at the origin. As mentioned above, in [18] the Szegö polynomials associated to the Wiener transfer function for pure noise were shown to have this form if the sample length is $N \gg n \gg 1$.

Defining a new rescaled coordinate $\xi = 2n \ln |z|$ and rescaling also the dispersion, $\delta^{-1} = N = n\lambda$, we obtain a simple asymptotic expression for the average density of roots valid for large n at points which satisfy

$$\xi \coth \frac{\xi}{2} \ll n|\sin \arg(z)|. \tag{1.19}$$

This is a small neighborhood of the unit circle which does not intersect the real axis and it contains nearly all the zeros. To leading order, the average density in this domain does not depend on $\arg(z)$. It approaches n^2 times a universal function of ξ and λ. For $\sin\theta \neq 0$,

$$\lim_{n\to\infty} \frac{1}{n^2} \mathcal{D}(\mathrm{e}^{\frac{\xi}{2n}}, \theta) = -\frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}\xi} \left\{ \left(\frac{1}{\xi} - \frac{1}{\mathrm{e}^{\xi} - 1} \right) \exp \left[-\lambda \frac{\xi}{1 - \mathrm{e}^{-\xi}} \right] \right\}. \tag{1.20}$$

In the case of homogeneous polynomials, which corresponds to $\lambda = 0$, (1.20) is equivalent to (1.2), the result of Shepp and Vanderbei [10].

We will finally remark that the asymptotic distribution of complex roots for polynomials with *real* Gaussian coefficients, (1.20), which do not have the rotational symmetry of the polynomials with *complex* coefficients, nevertheless coincides with the rotationally invariant one for the latter, which can be readily obtained using the methods we use. For the complex homogeneous case ($\lambda = 0$), the asymptotic estimate (1.2) is due to Arnold [21,5].

A part of the calculations for the algebraic polynomials will be presented in more detail in the Appendix. Before proceeding further, we will define in the next section some notations that will allow us to write in a compact way the joint distribution function (1.16) and our results when the random function is a polynomial with Gaussian coefficients.

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II. A DIGRESSION ON NOTATIONS

For any holomorphic function, h, we will write

$$\frac{dh}{dz} = \dot{h}. \tag{2.1}$$

for its derivative. The real and imaginary parts of a complex number or function h will be denoted by the subscripts *1* and *2* respectively.

$$h_1 = \mathrm{Re}(h),\ \ h_2 = \mathrm{Im}(h). \tag{2.2}$$

As remarked in the Introduction, for a given realization the coefficients define a point in $\mathbb{R}^n$, while the basis set of functions, ϕ and $f_k,\ k = 1 \ldots n$, maps $\mathbb{C} \sim R^2$ in $\mathbb{C}\times\mathbb{C}^n\ \sim \mathbb{R}^2\times\mathbb{R}^{2n}$. The values of the polynomial $P_n(z)$, Eq.(1.6), and its derivative $\dot{P}_n(z)$ are affine mappings (with complex coefficients) of the real n-vector of the coefficients into $\mathbb{C} \sim R^2$. Let us introduce more compact notations for the various vector structures we are dealing with.

We will use the *same* lower case *Greek* letter (*e.g.* ϕ) for *both* the *complex scalar* $\phi \in \mathbb{C}$ and the *real 2-vector* ϕ whose components $(\phi_1, \phi_2) \in \mathbb{R}^2$ are the real and imaginary parts of the complex scalar ϕ.

Boldface letters denote real n-vectors $\mathbb{R}^n \ni \mathbf{f} = (f_1, f_2, \ldots, f_n)$. When it does not lead to ambiguities, we might abuse the notation to denote a complex n-vector – like the vector made of the basis set of functions $f_k(z),\ k = 1, 2 \ldots, n$ – by $\mathbf{f}(z)$. In such cases, the real and imaginary parts are $\mathbf{f}_1(x,y),\ \mathbf{f}_2(x,y)$ and we may write $\mathbf{f}(z) = \mathbf{f}_1(x,y) + \mathrm{i}\mathbf{f}_2(x,y)$.

Latin letters, $f,\ g$, will be used for *real* $2n$-vectors with components $f_{k\alpha},\ g_{k\alpha},\ k = 1, 2, \ldots, n,\ \alpha = 1, 2$. We use *the same letter* in the typefaces mentioned above for related objects, like $\mathbf{f}(z),\ (\mathbf{f}_1(x,y),\ \mathbf{f}_2(x,y)),\ f(x,y)$ for the basis set of functions at point z considered as a complex n-vector made of two real n-vectors or as a real $2n$-vector.

Greek subscripts always run over the set $\{1,2\}$ while the Latin ones over $\{1,2,\ldots,n\}$.

For a Gaussian distribution of the coefficients, the joint distribution function of the values of the random polynomial $P_n(z)$, Eq.(1.6), and its derivative $\dot{P}_n(z)$ is determined by their correlation matrix. The calculation of averages can be done using Wick's theorem, replacing the products of coefficients by their expectation value. This reduces further to a contraction over the Latin indices if the Gaussian process is a direct sum of normal ones.[4] The compact notations defined in the rest of this section will help us with the necessary bookkeeping.

We use *bold square brackets* $\mathbf{[}\circ,\circ\mathbf{]}$ for the contractions over the Greek indices and *bold round brackets* $\mathbf{(}\circ,\circ\mathbf{)}$ for the contractions over the Latin ones. Using these conventions,

$$\mathbf{[}\phi,\psi\mathbf{]}=\sum_{\alpha=1}^{2}\phi_\alpha\psi_\alpha, \tag{2.3}$$

will be *the real scalar product* of the 2-vectors ϕ and ψ; $\mathbf{[}\phi,f\mathbf{]}$ will denote the real n-vector with components

$$\mathbf{[}\phi,f\mathbf{]}_k=\sum_{\alpha=1}^{2}\phi_\alpha f_{k\alpha}, \tag{2.4}$$

obtained by contacting the direct product between the 2-vector ϕ and the 2n-vector f over the Greek indices. In an analogous way, $\mathbf{[}f,g\mathbf{]}$ will be the second order tensor obtained by contracting over the Greek indices the direct product of the 2n-vectors f and g. Its components form the $n\times n$ matrix

$$\mathbf{[}f,g\mathbf{]}_{jk}=\sum_{\alpha=1}^{2}f_{j\alpha}g_{k\alpha}, \tag{2.5}$$

In a similar way, the scalar product between the real n-vectors $\mathbf{f}$ and $\mathbf{g}$ is

$$\mathbf{(f,g)}=\sum_{k=1}^{n}f_k g_k, \tag{2.6}$$

while

$$\mathbf{(}f,g\mathbf{)}_{\alpha\beta}=\sum_{k=1}^{n}f_{k\alpha}g_{k\beta}, \tag{2.7}$$

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[4] As noted in the introduction this is always possible choosing a suitable (1.7).

are the elements of the 2×2 (real) matrix (f, g), while

$$(\mathbf{c}, f)_\alpha = \sum_{k=1}^{n} c_k f_{k\alpha}, \tag{2.8}$$

are the components of a 2-vector.

