Dynamic around a chain of simple resonant tori in nearly integrable Hamiltonian systems

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Résumé : Grace à une application de la théorie K.A.M., nous construisons une forme normale permettant une étude fine de la dynamique au voisinage des tores invariants partiellement hyperboliques qui apparaissent dans les régions simplement résonantes associées à un système hamiltonien intégrable faiblement perturbé. Ceci donne des estimations précises sur les temps de transition au voisinage des variétés stable et instable liées à ces tores. Ainsi, ces formes normales fournissent un moyen de calculer la vitesse de dérive d'orbites qui "ombrent" une chaîne de tores hyperboliques associée à une courbe simplement résonante dans l'espace des actions.

Abstract : By an application of the K.A.M. theory, we derive an accurate normal form valid in the vicinity of partially hyperbolic tori which arise close to simple resonances in nearly integrable Hamiltonian systems. This allows precise estimates on the times of transition around the stable and unstable manifolds of these tori. Hence, these normal forms provide an efficient tool to compute the speed of drift of orbits shadowing a chain of hyperbolic tori associated to a simple resonant curve in the action space.

Mots clés : Systèmes Hamiltoniens – Stabilité – Flots Quasipériodiques – Perturbations – Formes Normales – Structures Hyperboliques.

Key words : Hamiltonian systems – Stability – Quasi periodic flows – Perturbations – Normal forms – Hyperbolic structures.

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I Introduction :

We look at perturbed integrable Hamiltonian systems which are governed by the classical Hamiltonian : $\mathcal{H}(I,\varphi) = h(I)+\varepsilon f(I,\varphi)$ with the action-angle variables $(I,\varphi) \in \mathbf{R}^n \times \mathbf{T}^n$ where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. As it is well known, K.A.M. theory states that under the (sufficient) assumption of analyticity of H and non-degeneracy of h ($|\nabla^2 h| \neq 0$), there remain many n-dimensional Lagrangian tori invariant for the perturbed flow which are slight deformations of the initial tori ($I = I_0$) located in non resonant area. Nevertheless, in case $n \geq 3$, this does not allow to prevent from a drift of the orbits over a large part of the phase space in the perturbed system because the complement of the invariant tori is a connected set. Actually, according to a theorem of Nekhorochev [N, 1977], this possible instability can only occur with a speed which is at most exponentially small with respect to the inverse of the size of the perturbation (ε). Hence, two questions arise : to prove the existence of unstable orbits and to estimate their times of instability.

In a famous paper of 1964, Arnold [A, 1964] (see also [A, 1992]) has given an example of a three degrees of freedom nearly integrable Hamiltonian system where a global instability of the action variables occurs. The mechanism which generate this instability is based on the existence in the perturbed system of arbitrary long chains of hyperbolic tori connected by heteroclinic orbits and Arnold finds orbits which drift along these lines of tori. Nevertheless, it should be pointed out that the existence of these chains is ensured by very specific properties of the Hamiltonian studied by Arnold. The generalisation of this result in a wider class of nearly integrable Hamiltonian systems

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is a difficult problem, a precise survey on this question has recently been written by Lochak ([L, 1996], see also the introduction of [RW2, 1997]).

In the general case, the first step to prove instability along the lines of Arnold's reasoning is to ensure the existence of enough hyperbolic tori invariant under the perturbed flow. The study of the lower dimensional tori which survive under perturbation was initiated by Moser [Mo, 1967], Brjuno [Br, 1971] and then by numerous authors. A refined result was obtained by Graff [Gr, 1973] (see also Zehnder [Ze, 1976]) in the case where the phase space is foliated by tori with a hyperbolic structure (stable/unstable manifolds) invariant under the unperturbed flow. In this initially hyperbolic setting, he showed that the tori with a strongly non resonant frequency and the associated manifolds persist in the perturbed problem. This result has been extended in the general case of nearly integrable Hamiltonian systems by Treschev [Tr, 1991] who showed that many tori of dimension $(n-d)$ with n-dimensional Lagrangian hyperbolic manifolds (the"whiskers") arise in the perturbed problem close to the manifolds associated to resonances of multiplicity d $(d < n)$. The two previous theorems come from the application of the K.A.M. theory which allows to normalize the Hamiltonian in the vicinity of an initial torus in order that the perturbed flow in the new coordinates is linearized on an invariant torus which admits a hyperbolic normal behaviour. Then, the use of adapted stable/unstable manifold theorems shows the persistence under the new perturbed flow of a hyperbolic structure associated to the considered torus.

The next question is the possibility of using the tori determined in the previous studies as the skeleton for Arnold's mechanism of instability. The results on the existence of hyperbolic tori connected by heteroclinic orbits will be specified farther on but we already point out that, *assuming* the existence of such a transition chain, Marco ([Ma, 1996]) has proved that the normal forms derived in the first part of Graff's and Treschev's papers allow to ensure the existence of orbits shadowing the considered chain. Nevertheless, the upper bound on the times of transition computed in the previous study is super exponential and much greater than Nekhorochev's time of stability (assuming the existence of a transition chain in a nearly integrable system). In the second part of his paper, Marco ([Ma, 1996]) has shown that the estimates on the transition times in Arnold's mechanism could be optimized if refined normal forms could be defined in the vicinity of the considered hyperbolic tori, this point will be specified in the following. It can also be noticed that the complete results of Graff and Treschev cannot be explicitly applied to build shadowing orbits because the straightening of the hyperbolic directions in these studies is obtained by means of transformations which are not derived.

Actually, in the preceding theorems, the dynamic around a surviving torus in multiplicity d resonant areas can be seen as the product of the dynamics around two objects : a $(n-d)$-dimensional torus in $\mathbf{R}^{n-d} \times \mathbf{T}^{n-d}$ and an hyperbolic fixed point in $\mathbf{R}^{2d}$. Here, the case of a simple resonance $(d = 1)$ allows a significant simplification because we can use a theorem of Moser [Mo, 1956] which shows that an Hamiltonian system around an hyperbolic fixed point in the plane could be *integrated* by means of a normalizing transformation.

The goal of this paper is precisely to carry out a quantitative study of the application of the K.A.M. theory in the case of simple resonant tori, this allows to use Moser's result and to get a very strong control on the dynamic around the persistent hyperbolic tori and their invariant manifolds.

K.A.M. theory in this setting have been constructed previously by Chierchia and Gallavotti [CG, 1994], Rudnev and Wiggins [RW1, 1996], Gentile [Ge, 1995], but these works were based on the construction of a normalizing transformation which is convergent only on the surviving tori and their invariant manifolds (hence, on a Cantor set). On the one hand, this allows to prove globally the existence of invariant hyperbolic tori and (in certain particular cases) that they are connected by heteroclinic orbits ([CG, 1994], [RW2, 1997]). On the other hand, these studies do not give any information on the dynamic around these invariant sets and, consequently, do not allow to detect orbits drifting along a given chain of connected tori. Indeed, the latters are obtained by showing that after a certain time, called obstruction time, the image by the perturbed flow of an open set around a heteroclinic orbit connecting two successive tori $\mathcal{T}_{i-1}$ and $\mathcal{T}_i$ intersects an arbitrary neighbourhood of a next heteroclinic orbit between $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$, then a simple recursive proof yields the desired shadowing orbits. The previous reasoning was given in a sketchy way in

Arnold's original paper and, as previously said, clarified by Marco in a general setting. Actually, the transition time growth linearly with the obstruction time and the number of tori in the considered chain which is a given parameter of the studied problem. Consequently, estimates on the speed of drift of shadowing orbits cannot be improved without accurate computation of the obstruction time and this imposes a refined analysis of the dynamic around the invariant tori and their invariant manifolds.

Here, we focus our attention on a single torus and derive a local normal form defined on an *open* set which linearize the perturbed flow on the considered torus and its linked manifolds. In that way, the normalized perturbed system is very close to the one considered by Marco in the second part of his paper and, consequently, it can be expected important improvements on the rate of instability for shadowing orbits with the results derived here.

We should mention that Jorba and Villanueva ([JV1, 1996], [JV2, 1996]) have recently made a similar quantitative study on the persistence and the dynamic around partially elliptic lower dimensional tori in nearly-integrable Hamiltonian systems in the case of a quasi-periodic perturbation.

The present paper is organised as follow. First, we deal with singular perturbation theory since we are looking at invariant tori in case of partial hyperbolicity and none is present in the limit integrable case ($\varepsilon = 0$). Hence, in section II, we carry out an application of the K.A.M. theory in an initially hyperbolic setting with two parameters which respectively correspond to the rate of hyperbolicity (Liapounov exponents) and the size of the perturbation. Then, in section III, we show that, under general assumptions, enough hyperbolicity is present in a nearly integrable Hamiltonian system around a simple resonant torus to apply the results of section II by linking the two parameters. This follows the lines of Treschev's reasonings but the estimates are much sharper here in a simple resonant case.

Finally, in section IV, the implementation of our results is considered in connection with the results of Gallavotti-Chierchia ([CG, 1994]) and Rudnev-Wiggins ([RW2, 1997]) on the existence of transition chains. At this step, we are faced to a basic problem : the gaps between the invariant tori provided by an application of K.A.M. theory are usually much greater than the mean distance between two tori connected by heteroclinic orbits. On the one hand, along the lines of Chierchia and Gallavotti's reasonings ([CG, 1994]), this latter difficulty can be overcomed for certain perturbations of initially hyperbolic Hamiltonian systems which allow to prove *globally* in the phase space the conservation of enough invariant tori ; hence, our normal forms can be derived around each torus of a given chain in these cases. On the other hand, for a general nearly-integrable Hamiltonian, we show (as a consequence of theorem 4.1.) an exponential accumulation of new invariant hyperbolic tori in the vicinity of each torus provided by the present study. Consequently, we point out areas in the phase space where the gaps between hyperbolic tori invariant for the perturbed flow are smaller than the mean distance between two tori connected by heteroclinic orbits in the cases studied by Rudnev and Wiggins ([RW2, 1997]). Because of this *local* result, our normal forms can be used to compute estimates on the speed of drift of orbits close to certain transition chains associated to a simple resonance. In the same section IV, we also discuss accurately the minimal distance to multiple resonances which is needed to derive our results.

The conclusion of the paper (section V) is devoted to possible extensions of this work in the case of partially hyperbolic tori linked to multiple resonances.

II The case of an initially hyperbolic system

II.1 Set-up and main result

Set up. Along the lines of Graff and Treschev's reasonings, we first study the case where the phase space is foliated in tori with hyperbolic directions invariant under the unperturbed flow. More specifically, we focus our attention at an Hamiltonian system governed by :

$$\mathcal{H}_0(J,\theta,p,q) = H_0(J,p,q) + \varepsilon\mu g(J,\theta,p,q) \text{ with } H_0(J,p,q) = h(J;\varepsilon) + \varepsilon P(J,p,q;\varepsilon)$$

where (J,θ,p,q) are in $\mathbf{R}^n \times \mathbf{T}^n \times \mathbf{R}^2$ equipped with the symplectic form $\sum_{j=0}^{n} d\theta_j \wedge dJ_j + dq \wedge dp$ and it is also assumed that :

(i) ε is a positive parameter (not necessarily small) and $0 < \mu << 1$.

(ii) The total Hamiltonian $\mathcal{H}_0$ is analytical over the complex domain :

$$V_\kappa = \left\{ (J,\theta,p,q) \in \mathbf{C}^{2(n+1)} \text{ such that } \mathrm{dist}(J,\mathcal{J}) < \kappa \, ; \, \Re e(\theta) \in \mathbf{T}^{n+1} \, ; \, ||\Im m\,(\theta_j)||_\infty < \kappa \, ; \, ||(p,q)||_\infty < \kappa \right\}$$

where κ is a positive constant, $||.||_\infty$ is the maximum modulus of the coordinates, $\mathcal{J}$ is a subset of $\mathbf{R}^n$ and the distance to $\mathcal{J}$ is given by the Euclidean norm in $\mathbf{C}^n$. We also denote $\mathcal{J}_\kappa$ the real domain $\{J \in \mathbf{R}^n \text{ such that } \mathrm{dist}(J,\mathcal{J}) < \kappa\}$

(iii) For a numerical function F defined on V_κ, we denote $||F||_\kappa$ the sup norm (L^∞) over V_κ and set $||\mathcal{H}_0||_\kappa < E_1$,$||P||_\kappa < E_1$ and $||g||_\kappa < E_1$. In the case of a vector valued function, $||.||_\kappa$ is defined as the supremum over V_κ of the Euclidean norm of its value.

(iv) uniformly over $\mathcal{J}_\kappa$, we have $||\partial_J^2 h(J)|| \leq p^{-1}$ with $0 < p < 1$ and the classical norm for the operator induced by the Euclidean norm.

(v) The unperturbed Hamiltonian h satisfies uniformly over $\mathcal{J}_\kappa$ a condition of isoenergetic non degeneracy with the preceding constant p :

$$\text{for all } \mathbf{V} \in \mathbf{R}^{n+1}, \text{ we have } \left|\left| \begin{pmatrix} \partial_J^2 h(J) & \partial_J h(J) \\ {}^t\partial_J h(J) & 0 \end{pmatrix} \mathbf{V} \right|\right| \geq \varepsilon p ||\mathbf{V}||.$$

(vi) $P(J,p,q) = \mathcal{O}_2(p,q;J,\varepsilon)$; in all the paper, $f(x;y)$ (resp. $\mathcal{O}_n(x;y)$) means a function of the argument x with the parameter y (resp. a function of the order of $||x||^n$ parametrized by y).

(vii) Uniformly over $\mathcal{J}_\kappa$, we have :

$$\frac{\partial^2 P}{\partial p^2}(J,0,0)\,\frac{\partial^2 P}{\partial q^2}(J,0,0) - \left(\frac{\partial^2 P}{\partial p \partial q}(J,0,0)\right)^2 < -\lambda^2 < 0$$

where λ is a positive constant.

The isoenergetic setting given in condition (v) arises naturally since, in Arnold's instability, we look at the dynamic along a chain of invariant tori at a prescribed energy ; moreover, the constant in this assumption is scaled in view of the application to the general nearly integrable case. Condition (vi) on $P(J,p,q)$ ensures that the embedding of $\mathcal{J} \times \mathbf{T}^n$ in the phase space $\mathbf{R}^n \times \mathbf{T}^n \times \mathbf{R}^2$ is foliated, for $\varepsilon \neq 0$ and $\mu = 0$, in n-dimensional tori $\{J = J_0, p = q = 0\}$ invariant for the flow linked to H_0 while the desired hyperbolic normal behaviour is given by condition (vii).

For all $J \in \mathcal{J}_\kappa$, we denote $\alpha(J) = \partial_{pp}^2 P(J,0,0)$, $\beta(J) = \partial_{pq}^2 P(J,0,0) = \partial_{qp}^2 P(J,0,0)$ and $\gamma(J) = \partial_{qq}^2 P(J,0,0)$. $\lambda_1(J)$, $\lambda_2(J)$ are the eigenvalues of the Hessian matrix of P with respect to p and q at the point $(J,0,0)$. Then, the transformation $\hat{\mathcal{T}}$ generated by the function

$$\hat{\mathcal{S}}\left(J,\hat{\theta},,p,\hat{q}\right) = J\hat{\theta} - \frac{\beta(J)}{2\gamma(J)}p^2 + \frac{\alpha(J)\gamma(J) - \beta^2(J)}{\gamma(J)\lambda_1(J)}p\hat{q} + \frac{\beta(J)\lambda_2(J)}{2\gamma(J)\lambda_1(J)}\hat{q}^2$$

yields the unperturbed Hamiltonian :

$$\hat{H}_0\left(\hat{J},\hat{p},\hat{q}\right) = h\left(\hat{J}\right) + \frac{\varepsilon}{2} {}^t\!\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} \begin{pmatrix} \lambda_1(\hat{J}) & 0 \\ 0 & \lambda_2(\hat{J}) \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} + \varepsilon \hat{P}\left(\hat{J},\hat{p},\hat{q};\varepsilon\right)$$

where $\hat{P}\left(\hat{J},\hat{p},\hat{q};\varepsilon\right) = \mathcal{O}_3\left(\hat{p},\hat{q};\hat{J},\varepsilon\right)$.

Now, a second transformation $\left(\hat{J},\hat{\theta},\hat{p},\hat{q}\right)=\overline{\mathcal{T}}\left(\bar{I},\bar{\varphi},\bar{s},\bar{u}\right)$ generated by the function

$$\overline{S}\left(\hat{J},\bar{\varphi},\hat{p},\bar{u}\right)=\hat{J}\bar{\varphi}-\sqrt{-\frac{\lambda_1\left(\hat{J}\right)}{\lambda_2\left(\hat{J}\right)}}\frac{\hat{p}^2}{2}+\sqrt{2}\hat{p}\bar{u}-\sqrt{-\frac{\lambda_2\left(\hat{J}\right)}{\lambda_1\left(\hat{J}\right)}}\frac{\bar{u}^2}{2}$$

gives the new unperturbed Hamiltonian

$$\overline{H}_0\left(\bar{I},\bar{s},\bar{u}\right)=h\left(\bar{I};\varepsilon\right)+\varepsilon\lambda\left(\bar{I}\right)\bar{s}\bar{u}+\varepsilon\overline{P}\left(\bar{I},\bar{s},\bar{u};\varepsilon\right)$$

where $\lambda\left(\bar{I}\right)=\sqrt{-\lambda_1\left(\bar{I}\right)\lambda_2\left(\bar{I}\right)}$ and $\overline{P}\left(\bar{I},\bar{s},\bar{u};\varepsilon\right)=\mathcal{O}_3\left(\bar{s},\bar{u};\bar{I}\right)$.

At this step, we can exploit the hyperbolicity at the origin for the one degree of freedom Hamiltonian $\overline{P}\left(\bar{I},\bar{s},\bar{u};\varepsilon\right)$ (with $\bar{I}$ as a parameter) by applying the theorem of Moser ([Mo, 1956]) stated in the introduction which allows an integration of the unperturbed Hamiltonian $\overline{H}_0\left(\bar{I},\bar{s},\bar{u}\right)$ in the vicinity of the embedding of the action space $\mathcal{J}\times\mathbf{T}^n$ in $\mathbf{R}^n\times\mathbf{T}^n\times\mathbf{R}^2$.

Lemma 2.1. (Moser's transformation)

We denote $V_{\bar{\kappa}}$ the complex domain of analyticity of $\overline{\mathcal{H}}_0\left(\bar{I},\bar{\varphi},\bar{s},\bar{u}\right)$ and recall that $\lambda\left(\bar{I}\right)>\lambda>0$, $||\bar{P}||_{\bar{\kappa}}\leq\bar{C}E_1$, $||\bar{P}-\lambda\left(\bar{I}\right)\bar{s}\bar{u}||_{\bar{\kappa}}\leq\bar{C}E_1\bar{\kappa}^3$ where $\bar{C}$ is a positive constant and $||.||_{\bar{\kappa}}$ is the supremum norm over $V_{\bar{\kappa}}$. Then, if

$$\bar{\kappa}\leq C\frac{\lambda}{E_1}\tag{1}$$

where λ is the uniform bound on the Liapounov exponent over the considered domain and C is a positive constant (we will assume that $C\leq1$), there exists a real analytic canonical transformation

$$\begin{array}{llll}\mathcal{T}: & V_{\bar{\kappa}/2} & \longrightarrow & V_{\bar{\kappa}} \quad \text{with } \bar{I}=I,\ \mathcal{T}\left(I,\varphi,s,u\right)-\left(I,\varphi,s,u\right)=\mathcal{O}_2(s,u;I,\varepsilon)\\ & \left(\bar{I},\bar{\varphi},\bar{s},\bar{u}\right) & \longmapsto & \left(I,\varphi,s,u\right)\end{array}$$

such that the unperturbed Hamiltonian becomes in the new variables :

$$H_0(I,s,u)=h\left(I;\varepsilon\right)+\varepsilon P\left(I,su;\varepsilon\right)=h\left(I\right)+\varepsilon\lambda\left(I\right)su+\varepsilon\mathcal{O}_2\left(su;I,\varepsilon\right).$$

A proof of this theorem based on an iterative quadratic scheme can be found in [CG, 1994] (annex A3), this allows simplifications in comparison with Moser's original reasoning which uses majoring series.

We also note that the previous transformations leave the actions I invariant and are defined on a domain whose size is independent of μ. Hence, they are fitted for the study of Arnold's instability where we want to estimate the speed of drift of orbits shadowing lines of hyperbolic tori whose lengths are μ-independent.

At present, the original Hamiltonian is properly reduced and, in order to apply the K.A.M. theory, we should select an origin I_0 in the action space linked to a torus whose associated frequency vector is strongly non resonant, that is $\omega_0=\nabla h(I_0)$ satisfies a Diophantine condition :

$$\omega_0\in\Omega_{\gamma,\tau}=\left\{\omega\in\mathbf{R}^n\text{ such that }|k.\omega|\geq\frac{\gamma}{||k||_\infty^\tau}\text{ for all }k\in\mathbf{Z}^n\right\}$$

where γ is an arbitrary positive constant and $\tau\in]n-1,+\infty[$. We recall that for $\tau>n-1$, the measure of the complementary set of $\Omega_{\gamma,\tau}$ is of the order of $\mathcal{O}(\gamma)$

With a translation of the origin at $(I_0,0,0,0)$, the Taylor expansion of the Hamiltonian can be written :

$$\mathcal{H}(I,\varphi,s,u)=\omega_0.I+\varepsilon\lambda_0su+\varepsilon f(I,su)+\varepsilon\mu g(I,\varphi,s,u)\tag{2}$$

where

(i) $\lambda_0 = \lambda(I_0)$, hence $\lambda_0 > \lambda > 0$

(ii) $f(I, su) = \mathcal{O}_2(I, su; \varepsilon)$ and $g(I, \varphi, s, u)$ is an arbitrary function.

(iii) With the previous notations, the total Hamiltonian $\mathcal{H}$ is supposed analytical over the complex domain V_K where K is a positive constant

(iv) We set $||\omega_0|| = \nu_0$ where $||.||$ denotes the Euclidean norm

(v) With the operator norm induced by the Euclidean norm, we have $||\partial_I^2 f(0,0)|| \leq p^{-1}$ where p is the constant defined previously for h and the unperturbed Hamiltonian H satisfies a condition of isoenergetic non degeneracy with the same constant p :

$$\text{for all } \mathbf{V} \in \mathbf{R}^{n+1}, \text{ we have } \left|\left| \begin{pmatrix} \varepsilon\partial_I^2 f(0,0) & \omega_0 \\ \omega_0 & 0 \end{pmatrix} \mathbf{V} \right|\right| \geq \varepsilon p ||\mathbf{V}||$$

(vi) With the sup norm (L^∞) over V_K, we have $||\mathcal{H}||_K < E_1$ and $||g||_K < E_1$. Here, we also assume that $E_1 \geq \sup(1, \nu_0)$

Now, we can state the main result :

Theorem 2.2. (main result)

Let $m_* = \inf\left(\frac{p}{2}, \frac{\lambda}{2}\right)$, $\sigma = 2^{4n+1}\left(\frac{n+1}{e}\right)^{n+1}$ *with* $1 = \ln(e)$ *; under the previous assumptions and also*

$$n-1 < \tau \leq n \; ; \; K \leq C\frac{m_*}{E_1} \; ; \; \gamma \leq \frac{\varepsilon}{30} \; ; \; \frac{\varepsilon\mu}{\gamma^4} < \left(\frac{m_*^3}{3\sigma^3 K}\right)^2 \left(\frac{K}{240}\right)^{4(2n+5)} \tag{3}$$

there exist a symplectic transformation $\mathcal{T} : V_{K/3} \longrightarrow V_K$ *such that :*

$$\mathcal{H} \circ \mathcal{T} = (1+\eta)\omega_0 \widetilde{I} + \varepsilon\lambda_1 \widetilde{s}\widetilde{u} + \varepsilon\widetilde{f}\left(\widetilde{I}, \widetilde{\varphi}, \widetilde{s}, \widetilde{u}\right) \text{ with } \widetilde{f}\left(\widetilde{I}, \widetilde{\varphi}, \widetilde{s}, \widetilde{u}\right) = \mathcal{O}_2\left(\widetilde{I}, \widetilde{s}\widetilde{u}; \widetilde{\varphi}, \widetilde{s}, \widetilde{u}, \varepsilon, \mu\right)$$

with $\frac{1}{2} < 1+\eta < \frac{3}{2}$, $\lambda_1 > \frac{\lambda_0}{2}$ *and*

$$\left|\left| \eta\omega_0 \widetilde{I} + \varepsilon(\lambda_1 - \lambda_0)\widetilde{s}\widetilde{u} + \varepsilon\left[\widetilde{f}\left(\widetilde{I}, \widetilde{\varphi}, \widetilde{s}, \widetilde{u}\right) - f(\widetilde{I}, \widetilde{s}\widetilde{u})\right] \right|\right|_{K/3} \leq \frac{E_1}{2^{4n+6}}\sqrt{\varepsilon\mu}$$

Hence, $\{\widetilde{I} = 0\}$ *becomes an hyperbolic invariant set for the perturbed flow with linear motions.*

Remark : The previous estimates are valid for $n-1 < \tau \leq n$, in the case of $\tau > n$ the main result is valid with the thresholds :

$$n < \tau \; ; \; K \leq C\frac{m_*}{E_1} \; ; \; \gamma \leq \frac{\varepsilon}{30} \; ; \; \frac{\varepsilon\mu}{\gamma^4} < \left(\frac{m_*^3}{3\sigma^3 K}\right)^2 \left(\frac{K}{240}\right)^{4(2\tau+5)} \quad \text{with } \widetilde{\sigma} = 2^{4\tau+1}\left(\frac{\tau+1}{e}\right)^{\tau+1} \tag{$\widetilde{3}$}$$

This follows from the estimates derived in the lemma A.3. and A.4. in the case of a Diophantine condition with $\tau > n$.

II.2 Formal scheme of the proof

We use the classical method to build a Kolmogorov normal form in the vicinity of an invariant torus. Actually, we follow the lines of the proof of Kolmogorov theorem given by Benettin, Galgani, Giorgilli and Strelcyn [BGGS, 1984].

One first expand the total Hamiltonian in the vicinity of $(I, s, u) = (0, 0, 0)$, hence

$$\mathcal{H}(I, \varphi, s, u) = H^{(0)}(I, \varphi, s, u) + \varepsilon\mu g^{(0)}(I, \varphi, s, u)$$

with : $H^{(0)}(I,\varphi,s,u) = \omega^{(0)}I + \varepsilon\lambda^{(0)}su + \varepsilon b^{(0)}(\varphi,s,u)Isu + \varepsilon c^{(0)}(\varphi,s,u)(su)^2$
$+\varepsilon d^{(0)}(\varphi,s,u)I^2 + \varepsilon e^{(0)}(I,\varphi,s,u)I^3$

and $g^{(0)}(I,\varphi,s,u) = \alpha_0^{(0)}(\varphi) + \alpha_{10}^{(0)}(\varphi,s)s + \alpha_{01}^{(0)}(\varphi,u)u + \alpha_{11}^{(0)}(\varphi)su$
$+\alpha_{21}^{(0)}(\varphi,s)s^2u + \alpha_{12}^{(0)}(\varphi,s)su^2 + \beta_0^{(0)}(\varphi)I + \beta_{10}^{(0)}(\varphi,s)Is + \beta_{01}^{(0)}(\varphi,u)Iu$

where $\omega^{(0)} = \omega_0$, $\lambda^{(0)} = \lambda_0$ and

(i) For all $\mathbf{V} \in \mathbf{R}^{n+1}$, we can write with $p_0 = p - \dfrac{4E_1}{K^2}\mu$:

$$\left|\left|\begin{pmatrix} 2\varepsilon\overline{d^{(0)}}(0,0) & \omega^{(0)} \\ \omega^{(0)} & 0 \end{pmatrix}\mathbf{V}\right|\right| \geq \left|\left|\begin{pmatrix} \varepsilon\partial_I^2 f(0,0,0) & \omega_0 \\ \omega_0 & 0 \end{pmatrix}\mathbf{V}\right|\right| - \left|\left|\begin{pmatrix} \varepsilon\mu\overline{\partial_I^2 g}(0,0,0) & 0 \\ 0 & 0 \end{pmatrix}\mathbf{V}\right|\right| \geq \varepsilon p_0||\mathbf{V}||$$

where the averaging over the angles is denoted by a bar : for any function f defined over V_K, we denote

$$\bar{\mathrm{f}}(I,s,u) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\dots\int_0^{2\pi}\mathrm{f}(I,\varphi,s,u)d\varphi_1\dots d\varphi_n$$

and the thresholds (1) gives $m_* \leq \dfrac{p}{2} < p_0 < 1$.

(ii) Finally, $||d^{(0)}(\varphi,0,0)|| \leq \dfrac{1}{p} + \mu\dfrac{4||g||}{K^2} \leq \dfrac{1}{p} + \mu\dfrac{4E_1}{K^2} \leq \dfrac{1}{p_0} < \dfrac{1}{m_*}$ since $0 < p < 1$ and $0 < \mu\dfrac{4E_1}{K^2}$.

Along the lines of the usual proof of Kolmogorov's theorem, we construct the transformation $\mathcal{T}$ of the main result as the limit of a sequence of symplectic transformations $(\mathcal{T}_l)_{l\in\mathbf{N}^*}$ such that the Hamiltonian in the intermediate variables is defined over $V_{(1-\Delta_l)K}$ with $\Delta_l < 2/3$ and can be written

$$\mathcal{H}^{(l)}(I,\varphi,s,u) = H^{(l)}(I,\varphi,s,u) + \varepsilon\mu_l g^{(l)}(I,\varphi,s,u)$$

where $\mathcal{H}^{(l)} = \mathcal{H}\circ\mathcal{T}_1\circ\dots\circ\mathcal{T}_l$ and

$$H^{(l)}(I,\varphi,s,u) = \omega^{(l)}I + \varepsilon\lambda^{(l)}su + \varepsilon b^{(l)}(\varphi,s,u)Isu + \varepsilon c^{(l)}(\varphi,s,u)(su)^2 + \varepsilon d^{(l)}(\varphi,s,u)I^2 + \varepsilon e^{(l)}(I,\varphi,s,u)I^3$$

$$g^{(l)}(I,\varphi,s,u) = \alpha_0^{(l)}(\varphi) + \alpha_{10}^{(l)}(\varphi,s)s + \alpha_{01}^{(l)}(\varphi,u)u + \alpha_{11}^{(l)}(\varphi)su + \alpha_{21}^{(l)}(\varphi,s)s^2u + \alpha_{12}^{(l)}(\varphi,s)su^2 + \beta_0^{(l)}(\varphi)I + \beta_{10}^{(l)}(\varphi,s)Is + \beta_{01}^{(l)}(\varphi,u)Iu$$

with $\left|\left|\mathcal{H}^{(l)}\right|\right|_{(1-\Delta_l)K} \leq E_1$; $\left|\left|g^{(l)}\right|\right|_{(1-\Delta_l)K} \leq E_1$.

We assume that the size of the perturbation (given by $\varepsilon\mu_l E_1$ over $V_{(1-\Delta_l)K}$) should decrease to zero as l goes to infinity and that the coefficients of the new Hamiltonian also satisfy :

(i) $\omega^{(l)} = (1+\eta_l)\omega_0 \in \Omega_{m_*\gamma,\tau}$ where $m_* < \dfrac{1}{2} < 1+\eta_l < \dfrac{3}{2}$ and $m_* < \dfrac{\lambda_0}{2} \leq \lambda^{(l)}$

(ii) for all $\mathbf{V}\in\mathbf{R}^n$, we have $\left|\left|\begin{pmatrix} 2\varepsilon\overline{d^{(l)}}(0,0) & \omega^{(l)} \\ \omega^{(l)} & 0 \end{pmatrix}\mathbf{V}\right|\right| \geq \varepsilon p_l||\mathbf{V}||$ with $m_* \leq \dfrac{p}{2} < p_l < 1$

(iii) $\left|\left|d^{(l)}(\varphi,0,0)\right|\right| \leq \dfrac{1}{p_l} \leq \dfrac{1}{m_*}$

Now, we present the formal construction of the transformation $\mathcal{T}_{l+1}$ at the each step of this iterative scheme.

With the formalism of the Lie series, we consider a transformation $\mathcal{T}$ defined as the time one map of an auxiliary Hamiltonian $\chi(I,\varphi,s,u)$ (see [BGGS, 1984] for some information about Lie series). Using the Poisson bracket :

$$L_\chi(f) = \{\chi, f\} = \frac{\partial\chi}{\partial I}.\frac{\partial f}{\partial\varphi} - \frac{\partial\chi}{\partial\varphi}.\frac{\partial f}{\partial I} + \frac{\partial\chi}{\partial s}.\frac{\partial f}{\partial u} - \frac{\partial\chi}{\partial u}.\frac{\partial f}{\partial s}$$

we have $\mathcal{T} = \exp(L_\chi)$ and $H\circ\mathcal{T} = \exp(L_\chi)(H)$.

Here, we consider a first transformation $\mathcal{T}_A$ generated by the auxiliary Hamiltonian :

$$\chi_A(\varphi) = \varepsilon\mu_l\left(X_0(\varphi) + \xi.\varphi\right) \text{ with a fixed vector } \xi \in \mathbf{R}^n$$

and one can write $\mathcal{H}^{(l)}\circ\mathcal{T}_A = H^{(l)} + \left[\left\{\chi_A, H^{(l)}\right\} + \varepsilon\mu_l g^{(l)}\right] + \mathcal{R}_A = H_A + \varepsilon\mu_l g_A + \mathcal{R}_A$ where :

$$\left\{\chi_A, H^{(l)}\right\} = -\varepsilon\mu_l\left(\omega^{(l)}.\xi + \partial_\varphi X_0(\varphi)\right) - 2\varepsilon^2\mu_n\left(d^{(l)}(\varphi,0,0).\xi + \partial_\varphi X_0(\varphi)\right)I + \mathcal{O}_2(I,s,u;I,\varphi,s,u)$$

and $\mathcal{R}_A(I,\varphi,s,u) = \varepsilon\mu_l\left\{\chi_A, g^{(l)}\right\} + \left[\mathcal{H}^{(l)}\circ\mathcal{T}_A - \mathcal{H}^{(l)} - \left\{\chi_A,\mathcal{H}^{(l)}\right\}\right] = \mathcal{O}_2(\varepsilon\mu_l; I,\varphi,s,u)$.

We choose $X_0(\varphi)$ solution of the equation

$$\left(\omega^{(l)}.\partial_\varphi X_0(\varphi)\right) = \alpha_0^{(l)}(\varphi) - \overline{\alpha_0^{(l)}} \qquad (\mathcal{A}_1)$$

and, denoting $d_0^{(l)}(\varphi) = d^{(l)}(\varphi,0,0)$, we look for $(\eta_A,\xi)\in\mathbf{R}\times\mathbf{R}^n$ such that :

$$\begin{cases} -\left(\omega^{(l)}.\xi\right) + \overline{\alpha_0^{(l)}} = 0 \\ -2\varepsilon\overline{\left(d_0^{(l)}.\xi + \partial_\varphi X_0\right)} + \overline{\beta_0^{(l)}} = \eta_A\omega^{(l)} \end{cases} \Longleftrightarrow \begin{cases} 2\varepsilon\left(\overline{d_0^{(l)}}.\xi\right) + \eta_A\omega^{(l)} = \overline{\beta_0^{(l)}} - 2\varepsilon\overline{\left(d_0^{(l)}.\partial_\varphi X_0\right)} \\ \omega^{(l)}.\xi = \overline{\alpha_0^{(l)}} \end{cases} \qquad (\mathcal{A}_2)$$

with the assumption of isoenergetic non degeneracy, the system $(\mathcal{A}_2)$ admits a solution close to $(0,0)\in\mathbf{R}\times\mathbf{R}^n$.