Finally, we use matrix notations like $(f, g)\psi$ for the product of the 2×2 matrix (f, g) and the 2-vector ψ,

$$((f, g)\psi)_\alpha = \sum_{\beta=0}^{2} (f, g)_{\alpha\beta} \psi_\beta. \tag{2.9}$$

III. AVERAGE DENSITY OF ROOTS FOR RANDOM GAUSSIAN GENERALIZED MONIC POLYNOMIALS

In this section we consider monic holomorphic polynomials of type (1.6) with coefficients distributed according to a Gaussian law. As we remarked in the Introduction, the case of homogeneous (non monic) polynomials may be obtained by setting $\phi \equiv 0$. There we also noted that it suffices to consider only the case of *iid* normal Gaussian coefficients (with zero expectation and unit dispersion):

$$\begin{aligned} \mathbb{E}\{c_k\} &= 0, \\ & \qquad\qquad j, k = 1, 2, \ \ldots, n. \\ \mathbb{E}\{c_k c_j\} &= \delta_{kj}, \end{aligned} \tag{3.1}$$

The calculation of the average density of roots, their two-point correlation function $\mathcal{D}$, $\mathcal{D}_2$ and even higher correlation functions $\mathcal{D}_m$ may be done in closed form since all the integrals will be Gaussian, although the formulae will tend to become rather encumbering with increasing m.

A. Average density of complex roots

The starting point will be the formula (1.17) for the average density of roots for the polynomial

$$P(z) = \phi(z) + F(z), \tag{3.2}$$

where ϕ is the deterministic (non random) part and

$$F(z) = \sum_{k=1}^{n} c_k f_k(z) = (\mathbf{c}, f(z)).$$

Here $\mathbf{c}$ is the real vector of the Gaussian coefficients, and we use the bold round bracket notation (2.8) introduced in the previous section.

For the sake of clarity of the main points of the calculation, we will first illustrate our approach in the case of homogeneous (non monic) polynomials, *i.e.*

$$\phi \equiv 0,$$

obtaining a generalization of the results derived by Shepp and Vanderbei [10] in the algebraic case ($f_j(z) = z^{j-1}$). Subsequently we will obtain the average density of complex roots in general case of monic polynomials.

In the homogeneous case the joint distribution function for the polynomial and its derivative, P and $\dot{P}$, at the point $z = x + \mathrm{i}y$ coincides with the one for F and $\dot{F}$ at the same point. Then,

$$\mathcal{D} = \int [\tilde{\alpha}, \tilde{\alpha}] \mathcal{P}(0, \tilde{\alpha}) \mathrm{d}^{(2)} \tilde{\alpha}, \tag{3.3}$$

Here we used the bold square bracket notation (2.3) for the contractions introduced in section II to rewrite $|\tilde{\alpha}|^2$ as $[\tilde{\alpha}, \tilde{\alpha}]$.

Now, $\mathcal{P}$ will be a Gaussian distribution determined by $\Delta(z)$, the 4×4 correlation matrix of $F(z)$ and $\dot{F}(z)$. Its elements are the expectation values of products of elements of the 4-vector $\Big(\mathrm{Re}(F), \mathrm{Im}(F), \mathrm{Re}(\dot{F}), \mathrm{Im}(\dot{F})\Big)^T = \Big(F_1, F_2, \dot{F}_1, \dot{F}_2\Big)^T$. For example:

$$\Delta_{14}(z) = \mathbb{E}\{\mathrm{Re}(F(z)) \ \mathrm{Im}(\dot{F}(z))\} = \sum_{k,j} f_{k1}(z) \ \dot{f}_{j2}(z) \mathbb{E}\{c_k c_j\}.$$

Using (3.1) this reduces to

$$\Delta_{14}(z) = \sum_{k} f_{k1}(z)) \ \dot{f}_{k2}(z).$$

We may use the bold round bracket notation (2.7) to write the $4{\times}4$ matrix $\Delta(z)$ as a $2{\times}2$ block matrix whose elements are $2{\times}2$ matrices

$$\Delta(z)=\begin{pmatrix}\big(f(z),f(z)\big) & \big(f(z),\dot f(z)\big)\\ \big(\dot f(z),f(z)\big) & \big(\dot f(z),\dot f(z)\big)\end{pmatrix}. \tag{3.4}$$

As a correlation matrix $\Delta(z)$ is symmetric and non-negative. The same is true for the diagonal blocks in (3.4). Since the functions $f_k(z)$ are linearly independent and of real type, (1.18), $\Delta(z)$ will be generically non singular off the real axis. For $\mathrm{Im}(z)\neq 0$, we may then write the joint distribution function of F and $\dot F$

$$\mathcal{P}(\alpha,\tilde\alpha;z)=\frac{1}{(2\pi)^2\sqrt{\det\,\Delta(z)}}\exp\Big[-\frac{1}{2}(\alpha^T,\tilde\alpha^T)\Delta^{-1}(z)\begin{pmatrix}\alpha\\ \tilde\alpha\end{pmatrix}\Big]. \tag{3.5}$$

Here $\alpha=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ and $\tilde\alpha=\begin{pmatrix}\tilde\alpha_1\\ \tilde\alpha_2\end{pmatrix}$ are 2-component vectors and we use ^{T} to denote the transposed.

For $\mathrm{Im}(z)\neq 0$, where the rank of Δ is equal to four, we may introduce the block-Cholesky decomposition of the positive matrix Δ into block-triangular $2{\times}2$ factors:

$$\Delta=\mathsf{K}^T\mathsf{K}, \tag{3.6}$$

where the upper block triangular matrix

$$\mathsf{K}=\begin{pmatrix}(f,f)^{\frac12} & (f,f)^{-\frac12}(f,\dot f)\\ 0 & \big[(\dot f,\dot f)_\perp\big]^{\frac12}\end{pmatrix}. \tag{3.7}$$

Here we introduced the notation

$$(\dot f,\dot f)_\perp=(\dot f-\mathsf{T}_\perp f,\dot f-\mathsf{T}_\perp f)=(\dot f,\dot f)-(\dot f,f)(f,f)^{-1}(f,\dot f), \tag{3.8}$$

with the matrix

$$\mathsf{T}_\perp=(\dot f,f)(f,f)^{-1}. \tag{3.9}$$

The square root of a matrix, M, is defined by the well known relation

$$\mathsf{M}^{\frac12}=\frac{2}{\pi}\int_0^\infty\frac{\mathrm{d}t}{1+t^2\mathsf{M}^{-1}},$$

which is true as long as M has no eigenvalues on the interval $(-\infty, 0]$. For brevity's sake we also dropped the dependence on $z = x + \mathrm{i}y$, which we will continue to do whenever this does not lead to ambiguities.