The Hamiltonian becomes in the new variables :

$$\mathcal{H}^{(l)}\circ\mathcal{T}_A(I,\varphi,s,u) = H_A(I,\varphi,s,u) + \varepsilon\mu_n g_A(I,\varphi,s,u) + \mathcal{R}_A(I,\varphi,s,u)$$

with

$$\begin{aligned} H_A(I,\varphi,s,u) = \omega^{(l+1)}I + \varepsilon\lambda^{(l+1)}su + \varepsilon b_A(\varphi,s,u)Isu + \varepsilon c_A(\varphi,s,u)(su)^2 \\ + \varepsilon d_A(\varphi,s,u)I^2 + \varepsilon e_A(I,\varphi,s,u)I^3 \end{aligned}$$

where $\omega^{(l+1)} = (1+\varepsilon\mu_l\eta_A)\,\omega^{(l)} = (1+\varepsilon\mu_l\eta_A)\,(1+\eta_l)\,\omega_0 = (1+\eta_{l+1})\,\omega_0$; $\lambda^{(l+1)} = \lambda^{(l)} + \mu_l\overline{\alpha_{11}^A}$

and

$$\begin{aligned} g_A(I,\varphi,s,u) =& \alpha_{10}^A(\varphi,s)s + \alpha_{01}^A(\varphi,u)u + \left(\alpha_{11}^A(\varphi) - \overline{\alpha_{11}^A}\right)su \\ &+ \alpha_{21}^A(\varphi,s)s^2u + \alpha_{12}^A(\varphi,s)su^2 + \beta_0^A(\varphi)I + \beta_{10}^A(\varphi,s)Is + \beta_{01}^A(\varphi,u)Iu \end{aligned}$$

$$\text{where } \beta_0^A = \beta_0^{(l)} - 2\varepsilon\left(d_0^{(l)}.\xi + \partial_\varphi X_0\right) - \overline{\beta_0^{(l)} - 2\varepsilon\left(d_0^{(l)}.\xi + \partial_\varphi X_0\right)}$$

hence, the coefficients of su and I in the main part of the perturbation have a zero mean value.

Then, we consider a second transformation $\mathcal{T}_B$ generated by the function :

$$\begin{aligned} \chi_B(I,\varphi,s,u) = \varepsilon\mu_l\,[X_{10}(\varphi,s)s + X_{01}(\varphi,u)u + X_{11}(\varphi)su + X_{21}(\varphi,s)s^2u \\ + X_{12}(\varphi,u)su^2 + Y_0(\varphi)I + Y_{10}(\varphi,s)Is + Y_{01}(\varphi,u)Iu] \end{aligned}$$

which gives $\mathcal{H}^{(l)} \circ \mathcal{T}_A \circ \mathcal{T}_B = H_A + [\{\chi_B, H_A\} + \varepsilon\mu_l g_A] + \widetilde{\mathcal{R}} = H^{(l+1)} + \varepsilon\mu_{l+1} g^{(l+1)}$ where

$$\varepsilon\mu_{l+1} g^{(l+1)} = \widetilde{\mathcal{R}} = \mathcal{R}_A + \varepsilon\mu_l \{\chi_B, g_A\} + \{\chi_B, \mathcal{R}_A\} + \left[\mathcal{H}^{(l)} \circ \mathcal{T}_A \circ \mathcal{T}_B - \mathcal{H}^{(l)} \circ \mathcal{T}_A - \left\{\chi_B, \mathcal{H}^{(l)} \circ \mathcal{T}_A\right\}\right]$$

Hence $\varepsilon\mu_{l+1} g^{(l+1)} (I, \varphi, s, u) = \mathcal{O}_2 (\varepsilon\mu_l; I, \varphi, s, u)$ and

$$\begin{aligned}\{\chi_B, H_A\} (I, \varphi, s, u) = &\varepsilon\mu_l \left[u_{10} (\varphi, s)s + u_{01}(\varphi, u)u + u_{11}(\varphi)su + u_{21}(\varphi, s)s^2 u + u_{21}(\varphi, s)s^2 u\right. \\ &\left. + u_{12}(\varphi, u)su^2 + v_0(\varphi)I + v_{10}(\varphi, s)Is + v_{01}(\varphi, u)Iu + \mathcal{O}_2(I, su; I, \varphi, s, u)\right]\end{aligned}$$

with

$$\begin{aligned}
&u_{10}(\varphi, s) = \partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{10}(\varphi, s) \ ; \ u_{01}(\varphi, u) = -\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{01}(\varphi, u) \ ; \\
&u_{11}(\varphi) = -\left(\omega^{(l+1)}.X_{11}(\varphi)\right) = -(1 + \eta_{l+1}) . (\omega_0.X_{11}(\varphi)) \ ; \\
&u_{21}(\varphi, s) = \partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{21}(\varphi, s) - \varepsilon (\partial_\varphi X_{10}(\varphi, s).b_A(\varphi, s, 0)) + 2\varepsilon c_A(\varphi, s, 0)\partial_{1,0} X_{10}(\varphi, s) \ ; \\
&u_{12}(\varphi, u) = -\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{12}(\varphi, u) - \varepsilon (\partial_\varphi X_{01}(\varphi, u).b_A(\varphi, 0, u)) - 2\varepsilon c_A(\varphi, 0, u)\partial_{1,0} X_{01}(\varphi, u) \ ; \\
&v_0(\varphi) = -\left(\omega^{(l+1)}.Y_0(\varphi)\right) = -(1 + \eta_{l+1}) . (\omega_0.Y_0(\varphi)) \ ; \\
&v_{10}(\varphi, s) = \partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} Y_0(\varphi, s) - 2\varepsilon (\partial_\varphi X_{10}(\varphi, s).d_A(\varphi, s, 0)) + \varepsilon b_A(\varphi, s, 0)\partial_{1,0} X_{10}(\varphi, s) \ ; \\
&v_{01}(\varphi, u) = -\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} Y_{01}(\varphi, u) - 2\varepsilon (\partial_\varphi X_{01}(\varphi, u).d_A(\varphi, 0, u)) - \varepsilon b_A(\varphi, 0, u)\partial_{1,0} X_{01}(\varphi, u) \ ;
\end{aligned}$$

where $\partial^\pm_{\lambda,\omega}$ denotes the operator :

$$\partial^\pm_{\lambda,\omega} \ : \ f(\varphi, x) = \sum_{k\in\mathbf{Z}^n, l\in\mathbf{N}} a_{k,l} x^l \exp(i(k.\varphi)) \longmapsto \partial^\pm_{\lambda,\omega} f = \sum_{k\in\mathbf{Z}^n, l\in\mathbf{N}} ((l+1)\lambda \pm i(\omega.l)) a_{k,l} x^l \exp(i(k.\varphi))$$

We choose $X_{11}(\varphi)$, $Y_0(\varphi)$, $X_{10}(\varphi, s)$, $X_{01}(\varphi, u)$, $X_{21}(\varphi, s)$, $X_{12}(\varphi, u)$, $Y_{10}(\varphi, s)$ and $Y_{01}(\varphi, u)$ solutions of the following equations :

$$(\mathcal{B}) \begin{cases}
(1 + \eta_{l+1}) . (\omega_0.\partial_\varphi X_{11}(\varphi)) = \alpha^A_{11}(\varphi) - \overline{\alpha^A_{11}} \\
(1 + \eta_{l+1}) . (\omega_0.\partial_\varphi Y_0(\varphi)) = \beta^A_0(\varphi) \\
\partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{10}(\varphi, s) = -\alpha^A_{10}(\varphi, s) \\
\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{01}(\varphi, u) = \alpha^A_{01}(\varphi, u) \\
\partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{21}(\varphi, s) = -\alpha^A_{21}(\varphi, s) + \varepsilon (b_A(\varphi, s, 0).\partial_\varphi X_{10}(\varphi, s)) - 2\varepsilon c_A(\varphi, s, 0)\partial_{1,0} X_{10}(\varphi, s) \\
\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} X_{12}(\varphi, u) = \alpha^A_{12}(\varphi, u) - \varepsilon (b_A(\varphi, 0, u).\partial_\varphi X_{01}(\varphi, u)) - 2\varepsilon c_A(\varphi, 0, u)\partial_{1,0} X_{01}(\varphi, u) \\
\partial^-_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} Y_{10}(\varphi, s) = -\beta^A_{10}(\varphi, s) + 2\varepsilon (d_A(\varphi, s, 0).\partial_\varphi X_{10}(\varphi, s)) - \varepsilon b_A(\varphi, s, 0)\partial_{1,0} X_{10}(\varphi, s) \\
\partial^+_{\varepsilon\lambda^{(l+1)},\omega^{(l+1)}} Y_{01}(\varphi, u) = \beta^A_{01}(\varphi, u) - 2\varepsilon (d_A(\varphi, 0, u).\partial_\varphi X_{01}(\varphi, u)) - \varepsilon b_A(\varphi, 0, u)\partial_{1,0} X_{01}(\varphi, u)
\end{cases}$$

Hence, the next transformation $\mathcal{T}_{l+1}$ is given by $\mathcal{T}_{l+1} = \mathcal{T}_A \circ \mathcal{T}_B$ and the Hamiltonian becomes in the new variables :

$$\begin{aligned}\mathcal{H} \circ \mathcal{T}_1 \circ \ldots \circ \mathcal{T}_l \circ \mathcal{T}_{l+1}(I, \varphi, s, u) = &H^{(l+1)}(I, \varphi, s, u) + \varepsilon\mu_{l+1} g^{(l+1)}(I, \varphi, s, u) \\ = &H_A(I, \varphi, s, u) + \varepsilon\mu_l \mathcal{O}_2 (I, su; I, \varphi, s, u) + \mathcal{O}_2 (\varepsilon\mu_l; I, \varphi, s, u)\end{aligned}$$

which gives a decreasing magnitude of the perturbation at each step.

II.3 Convergence estimates :

The successive use of the previous procedure will give the desired normal form. Now, we specify the rate of decrease of the perturbation at each step of this iterative scheme.

Lemma 2.3. (iterative lemma)

We consider an Hamiltonian $\mathcal{H}^{(l)}$ *and the parameters* Δ_l, μ_l, η_l, $\lambda^{(l)}$, p_l *which satisfy the assumptions given in the previous section.*

Let $0 < \delta_{l+1} < \dfrac{1}{12}$ *; with the constants given in main result, we assume that :*

$$\delta_{l+1}^{2n+1} \leq \frac{\varepsilon\sigma}{m_*^2\gamma}\left(\frac{3}{K}\right)^{2n} \text{ and } \frac{\varepsilon\mu_l}{\gamma^2} \leq \frac{m_*^6\delta_{l+1}^4}{21\sigma^3E_1^3}\left(\frac{\delta_{l+1}K}{3}\right)^{4n+6} \tag{4}$$

then, there exist a symplectic transformation $\mathcal{T}_{l+1}$ *such that*

$$V_{(1-12\delta_{l+1})K_l} \subset \mathcal{T}\left(V_{K_{l+1}}\right) \subset V_{(1-8\delta_{l+1})K_l}$$

where $K_l = (1-\Delta_l)K$ *and* $K_{l+1} = (1-10\delta_{l+1})(1-\Delta_l)K$.

The transformed Hamiltonian can be written with the previous notations

$$\mathcal{H}^{(l)} \circ \mathcal{T}_{l+1}(I,\varphi,s,u) = H^{(l+1)}(I,\varphi,s,u) + \varepsilon\mu_{l+1}g^{(l+1)}(I,\varphi,s,u)$$

with $\mu_{l+1} = \left(\dfrac{54\sigma^3E_1^3}{m_*^6\delta_{l+1}^4}\right)^2\left(\dfrac{3}{\delta_{l+1}K}\right)^{4n+6}\dfrac{\varepsilon\mu_l^2}{\gamma^4}$ *and*

$$\left|\left|H^{(l+1)} - H^{(l)}\right|\right|_{K_{l+1}} \leq \frac{150\sigma^3E_1^4}{m_*^6\delta_{l+1}^4}\left(\frac{3}{\delta_{l+1}K}\right)^{4n+6}\frac{\varepsilon\mu_l}{\gamma^2} \; ; \; \left|\left|g^{(l+1)}\right|\right|_{K_{l+1}} \leq E_1$$

The coefficients of the Hamiltonian also satisfy :

(i) $\omega^{(l+1)} = (1+\eta_{l+1})\omega_0 \in \Omega_{m_*\gamma,\tau}$ *where*

$$m_* < \frac{1}{2} < 1 - |\eta_l| - \frac{6\sigma E_1\varepsilon\mu_l}{m_*^3\gamma}\left(\frac{3}{\delta_{l+1}K}\right)^{2n+1} \leq 1+\eta_{l+1} \leq 1+|\eta_l| + \frac{6\sigma E_1\varepsilon\mu_l}{m_*^3\gamma}\left(\frac{3}{\delta_{l+1}K}\right)^{2n+1} < \frac{3}{2}$$

(ii) $m_* < \dfrac{\lambda_0}{2} < \lambda - \dfrac{7\sigma E_1^2\delta_{l+1}^2}{m_*^3\gamma}\left(\dfrac{3}{\delta_{l+1}K}\right)^{2(l+2)}\mu_l \leq \lambda^{(l+1)}$

(iii) $\left|\left|\begin{pmatrix} 2\varepsilon\overline{d^{(l+1)}}(0,0) & \omega^{(l+1)} \\ \omega^{(l+1)} & 0\end{pmatrix}\mathbf{V}\right|\right| \geq \varepsilon p_{l+1}||\mathbf{V}||$ *and* $\left|\left|d^{(l+1)}(\varphi,0,0)\right|\right|_{K_{l+1}} \leq \dfrac{1}{p_{l+1}} \leq \dfrac{1}{m_*}$

$$\text{where } m_* \leq \frac{p}{2} < p_{l+1} = p_l - \frac{(16\sigma E_1)^3}{m_*^6\gamma^2\delta_{l+1}^2}\left(\frac{3}{\delta_{l+1}K}\right)^{4(n+2)}\mu_l < 1 \tag{5}$$

The proof of this lemma is deferred to the appendix A (replace K_* by $K/3$ in the lemma A.5. of the appendix)

Conclusion of the proof of the main result

Now, we apply successively the previous lemma in order to eliminate the perturbation $g(I,\varphi,s,u)$, this gives the sequences of parameters μ_l, p_l, η_l, λ_l where $l \in \mathbf{N}$ which are linked with the relations in the previous lemma. As previously said, the starting parameters are $\eta_0 = 0$, $\lambda_0 = \lambda_0$, $p_0 = p - \mu\frac{4E_1}{K^2}$ and the reasoning used in the proof of the iterative lemma 2.3. to compute the size of the perturbation g_A allows to write

$$\varepsilon\mu\left|\left|g^{(0)}\right|\right|_{\left(1-\frac{1}{3}\right)K} \leq 3\left(1+\frac{2}{\frac{1}{3}}\right)\varepsilon\mu||g||_K \leq \varepsilon\mu_0E_1 \text{ where } \mu_0 = 21\mu$$

we can also write $\left|\left|H^{(0)}(I,\varphi,s,u)-[\omega_0 I+\varepsilon\lambda_0 su+\varepsilon f(I,su)]\right|\right|_{\frac{2K}{3}}\leq\varepsilon\mu\left|\left|g^{(0)}-g\right|\right|_{\frac{2K}{3}}\leq 22\varepsilon\mu E_1$.

If we denote $\mu_l=c_l\mu_0$, then

$$\delta_l^{4(2n+5)}=\left(\frac{54\sigma^3E_1^3}{m_*^6}\right)^2\left(\frac{3}{K}\right)^{4(2n+3)}\frac{\varepsilon\mu_0}{\gamma^4}\frac{c_{l-1}^2}{c_l}=\left(\frac{250\sigma^3E_1^3}{m_*^6}\right)^2\left(\frac{3}{K}\right)^{4(2n+3)}\frac{\varepsilon\mu}{\gamma^4}\frac{c_{l-1}^2}{c_l}\text{ for }l\in\mathbf{N}^*$$

We impose a rate of decrease of the perturbation in the iterative scheme given by the ratio $c_l=2^{-4(2n+5)l}$ $(l\in\mathbf{N})$ which implies :

$$\delta_l=\left(\frac{250\sigma^3E_1^3}{m_*^6}\right)^{\frac{1}{2(2n+5)}}\left(\frac{3}{K}\right)^{\frac{2n+3}{2n+5}}\left(\frac{\varepsilon\mu}{\gamma^4}\right)^{\frac{1}{4(2n+5)}}\left(\frac{4}{2^l}\right)\text{ for }l\geq 1$$

then, the condition (3) of the main result implies the assumption (4) of the iterative lemma 2.3.

The size of the domains $V_{(1-\Delta_n)K}$ should not decrease too rapidly ; more specifically, the following condition must be satisfied :

$$\frac{2}{3}K\prod_{l=1}^{\infty}(1-10\delta_l)\leq\frac{K}{3}$$

which is imposed by $\displaystyle\sum_{l=1}^{\infty}\delta_l=4\left(\frac{250\sigma^3E_1^3}{m_*^6}\right)^{\frac{1}{2(2n+5)}}\left(\frac{3}{K}\right)^{\frac{2n+3}{2n+5}}\left(\frac{\varepsilon\mu}{\gamma^4}\right)^{\frac{1}{4(2n+5)}}\left(\sum_{l=1}^{\infty}\frac{1}{2^l}\right)<\frac{1}{20}$

$$\Longleftrightarrow\frac{\varepsilon\mu}{\gamma^4}<\left(\frac{m_*^6}{250\sigma^3E_1^3}\right)^2\left(\frac{K}{3}\right)^{4(2n+3)}\left(\frac{1}{80}\right)^{4(2n+5)}=\left(\frac{m_*^6}{3\sigma^3E_1^3K^4}\right)^2\left(\frac{K}{240}\right)^{4(2n+5)}$$

and this last threshold gives also small enough variations on the frequency, the hyperbolic exponent and the Hessian matrix to carry out the iterative scheme.

Finally, we can write :

$$\begin{aligned}\left|\left|(1+\eta)\omega_0\widetilde{I}+\varepsilon\lambda_1\widetilde{s}\widetilde{u}+\varepsilon\widetilde{f}\left(\widetilde{I},\widetilde{\varphi},\widetilde{s},\widetilde{u}\right)-H^{(0)}\left(\widetilde{I},\widetilde{\varphi},\widetilde{s},\widetilde{u}\right)\right|\right|_{K/3}&\leq\frac{150\sigma^3E_1^4}{m_*^6\gamma^2}\left(\frac{3}{K}\right)^{4n+6}\sum_{l\leq 0}\frac{\varepsilon\mu_l}{\delta_{l+1}^{4n+10}}\\&\leq\frac{3E_1}{5\sqrt{\varepsilon\mu}}\sum_{l\leq 0}\left(\frac{1}{2^{l+1}}\right)^{4n+10}\varepsilon\mu_0\end{aligned}$$

then the value of δ_{l+1} and the upper bound for $\left|\left|H^{(0)}-[\omega_0I+\varepsilon\lambda_0su+\varepsilon f]\right|\right|_{\frac{2K}{3}}$ computed previously allows to derive the estimate given in the main result

III The general case of a nearly integrable system

As said in the introduction, Treschev [Tr, 1991] showed that, under general assumptions, a nearly integrable Hamiltonian $\mathcal{H}(J,\theta)=h(J)+\varepsilon f(J,\theta)$ in a resonant area could be reduced to a perturbed initially hyperbolic Hamiltonian. In that way, Treschev proved the existence of many partially hyperbolic tori invariant under the perturbed flow. More specifically, at a given point in the phase space, the perturbation includes a non-resonant part composed of harmonics $k\in\mathbf{Z}^n$ which satisfy $k.\nabla h(J)\neq 0$ and there exist a symplectic transformation such that the Hamiltonian expressed with the new variables is reduced to a resonant normal form where most of this non resonant part have been removed. In this setting, the resonant part of the perturbation can yield the required hyperbolicity and we can apply the results of the previous section.