If the matrix Δ is strictly positive, its diagonal block (f, f) has the same property. Then,

$$\Delta^{-1} = \mathsf{K}^{-1}\left(\mathsf{K}^{-1}\right)^T$$

where

$$\mathsf{K}^{-1} = \begin{pmatrix} (f,f)^{-\frac{1}{2}} & -(f,f)^{-1}(f,\dot{f})\left[(\dot{f},\dot{f})_\perp\right]^{-\frac{1}{2}} \\ 0 & \left[(\dot{f},\dot{f})_\perp\right]^{-\frac{1}{2}} \end{pmatrix}. \tag{3.10}$$

The matrix $(\dot{f}, \dot{f})_\perp$ was defined above, (3.8).

Substituting (3.5) into (3.3) we may rewrite (3.3) as

$$\mathcal{D} = \frac{1}{(2\pi)^2\sqrt{\det\,\Delta}} \int [\tilde{\alpha}, \tilde{\alpha}] \exp\left\{-\frac{1}{2}\left[\tilde{\alpha}, ((\dot{f},\dot{f})_\perp)^{-1}\tilde{\alpha}\right]\right\} \mathrm{d}^{(2)}\tilde{\alpha}. \tag{3.11}$$

This Gaussian integral is readily evaluated yielding

$$\mathcal{D} = \frac{\sqrt{\det\,(\dot{f},\dot{f})_\perp}}{2\pi\sqrt{\det\,\Delta}} \mathrm{Tr}\,(\dot{f},\dot{f})_\perp.$$

But from (3.6) and (3.7) and taking into account the block-Cholesky decomposition introduced above

$$\det\,\Delta = \det\,\mathsf{K}^2 = \det\,(f,f)\ \det\,(\dot{f},\dot{f})_\perp.$$

Thus, we get a simple formula for the average density of complex roots extending the Shepp and Vanderbei one [10] to random Gaussian homogeneous generalized polynomials:

$$\mathcal{D} = \frac{\mathrm{Tr}\,(\dot{f},\dot{f})_\perp}{2\pi\sqrt{\det\,(f,f)}}. \tag{3.12}$$

Let us now return to the general case of monic polynomials, when the deterministic part ϕ is not identically zero. The joint distribution function of the values of the polynomial and its derivative P and $\dot{P}$ coincides with the one for F and $\dot{F}$ shifted by their deterministic parts ϕ and, respectively, $\dot{\phi}$. Thus, for $\mathrm{Im}(z) \neq 0$,

$$\mathcal{D} = \int [\tilde{\alpha}, \tilde{\alpha}] \mathcal{P}(-\phi, \tilde{\alpha} - \dot{\phi}) \mathrm{d}^{(2)} \tilde{\alpha}, \tag{3.13}$$

where the function $\mathcal{P}$ is again given by (3.5):

$$\mathcal{P}(-\phi, \tilde{\alpha} - \dot{\phi}) = \frac{1}{(2\pi)^2 \sqrt{\det\,\Delta}} \exp\Big[-\frac{1}{2} \big(-\phi^T, \tilde{\alpha}^T - \dot{\phi}^T \big) \Delta^{-1} \begin{pmatrix} -\phi \\ \tilde{\alpha} - \dot{\phi} \end{pmatrix} \Big]. \tag{3.14}$$

The inhomogeneous Gaussian integral (3.13) with $\mathcal{P}$ given by (3.14) is calculated in a similar way to the previous one by using the block-Cholesky decomposition (3.7-3.10) and introducing a new integration variable

$$\tilde{\beta} = \tilde{\alpha} - \dot{\phi}_\perp,$$

where

$$\dot{\phi}_\perp = \dot{\phi} - \mathsf{T}_\perp \phi. \tag{3.15}$$

Finally, the expected density of roots in the monic case is

$$\mathcal{D} = \frac{\exp\left\{ -\frac{1}{2} [\phi, (f, f)^{-1} \phi] \right\}}{2\pi \sqrt{\det\,(f, f)}} \Big\{ \mathrm{Tr}\,(\dot{f}, \dot{f})_\perp + [\dot{\phi}_\perp, \dot{\phi}_\perp] \Big\}, \tag{3.16}$$

where $(\dot{f}, \dot{f})_\perp$ and $\mathsf{T}_\perp$ were defined above in (3.8) and (3.9).

B. Average density of real roots

On the real axis the situation is rather different. There the imaginary parts of all f_k and $\dot{f}_k$ vanish due to the reality condition (1.18). The rank of the matrix Δ is equal to 2 on the real axis so that the above calculations are invalid there.

For small values of $y = \mathrm{Im}(z)$ the real and imaginary parts of the homogeneous polynomial F are

$$F_1(x, y) = F(x, 0) + \mathcal{O}(y^2), \quad F_2(x, y) = y F'(x, 0) + \mathcal{O}(y^3),$$

where $F(x,0)$ is real[5] and we use the prime to denote the derivative with respect to x. Introducing this into (1.16) — the definition of $\mathcal{P}$

$$\begin{aligned}\mathcal{P}(\alpha,\tilde{\alpha};z) = \ & \mathbb{E}\Big\{\delta(\alpha_1 - F_1(x,0;\circ) + \mathcal{O}(y^2))\delta(\alpha_2 - yF'(x,0;\circ) + \mathcal{O}(y^3)) \\ & \times \delta(\tilde{\alpha}_1 - F'(x,0;\circ) + \mathcal{O}(y^2))\delta(\tilde{\alpha}_2 - yF''(x,0;\circ) + \mathcal{O}(y^3))\Big\},\end{aligned} \tag{3.17}$$

and comparing the arguments of the second and third deltas we see that the second one may be written as $\delta(\alpha_2 - y\tilde{\alpha}_1 + \mathcal{O}(y^3))$. Setting the argument $\alpha_2 = 0$, this becomes

$$|\tilde{\alpha}_1|^{-1}\delta(y) + |y|^{-1}\delta(\tilde{\alpha}_1 - \mathcal{O}(y^2)).$$

Substituting the second term in the above sum into (3.17) would yield the small $|y|$ asymptotic behaviour of the density of complex roots (3.12) which we will obtain a little further on in III C.

The first term, which is zero off the real axis, generates the singular component of $\mathcal{P}(\alpha_1, 0, \tilde{\alpha}_1, \tilde{\alpha}_2; x, y)$

$$\mathcal{P}_{sing}(\alpha_1, 0, \tilde{\alpha}_1, \tilde{\alpha}_2; z) = |\tilde{\alpha}_1|^{-1}\mathcal{P}_0(\alpha_1, \tilde{\alpha}_1; x)\delta(\tilde{\alpha}_2)\delta(y), \tag{3.18}$$

where $\mathcal{P}_0$ is the density for the joint distribution function of the *real* values of $F(x,0)$ and $F'(x,0)$.

In the Gaussian case $\mathcal{P}_0$ is also a Gaussian, determined by the 2×2 correlation matrix of the (real) values of F and F':

$$\Lambda(x) = \begin{pmatrix} (\mathbf{f}(x), \mathbf{f}(x)) & (\mathbf{f}(x), \mathbf{f}'(x)) \\ (\mathbf{f}'(x), \mathbf{f}(x)) & (\mathbf{f}'(x), \mathbf{f}'(x)) \end{pmatrix}. \tag{3.19}$$

We use here the notations introduced in the previous section for the real n-vector $\mathbf{f}(x)$, which has components $f_{k1}(x,0)$, and the bold round brackets (2.6) for the contraction over the Latin indices.

---

[5]We remind the reader that the real and imaginary parts of a complex quantity are denoted by the indices 1, respectively 2.

Thus, in the Gaussian case:

$$\mathcal{P}_0(\alpha_1, \tilde{\alpha}_1; x) = \frac{1}{2\pi\sqrt{\det\,\Lambda(x)}} \exp\Big[-\frac{1}{2}(\alpha_1, \tilde{\alpha}_1)\Lambda^{-1}(x) \begin{pmatrix} \alpha_1 \\ \tilde{\alpha}_1 \end{pmatrix}\Big]. \tag{3.20}$$

Substituting the singular component $\mathcal{P}_{sing}$, with $\mathcal{P}_0$ given by (3.20), into (3.3) and performing the integral leads to Kac's formula [4] for the average density of real roots in the homogeneous case

$$\mathcal{D}_0(x, y) = \delta(y) \frac{\sqrt{(\mathbf{f}(x), \mathbf{f}(x))(\mathbf{f}'(x), \mathbf{f}'(x)) - (\mathbf{f}(x), \mathbf{f}'(x))^2}}{\pi(\mathbf{f}(x), \mathbf{f}(x))}. \tag{3.21}$$

Let us now obtain the singular component of the average density of roots for monic polynomials. Substituting $\mathcal{P}_{sing}$, (3.18-3.20), shifted by the deterministic part into (3.3) and performing the integration yields

$$\mathcal{D}_0(x, y) = \delta(y) \frac{\sqrt{\det\Lambda(x)}}{\pi(\mathbf{f}(x), \mathbf{f}(x))} H[w(x)] \exp\left\{-\frac{[\phi(x, 0), \phi(x, 0)]}{2(\mathbf{f}(x), \mathbf{f}(x))}\right\}, \tag{3.22}$$

where

$$H(w) = \exp\left(-\frac{w^2}{2}\right) + \sqrt{\frac{\pi}{2}} w\ \mathrm{erf}(w), \tag{3.23}$$

$$w(x) = \sqrt{\frac{(\mathbf{f}(x), \mathbf{f}(x))}{\det\Lambda(x)}} \left|\phi'(x, 0) - \frac{(\mathbf{f}(x), \mathbf{f}'(x))}{(\mathbf{f}(x), \mathbf{f}(x))}\phi(x, 0)\right| \tag{3.24}$$

and the error function is

$$\mathrm{erf}(s) = \sqrt{\frac{2}{\pi}} \int_0^s \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t.$$

The function $H(w)$ is monotonically increasing. $H(0) = 1$ and $H(w) \approx \sqrt{\pi/2}w$ as $w \to +\infty$.

For homogeneous polynomials with non-central Gaussian distributions the density of real roots was obtained by Edelman and Kostlan [6]. Taking a singular limit of the covariance matrix in section 5 of [6] will also yield (3.22).

Thus, in the Gaussian case the average density of zeros has a regular component given by (3.16) in terms of the 2×2 matrices (f, f), $(\dot{f}, f)$ and $(\dot{f}, \dot{f})$. For the real type, (1.18), polynomials considered in this paper the average density of roots has also a singular component, localized on the real line, given by the generalized Kac type formula (3.22).

C. Asymptotic behavior of the average density of complex roots near the real axis

Let us now study the asymptotic behavior of the density of complex roots near the real axis. For small $y = \mathrm{Im}(z)$, the smallest eigenvalue of the matrix

$$(f, f) \approx \begin{pmatrix} (\mathbf{f}, \mathbf{f}) & y(\mathbf{f}, \mathbf{f}') \\ y(\mathbf{f}, \mathbf{f}') & y^2(\mathbf{f}', \mathbf{f}') \end{pmatrix}$$

goes to zero as $\left[(\mathbf{f}', \mathbf{f}') - (\mathbf{f}, \mathbf{f}')^2/(\mathbf{f}, \mathbf{f})\right] y^2$. The argument of the exponential in the density of complex roots (3.16) goes to a finite limit; while the other factor in the numerator is $\mathcal{O}(y^2)$. Thus, the average density of complex roots goes to zero as $\mathcal{O}(|y|)$:

$$\mathcal{D}(x, y) = \frac{|y| \mathrm{e}^{-\frac{1}{2}\left(\phi(x,0)\ \ \phi'(x,0)\right)\Lambda(x)\left(\begin{smallmatrix}\phi(x,0)\\ \phi'(x,0)\end{smallmatrix}\right)}}{2\pi\sqrt{\det \Lambda(x)}} \left\{[(\hat{Q}\mathbf{f}, \hat{Q}\mathbf{f})] + [\hat{Q}\phi, \hat{Q}\phi]\right\} + \mathcal{O}(|y^3|), \tag{3.25}$$

where the 2×2 matrix $\Lambda(x)$ was defined above in (3.19) and the linear operator $\hat{Q}$ is defined by

$$\hat{Q}h(x) = h'' - \frac{[(\mathbf{f}, \mathbf{f})(\mathbf{f}', \mathbf{f}'') - (\mathbf{f}, \mathbf{f}')(\mathbf{f}, \mathbf{f}'')]\, h' + [(\mathbf{f}', \mathbf{f}')(\mathbf{f}, \mathbf{f}'') - (\mathbf{f}, \mathbf{f}')(\mathbf{f}', \mathbf{f}'')]\, h}{\det \Lambda}. \tag{3.26}$$

We omitted all the arguments (x) of the functions appearing in the right hand side of (3.26).