The construction of these normal forms is also a central tool to obtain the Nekhorochev upper bound on the speed of drift of the orbits in the perturbed system. Actually, Lochak ([L, 1990], [LN, 1992]) and Pöschel [Pö, 1993] have obtained exponentially small bounds on the rates of instability which are likely to be optimal. This follows notably from the use of a very accurate perturbative scheme of Neistadt ([Ne, 1984]) in the construction of the resonant normal forms (see also [RS, 1996]).

Since the goal of this paper is to provide tools for computing sharp estimates on the times of instability and, ultimately, trying to compare the latter quantities with the Nekhorochev's bounds, it is relevant to use in our study the same resonant normal forms as in Lochak or Pöschel's proof (see also the section IV of this paper). The reasoning of Lochak is based on a refined study of the dynamic in the areas linked with resonances of maximal multiplicity –i.e. : n for a $n+1$ degrees of freedom system– whereas, here, we are only looking for simple resonances. In Pöschel's proof, all the resonances are considered and the linked normal forms are built. Hence, we will use the latter for a resonance of multiplicity one.

III.1 Construction of a resonant normal form

This paragraph is devoted to a presentation of the construction of Pöschel's normal form in the case of a simple resonance.

Set up. We consider a nearly integrable Hamiltonian

$$\mathcal{H}(J,\theta) = h(J) + \varepsilon f(J,\theta) \text{ with } (J,\theta) \in \mathbf{R}^{n+1} \times \mathbf{T}^{n+1},\ \mathbf{T} = \mathbf{R}/\mathbf{Z},$$

where (J,θ) are the action-angle variables of the integrable Hamiltonian h. We assume that $\mathcal{H}$ is analytical over a complex neighbourhood $V_{r_0,s_0}\mathcal{P} \subset \mathbf{C}^{2n+2}$ of a real domain $\mathcal{P} \times \mathbf{T}^{n+1}$ where $\mathcal{P} \subset \mathbf{R}^{n+1}$ and

$$V_{r_0,s_0}\mathcal{P} = \left\{(J,\theta) \in \mathbf{C}^{2n+2} \text{ such that } \operatorname{dist}(J,\mathcal{P}) \leq r_0 \ ;\ \Re e(\theta) \in \mathbf{T}^{n+1} \ ;\ ||\Im m(\theta_j)||_\infty \leq s_0\right\}$$

with $s_0 > 0$, $r_0 > 0$ and the distance to $\mathcal{P}$ given by the Euclidean norm in $\mathbf{C}^{n+1}$.

Denoting $||.||_{r_0,s_0}$ the sup norm (L^∞) for a numerical or a vector valued function defined over $V_{r_0,s_0}\mathcal{P}$, we assume that $||\mathcal{H}||_{r_0,s_0} \leq E$ and $||f||_{r_0,s_0} \leq E$ with $E > 1$. For the norm on the operators induced by the Euclidean norm, we assume that the Hessian matrix $\partial_J^2 h$ satisfies $\left|\left|\partial_J^2 h\right|\right|_{r_0,s_0} \leq p^{-1}$ with $0 < p < 1$.

Because of the exponential decrease of the coefficients in the Fourier expansion of an analytical function, it is relevant to consider the resonances only up to a certain order and we specify the notion of simple resonance in this setting. Let $\Lambda_0 = \mathbf{Z}k_0$ be a sub lattice of $\mathbf{Z}^{n+1}$ generated by $k_0 \in \mathbf{Z}_L^{n+1} = \left\{k \in \mathbf{Z}^{n+1} \text{ such that } ||k|| \leq L\right\}$ where $L > 0$. Let $\alpha > 0$, a subset $\mathcal{D} \subset \mathcal{P} \subset \mathbf{R}^{n+1}$ is said to be α, L non resonant modulo Λ_0, if we have :

$$\left|k.\partial_J^2 h(J)\right| = |k.\omega(J)| \geq \alpha \text{ for all } J \in \mathcal{D} \text{ and } k \in \mathbf{Z}_L^{n+1} \backslash \Lambda_0$$

For the time being, α and L will be considered as independent parameters but they will be chosen as suitable functions of ε in lemma 3.2.

In the vicinity of such a set $\mathcal{D}$, the perturbed Hamiltonian is transformed in a Λ_0-resonant normal form $h + f_* + g$ up to g which means that the expansion $f_*(J,\varphi) = \sum_{k\in\Lambda_0} f_{*k}(J)\exp(ik.\theta)$ contains only harmonics in $\mathbf{Z}_L^{n+1} \cap \Lambda_0$ while g is a general term.

Lemma 3.1. (normal form)
Suppose that $\mathcal{D} \subset \mathcal{P}$ is α, L-non resonant modulo Λ_0. If

$$6 \leq Ls_0 \ ;\ \varepsilon \leq C_1\frac{\alpha r}{L} \ ;\ r \leq \inf\left(C_2\frac{p\alpha}{L}, r_0\right) \quad \textit{where } C_1 > 0 \textit{ and } C_2 > 0$$

then, there exist a real analytic, symplectic coordinate transformation $\Phi : V_{r_*,s_*}\mathcal{D} \longmapsto V_{r,s_0}\mathcal{D}$, where $r_* = \frac{r}{2}$ and $s_* = \frac{s_0}{6}$, such that $\mathcal{H}\circ\Phi = h + f_* + g$ is in Λ_0-resonant normal form up to g with

$$\left|\left|f_* - f^{(0)}\right|\right|_{r_*,s_*} \leq C_3\frac{\varepsilon^2 L}{\alpha r} \ (C_3 > 0) \ ;\ ||g||_{r_*,s_*} \leq \varepsilon E \exp\left(-\frac{Ls_0}{6}\right)$$

where $f^{(0)}$ is the partial sums of the Fourier expansion of f containing only harmonics in $\mathbf{Z}_L^{n+1} \cap \Lambda_0$. Moreover :

$$||\Pi_J\Phi - Id_J||_{r_*,s_*} \leq C_3\frac{L}{\alpha}\varepsilon$$

uniformly on $V_{r_*,s_*}P$ where Π_J denotes the projection onto the action space and Id_J is the identity in the action space.

Now, we specify a domain $\mathcal{D}_0$ which is α, L non resonant modulo Λ_0. This set is first constructed in the frequency space and then pulled back in the action space via the frequency map.

Let $\omega \in \mathbf{R}^{n+1}$ and $k \in \mathbf{Z}_L^{n+1}$, we assume that

$$\left\| \omega - \Pi_0(\omega) - \langle k \rangle^\perp \cap \langle k_0 \rangle^\perp \right\| \geq \frac{2\alpha}{\|k - \Pi_0(k)\|} \text{ and } \|\Pi_0(\omega)\| \leq \frac{\alpha}{L}$$

where Π_0 denotes the orthogonal projection over the vectorial line $\langle k_0 \rangle \subset \mathbf{R}^{n+1}$ generated by k_0 and $\left\| \omega - \Pi_0(\omega) - \langle k \rangle^\perp \cap \langle k_0 \rangle^\perp \right\|$ is the distance to the set $\langle k \rangle^\perp \cap \langle k_0 \rangle^\perp$ given by the Euclidean norm, then :

$$\begin{aligned} |k.\omega| &\geq |k - \Pi_0(k).\omega - \Pi_0(\omega)| - |\Pi_0(k).\Pi_0(\omega)| \\ &\geq \|k - \Pi_0(k)\| \left\| \omega - \Pi_0(\omega) - \langle k \rangle^\perp \cap \langle k_0 \rangle^\perp \right\| - \|\Pi_0(k)\| \, \|\Pi_0(\omega)\| \\ &\geq \|k - \Pi_0(k)\| \frac{2\alpha}{\|k - \Pi_0(k)\|} - \|\Pi_0(k)\| \frac{\alpha}{L} \geq \alpha \text{ since } \|\Pi_0(k)\| \leq L. \end{aligned}$$

We denote $\mathcal{Z}_L^{n+1}$ the set of vectors in $\mathbf{Z}_L^{n+1}$ which generate a maximal L-lattice in $\mathbf{Z}^{n+1}$, that means a lattice not properly contained in any other lattice of dimension one. The precedent reasoning ensure that the following set defined with the previous notations :

$$D_0 = \left\{ \omega \in \mathbf{R}^{n+1} \text{such that } \|\Pi_0(\omega)\| \leq \frac{\alpha}{L} \text{ and } \left\| \omega - \Pi_0(\omega) - \langle k \rangle^\perp \cap \langle k_0 \rangle^\perp \right\| \geq \frac{2\alpha}{\|k - \Pi_0(k)\|} \text{ for all } k \in \mathcal{Z}_L^{n+1} \right\}$$

is α, L non resonant modulo Λ_0.

The use of Dirichlet's theorem on Diophantine approximation (see [LM, 1988] and the section 4 of [Pö, 1993]) shows that α must be at most of the order of L^{-n+1}. In this case, measure theoretical considerations exposed in [Pö, 1993] ensure that the complementary of D_0 in the resonant area linked with Λ_0 (i.e. : $\Pi_0(\omega) = 0$) is of the order of αL^{n-1}.

Gathering everything together, we arrive at the following :

Lemma 3.2. (simple resonance normal form)

We consider the subset $\mathcal{D}_0 = \{J \in \mathcal{P}$ such that $\omega(J) \in D_0\} \subset \mathcal{P}$ where D_0 is defined with $\alpha = \dfrac{A}{L^{n-1}}$ and A is a strictly positive function of ε. If

$$6 \leq Ls_0 \ ; \ \varepsilon < \frac{C_1 r_0^2}{C_2 p} \textit{ and } r = \sqrt{\frac{C_2}{C_1} p\varepsilon} \ ; \ L^n = A\sqrt{\frac{C_1 C_2 p}{\varepsilon}} \textit{ with } C_1 > 0, \ C_2 > 0 \tag{6}$$

then there exist a real analytic, symplectic coordinate transformation $\Phi : V_{r_*,s_*}\mathcal{D}_0 \longmapsto V_{r,s_0}\mathcal{D}_0$, where $r_* = \frac{r}{2} = \sqrt{\frac{pC_2}{4C_1}\varepsilon}$ and $s_* = \frac{s_0}{6}$, such that $\mathcal{H} \circ \Phi = h + f_* + g$ is in $< k_0 >$−resonant normal form up to g with

$$\left\| f_* - f^{(0)} \right\|_{r_*,s_*} \leq \widetilde{C}_3 \varepsilon \ (\widetilde{C}_3 > 0) \textit{ and } \|g\|_{r_*,s_*} \leq \varepsilon E \exp\left(-\frac{Ls_0}{6}\right)$$

where $f^{(0)}$ is the partial Fourier expansion of f containing only harmonics in $\mathbf{Z}_L^{n+1} \cap \Lambda_0$. Moreover $\|\Pi_J \Phi - Id_J\|_{r_*,s_*} \leq \widetilde{C}_3 \sqrt{\varepsilon}$ uniformly on $V_{r_*,s_*}\mathcal{D}_0$ with the notations of lemma 3.1.

Remarks : 1) By means of an unimodular transformation in the phase space, we can always assume that Λ_0 is generated by the first vector of the basis in the frequency space, then we can write in the new variables :

$$\widehat{\mathcal{H}}\left(\widehat{J}, \widehat{\theta}\right) = \mathcal{H} \circ \Phi \left(\widehat{J}, \widehat{\theta}\right) = h\left(\widehat{J}\right) + \varepsilon \widehat{f}_*\left(\widehat{J}, \widehat{\theta}_1\right) + \varepsilon\mu\widehat{g}\left(\widehat{J}, \widehat{\theta}\right) \text{ with } \mu = \exp\left(-\frac{Ls_0}{6}\right) \tag{7}$$

2) If we choose $A = A_0$ where A_0 is a positive constant, then $L = \left(A_0\sqrt{C_1C_2p}\right)^{\frac{1}{n}}\varepsilon^{-\frac{1}{2n}}$ and we find the normal form of Pöschel for a simple resonance with a remainder of the order of $\exp\left(-C\varepsilon^{-\frac{1}{2n}}\right)$.

III.2 Reduction to an initially hyperbolic Hamiltonian and main result.

Now, we reduce the normalized Hamiltonian of the previous paragraph to an initially hyperbolic Hamiltonian studied in section II.

Hence, we start with the expression (7) of the averaged Hamiltonian

$$\mathcal{H}(J,\theta) = h\left(J_1,\widetilde{J}\right) + \varepsilon f_*\left(J_1,\widetilde{J},\theta_1\right) + \varepsilon\mu g\left(J_1,\widetilde{J},\theta_1,\widetilde{\theta}\right)\ ;\ \left(J_1,\widetilde{J}\right)\in\mathbf{R}\times\mathbf{R}^n\ ;\ \left(\theta_1,\widetilde{\theta}\right)\in\mathbf{T}\times\mathbf{T}^n$$

with $\mathcal{H}(J,\theta)$ analytical over $V_{r_*,s_*}\mathcal{D}_0$ where r_* is of the order of $\sqrt{\varepsilon}$.

We consider $\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)\in\mathcal{D}_0\times\mathbf{T}$ such that :

$$\left(J_1^{(0)},\widetilde{J}^{(0)}\right) \text{ is on the resonant manifold : } \frac{\partial h}{\partial J_1}\left(J_1^{(0)},\widetilde{J}^{(0)}\right) = 0 \qquad (\mathcal{C}_1)$$

$$\theta_1^{(0)} \text{ is a critical point of } f_*\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1\right),\text{ hence : } \frac{\partial f_*}{\partial\theta_1}\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right) = 0 \qquad (\mathcal{C}_2)$$

$\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)$ satisfy the following hypothesis of hyperbolicity :

$$-\lambda_{(0)}^2 = \frac{\partial^2 h}{\partial J_1^2}\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)\frac{\partial^2 f_*}{\partial\theta_1^2}\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right) < 0 \qquad (\mathcal{C}_3)$$

We also assume that the unperturbed Hamiltonian h satisfies a condition of isoenergetic nondegeneracy with the constant p considered in the previous paragraph :

$$\text{for all } \mathbf{V}\in\mathbf{R}^{n+2},\text{ we have } \left\|\begin{pmatrix} \partial^2_{J_1}h\left(J_1,\widetilde{J}\right) & \partial^2_{\widetilde{J}J_1}h\left(J_1,\widetilde{J}\right) & \partial_{J_1}h\left(J_1,\widetilde{J}\right) \\ \partial^2_{J_1\widetilde{J}}h\left(J_1,\widetilde{J}\right) & \partial^2_{\widetilde{J}\widetilde{J}}h\left(J_1,\widetilde{J}\right) & \partial_{\widetilde{J}}h\left(J_1,\widetilde{J}\right) \\ \partial_{J_1}h\left(J_1,\widetilde{J}\right) & \partial_{\widetilde{J}}h\left(J_1,\widetilde{J}\right) & 0 \end{pmatrix}\mathbf{V}\right\| \geq p\|\mathbf{V}\|$$

uniformly for $\left(J_1,\widetilde{J}\right)\in\mathcal{D}_0$.

For the reduction of the averaged Hamiltonian $H_\varepsilon\left(J_1,\widetilde{J},\theta_1\right) = h\left(J_1,\widetilde{J}\right) + \varepsilon f_*\left(J_1,\widetilde{J},\theta_1\right)$ to an initially hyperbolic Hamiltonian, we must find a family of points in $\mathcal{D}_0\times\mathbf{T}$ parametrized by $\widetilde{J}$ which satisfy the hypothesis $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ with H_ε instead of h and f_*.

First, condition $(\mathcal{C}_3)$ of hyperbolicity allows to use an analytical implicit function theorem on:

$$\mathcal{F}_0\left(J_1,\widetilde{J},\theta_1\right) = \left(\partial_{J_1}h\left(J_1,\widetilde{J}\right),\partial_{\theta_1}f_*\left(J_1,\widetilde{J},\theta_1\right)\right)$$

at the point $\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)$; hence, there exist a neighbourhood of $\widetilde{J}^{(0)}$ denoted $\mathcal{V}_{\widetilde{J}^{(0)}}$ and a function :

$$\begin{aligned}\mathcal{G}_0 : \mathcal{V}_{\widetilde{J}^{(0)}} &\longrightarrow \Pi_{J_1}\left(\mathcal{D}_0\right)\times\mathbf{T} \\ \widetilde{J} &\longmapsto \mathcal{G}_0\left(\widetilde{J}\right) = \left(J_{10}\left(\widetilde{J}\right),\theta_{10}\left(\widetilde{J}\right)\right)\end{aligned}$$

such that $\mathcal{G}_0\left(\widetilde{J}^{(0)}\right) = \left(J_1^{(0)},\theta_1^{(0)}\right)$ and $\mathcal{F}_0\left(\widetilde{J},\mathcal{G}_0\left(\widetilde{J}\right)\right) = (0,0)$ for all $\widetilde{J}\in\mathcal{V}_{\widetilde{J}^{(0)}}$.