Thus, the real axis attracts the roots in its vicinity to the singular component located on it, depleting the density of roots in its neighborhood.

D. High– and low-disorder limits

With our definition of the monic random polynomial (1.6) we may always consider that the random part of the polynomial (the coefficients c_k in (1.6)) has expectation value equal to zero. Let us rescale the deterministic part ϕ of the polynomial to $\Gamma\phi$, introducing a real parameter $\Gamma \geq 0$. The random polynomial is now

$$P(z) = \Gamma\phi(z) + F(z). \tag{3.27}$$

The form (3.27) allows us to interpolate from the monic to the homogeneous case. In the Gaussian case this is equivalent to considering the coefficients c_k of the random polynomial as realizations of a Gaussian process with zero average and dispersion

$$\delta = \Gamma^{-1}, \tag{3.28}$$

instead of (3.1).

For small values of $\Gamma \to 0$, (the large randomness limit, $\delta \to \infty$) the average density of roots $\mathcal{D}$, (3.16), approaches the density for homogeneous polynomials, (3.12). In the large Γ limit, equivalent to small randomness, $\delta \to 0$ in Eq.(3.28), the average density of roots concentrates near the roots of the deterministic part $\phi(z)$ and decays exponentially away from them. Indeed, inspection of (3.16) shows that at fixed $z = x + \mathrm{i}y$, which is not a zero of ϕ, the average density of roots $\mathcal{D}$, goes to zero exponentially when $\Gamma \to \infty$.

Let now $z_0 = x_0 + \mathrm{i}y_0$ be a complex zero of ϕ having multiplicity $k \geq 1$:

$$\phi(z) = c(z - z_0)^k + \mathcal{O}((z - z_0)^{k+1}), \tag{3.29}$$

and assume that the matrix $\Delta(z_0)$ is non-singular. Then, to leading order in $\Gamma \to +\infty$ and for small values of $\rho = |z - z_0|$, the average density of roots is

$$\mathcal{D}(x, y) \approx Ak^2\Gamma^2\rho^{2k-2}\mathrm{e}^{-B(\theta_0)\Gamma^2\rho^{2k}}, \tag{3.30}$$

where

$$A = \frac{c^2}{2\pi\sqrt{\lambda_1\lambda_2}}, \tag{3.31}$$

$$B(\theta_0) = \frac{1}{2}c^2[\lambda_1 + \lambda_2 + |\lambda_1 - \lambda_2|\cos(2k\theta_0 + \chi_0)], \tag{3.32}$$

$\theta_0 = \arg(z - z_0)$,

$$\chi_0 = \arctan\frac{2(f, f)_{12}}{(f, f)_{11} - (f, f)_{22}}, \tag{3.33}$$

and $\lambda_1,\ \lambda_2$ are the (positive) eigenvalues of the 2×2 matrix (f, f).

Now, it is readily seen that for $\Gamma \to \infty$, (3.30) goes to $k\delta^2(z - z_0)$ in the sense of distributions. For large but finite values of Γ the average density of roots is a Gaussian centered at z_0 for $k = 1$.

For $k>1$ if the eigenvalues of (f,f) are equal, $\lambda_1=\lambda_2$, then the surface $S=\mathcal{D}(x,y)$ has an annular maximum for $\rho^{2k}=(1-k^{-1})/B\Gamma^2$. If $\lambda_1\neq\lambda_2$, the annular maximum splits into $2k$ individual maxima located at $2k\theta_0+\chi_0=(2M+1)\pi,\ M=0,\dots,2k-1$ and $\rho^{2k}=(1-k^{-1})/B_{min}\Gamma^2$, where $B_{min}=c^2\lambda_{min}=\min_\theta B(\theta)$ is the minimal value of (3.32).

IV. ASYMPTOTIC DENSITY OF ROOTS FOR MONIC ALGEBRAIC POLYNOMIALS

In this section we investigate the average density of roots for algebraic monic polynomials,

$$P_n(z)=z^n+\delta\sum_{k=1}^{n}c_kz^{k-1}, \tag{4.1}$$

in the large n limit. Here the parameter $\delta=\Gamma^{-1}\geq 0$ is the dispersion of the original Gaussian distribution which was transformed to the normal one, Eq.(3.1), as mentioned in the preceding section.

The roots of the polynomial (4.1) coincide with the roots of

$$\Gamma P_n(z)=\Gamma z^n+\sum_{k=1}^{n}c_kz^{k-1}, \tag{4.2}$$

so that we may use the rescaled ϕ, (3.27), as in the preceding section. The large Γ (low disorder) asymptotic obtained there near complex roots of the deterministic part ϕ cannot be used straightforwardly since in our case ϕ has a highly degenerate *real* zero at the origin.

The calculation of the 2×2 matrices (f,f), $(f,\dot f)$ and $(\dot f,\dot f)$ in this case is presented in some detail in Appendix A. There we show that the matrix elements may be expressed in terms of the functions

$$C_n(r^2,2\theta)=\frac{1-r^2\cos 2\theta-r^{2n}\cos 2n\theta+r^{2n+2}\cos(2n-2)\theta}{1-2r^2\cos 2\theta+r^4}, \tag{4.3}$$

$$S_n(r^2,2\theta)=\frac{r^2\sin 2\theta-r^{2n}\sin 2n\theta+r^{2n+2}\sin(2n-2)\theta}{1-2r^2\cos 2\theta+r^4}, \tag{4.4}$$

and their derivatives with respect to r. Here

$$z=x+iy=r\mathrm{e}^{i\theta}.$$

The matrix (f,f) is thus given by

$$(f,f) = \frac{1}{2}\begin{pmatrix} C_n(r^2,0)+C_n(r^2,2\varphi) & S_n(r^2,2\varphi) \\ S_n(r^2,2\varphi) & C_n(r^2,0)-C_n(r^2,2\varphi) \end{pmatrix}. \tag{4.5}$$

Its eigenvalues are $C_n(r^2,0) \pm \sqrt{C_n^2(r^2,2\theta)+S_n^2(r^2,2\theta)}$. The smallest is positive for all θ such that $\sin\theta \neq 0$, as shown in A.

In the Appendix we expressed the matrix elements of $(\dot{f},f)$ and $(\dot{f},\dot{f})$ in terms of derivatives of the matrix (f,f) with respect to r. We may now rewrite Eqs.(A10 - A11) in matrix form using the orthogonal matrix

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.6}$$

$$(\dot{f},f) = \frac{1}{2}U(\theta)\frac{\partial(f,f)}{\partial r}, \tag{4.7}$$

$$(\dot{f},\dot{f}) = \frac{1}{4r}U(\theta)\frac{\partial}{\partial r}\left[r\frac{\partial(f,f)}{\partial r}\right]U^T(\theta). \tag{4.8}$$

Here U^T is the transposed of U.

For examining the behaviour of the density of complex roots (3.16) in the vicinity of the unit circle, let us introduce a new, logarithmically rescaled variable

$$\mathrm{e}^{\xi} = r^{2n}. \tag{4.9}$$

Inspection of (3.16) and comparison with (4.7), (4.8), (3.8), (3.9) and (3.15) shows us that we need to estimate the matrices (f,f) and its inverse, the matrix

$$U^T(\theta)\mathsf{T}_\perp = \frac{1}{2}\frac{\partial(f,f)}{\partial r}(f,f)^{-1},$$

and the second derivative with respect to r of the function $C_n(r^2,0)$. The functions $C_n(r^2,2\varphi)$ and $S_n(r^2,2\varphi)$, (4.3) and (4.4), have rapid oscillations with θ.

Let us study the quotient

$$Q = \frac{C_n(r^2,2\varphi)}{C_n(r^2,0)} = \frac{(1-r^2)\,(1-r^2\cos 2\theta - r^{2n}\cos 2n\theta + r^{2n+2}\cos(2n-2)\theta)}{(1-r^{2n})\,(1-2r^2\cos 2\theta + r^4)},$$

for $|\xi| \ll n$. The absolute value of the sum of terms proportional to r^{2n} in the numerator, which are fast oscillating with θ, does not exceed $r^{2n}\sqrt{1-2r^2\cos 2\theta + r^4}$. Noting that $1-2r^2\cos 2\theta + r^4 = (1-r^2)^2 + (2r\sin\theta)^2$, we obtain

$$|Q| < \frac{1-r^2}{2r|\sin\theta|}\left(\frac{1}{1-r^{2n}} + \frac{1}{r^{-2n}-1}\right) \approx \frac{\xi}{2n|\sin\theta|}\coth\frac{\xi}{2},$$

where we used the obvious inequalities $2ab < a^2+b^2$ and $b < \sqrt{a^2+b^2}$ to estimate the θ dependent terms. A similar estimate can be obtained for the quotient $S_n(r^2, 2\varphi)/C_n(r^2, 0)$.

Thus, for

$$\tau = \frac{\xi}{2n|\sin\theta|}\coth\frac{\xi}{2} \ll 1, \tag{4.10}$$

the matrix $\boldsymbol{(}f, f\boldsymbol{)}$ is asymptotically proportional to the unit one times $C_n(r^2, 0)$. Substituting the $|\xi| \ll n$ asymptotic behavior of $C_n(r^2, 0)$, (A20) we obtain that $\boldsymbol{(}f, f\boldsymbol{)}$ is equal to n times a universal function of ξ, which does not depend on θ and n:

$$\boldsymbol{(}f, f\boldsymbol{)} = \frac{n}{2}\,\frac{\mathrm{e}^{\xi}-1}{\xi}\Big[1+\mathcal{O}(\tau)\Big]. \tag{4.11}$$

Noting that (4.10) implies $\xi \ll n$, we see that we may use (4.11) also for estimating the derivatives of (f,f).

Let us first look at the homogeneous case, $P_n(z) = \sum_{k=0}^{n-1} \mathbf{c}_k z^{n-k}$, corresponding to $\Gamma = 0$ and recover the result obtained by Shepp and Vanderbei [10]. Substituting (4.11) into (4.7), (4.8), we obtain after a little algebra on (3.12)

$$\mathcal{D}(\mathrm{e}^{\frac{\xi}{2n}}, \theta) \approx -\frac{n^2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi} - \frac{1}{\mathrm{e}^{\xi}-1}\right). \tag{4.12}$$

Thus, for $\tau \ll 1$, (4.10), the expected density of complex roots for homogeneous ($\Gamma = 0$) Gaussian polynomials is asymptotically equal to n^2 times a simple universal symmetric function of ξ. For large values of $|\xi| \ll n$ the density has inverse power behavior, $\mathcal{D} \sim \xi^{-2}$.

In Fig. 1 we plot the renormalized average density of complex roots, $n^{-2}D$, for homogeneous real Gaussian polynomials of degree 99 as a function of $\xi = 2n\ln|z|$ and $\theta = \arg(z)$. We see that at $n = 100$ the concordance with the asymptotic formula (4.12) is good, excepting in the vicinity of the real axis where the condition (4.10) is invalid.

Performing the same substitutions on (3.16) and noting that for $\phi = \Gamma z^n$ the 2-vector $\dot{\phi}$ satisfies

$$\dot{\phi} = \frac{n}{r} U(\theta)\phi,$$

we obtain after a little algebra a simple asymptotic formula, valid for $\tau \ll 1$, for the average density of complex roots $\mathcal{D}$ of monic algebraic polynomials:

$$\mathcal{D}(\mathrm{e}^{\frac{\xi}{2n}}, \theta) = -\frac{n^2}{\pi}\frac{d}{d\xi}\Big\{\Big(\frac{1}{\xi} - \frac{1}{\mathrm{e}^{\xi} - 1}\Big)\exp\Big[-\frac{\Gamma^2}{n}\frac{\xi}{1-\mathrm{e}^{-\xi}}\Big]\Big\}. \tag{4.13}$$

In the monic case the expected density of roots is also asymptotically equal to n^2 times a universal function of ξ and Γ^2/n. For nonzero values of $\Gamma/\sqrt{n}$ the large $|\xi| \ll n$ asymptotic behavior of (4.13) is exponential decay for large positive ξ and remains inverse-power for large negative values of ξ. This asymmetry becomes rather pronounced in the case of large values of the parameter Γ^2/n, which was investigated numerically in [18].

Let us estimate $n^{-1}\mathcal{N}_{out}(R)$, the fraction of the expected number of roots outside a disk of radius R centered at $z = 0$. For large n and $|\ln R| \ll 2n$ the total number of real roots is $\mathcal{O}(\log n)$ and since the sectors of angle $\mathcal{O}(\pi/n)$ near the real axis contain a number of roots comparable to the error of the asymptotic relation (4.13), we may also use it there:

$$\mathcal{N}_{out}(R) = \frac{1}{n}\int_{\xi_R}^{\infty} \mathrm{d}\xi \mathrm{e}^{\xi/n} \int_0^{\pi} \mathrm{d}\theta \mathcal{D}(\mathrm{e}^{\frac{\xi}{2n}}, \theta).$$

Here $\xi_R = -\ln R/(2n)$ and we may exploit the exponential decay of $\mathcal{D}$ for large ξ and replace $\mathrm{e}^{\xi/n}$ by 1 under the integral if $|\xi_R| \ll n$.