Then, we consider

$$\mathcal{F}\left(J_1,\widetilde{J},\theta_1,\varepsilon,x,y,z,s\right)=\begin{pmatrix}\partial_{J_1}H_\varepsilon(x,y,z)-(1+s)\partial_{J_1}h\left(J_1,\widetilde{J}\right)\\ \partial_{\widetilde{J}}H_\varepsilon(x,y,z)-(1+s)\partial_{\widetilde{J}}h\left(J_1,\widetilde{J}\right)\\ \partial_{\theta_1}f_*(x,y,z)-(1+s)\partial_{\theta_1}f_*\left(J_1,\widetilde{J},\theta_1\right)\\ H_\varepsilon(x,y,z)-h\left(J_1,\widetilde{J}\right)\end{pmatrix}$$

the isoenergetic non-degeneracy condition and the assumption that $\partial^2_{\theta_1}f_*\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)\neq 0$ allow to use the implicit function theorem on $\mathcal{F}$ at the point $\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)},0,J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)},0\right)$. Hence, we find a neighbourhood $\mathcal{V}$ of $\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)$ and $\varepsilon_0>0$ such that :

$$\begin{aligned}\mathcal{G}:\mathcal{V}\times[0,\varepsilon_0[&\longrightarrow\mathcal{D}_0\times\mathbf{T}\times\mathbf{R}\\ \left(J_1,\widetilde{J},\theta_1,\varepsilon\right)&\longmapsto(x,y,z,s)=\left(\mathcal{G}_\varepsilon\left(J_1,\widetilde{J},\theta_1\right),s\right)\end{aligned}$$

with $\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)},0\right)=\mathcal{G}\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)},0\right)$ and $\mathcal{F}\left(\mathcal{G}\left(J_1,\widetilde{J},\theta_1,\varepsilon\right),J_1,\widetilde{J},\theta_1,\varepsilon\right)=(0,0,0,0)$, moreover $||\mathcal{G}_\varepsilon-Id||_{\mathcal{V}}$ is of the order of ε.

We denote $\mathcal{G}_\varepsilon\left(J_{10}\left(\widetilde{J}\right),\widetilde{J},\theta_{10}\left(\widetilde{J}\right)\right)=\left(J_{1\varepsilon}\left(\widetilde{J}\right),\widetilde{J}_\varepsilon\left(\widetilde{J}\right),\theta_{1\varepsilon}\left(\widetilde{J}\right)\right)=\mathcal{M}_\varepsilon\left(\widetilde{J}\right)$ for $\widetilde{J}\in\mathcal{V}_{\widetilde{J}^{(0)}}$ such that $\left(J_{10}\left(\widetilde{J}\right),\widetilde{J},\theta_{10}\left(\widetilde{J}\right)\right)\in\mathcal{V}$, then :

(i) $\mathcal{M}_\varepsilon\left(\widetilde{J}^{(0)}\right)=\left(J_1^{(0)},\widetilde{J}^{(0)},\theta_1^{(0)}\right)$, (ii) $\dfrac{\partial H_\varepsilon}{\partial\widetilde{J}}\left(\mathcal{M}_\varepsilon\left(\widetilde{J}\right)\right)=(1+s)\dfrac{\partial h}{\partial\widetilde{J}}\left(J_{10}\left(\widetilde{J}\right),\widetilde{J}\right)$,

(iii) $\dfrac{\partial H_\varepsilon}{\partial J_1}\left(\mathcal{M}_\varepsilon\left(\widetilde{J}\right)\right)=\dfrac{\partial H_\varepsilon}{\partial\theta_1}\left(\mathcal{M}_\varepsilon\left(\widetilde{J}\right)\right)=0$, (iv) $H_\varepsilon\left(\mathcal{M}_\varepsilon\left(\widetilde{J}\right)\right)=h\left(J_{10}\left(\widetilde{J}\right),\widetilde{J}\right)$.

Now, as in [CG, 1994], we make the change of variables generated by the function

$$\mathcal{S}_\varepsilon\left(J_1',\theta_1,\widetilde{J}',\widetilde{\theta}\right)=\left(\theta_1-\theta_{1\varepsilon}\left(\widetilde{J}'\right)\right)J_1'+J_{1\varepsilon}\left(\widetilde{J}'\right)\sin\left(\theta_1-\theta_{1\varepsilon}\left(\widetilde{J}'\right)\right)+\widetilde{\theta}.\widetilde{J}_\varepsilon\left(\widetilde{J}'\right)$$

and the averaged Hamiltonian becomes :

$$H_\varepsilon'\left(J_1',\widetilde{J}',\theta_1'\right)=H_\varepsilon\left(J_1'+J_{1\varepsilon}\left(\widetilde{J}'\right)\cos\left(\theta_1'\right),\widetilde{J}_\varepsilon\left(\widetilde{J}'\right),\theta_1'+\theta_{1\varepsilon}\left(\widetilde{J}'\right)\right).\tag{8}$$

Hence, for $\widetilde{J}'\in\mathcal{V}_{\widetilde{J}^{(0)}}$ such that $\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}',\theta_{10}\left(\widetilde{J}'\right)\right)\in\mathcal{V}$, the point $\left(0,\widetilde{J}',0\right)$ correspond to $\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)$ in the old variables ; moreover, the total Hamiltonian $\mathcal{H}'$ is analytical over the complex domain $V_{r'\sqrt{\varepsilon},s'}\mathcal{D}_0'$ where $\mathcal{D}_0'$ is a neighbourhood of $\mathcal{M}_\varepsilon\left(\mathcal{V}_{\widetilde{J}^{(0)}}\right)$.

Then, we look at the derivatives of H_ε', condition (iii) implies :

$$\frac{\partial H_\varepsilon'}{\partial J_1'}\left(0,\widetilde{J}',0\right)=\frac{\partial H_\varepsilon'}{\partial\theta_1'}\left(0,\widetilde{J}',0\right)=0,$$

and the conditions (iii) and (iv) gives :

$$\frac{\partial H_\varepsilon'}{\partial\widetilde{J}'}\left(0,\widetilde{J}',0\right)=\frac{\partial h}{\partial J_1}\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)\frac{\partial J_{10}}{\partial\widetilde{J}}\left(\widetilde{J}'\right)+\frac{\partial h}{\partial\widetilde{J}}\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)=\frac{\partial h}{\partial\widetilde{J}}\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right).$$

For the second derivatives, condition (iv) yields :

$$\partial^2_{\widetilde{J}'\widetilde{J}'} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \partial^2_{\widetilde{J}\widetilde{J}} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right) - \frac{{}^t\partial^2_{\widetilde{J}J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)\partial^2_{\widetilde{J}J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)}{\partial^2_{J_1J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)}$$

$$\Longrightarrow \begin{pmatrix} \partial^2_{\widetilde{J}'\widetilde{J}'} H'_\varepsilon\left(0,\widetilde{J}',0\right) & {}^t\partial_{\widetilde{J}'} H'_\varepsilon\left(0,\widetilde{J}',0\right) \\ \partial_{\widetilde{J}'} H'_\varepsilon\left(0,\widetilde{J}',0\right) & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \partial^2_{\widetilde{J}\widetilde{J}} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right) - \dfrac{{}^t\partial^2_{\widetilde{J}J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)\partial^2_{\widetilde{J}J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)}{\partial^2_{J_1J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)} & {}^t\partial_{\widetilde{J}} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right) \\ \partial_{\widetilde{J}} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right) & 0 \end{pmatrix}$$

The previous estimate, the analyticity of H'_ϵ and the lemma B.1. of the appendix B ensure that, for a small enough neighbourhood $\mathcal{V}_{\widetilde{J}^{(0)}}$ and $\varepsilon < \varepsilon_0$, the Hamiltonian $H'_\varepsilon\left(0,\widetilde{J}',0\right)$ satisfies uniformly an isoenergetic non degeneracy condition over $\mathcal{V}_{\widetilde{J}^{(0)}}$ with a constant p' of the order of $p\partial^2_{J_1J_1} h\left(J_{10}\left(\widetilde{J}'\right),\widetilde{J}'\right)$.

Finally $\dfrac{\partial H'_\varepsilon}{\partial J'_1}\left(0,\widetilde{J}',0\right) = \dfrac{\partial H'_\varepsilon}{\partial \theta'_1}\left(0,\widetilde{J}',0\right) = 0 \Longrightarrow \partial^2_{\widetilde{J}'J'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \partial^2_{\widetilde{J}'\theta'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = 0$ and

$$\partial^2_{J'_1J'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \partial^2_{J_1J_1} H_\varepsilon\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right) = \alpha(\widetilde{J}')\ ;\ \partial^2_{J'_1\theta'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \varepsilon\partial^2_{J_1\theta_1} f_*\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right) = \varepsilon\beta(\widetilde{J}')$$

$$\partial^2_{\theta'_1J'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \varepsilon\partial^2_{\theta_1J_1} f_*\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right) = \varepsilon\beta(\widetilde{J}')\ ;\ \partial^2_{\theta'_1\theta'_1} H'_\varepsilon\left(0,\widetilde{J}',0\right) = \varepsilon\partial^2_{\theta_1\theta_1} f_*\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right) = \varepsilon\gamma(\widetilde{J}')$$

Now, the Taylor expansion of H'_ε at $\left(0,\widetilde{J}',0\right)$ can be written :

$$H'_\varepsilon\left(J'_1,\widetilde{J}',\theta'_1\right) = H'_\varepsilon\left(0,\widetilde{J}',0\right) + \frac{1}{2}{}^t\!\begin{pmatrix} J'_1 \\ \theta'_1 \end{pmatrix}\begin{pmatrix} \alpha(\widetilde{J}') & \varepsilon\beta(\widetilde{J}') \\ \varepsilon\beta(\widetilde{J}') & \varepsilon\gamma(\widetilde{J}') \end{pmatrix}\begin{pmatrix} J'_1 \\ \theta'_1 \end{pmatrix} + \mathcal{O}_3(J'_1;\widetilde{J}') + \varepsilon\mathcal{O}_3(J'_1,\theta'_1;\widetilde{J}').$$

The eigenvalues of the Hessian matrix with respect to J'_1 and θ'_1 are :

$$\lambda_1(\widetilde{J}') = \alpha(\widetilde{J}') + \mathcal{O}_1(\varepsilon;\widetilde{J}') \text{ and } \varepsilon\lambda_2(\widetilde{J}') = \varepsilon\gamma(\widetilde{J}') + \mathcal{O}_2(\varepsilon;\widetilde{J}')$$

and we assume that $\mathcal{V}_{\widetilde{J}^{(0)}}$, the neighbourhood around $\widetilde{J}^{(0)}$, is small enough to ensure that the hyperbolicity condition $\alpha(\widetilde{J}')\gamma(\widetilde{J}') = \partial^2_{J'_1J'_1} H_\varepsilon\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right)\partial^2_{\theta_1\theta_1} g_\varepsilon\left(\mathcal{M}_\varepsilon\left(\widetilde{J}'\right)\right) < -\frac{\lambda^2_{(0)}}{2} < 0$ is uniformly satisfied over $\mathcal{V}_{\widetilde{J}^{(0)}}$.

Hence, we see that the original Hamiltonian is nicely amenable to hyperbolic case studied in the second section except for the dissymetry in the eigenvalues of the Hessian (since λ_1 and λ_2 have respective sizes of the order of 1 and ε). Consequently, the reduction made in section II cannot be carried out in a suitable way. This difficulty is overcomed by a rescaling of the action variables but, now, we have to restrict our study to an area whose size is $\varepsilon-$dependent.

More specifically, we first select an action linked to an hyperbolic torus with a Diophantine frequency as the origin in the action space. Hence, we consider $\widetilde{J}'^{(1)} \in \mathcal{V}_{\widetilde{J}^{(0)}}$ such that

$$\omega_1 = \frac{\partial H'_\varepsilon}{\partial \widetilde{J}'}\left(0,\widetilde{J}'^{(1)},0\right) = \frac{\partial h}{\partial \widetilde{J}}\left(J_{10}\left(\widetilde{J}'^{(1)}\right),\widetilde{J}'^{(1)}\right) \in \Omega_{\gamma,\tau} \qquad \left(\mathcal{C}_1^{(1)}\right)$$

where $\Omega_{\gamma,\tau}$ still denotes the set $\left\{\omega \in \mathbf{R}^n \text{ such that } |k.\omega| \geq \frac{\gamma}{||k||_\infty^\tau} \text{ for all } k \in \mathbf{Z}^n\right\}$ with $\gamma > 0$ and $\tau \in]n-1, +\infty[$.

Then, the energy is settled to zero

$$H'_\varepsilon\left(0, \widetilde{J}'^{(1)}, 0\right) = h\left(J_{10}\left(\widetilde{J}'^{(1)}\right), \widetilde{J}'^{(1)}\right) = 0 \qquad \left(\mathcal{C}_2^{(1)}\right)$$

and the origin is translated to $\left(0, \widetilde{J}'^{(1)}, 0, 0\right)$.

The change of variables :

$$\left(J'_1, \widetilde{J}', \theta'_1, \widetilde{\theta}'\right) = \widehat{\mathcal{T}}\left(\widehat{J}, \widehat{\theta}, \widehat{p}, \widehat{q}\right) \text{ where } J'_1 = \sqrt{\varepsilon}\widehat{p},\ \widetilde{J}' = \sqrt{\varepsilon}\widehat{J},\ \theta'_1 = \widehat{q},\ \widetilde{\theta}' = \widehat{\theta},$$

is symplectic up to a multiplier equal to $\sqrt{\varepsilon}$ and, with a fitted time scaling by $\sqrt{\varepsilon}$, the Hamiltonian becomes in the new variables $\widehat{\mathcal{H}}\left(\widehat{J}, \widehat{\theta}, \widehat{p}, \widehat{q}\right) = \widehat{H}_\varepsilon\left(\widehat{J}, \widehat{p}, \widehat{q}\right) + \sqrt{\varepsilon}\mu\widehat{g}\left(\widehat{J}, \widehat{\theta}, \widehat{p}, \widehat{q}\right)$ where the averaged Hamiltonian can be written $\widehat{H}_\varepsilon\left(\widehat{J}, \widehat{p}, \widehat{q}\right) = \widehat{h}\left(\widehat{J}; \sqrt{\varepsilon}\right) + \sqrt{\varepsilon}\widehat{P}\left(\widehat{J}, \widehat{p}, \widehat{q}\right)$ with

$$\widehat{h}\left(\widehat{J}\right) = \frac{1}{\sqrt{\varepsilon}} H'_\varepsilon\left(0, \sqrt{\varepsilon}\widehat{J}, 0\right) = \omega_1 \widehat{J} + \frac{\sqrt{\varepsilon}}{2} {}^t\widehat{J}\Gamma_1 \widehat{J} + \varepsilon \mathcal{O}_3\left(\widehat{J}; \sqrt{\varepsilon}\right)$$

and

$$\widehat{P}\left(\widehat{J}, \widehat{p}, \widehat{q}\right) = \frac{1}{2} {}^t\!\begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} \begin{pmatrix} \widehat{\alpha}(\widehat{J}) & \widehat{\beta}(\widehat{J}) \\ \widehat{\beta}(\widehat{J}) & \widehat{\gamma}(\widehat{J}) \end{pmatrix} \begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} + \mathcal{O}_3\left(\widehat{p}, \widehat{q}; \widehat{J}, \sqrt{\varepsilon}\right)$$

where $\Gamma_1 = \partial^2_{\widetilde{J}'\widetilde{J}'} H'_\varepsilon\left(0, \widetilde{J}'^{(1)}, 0\right)$, $\widehat{\alpha}(\widehat{J}) = \alpha\left(\sqrt{\varepsilon}\widehat{J}\right)$, $\widehat{\beta}(\widehat{J}) = \beta\left(\sqrt{\varepsilon}\widehat{J}\right)$, $\widehat{\gamma}(\widehat{J}) = \gamma\left(\sqrt{\varepsilon}\widehat{J}\right)$. Moreover, using the notations of section II, we can ensure that the total Hamiltonian $\widehat{\mathcal{H}}$ is analytical on the complex domain $V_{\widehat{K}}$ where $\widehat{K} = \inf(r', s')$ (one should notice that, in the old variables, the size of $\widehat{\mathcal{T}}\left(V_{\widehat{K}}\right)$ is of the order of $\sqrt{\varepsilon}$).

Finally, the lemma B.2 of the appendix B ensures that, for a small enough neighbourhood $V_{\widehat{K}}$ of the origin and $\varepsilon < \varepsilon_0$, the Hamiltonian $\widehat{h}$ satisfies uniformly an isoenergetic non degeneracy condition over $V_{\widehat{K}}$ with a constant $\widehat{p} = p_0\sqrt{\varepsilon}$.

We see that, now, we can directly apply the results of section II on $\widehat{\mathcal{H}}$ and, just replacing ε by $\sqrt{\varepsilon}$, K by $\widehat{K}$, E_1 by E, λ by $\frac{\lambda_{(0)}}{\sqrt{2}}$ and ν_0 by $\nu_1 = ||\omega_1||$ in theorem 2.2., we obtain :

Theorem 3.3. (Main result in the nearly integrable case).

Let $E \geq \sup(1, \nu_1)$, $m_* = \inf\left(\frac{p_0}{2}, \frac{\lambda_{(0)}}{\sqrt{2}}\right)$, $\sigma = 2^{4n+1}\left(\frac{n+1}{e}\right)^{n+1}$ *with* $1 = \ln(e)$ *and assume that :*

$$n - 1 < \tau \leq n\ ;\ \widehat{K} \leq C\frac{m_*}{E}\ ;\ \gamma \leq \frac{\sqrt{\varepsilon}}{30}\ ;\ \frac{\sqrt{\varepsilon}\mu}{\gamma^4} < \left(\frac{m_*^3}{3\sigma^3\widehat{K}}\right)^2 \left(\frac{\widehat{K}}{240}\right)^{4(2n+5)} \qquad (9)$$

then there exist a symplectic transformation $\mathcal{T}^{(1)} : V_{\widehat{K}/6} \longrightarrow V_{\widehat{K}/2}$ *such that :*

$$\mathcal{H} \circ \mathcal{T}^{(1)} = (1 + \eta^{(1)})\omega_1 I^{(1)} + \sqrt{\varepsilon}\lambda_1 s^{(1)} u^{(1)} + \sqrt{\varepsilon} f^{(1)}\left(I^{(1)}, \varphi^{(1)}, s^{(1)}, u^{(1)}\right)$$

with $f^{(1)}\left(I^{(1)}, \varphi^{(1)}, s^{(1)}, u^{(1)}\right) = \mathcal{O}_2\left(I^{(1)}, s^{(1)}u^{(1)}; I^{(1)}, \varphi^{(1)}, s^{(1)}, u^{(1)}\right)$ *and*

$$\frac{1}{2} < 1 + \eta^{(1)} < \frac{3}{2}\ ;\ \lambda_1 > \frac{\lambda_{(1)}}{2} > \frac{\lambda_{(0)}}{4}$$

moreover

$$\left\|\left|\eta^{(1)}\omega_1 I^{(1)} + \sqrt{\varepsilon}\left(\lambda_1 - \lambda_{(1)}\right) s^{(1)}u^{(1)} + \sqrt{\varepsilon}\left[f^{(1)}(I^{(1)},\varphi^{(1)},s^{(1)},u^{(1)}) - f(I^{(1)},s^{(1)}\ u^{(1)})\right]\right|\right\|_{\widehat{K}/6} \leq \frac{E}{2^{4n+6}}\sqrt[4]{\varepsilon}\sqrt{\mu}$$

Hence, $\{I^{(1)} = 0\}$ *becomes an hyperbolic invariant set for the perturbed flow with linear motions.*

Remark : The extension in the case of $\tau > n$ is still valid with the modifications given in section II.

IV Implementation on a transition chain

In [CG, 1994] and [RW1, 1996], many hyperbolic tori and their invariant manifolds are straightened by means of a *single chart* built by applying the K.A.M. theory with Arnold's scheme to an initially hyperbolic Hamiltonian. This allowed Chierchia and Gallavotti ([CG, 1994]) to prove, in the initially hyperbolic case, the existence of transition chains (i.e. : families of tori invariant under the perturbed flow and linked by heteroclinic orbits) which are fitted for the construction of Arnold's mechanism. Rudnev and Wiggins ([RW2, 1997]) have obtained the same result in the (much more delicate) case of a nearly integrable Hamiltonian with some specific perturbations. On the other hand, as it was said in the introduction, the charts considered in the previous studies is convergent only on a Cantor set and cannot be used for the study of the dynamics of orbits shadowing these lines of tori.

On the contrary, the changes of variables given in this paper are defined on an open set and, consequently, gives the desired control around the invariant manifolds of a partially hyperbolic torus. The price to pay for this extension is the non-integrability of the normalized Hamiltonian whose associated flow is linear only on a *single* torus and its linked hyperbolic directions. Now, using the results of [CG, 1994] and [RW2, 1997], we assume the existence of a transition chain and study the problem of building our normal forms around each torus of the considered family. We first restrict our study to an initially hyperbolic Hamiltonian, the equivalent results in the general nearly-integrable case will be specified at the end of this paragraph.

We denote $\mathbf{T}$ the set of tori and their invariant manifolds which can be straightened with the use of theorem 2.2. A crude application of the latter shows that the tori in $\mathbf{T}$ admit a frequency which is Diophantine with a constant γ of the order of $\sqrt[4]{\varepsilon\mu}$. Since we assume $\tau > n-1$, the complementary of $\Omega_{\gamma,\tau}$ have a measure of the order of $\mathcal{O}(\gamma) = \mathcal{O}(\sqrt[4]{\varepsilon\mu})$. Consequently, because of the isoenergetic nondegeneracy condition and the local invertibility of the frequency map on a constant energy manifold, we can ensure that the width of the gaps between tori in $\mathbf{T}$ is $\mathcal{O}(\sqrt[4]{\mu/\varepsilon^3})$.

Actually, in Arnold's instability we look for orbits around a family of tori associated to a curve $\mathcal{C}$ in the non resonant action space $\mathbf{R}^n$ (see [L, 1996]). Hence, the average distance between two tori in $\mathbf{T}$ with a linked action in $\mathcal{C}$ is given by the measure of $\omega(\mathcal{C}) \cap \Omega^c_{\gamma,\tau}$ where ω is the frequency map in the unperturbed problem and $\Omega^c_{\gamma,\tau}$ is the complement of $\Omega_{\gamma,\tau}$ in $\mathbf{R}^n$. If $\mathcal{C}$ has a full torsion (i.e. : the tangent vector of $\mathcal{C}$ and its first derivatives span the action space $\mathbf{R}^n$), Pyartli ([Py, 1969]) has given estimates for the size of $\omega(\mathcal{C}) \cap \Omega^c_{\gamma,\tau}$ which are explicitly computed in [CG, 1994] (section 3). They give a measure of the considered set of the order of $\mathcal{O}\left(\gamma^{1/n}\right)$ for $\tau = n^2$ and, here, the width of the gaps between tori in $\mathbf{T}$ associated to the curve $\mathcal{C}$ is $\mathcal{O}\left(\varepsilon^{-1}\left(\varepsilon\mu\right)^{\frac{1}{4n}}\right)$.

On the other hand, the results of [CG, 1994] show that hyperbolic tori can be connected by heteroclinic orbits only if their mutual distance is of the order of $\mathcal{O}\left((\varepsilon\mu)^{\delta}\right)$ with $\delta > 1$ and, consequently, we cannot directly apply the theorem 2.2. on a transition chain in a general case since the gaps between invariant tori are too important. As it can be seen in [CG, 1994] (section 7), this difficulty can be overcomed by making p steps of perturbation theory which reduce the studied Hamiltonian in section II to $\breve{h}\left(\breve{I},\breve{s}\breve{u};\varepsilon\right) + (\varepsilon\mu)^p\breve{g}\left(\breve{I},\breve{\varphi},\breve{s},\breve{u}\right)$ and the distance between invariant tori is small enough for a high exponent p but this procedure imposes to consider perturbations given by trigonometric polynomial of finite order in order to get a global result.

For an arbitrary perturbation, the results derived in this paper are still relevant because they allow the construction of a Birkhoff normal form around each persistent torus. Studies of stability in the vicinity of an invariant torus provided by the K.A.M. theory have been made by Morbidelli and Giorgilli ([MG, 1995]) in the Lagrangian case, by Jorba and Villanueva ([JV2, 1996]) in the case of a lower-dimensional elliptic torus and these results can be easily extended here in the partially hyperbolic case.

Indeed, by application of the theorem 2.2. we can locally consider the Hamiltonian

$$\mathcal{H}^\flat = \omega^\flat I^\flat + \varepsilon\lambda^\flat s^\flat u^\flat + \varepsilon f^\flat\left(I^\flat, s^\flat u^\flat\right) + \sqrt{\varepsilon\mu} g^\flat\left(I^\flat, \varphi^\flat, s^\flat, u^\flat\right) \text{ where } \omega^\flat \in \Omega_{\gamma,\tau},\ \lambda^\flat > 0$$

and $f^\flat\left(I^\flat, s^\flat u^\flat\right)$ (resp. $g^\flat\left(I^\flat, \varphi^\flat, s^\flat, u^\flat\right)$) is on the order of $\mathcal{O}_2\left(I^\flat, s^\flat u^\flat; \varepsilon\right)$ (resp. on the order of $\mathcal{O}_2\left(I^\flat, s^\flat u^\flat; I^\flat, \varphi^\flat, s^\flat, u^\flat, \varepsilon\right)$). Since, the small denominators in the construction of the Birkhoff's normal form with the elliptic-saddle Hamiltonian $\omega^\flat I^\flat + \varepsilon\lambda^\flat s^\flat u^\flat$ can be written $i\left(k^\flat.\omega^\flat\right) + \varepsilon l^\flat.\lambda^\flat$ for $k^\flat \in \mathbf{Z}^n$ and $l^\flat \in \mathbf{Z}$, the convergence of the normalizing transformation is only prevented by the arithmetical properties of $\omega^\flat$. Using the estimates given by Pöschel ([Pö, 1993], theorem 5), we can normalize the Hamiltonian up to an exponentially small remainder and obtain the following :

Theorem 4.1.

Let $\mathcal{H}^\flat = \omega_0^\flat I^\flat + \varepsilon\lambda^\flat s^\flat u^\flat + \varepsilon f^\flat\left(I^\flat, s^\flat u^\flat\right) + \sqrt{\varepsilon\mu} g^\flat\left(I^\flat, \varphi^\flat, s^\flat, u^\flat\right)$ be the Hamiltonian obtained after the application of theorem 2.2. around a first torus $\mathcal{T}_0$ with a linked frequency $\omega_0^\flat$ in $\Omega_{\gamma_0,\tau}$ where $\tau > n-1$. We denote $V_{K^\flat}$ the complex domain of analyticity of $\mathcal{H}^\flat$, then there exist $K_0^\flat > 0$ such that, provided $\varepsilon K^\flat < \gamma_0 K_0^\flat$, we can find a symplectic transformation $\mathcal{T} : V_{K^{\flat\prime}} \longrightarrow V_{K^\flat}$ for some $K^{\flat\prime} = \rho K^\flat$ $(0 < \rho < 1)$ with $\left(I^\flat, \varphi^\flat, s^\flat, u^\flat\right) = \mathcal{T}(I, \varphi, s, u)$ which cast the original Hamiltonian $\mathcal{H}^\flat$ to the following :

$$\mathcal{H}^\flat \circ \mathcal{T}^\flat = \omega_0^\flat I + \varepsilon\lambda^\flat su + \varepsilon f(I, su) + \sqrt{\varepsilon\mu}\exp\left(-C\left(\frac{\gamma_0 K_0^\flat}{\varepsilon K^\flat}\right)^{\frac{1}{1+\tau}}\right) g(I, \varphi, s, u)$$

with $g(I, \varphi, s, u) = \mathcal{O}_2(I, su; I, \varphi, s, u)$.

Hence, in these new coordinates, the normalized Hamiltonian agrees with the expression (2) of section II and the theorem 2.2. can be applied on a torus with a linked frequency in $\Omega_{\gamma,\tau}$ if :

$$\sqrt{\varepsilon\mu}\exp\left(-C\left(\frac{\gamma_0 K_0^\flat}{\varepsilon K^\flat}\right)^{\frac{1}{1+\tau}}\right) < C_1\gamma^4 \text{ with } C_1 > 0.$$

Consequently, if γ_0 is ε-independent, we find a neighbourhood of $\mathcal{T}_0$ whose size is of the order of $\mathcal{O}\left(|\ln(\mu)|^{-(\tau+1)}\right)$ where the hyperbolic tori with a linked Diophantine frequency in $\Omega_{\gamma,\tau}$ for γ of the order of $\mathcal{O}\left((\varepsilon\mu)^{4n\delta}\right)$ can be straightened by means of theorem 2.2. In that way, we obtain open sets where our normal forms can be used around two tori connected by heteroclinic orbits in the cases studied by Gallavotti-Chierchia ([CG, 1994]).

In the general nearly-integrable case, the perturbation in Pöschel's normal form is exponentially small with respect to $\varepsilon^{\frac{1}{2n}}$ (c.f. lemma 3.2.) and, at least in the cases studied by Rudnev and Wiggins ([RW2, 1997]), the mean distance between two tori connected by heteroclinic orbits is also of the order of $\mathcal{O}\left(\exp(-C\varepsilon^{-\frac{1}{2n}})\right)$. Consequently, the problem encountered previously with an initially hyperbolic Hamiltonian is still present in a nearly-integrable system where we replace ε and μ by $\sqrt{\varepsilon}$ and $\exp\left(-C\varepsilon^{-\frac{1}{2n}}\right)$. Hence, if the frequency associated to the initial torus $\mathcal{T}_0$ is

Diophantine with a constant γ_0 independent of ε, we can straighten close enough tori in a neighbourhood of $\mathcal{T}_0$ whose size is of the order of $\mathcal{O}\left(\varepsilon^{\frac{\tau+1}{2n}-\frac{1}{2}}\right)$ for $\tau > n-1$. Consequently, in the original coordinates, our theorems can be applied to study the dynamic around transition chains close to $\mathcal{T}_0$ whose lengths are of the order of $\mathcal{O}\left(\varepsilon^{\frac{1}{2}+\zeta}\right)$ where $\zeta > 0$.

At this step, we can also specify the minimal distance to multiple resonances needed to derive our results in the nearly-integrable case. An examination at the lower bound for resonances α given in lemma 3.2. and at the size of the perturbation $\mu = \mathcal{O}\left(\varepsilon^{3/2}\right)$ needed to apply theorem 3.3. (this comes from the last two thresholds in (9)) shows that our study is valid in a strip of width $\sqrt{\varepsilon}$ around a manifold in the action space linked to a simple resonance where we exclude gaps around multiple resonances whose total measure is of the order of $\mathcal{O}\left(\sqrt{\varepsilon}|\ln(\varepsilon)|^n\right)$.

V Conclusion and prospects.

Despite the restrictions given at the end of the previous section, estimating the speed of drift of orbits contained in the domain of application of our theorems is already relevant to look at the optimality of the bounds on instability obtained in certain Nekhorochev-like results. Indeed, assuming the existence of transition chains, we give tools for an accurate analysis of the dynamics of orbits shadowing relatively long lines of tori. First, Pöschel ([Pö, 1993], theorem 2) has shown that, except on a set of small measure, each action is confined over exponentially long times in an $\mathcal{O}(\sqrt{\varepsilon})$-area and this is almost the length of the chains considered at the end of section IV. More generally, the theorems of exponential stability given by Lochak [LN, 1992] and Pöschel [Pö, 1993] rely on the properties of resonant normal forms which, under suitable assumptions, ensure a drift of the actions along the considered resonant manifold at most of the order of $\mathcal{O}(\sqrt{\varepsilon})$ over exponentially long times. Here, the combination of our theorems and of Marco's geometrical methods ([Ma, 1996]) might ensure (in certain cases) the existence of orbits moving over exponentially long times along pieces of resonant manifolds whose lengths are also of the order of $\mathcal{O}(\sqrt{\varepsilon})$. Hence, one can expect results on the limits of the previous reasonings to confine the orbits in nearly-integrable Hamiltonian systems. Moreover, we should notice that the simple resonant case studied here allows only to prove the"worst"estimate of stability over exponentially long times (see Lochak [L, 1996]) and, consequently, it seems to be the best setting to look for orbits with a speed of drift comparable with the upper bound on the rate of instability predicted in Nekhorochev-like theorems.

On the other hand, the initial Arnold's conjecture (i.e. : the existence of orbits in a generic nearly-integrable systems connecting two actions separated by an arbitrary distance) cannot be tackled without accurate informations on the dynamic close to multiple resonances. First, we should mention that Chierchia [Ch, 1996] have recently studied instability of orbits around but not too close from double resonances by means of a reduction to a perturbation of the Hamiltonian of section II with a single hyperbolic degree of freedom ; hence, the results obtained here are still applicable in this setting.

Close to arbitrary multiple resonances, following the lines of Treschev's reasonings [Tr, 1991], we can still reduce a generic nearly-integrable Hamiltonian to the initially hyperbolic case of section II (now, with several degrees of freedom) but, unfortunately, we do not have anymore tools like the Moser's transformation used here because of the generic divergence of Birkhoff's normal forms in higher dimension. Actually, denoting by T an hyperbolic torus and its linked manifolds, the results of Brjuno ([Br, 1971] and [Br, 1989]) ensure the existence of a normal form with a linear flow on T but the normalizing transformation is convergent only on T. On the other hand, Graff ([Gr, 1973], theorem 7) has given a formal scheme for the construction of the desired normalizing transformation but the question of convergence is not discussed.

A first track for a generalisation of our results is the use of the work of Jorba and Villanueva [JV1, 1996] who proved the persistence under quasi-periodic perturbation of lower-dimensional elliptic tori. Here, we can focus our attention on tori associated to resonances of order k where the averaged Hamiltonian admits $n-k-1$ elliptic degrees of freedom and one hyperbolic degree of freedom when we carry out a reduction similar to Treschev's reasoning in the complete hyperbolic case. Hence, we are almost in the setting of this study and similar results can be expected in some cases of multiple resonances. Another clue for a generalisation of the theorems obtained here is the possibility of making our K.A.M. iterative scheme in the case where the unperturbed flow is only linear *on* the hyperbolic tori and their linked invariant manifolds (i.e. : the unperturbed Hamiltonian can be written $H(I,s,u)=\omega.I+\varepsilon\lambda su+\varepsilon P(I,s,u)$ where $P(I,s,u)=\mathcal{O}_2(I,su;s,u,\varepsilon)$). Hence, the integration of the initially hyperbolic Hamiltonian by means of Moser's theorem is not necessary and, along the lines of the reasoning in section II, we only need a symplectic transformation which straighten and linearize the flow *on* the invariant manifolds associated to an hyperbolic fixed point in $\mathbf{R}^{2n}$. Actually, the conditions needed to prove the existence of such a transformation are much less stringent than those imposed to ensure the convergence of Birkhoff's normal forms (see Brjuno [Br, 1989]). Then, in certain cases, we can certainly carry out a multidimensional K.A.M. scheme analogous to the unidimensional K.A.M. scheme made in section II.

Appendix A : Proof of the iterative lemma 2.3.

We omit the indexes (n) in this paragraph and use the same notations as in the section $II.2$.

First, we need some preliminary analytical lemmas. Let $(\mathcal{A}_K;||.||_K)$ be the normed space of numerical or vector valued functions which are analytical over V_K.

Lemma A.1.
Let $(f,g)\in\mathcal{A}_K^2$, $0<\delta<1$ and two integers $(k,l)\in\mathbf{N}^2$, then :

1) $\left|\left|\dfrac{\partial^{k+l}f}{\partial s^k\partial u^l}(I,\varphi,s,u)\right|\right|_{(1-\delta)K}\leq\dfrac{k!l!}{(\delta K)^{k+l}}||f||_K$ *and* $\left|\left|\dfrac{\partial^{k+l}f}{\partial s^k\partial u^l}(0,\varphi,0,0)\right|\right|_K\leq\dfrac{k!l!}{K^{k+l}}||f||_K$

2) $\left|\left|\dfrac{\partial f}{\partial I}(I,\varphi,s,u)\right|\right|_{(1-\delta)K}\leq\dfrac{||f||_K}{\delta K}$ *and* $\left|\left|\dfrac{\partial^{1+k+l}f}{\partial I\partial s^k\partial u^l}f(0,\varphi,0,0)\right|\right|_{(1-\delta)K}\leq\dfrac{k!l!}{(\delta K)^{1+k+l}}||f||_K$

3) $\left|\left|\dfrac{\partial^2 f}{\partial I^2}(I,\varphi,s,u)\right|\right|_{(1-\delta)K}\leq\dfrac{4||f||_K}{(\delta K)^2}$

4) $||\{f,g\}||_{(1-\delta)K}\leq\dfrac{||f||_K||g||_K}{(\delta K)^2}$ *and* $||\{f,g\}||_{(1-\delta)K}\leq\dfrac{2||f||_K||g||_{(1-\frac{\delta}{2})K}}{(\delta K)^2}$

Proof 1) is a direct application of the Cauchy inequality (see [BGGS, 1984]).

To obtain 2), we write $\left|\left|\frac{\partial f}{\partial I}(I,\varphi,s,u)\right|\right|_{(1-\delta)K}=\mathrm{Sup}_{||e||=1}\left|\left|\frac{df}{dt}_{|t=0}f(I+te,\varphi,s,u)\right|\right|_{(1-\delta)K}$ and apply Cauchy formula to the function $t\mapsto f(I+te,\varphi,s,u)$ of the complex variable t defined for $|t|\leq\delta K$ when $(I,\varphi,s,u)\in V_{(1-\delta)K}$, this yields the desired estimate. It also allows to deduce the second part of 2).