The fraction of roots outside a disk of radius R is thus asymptotically equal to

$$\frac{1}{n}\mathcal{N}_{out}(R) \approx \Big(\frac{1}{2n\ln(R)} - \frac{1}{R^{2n} - 1}\Big)\exp\Big(-\frac{2\Gamma^2\ln(R)}{1 - R^{-2n}}\Big). \tag{4.14}$$

For $R \to 0$ this goes to n.

In Fig. 2 we plot the (re)normalized density of roots for $n = 10$ and $N = \Gamma^2 = 100$. It has nine sharp peaks in each of the half-planes $\mathrm{Im}(z) \gtrless 0$. Our analysis n III D predicts the splitting of a *complex* n times degenerate zero of the deterministic part into $2n$ maxima. In the present case we have only $2n-2$ peaks because the zero of the deterministic part is *real* and the other two peaks of the distribution are on the singular component. An interesting feature of this splitting is the fact that there are twice as many peaks as there are roots. Thus, on a typical realization of the random process we expect to find the n roots of the polynomial located near *half* of the positions of the maxima.

At $n = 30$ and $N = 1024$ there is still some oscillation in the expected number of roots in an angular sector as can be seen in Fig. 3. For larger values of n, the density of complex roots approaches the θ independent asymptotic form (4.13). In Fig. 4 this convergence is shown for $\frac{2\pi}{5} < \theta < \frac{\pi}{2}$ at $N = \Gamma^2 = 10n$. The number of extrema increases with n while their values become closer. The first several maxima near the real axis have a larger amplitude than those in the domain of universality. Since these are at a distance $\sim n^{-1}$ from the real axis, the linear with $|\mathrm{Im}(z)|$ fall to zero of the density of roots may become rather steep as can be seen in Fig. 5.

APPENDIX A: MATRIX ELEMENTS – ALGEBRAIC CASE

The density of complex roots is given by Eq.(3.16) in terms of the 2×2 matrices (f, f), $(f, \dot{f})$ and $(\dot{f}, \dot{f})$ defined in Section II. Let us calculate them explicitly in the case when the polynomials are algebraic, *i.e.* when the basis functions are given by:

$$f_k(z) = z^{k-1}, \quad k = 1, \ldots, n. \tag{A1}$$

Taking $z = r\mathrm{e}^{i\theta}$, the components of the $2n$-vectors f and $\dot{f}$ are

$$\begin{aligned} f_{k1} &= r^{k-1}\cos(k-1)\theta, \\ f_{k2} &= r^{k-1}\sin(k-1)\theta, \end{aligned} \tag{A2}$$

$$\begin{aligned} \dot{f}_{k1} &= (k-1)r^{k-2}\cos(k-2)\theta, \\ \dot{f}_{k2} &= (k-1)r^{k-2}\sin(k-2)\theta \end{aligned} \tag{A3}$$

Let us start with the 2×2 real symmetric matrix (f, f).

$$
\begin{aligned}
(f,f)_{11} &= \sum_{k=0}^{n-1} r^{2k}\cos^2 k\theta = \frac{1}{2}\sum_{k=0}^{n-1} r^{2k}(1+\cos 2k\theta) \\
&= \frac{1}{2}[C_n(r^2,0) + C_n(r^2,2\varphi)].
\end{aligned} \tag{A4}
$$

Here, we defined the function $C_n(x,\chi)$ as the real part of the sum

$$
\sum_{k=0}^{n-1} x^k \mathrm{e}^{ik\chi} = C_n(x,\chi) + iS_n(x,\chi), \tag{A5}
$$

$$
C_n(x,\chi) = \frac{1 - x\cos\chi - x^n\cos n\chi + x^{n+1}\cos(n-1)\chi}{1 - 2x\cos\chi + x^2}, \tag{A6}
$$

$$
S_n(x,\chi) = \frac{x\sin\chi - x^n\sin n\chi + x^{n+1}\sin(n-1)\chi}{1 - 2x\cos\chi + x^2}. \tag{A7}
$$

In a similar way, the other elements of the symmetric matrix (f,f) are given by

$$
\begin{aligned}
(f,f)_{12} &= \tfrac{1}{2}S_n(r^2,2\varphi), \\
(f,f)_{22} &= \tfrac{1}{2}[C_n(r^2,0) - C_n(r^2,2\varphi)],
\end{aligned} \tag{A8}
$$

where the function $S_n(x,\chi)$ is defined in (A7). The determinant of the matrix (f,f) is

$$
\begin{aligned}
\det\,(f,f) &= \tfrac{1}{4}[C_n^2(r^2,0) - C_n^2(r^2,2\theta) - S_n^2(r^2,2\theta)] \\
&= \tfrac{1}{4}\left[\left(\frac{1-r^{2n}}{1-r^2}\right)^2 - \frac{1-2r^{2n}\cos 2n\theta + r^{4n}}{1-2r^2\cos 2\theta + r^4}\right].
\end{aligned} \tag{A9}
$$

It is readily seen that the determinant is greater than zero if $\sin\theta \neq 0$ *i.e. off the real axis.*

The sums appearing in the definitions of $(f,\dot f)$ and $(\dot f,\dot f)$ are similar. The factors k and k^2 may be dealt with using the relation $\frac{dx^k}{dx} = kx^{k-1}$. Thus, the elements of the matrices $(f,\dot f)$ and $(\dot f,\dot f)$ can be expressed in terms of derivatives with respect to the first argument of the functions C_n and S_n defined by (A6) and (A7) above.