For 3), we write

$$\left|\left|\frac{\partial^2 f}{\partial I^2}(I,\varphi,s,u)\right|\right|_{(1-\delta)K}=\mathrm{Sup}_{||e||=1,||f||=1}\left|\left|\frac{\partial f}{\partial s\partial t}_{|(s,t)=(0,0)}f(I+se+tf,\varphi,s,u)\right|\right|_{(1-\delta)K}$$

and apply Cauchy formula to the function of the complex variables s and t in the right hand side, defined for $|s|$ and $|t|\leq\frac{\delta K}{2}$ when $(I,\varphi,s,u)\in V_{(1-\delta)K}$, this gives the estimate 3).

To obtain the size of the Poisson brackets for two analytical functions on V_K given in 4), we write in a similar way that $\{f,g\}(I,\varphi,s,u)=\frac{d}{dt}_{|t=0}\left[g\left(I-t\frac{\partial f}{\partial\varphi},\varphi+t\frac{\partial f}{\partial I},s-t\frac{\partial f}{\partial u},u+t\frac{\partial f}{\partial s}\right)\right]$ and the function of the complex variable t in the right hand side is defined for $|t|\leq\frac{(\delta K)^2}{||f||_K}$ (we apply the Cauchy formula to f), this yields the estimates 4).

Lemma A.2.
1) Let $f(I,\varphi,s,u)\in\mathcal{A}_K$ with the Fourier expansion $f(I,\varphi,s,u)=\sum_{k\in\mathbf{Z}^n}f_k(I,s,u)\exp(ik.\varphi)$ then

$$||f_k||_K<||f||_K\exp\left[-\left(|k_1|+\ldots+|k_n|\right)K\right]$$

2) Let $f_k(I,s,u) \in \mathcal{A}_K$ where $k \in \mathbf{Z}^n$ such that $||f_k||_K < C\exp\left[-\left(|k_1|+\ldots+|k_n|\right)K\right]$, then

$$f(I,\varphi,s,u) = \sum_{k\in\mathbf{Z}^n} f_k(I,s,u)\exp(ik.\varphi) \in \mathcal{A}_{(1-\delta)K} \text{ with } 0<\delta<1 \text{ and } ||f||_{(1-\delta)K} < C\left(\frac{4}{\delta K}\right)^n$$

3) For $x,y,s>0$, we have the following inequality : $x^s \leq \left(\dfrac{s}{ey}\right)^s \exp(xy)$ where $1=\ln(e)$.

The proof of this lemma is given in the appendix of [BGGS, 1984].

Lemma A.3.

Let $0<\delta<1$, $g(I,\varphi,s,u) \in \mathcal{A}_K$ such that $\overline{g}(I,s,u) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\ldots\int_0^{2\pi} g(I,\varphi,s,u)d\varphi_1\ldots d\varphi_n = 0$ and $\omega \in \Omega_{\gamma,\tau}$ where $\tau \leq n$, there exist $f \in \mathcal{A}_{(1-\delta)K}$ solution of the equation :

$$(1+\eta)\sum_{l=1}^{n}\omega_l\frac{\partial f}{\partial\varphi_l}(I,\varphi,s,u) = g(I,\varphi,s,u) \text{ where } 1+\eta > m_*$$

such that :

$$||f||_{(1-\delta)K} \leq \frac{\sigma}{m_*\gamma}\frac{||g||_K}{(\delta K)^{2n}} \ ; \ \left|\left|\frac{\partial f}{\partial\varphi}\right|\right|_{(1-\delta)K} \leq \frac{\sigma}{m_*\gamma}\frac{||g||_K}{(\delta K)^{2n+1}} \text{ with } \sigma = 2^{4n+1}\left(\frac{n+1}{e}\right)^{n+1}$$

Proof We are looking for f solution of the equation $\widetilde{\omega}\partial_\varphi f = g$ where $\widetilde{\omega} = (1+\eta)\omega \in \Omega_{m_*\gamma,\tau}$ and the existence of a function which satisfies the desired properties is the content of the lemma 2 in [BGGS, 1984].

Lemma A.4.

We consider $g(x,\varphi) \in \mathcal{A}_K$ where $x=s$ or u ; $0<\delta<1$ and $0<m_*<\frac{\lambda_0}{2}<\lambda$; $\omega \in \mathbf{R}^n$. There exist $f^\pm(x,\varphi) \in \mathcal{A}_{(1-\delta)K}$ solution of the equation

$$\partial^\pm_{\varepsilon\lambda,\omega}f^\pm(x\varphi) = g(x,\varphi) \text{ with the operator } \partial^\pm_{\varepsilon\lambda,\omega} \text{ defined in section II.2}$$

such that

$$||f^\pm||_{(1-\delta)K} \leq \frac{\sigma}{\varepsilon m_*}\frac{||g||_K}{\delta^{n+1}K^n} \ ; \ \left|\left|\frac{\partial f^\pm}{\partial\varphi}\right|\right|_{(1-\delta)K} \leq \frac{\sigma}{\varepsilon m_*}\frac{||g||_K}{\delta^{n+2}K^{n+1}} \text{ with the previous constant } \sigma$$

Proof We denote the expansion $g(x,\varphi) = \sum_{k\in\mathbf{N}}\sum_{l\in\mathbf{Z}^n} g_{kl}x^k\exp(il.\varphi) = \sum_{k\in\mathbf{N}} g_k(\varphi)x^k$, then :

$$f^\pm(x,\varphi) = \sum_{k\in\mathbf{N}}\sum_{l\in\mathbf{Z}^n}\frac{g_{kl}}{(k+1)\varepsilon\lambda \pm i(\omega.l)}x^k\exp(il.\varphi) = \sum_{k\in\mathbf{N}} f^\pm_k(\varphi)x^k$$

Lemma A.2.$\Longrightarrow |f^\pm_{kl}| \leq \dfrac{|g_{kl}|}{\varepsilon\lambda} \leq \dfrac{||g_k||_K}{\varepsilon m_*}\exp\left[-\left(|l_1|+\ldots+|l_n|\right)K\right] \Longrightarrow ||f^\pm_k||_{(1-\delta)K} \leq \dfrac{||g_k||_K}{\varepsilon m_*}\left(\dfrac{4}{\delta K}\right)^n$

then $|g_k|_K \leq \dfrac{||g||}{K^k}$ implies $||f^\pm||_{(1-\delta)K} \leq \dfrac{||g||_K}{\varepsilon m_*}\dfrac{4^n}{\delta^{n+1}K^n} \leq \dfrac{\sigma}{\varepsilon m_*}\dfrac{||g||_K}{\delta^{n+1}K^n}$

For the derivatives $\partial_{\varphi_j} f^{\pm}(x,\varphi) = \sum_{k\in\mathbf{N}}\sum_{l\in\mathbf{Z}^n} f_{kl}^{\pm} x^k i l_j \exp(il.\varphi) = \sum_{k\in\mathbf{N}} \partial_{\varphi_j} f_k^{\pm}(\varphi) x^k$, we have :

$$\begin{aligned}
\left|\left|f_{kl}^{\pm} l\right|\right| &= \left|f_{kl}^{\pm}\right| \left|\left|l\right|\right| \text{ with the Euclidian norm for } l \in \mathbf{Z}^n \\
&\leq \frac{||g_k||_K}{\varepsilon m_*} \exp\left[-\left(|l_1| + \ldots + |l_n|\right) K\right] \left(|l_1| + \ldots + |l_n|\right) \\
\text{Lemma A.2.} \Longrightarrow \left|\left|f_{kl}^{\pm} l\right|\right| &\leq \frac{||g_k||_K}{\varepsilon m_*} \frac{1}{e\delta K} \exp\left[\left(|l_1| + \ldots + |l_n|\right) \delta K\right] \exp\left[-\left(|l_1| + \ldots + |l_n|\right) K\right] \\
&\leq \frac{||g_k||_K}{\varepsilon m_*} \frac{\exp\left[-\left(|l_1| + \ldots + |l_n|\right)(1-\delta) K\right]}{e\delta K}
\end{aligned}$$

$$\begin{aligned}
\text{Lemma A.2.} \Longrightarrow \left|\left|\partial_\varphi f_k^{\pm}\right|\right|_{(1-\delta)K} &\leq \frac{||g_k||_K}{\varepsilon m_*} \frac{1}{e\delta K} \left(\frac{4}{\delta K}\right)^n \\
||g_k||_K \leq \frac{||g||_K}{K^k} \Longrightarrow \left|\left|\partial_\varphi f^{\pm}\right|\right|_{(1-2\delta)K} &\leq \frac{||g||_K}{\varepsilon m_*} \frac{4^n}{e\left(\delta K\right)^{n+1}} \sum_{k\in\mathbf{N}} \frac{\left(1-2\delta\right)^k K^k}{K^k} \\
\Longrightarrow \left|\left|\partial_\varphi f^{\pm}\right|\right|_{(1-2\delta)K} &\leq \frac{||g||_K}{\varepsilon m_*} \frac{2^{2n-1}}{e\delta\left(\delta K\right)^{n+1}}
\end{aligned}$$

We replace δ by $\frac{\delta}{2}$ and obtain the given estimate :

$$\left|\left|\partial_\varphi f^{\pm}\right|\right|_{(1-\delta)K} \leq \frac{||g||_K}{\varepsilon m_*} \frac{2^{3n+1}}{e\delta\left(\delta K\right)^{n+1}} \leq \frac{\sigma}{\varepsilon m_*} \frac{||g||_K}{\delta^{n+2} K^{n+1}}$$

Now, we recall the statement of the iterative lemma 2.3. :

Lemma A.5. (iterative lemma).

We consider an Hamiltonian $\mathcal{H}$ analytical over V_K, where $0 < K_* < K < 1$ and the parameters μ, η, λ, p which satisfy the assumptions given in section II.2. ; hence we can be write $\mathcal{H}(I,\varphi,s,u) = H(I,\varphi,s,u) + \varepsilon\mu g(I,\varphi,s,u)$ with $||\mathcal{H}||_K \leq E_1$ and $||g||_K \leq E_1$ where $E_1 \geq \sup\left(1,\nu_0\right)$ (we recall that $\nu_0 = ||\omega_0||$).

Let $0 < \delta < \frac{1}{12}$, if we assume :

$$\delta^{2n+1} \leq \frac{\varepsilon\sigma}{m_*^2\gamma K_*^{2n}} \text{ and } \frac{\varepsilon\mu}{\gamma^2} \leq \frac{m_*^6\delta^4}{21\sigma^3 E_1^3}\left(\delta K_*\right)^{4n+6} \tag{A_1}$$

there exist a symplectic transformation $\mathcal{T}\ :\ V_{(1-12\delta)K} \subset \mathcal{T}\left(V_{(1-10\delta)K}\right) \subset V_{(1-8\delta)K}$ such that the transformed Hamiltonian can be expanded with the parameters $\widetilde{\mu}$, $\widetilde{\eta}$, $\widetilde{\lambda}$ and $\widetilde{p}$ connected to μ, η, λ, p with the relations given in lemma 2.3. (replacing $K/3$ by K_*), thus

$$\widetilde{\mathrm{p}} = \mathrm{p} - \frac{\left(16\sigma E_1\right)^3}{m_*^6\gamma^2\delta^2\left(\delta K_*\right)^{4(n+2)}}\mu > m_* \tag{A_2}$$

Proof We use the same notations and the same transformations as in section II.

First, the size of the coefficients in the perturbation g is estimated. We write the expansion $g(I,\varphi,s,u) = \sum_{k\in\mathbf{N}^n}\sum_{(l,m)\in\mathbf{N}^2} g_{klm}(\varphi) I^k s^l u^m$ where $g_{klm}(\varphi) = \frac{1}{k!l!m!}\frac{\partial^{k+l+m} g}{\partial I^k \partial s^l \partial u^m}(0,\varphi,0,0)$ and :

$$||\alpha_0(\varphi)||_K = ||g(0,\varphi,0,0)||_K < E_1$$

$$||\alpha_{10}(\varphi,s)||_{(1-\delta)K} \leq \left|\left|\sum_{l\geq 1} g_{0l0}(\varphi)s^{l-1}\right|\right|_{(1-\delta)K} \leq \sum_{l\geq 1}\frac{||g||_K}{K^l}((1-\delta)K)^{l-1} < \frac{E_1}{\delta K} \text{ and } ||\alpha_{01}(\varphi,u)||_{(1-\delta)K} < \frac{E_1}{\delta K}$$

$$||\alpha_{11}(\varphi)||_K = ||\partial_{su}g(0,\varphi,0,0)||_K < \frac{E_1}{K^2}$$

$$||\alpha_{21}(\varphi,s)||_{(1-\delta)K} \leq \left|\left|\sum_{l\geq 2} g_{0l1}(\varphi)s^{l-2}\right|\right|_{(1-\delta)K} \leq \sum_{l\geq 2}\frac{||g||_K}{K^{l+1}}((1-\delta)K)^{l-2} < \frac{E_1}{\delta K^3} \text{ and } ||\alpha_{12}(\varphi,u)||_{(1-\delta)K} < \frac{E_1}{\delta K^3}$$

$$||\beta_0(\varphi)||_K = ||\partial_I g(0,\varphi,0,0)||_K \leq \frac{||g||_K}{K} < \frac{E_1}{K}$$

$$||\beta_{10}(\varphi,s)||_{(1-\delta)K} \leq \left|\left|\sum_{l\geq 1} g_{1l0}(\varphi)s^{l-1}\right|\right|_{(1-\delta)K} \leq \sum_{l\geq 1}\frac{||g||_K}{K^{l+1}}((1-\delta)K)^{l-1} < \frac{E_1}{\delta K^2} \text{ and } ||\beta_{01}(\varphi,u)||_{(1-\delta)K} < \frac{E_1}{\delta K^2}$$

With $\omega.\partial_\varphi X_0 = \alpha_0 - \overline{\alpha_0}$, the lemma A.3. of this appendix and $1+\eta > m_*$, we find :

$$||X_0||_{(1-\delta)K} \leq \frac{\sigma E_1}{m_*\gamma\,(\delta K)^{2n}} \text{ and } ||\partial_\varphi X_0||_{(1-\delta)K} \leq \frac{\sigma E_1}{m_*\gamma\,(\delta K)^{2n+1}}$$

then, with the assumption (A_1), one can write

$$\left|\left|\overline{\beta_0} - 2\varepsilon\overline{(d_0.\partial_\varphi X_0)}\right|\right| \leq ||\beta_0||_K + 2\varepsilon\frac{||\partial_\varphi X_0||_{(1-\delta)K}}{m} \leq \frac{E_1}{K_*} + \frac{2\varepsilon\sigma E_1}{m_*^2\gamma\,(\delta K_*)^{2n+1}} \leq \frac{3\varepsilon\sigma E_1}{m_*^2\gamma\,(\delta K_*)^{2n+1}}$$

In the same way, we find $||\overline{\alpha_0}||_K \leq E_1 \leq \frac{\sigma E_1\varepsilon}{m_*^2\gamma(\delta K_*)^{2n+1}}$ and

$$\left|\left|\begin{pmatrix} 2\varepsilon\overline{d}(0,0) & \omega \\ \omega & 0\end{pmatrix}^{-1}\right|\right| \leq \frac{1}{\varepsilon m_*} \Longrightarrow |\eta_A| \text{ and } ||\xi|| \leq \frac{4\sigma E_1}{m_*^3\gamma\,(\delta K_*)^{2n+1}}$$

which give $||\partial_\varphi\chi_A||_{(1-\delta)K} \leq \varepsilon\mu\left(||\partial_\varphi X_0||_{(1-\delta)K} + ||\xi||\right) \leq \dfrac{5\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}}$

$$\Longrightarrow V_{(1-3\delta)K} \subset \mathcal{T}_A\left(V_{(1-2\delta)K}\right) \subset V_{(1-\delta)K} \text{ under the assumption } (A_1)$$

To estimate the size of the new perturbation g_A, we use $||H||_K \leq ||\mathcal{H}||_K + \varepsilon\mu||g||_K \leq \frac{6E_1}{5}$ and, under the assumption (A_1), one can write in the new variables :

$$||\{\chi_A, H\}||_{(1-2\delta)K} \leq ||\partial_\varphi\chi_A||_{(1-2\delta)K}\,||\partial_I H||_{(1-2\delta)K} \leq \frac{5\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}}\frac{6E_1}{5\delta K_*}$$

then

$$\left|\left|\alpha_0^A(\varphi)\right|\right|_{(1-2\delta)K} \leq ||\alpha_0(\varphi)||_{(1-2\delta)K} + \frac{1}{\varepsilon\mu}\,||\{\chi_A, H\}\,(0,\varphi,0,0)||_{(1-2\delta)K} \leq \frac{7\sigma E_1^2}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}}$$

The same reasonings give the other coefficients of g_A and, using $\delta < \frac{1}{12}$, we can write

$$||g_A||_{(1-3\delta)K} \leq ||g||_{(1-2\delta)K} + 3\left(1+\frac{2}{\delta}\right)\frac{1}{\varepsilon\mu}\,||\{\chi_A, H\}||_{(1-2\delta)K} \leq \frac{50\sigma E_1^2}{m_*^3\gamma\delta\,(\delta K_*)^{2(n+1)}}$$

Finally, we must estimate the size of the remainder $\mathcal{R}_A$, we have

$$||\{\chi_A, g\}||_{(1-2\delta)K} \leq ||\partial_\varphi\chi_A||_{(1-2\delta)K}\,||\partial_I g||_{(1-2\delta)K} \leq \frac{5\sigma E_1\varepsilon\mu}{m_*^3\gamma\delta^{2n+1}K_*^{2n+1}}\,\frac{E_1}{\delta K_*} \leq \frac{5\sigma E_1^2\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}}$$

and the use of second order Taylor formulas along solutions of the system linked to χ_A gives

$$\begin{aligned}||\mathcal{H}\circ\mathcal{T}_A - \mathcal{H} - \{\chi_A, \mathcal{H}\}||_{(1-3\delta)K} &\leq \frac{1}{2}\,||\{\chi_A, \{\chi_A, \mathcal{H}\}\}||_{(1-2\delta)K} \leq \frac{1}{2}\left|\left|(\partial_\varphi\chi_A)^2\right|\right|_{(1-2\delta)K}\left|\left|\partial_I^2\mathcal{H}\right|\right|_{(1-2\delta)K}\\ &\leq \frac{1}{2}\left(\frac{5\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}}\right)^2\frac{4E_1}{\delta^2K_*^2} = \frac{50\sigma^2E_1^3\varepsilon^2\mu^2}{m_*^6\gamma^2\,(\delta K_*)^{4(n+1)}}\end{aligned}$$

hence, the assumption (A_1) yields

$$||\mathcal{R}_A||_{(1-3\delta)K} \leq \frac{55\sigma^2E_1^3\varepsilon^2\mu^2}{m_*^6\gamma^2\,(\delta K_*)^{4(n+1)}}$$

We should also estimate the variations of the frequency $(\omega - \widetilde{\omega})$ and the exponent $(\lambda - \widetilde{\lambda})$:

(i) Since $1+\widetilde{\eta} = (1+\varepsilon\mu\eta_A)(1+\eta)$, with $\varepsilon\mu\,|\eta_A| \leq \dfrac{4\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}}$ and $\dfrac{1}{2} < 1+\eta < \dfrac{3}{2}$, we find

$$1+\eta-\frac{3}{2}\frac{4\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}} < 1+\widetilde{\eta} < 1+\eta+\frac{3}{2}\frac{4\sigma E_1\varepsilon\mu}{m_*^3\gamma\,(\delta K_*)^{2n+1}}$$

which gives the bounds on the variation of the frequency in the iterative lemma A.5.