$$
\begin{aligned}
(\dot f,f)_{11} &= \sum_{k=0}^{n-1} kr^{2k-1}\cos k\theta\cos(k-1)\theta = \sum_{k=0}^{n-1} kr^{2k-1}[\cos\theta\cos^2 k\theta + \sin\theta\sin k\theta\cos k\theta] \\
&= \frac{1}{2}\frac{\partial}{\partial r}\Big[\cos\theta(f,f)_{11} + \sin\theta(f,f)_{21}\Big], \\
(\dot f,f)_{12} &= \frac{1}{2}\frac{\partial}{\partial r}\Big[\cos\theta(f,f)_{12} + \sin\theta(f,f)_{22}\Big], \\
(\dot f,f)_{21} &= \frac{1}{2}\frac{\partial}{\partial r}\Big[\cos\theta(f,f)_{21} - \sin\theta(f,f)_{11}\Big], \\
(\dot f,f)_{22} &= \frac{1}{2}\frac{\partial}{\partial r}\Big[\cos\theta(f,f)_{22} - \sin\theta(f,f)_{12}\Big].
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
(\dot f,\dot f)_{11} &= \sum_{k=0}^{n-1} k^2 r^{2k-2}\cos^2(k-1)\theta \\
&= \sum_{k=0}^{n-1} k^2 r^{2k-2}\Big[\cos^2\theta\cos^2 k\theta + 2\sin\theta\cos\theta\cos k\theta\sin k\theta + \sin^2\theta\sin^2 k\theta\Big] \\
&= \frac{1}{4r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Big[\cos^2\theta (f,f)_{11} + 2\sin\theta\cos\theta (f,f)_{12} + \sin^2\theta (f,f)_{22}\Big], \\
(\dot f,\dot f)_{12} &= \frac{1}{4r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Big[(\cos^2\theta - \sin^2\theta)(f,f)_{12} + \sin\theta\cos\theta\Big((f,f)_{22} - (f,f)_{11}\Big)\Big], \qquad \text{(A11)} \\
(\dot f,\dot f)_{22} &= \frac{1}{4r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Big[\cos^2\theta (f,f)_{22} - 2\sin\theta\cos\theta (f,f)_{12} + \sin^2\theta (f,f)_{11}\Big].
\end{aligned}
\tag{A12}
$$

After expressing all the relevant matrix elements in terms of the functions C_n and S_n, let us now consider their asymptotic behavior when $n \gg 1$. The sum in Eq.(A5) and its derivatives with respect to x are truncated power series in x with power bounded coefficients. Their radius of convergence is 1. For $x < 1$ the leading term of its large n asymptotic expansion is given by the $n \to \infty$ limit of Eq.(A5) yielding a result n-independent result with exponentially small corrections:

$$C_n(x,\chi) \approx \frac{1 - x\cos\chi}{1 - 2x\cos\chi + x^2} + \mathcal{O}(x^n), \tag{A13}$$

$$S_n(x,\chi) \approx \frac{x\sin\chi}{1 - 2x\cos\chi + x^2} + \mathcal{O}(x^n). \tag{A14}$$

If $x > 1$ the series the sum Eq.(A5) diverges as $n \to \infty$. The leading terms for C_n and S_n are now exponentially large in absolute value and rapidly oscillating functions of χ. The corrections are again rather small:

$$C_n(x,\chi) \approx x^n\left[\frac{x\cos(n-1)\chi - \cos n\chi}{1 - 2x\cos\chi + x^2} + \mathcal{O}(x^{-n})\right], \tag{A15}$$

$$S_n(x,\chi) \approx x^n\left[\frac{x\sin(n-1)\chi - \sin n\chi}{1 - 2x\cos\chi + x^2} + \mathcal{O}(x^{-n})\right]. \tag{A16}$$

Thus, in the large n limit the asymptotic behavior of the functions C_n and S_n changes dramatically in a narrow annulus of width $\mathcal{O}(n^{-1})$ near the unit circle, crossing over from the behavior Eq.(A13) to Eq.(A15) (respectively from Eq.(A14) to Eq.(A16)).

For investigating the behavior in the crossover region, let us go to a logarithmic scale for the variable x:

$$x = \exp\left(\frac{\xi}{n}\right). \tag{A17}$$

For $|\xi| \ll n\chi$, the leading asymptotic terms are

$$C_n(\mathrm{e}^{\frac{\xi}{n}}, \chi) \approx \frac{1}{2}\Big[1 + \frac{\sin(n-\frac{1}{2})\chi}{\sin\frac{\chi}{2}}\mathrm{e}^{\xi}\Big] + \mathcal{O}\Big(\frac{\xi}{n\chi}\Big), \tag{A18}$$

$$S_n(\mathrm{e}^{\frac{\xi}{n}}, \chi) \approx \frac{1}{2}\Big[\cot\frac{\chi}{2} - \frac{\cos(n-\frac{1}{2})\chi}{\sin\frac{\chi}{2}}\mathrm{e}^{\xi}\Big] + \mathcal{O}\Big(\frac{\xi}{n\chi}\Big). \tag{A19}$$

Inspection of Eqs.(A18-A19) shows that the corrections are small outside a neighborhood of the point $x = 1,\ \chi = 0$. For $\chi = 0$ and $|\xi| \ll n$, the leading term is[6]

$$C_n(\mathrm{e}^{\frac{\xi}{n}}, 0) \approx n\frac{\mathrm{e}^{\xi} - 1}{\xi} + \mathcal{O}(1). \tag{A20}$$

The asymptotic expansions given by Eqs.(A13 - A16), (A18 - A20) may be now used to obtain the large n asymptotic behavior of the matrices (f, f), $(f, \dot{f})$ and $(\dot{f}, \dot{f})$ in the domains $|z^n| \ll 1,\ \ |z^n| \gg 1$ and the thin annulus which separates them.

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[6]We will not write here the bulkier expressions which are valid for $|\xi| \ll n$ without restrictions on χ.

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FIGURES

FIG. 1. The renormalized density of complex roots, $n^{-2}D(z)$, for the homogeneous Gaussian polynomial $c_1 z^{99} + c_2 z^{98} + \ldots + c_{100}$, $n = 100$, as a function of $\xi = 2n \ln |z|$ and $\theta = \arg(z)$. (a) exact, Eq. (3.12); (b) asymptotic formula, Eq. (4.12); (c) contour plots: full line – exact, dotted line – asymptotic.

FIG. 2. Renormalized density of complex roots for the polynomial $N^{\frac{1}{2}} z^{10} + c_1 z^9 + \ldots + c_{10}$, $n = 10$, and $N = \Gamma^2 = 100$, as function of ξ and θ.

FIG. 3. Average number of roots in angular sectors for $n = 30$ and $N = 1024$. Full line – histogram of the averages for 1000 polynomials; dashed line – angular density of roots, $\int_0^\infty \mathrm{d}r r D(r, \theta)$, obtained by numerical integration of Eq. (3.16).

FIG. 4. Renormalized density of roots for $N = \Gamma^2 = 10n$: (a) $n = 100$, exact; (b) $n = 200$, exact; (c) asymptotic formula (4.13) for $N/n = 10$.

FIG. 5. Detail of the asymptotic behavior of the expected density of roots near the real axis for $n = 30$ and $N = 1000$.