(ii) Since $\widetilde{\lambda} = \lambda + \mu\overline{\alpha_{11}^A}$ with $||\alpha_{11}(\varphi)||_K = ||\partial_{su}g(0,\varphi,0,0)||_K < \dfrac{E_1}{K^2}$ and

$$\left|\left|\alpha_{11}^A - \alpha_{11}\right|\right|_{(1-2\delta)K} = \frac{1}{\varepsilon}\,||\partial_{su}\,\{\chi_A, H\}\,(0,\varphi,0,0)||_{(1-2\delta)K} \leq \frac{6\sigma E_1^2(1-2\delta)^{-2}\mu}{m_*^3\gamma\delta^{2(n+1)}K_*^{2(n+2)}}$$

the assumption (A_1) and $\left|\left|\alpha_{11}^A\right|\right|_{(1-2\delta)K} \leq ||\alpha_{11}||_{(1-2\delta)K} + \left|\left|\alpha_{11}^A - \alpha_{11}\right|\right|_{(1-2\delta)K}$ allows to write

$$\left|\left|\alpha_{11}^A\right|\right|_{(1-2\delta)K} \leq \frac{E_1}{K^2} + \frac{6\sigma E_1^2\mu(1-2\delta)^{-2}}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}\,K^2} \leq \frac{7\sigma E_1^2}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}\,K^2}$$

hence we find the lower bound on the exponent in the iterative lemma A.5.

Now, we look at the second transformation.

First, we have $\widetilde{\omega}.\partial_\varphi X_{11} = \alpha_{11}^A - \overline{\alpha_{11}^A}$ then the above estimate of $\left|\left|\alpha_{11}^A\right|\right|_{(1-2\delta)K}$ and the lemma A.3. gives :

$$\left|\left|X_{11}\right|\right|_{(1-3\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\gamma^2\left(\delta K_*\right)^{2(2n+1)}K^2} \text{ and } \left|\left|\partial_\varphi X_{11}\right|\right|_{(1-3\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\gamma^2\left(\delta K_*\right)^{4n+3}K^2} \qquad (A)$$

In the same way, we have $\widetilde{\omega}.\partial_\varphi Y_0 = \beta_0^A$ and $\left|\left|\beta_0^A\right|\right|_{(1-2\delta)K} \leq \dfrac{7\sigma E_1^2}{m_*^3\gamma\left(\delta K_*\right)^{2(n+1)}K}$ which yields :

$$\left|\left|Y_0\right|\right|_{(1-3\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\gamma^2\left(\delta K_*\right)^{2(2n+1)}K} \text{ and } \left|\left|\partial_\varphi Y_0\right|\right|_{(1-3\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\gamma^2\left(\delta K_*\right)^{4n+3}K} \qquad (B)$$

Now, $\partial^-_{\varepsilon\widetilde{\lambda},\widetilde{\omega}}X_{10} = -\alpha_{10}^A$ and $\left|\left|\alpha_{10}^A\right|\right|_{(1-3\delta)K} \leq \dfrac{7\sigma E_1^2}{m_*^3\gamma\left(\delta K_*\right)^{2(n+1)}\delta K}$ in the lemma A.4. give

$$\left|\left|X_{10}\right|\right|_{(1-4\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\varepsilon\gamma\left(\delta K_*\right)^{3n+2}\delta^2 K} \text{ and } \left|\left|\partial_\varphi X_{10}\right|\right|_{(1-4\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\varepsilon\gamma\left(\delta K_*\right)^{3(n+1)}\delta^2 K} \qquad (C)$$

the same reasonings give also these estimates for $\left|\left|X_{01}\right|\right|_{(1-4\delta)K}$ and $\left|\left|\partial_\varphi X_{01}\right|\right|_{(1-4\delta)K}$.

We should also estimate $\partial_{10}X_{10}$ and $\partial_{10}X_{01}$, with

$$X_{10}(\varphi,s) = \sum_{k\in\mathbf{N}}(k+1)X_{10}^{(k)}(\varphi)s^k \text{and } \left|\left|X_{10}^{(k)}\right|\right|_{(1-4\delta)K} \leq \frac{\left|\left|X_{10}\right|\right|_{(1-4\delta)K}}{((1-4\delta)K)^k}$$

and the analogous formulas for X_{01}, we find :

$$\left|\left|\partial_{10}X_{01}\right|\right|_{(1-5\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\varepsilon\gamma\left(\delta K_*\right)^{3n+2}\delta^4 K} \text{ and } \left|\left|\partial_{10}X_{10}\right|\right|_{(1-5\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^4\varepsilon\gamma\left(\delta K_*\right)^{3n+2}\delta^4 K}$$

Then $\partial^-_{\varepsilon\widetilde{\lambda},\widetilde{\omega}}X_{21}(\varphi,s) = -\alpha_{21}^A(\varphi,s) + \varepsilon\left(b_A(\varphi,s,0).\partial_\varphi X_{10}(\varphi,s)\right) - 2\varepsilon c_A(\varphi,s,0)\partial_{1,0}X_{10}(\varphi,s) = \mathcal{D}$, we have

$$\left|\left|\alpha_{10}^A\right|\right|_{(1-3\delta)K} \leq \frac{7\sigma E_1^2}{m_*^3\gamma\left(\delta K_*\right)^{2(n+1)}\delta K}$$

and $\left|\left|b_A(\varphi,s,0)\right|\right|_{(1-3\delta)K} \leq \left|\left|b(\varphi,s,0)\right|\right|_{(1-3\delta)K} + \left|\left|b_A(\varphi,s,0) - b(\varphi,s,0)\right|\right|_{(1-3\delta)K}$ with

$$\left|\left|b(\varphi,s,0)\right|\right|_{(1-3\delta)K} \leq \left|\left|\sum_{k\geq 1}H_{1k1}(\varphi)s^{k-1}\right|\right|_{(1-3\delta)K} \leq \sum_{k\geq 1}\frac{6E_1}{5((1-2\delta)K)^{k+2}}((1-3\delta)K)^{k-1} \leq \frac{6E_1}{5(1-2\delta)^2\delta K^3}$$

where we write the expansion $H(I,\varphi,s,u) = \displaystyle\sum_{k\in\mathbf{N}^n}\sum_{(l,m)\in\mathbf{N}^2}H_{klm}(\varphi)I^k s^l u^m$; in the same way

$$\left|\left|b_A(\varphi,s,0) - b(\varphi,s,0)\right|\right|_{(1-3\delta)K} \leq \frac{1}{(1-2\delta)^2\delta K^3}\frac{6\sigma E_1^2\mu}{m_*^3\gamma\left(\delta K_*\right)^{2(n+1)}}$$

Thus, under the assumption (A_1) and $\delta < \frac{1}{12}$, we can write

$$\left|\left|b_A(\varphi,s,0)\right|\right|_{(1-3\delta)K} \leq \frac{6E_1}{(1-2\delta)^2\delta K^3}\left(\frac{1}{5} + \frac{\sigma E_1\mu}{m_*^3\gamma\left(\delta K_*\right)^{2(n+1)}}\right) \leq \frac{2E_1}{\delta K^3}$$

hence $||b_A(\varphi,s,0).\partial_\varphi X_{10}(\varphi,s)||_{(1-4\delta)K} \leq \dfrac{14\sigma^2 E_1^3}{m_*^4 \varepsilon\gamma\,(\delta K_*)^{3n+4}\,\delta^2 K^3}$

For the third term of $\mathcal{D}$, we have :

$$||c(\varphi,s,0)||_{(1-3\delta)K} \leq \left|\left|\sum_{k\geq 2} H_{0k2}(\varphi)s^{k-2}\right|\right|_{(1-3\delta)K} \leq \sum_{k\geq 2}\frac{6E_1}{5((1-2\delta)K)^{k+2}}((1-3\delta)K)^{k-2} \leq \frac{6E_1}{5(1-2\delta)^3\delta K^4}$$

and $||c_A(\varphi,s,0)-c(\varphi,s,0)||_{(1-3\delta)K} \leq \dfrac{1}{(1-2\delta)^3\delta K^4}\dfrac{6\sigma E_1^2\mu}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}}$.

Under the assumption (A_1) and $\delta < \frac{1}{12}$, we can write

$$||c_A(\varphi,s,0)||_{(1-3\delta)K} \leq \frac{6E_1}{(1-2\delta)^3\delta K^4}\left(\frac{1}{5}+\frac{\sigma E_1\mu}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}}\right) \leq \frac{3E_1}{\delta K^4}$$

hence $||c_A(\varphi,s,0).\partial_{10}X_{10}(\varphi,s)||_{(1-5\delta)K} \leq \dfrac{21\sigma^2 E_1^3}{m_*^4\varepsilon\gamma\,(\delta K_*)^{3n+4}\,\delta^3 K^3}$

Finally, we obtain $||\mathcal{D}(\varphi,s)||_{(1-5\delta)K} \leq \dfrac{63\sigma^2 E_1^3}{m_*^4\gamma\,(\delta K_*)^{3n+4}\,\delta^3 K^3}$ and the lemma A.4. gives

$$||X_{21}||_{(1-6\delta)K} \leq \frac{63\sigma^3 E_1^3}{m_*^5\varepsilon\gamma\,(\delta K_*)^{4(n+1)}\,\delta^4 K^3} \text{ and } ||\partial_\varphi X_{21}||_{(1-6\delta)K} \leq \frac{63\sigma^3 E_1^3}{m_*^5\varepsilon\gamma\,(\delta K_*)^{4n+5}\,\delta^4 K^3} \qquad (D)$$

the same reasonings give also these estimates for $||X_{12}||_{(1-6\delta)K}$ and $||\partial_\varphi X_{12}||_{(1-6\delta)K}$.

In the same way $\partial^-_{\varepsilon\tilde{\lambda},\tilde{\omega}}Y_{10}(\varphi,s) = -\beta^A_{10}(\varphi,s)+2\varepsilon\,(d_A(\varphi,s,0).\partial_\varphi X_{10}(\varphi,s))-\varepsilon b_A(\varphi,s,0)\partial_{1,0}X_{10}(\varphi,s) = \mathcal{E}$,

we have $||\beta^A_{10}||_{(1-3\delta)K} \leq \dfrac{7\sigma E_1^2}{m_*^3\gamma\,(\delta K_*)^{2(n+1)}\,\delta K^2}$ and

$$||d_A(\varphi,s,0)||_{(1-2\delta)K} \leq ||d(\varphi,s,0)||_{(1-2\delta)K} + ||d_A(\varphi,s,0)-d(\varphi,s,0)||_{(1-2\delta)K}$$

with $||d_A(\varphi,s,u)-d(\varphi,s,u)||_{(1-3\delta)K} \leq \dfrac{1}{2\varepsilon}\left|\left|\partial^2_{II}\{\chi_A,H\}(0,\varphi,s,u)\right|\right|_{(1-2\delta)K} \leq \dfrac{12\sigma E_1^2(1-2\delta)^{-2}\mu}{m_*^3\gamma\delta^{2(n+1)}K_*^{2(n+2)}}$

$$\leq \frac{18\sigma E_1^2\mu}{m_*^3\gamma\delta^{2(n+1)}K_*^{2(n+2)}}$$

and $||d(\varphi,s,u)||_{(1-3\delta)K} \leq \dfrac{1}{\mathrm{p}}$; the assumption (A_2) yields $||d_A(\varphi,s,u)||_{(1-2\delta)K} < \dfrac{1}{m_*}$, hence

$$||d_A(\varphi,s,0).\partial_\varphi X_{10}(\varphi,s)||_{(1-4\delta)K} \leq \frac{7\sigma^2 E_1^2}{m_*^5\varepsilon\gamma\,(\delta K_*)^{3(n+1)}\,\delta^2 K}$$

then $||b_A(\varphi,s,0)||_{(1-3\delta)K} \leq \dfrac{2E_1}{\delta K^3}$ yields

$$||b_A(\varphi,s,0).\partial_{10}X_{10}(\varphi,s)||_{(1-5\delta)K} \leq \frac{14\sigma^2 E_1^3}{m_*^4\varepsilon\gamma\,(\delta K_*)^{3n+4}\,\delta^3 K^2}$$

and finally, we obtain $||\mathcal{E}(\varphi,s)||_{(1-5\delta)K} \leq \dfrac{35\sigma^2 E_1^3}{m_*^5\gamma\,(\delta K_*)^{3n+4}\,\delta^3 K^2}$ and the lemma A.4. gives

$$||Y_{10}||_{(1-6\delta)K} \leq \frac{35\sigma^3 E_1^3}{m_*^6\varepsilon\gamma\,(\delta K_*)^{4(n+1)}\,\delta^4 K^2} \text{ and } ||\partial_\varphi Y_{10}||_{(1-6\delta)K} \leq \frac{35\sigma^3 E_1^3}{m_*^6\varepsilon\gamma\,(\delta K_*)^{4n+5}\,\delta^4 K^2} \qquad (E)$$

the same reasonings give also these estimates for $||Y_{01}||_{(1-6\delta)K}$ and $||\partial_\varphi Y_{01}||_{(1-6\delta)K}$.

The assumption (3) of the main result allows to write

$$||\chi_B||_{(1-6\delta)K} \leq \frac{14\sigma^3 E_1^3\varepsilon\mu}{m_*^6\gamma\delta^4\,(\delta K_*)^{4(n+1)}}\left(\frac{15}{\varepsilon}+\frac{1}{\gamma}\right) \leq \frac{21\sigma^3 E_1^3\varepsilon\mu}{m_*^6\gamma^2\delta^4\,(\delta K_*)^{4(n+1)}}$$

and the assumption (A_1) gives

$$||\partial\chi_B||_{(1-7\delta)K} \leq \delta K \Longrightarrow V_{(1-9\delta)K} \subset \mathcal{T}_B\left(V_{(1-8\delta)K}\right) \subset V_{(1-7\delta)K}$$

To estimate the size of the remainder $\widetilde{\mathcal{R}}$, we write in the new variables :

$$||\{\chi_B, g_A\}||_{(1-9\delta)K} \leq \frac{||\chi_B||_{(1-8\delta)K}\,||g_A||_{(1-8\delta)K}}{(\delta K_*)^2} \leq \frac{1050\sigma^4 E_1^5}{m_*^9\gamma^3\delta^5\,(\delta K_*)^{6n+8}}\frac{\varepsilon^2\mu^2}{\gamma^3}$$

the use of second order Taylor formulas along solutions of the system linked to χ_B gives

$$\begin{aligned}
||\mathcal{H}\circ\mathcal{T}_A\circ\mathcal{T}_B - \mathcal{H}\circ\mathcal{T}_A - \{\chi_B, \mathcal{H}\circ\mathcal{T}_A\}||_{(1-10\delta)K} &\leq \frac{1}{2}\,||\{\chi_B,\{\chi_B,\mathcal{H}\circ\mathcal{T}_A\}\}||_{(1-9\delta)K} \\
&\leq \frac{4\,||\chi_B||^2_{(1-8\delta)K}||\mathcal{H}\circ\mathcal{T}_A||_{(1-8\delta)K}}{\delta^4 K_*^4} \\
&\leq \left(\frac{42\sigma^3 E_1^3}{m_*^6\delta^4\,(\delta K_*)^{4n+6}}\right)^2\frac{\varepsilon^2\mu^2}{\gamma^4}E_1
\end{aligned}$$

and the assumption (A_1) yields $||\{\chi_B,\mathcal{R}_A\}||_{(1-9\delta)K} \leq ||\mathcal{R}_A||_{(1-8\delta)K}\,(\delta K_*)^2 \leq \dfrac{55\sigma^2 E_1^3}{m_*^6\,(\delta K_*)^{4n+2}}\dfrac{\varepsilon^2\mu^2}{\gamma^2}$ hence

$$\left|\left|\widetilde{\mathcal{R}}\right|\right|_{(1-10\delta)K} \leq \left(\frac{54\sigma^3 E_1^3}{m_*^6\delta^4\,(\delta K_*)^{4n+6}}\right)^2\left(\frac{\varepsilon\mu}{\gamma^2}\right)^2 E_1$$

and we find the estimate of the iterative lemma A.5.

We should specify the variations of the Hessian matrix. We have computed previously the bound :

$$||d_A(\varphi,s,u) - d(\varphi,s,u)||_{(1-8\delta)K} \leq \frac{18\sigma E_1^2\mu}{m_*^3\gamma\delta^{2(n+1)}K_*^{2(n+2)}}$$

and
$$\begin{aligned}
\left|\left|\widetilde{d}(\varphi,s,u) - d_A(\varphi,s,u)\right|\right|_{(1-9\delta)K} &\leq \frac{1}{2\varepsilon}\,||\partial_{II}^2\{\chi_B, H_A\}(0,\varphi,s,u)||_{(1-9\delta)K} \\
&\leq \frac{2\,||\chi_B||_{(1-8\delta)K}\,||H_A||_{(1-8\delta)K}}{\varepsilon(1-9\delta)^2\delta^2 K_*^4}
\end{aligned}$$

With $H_A - H = \varepsilon\mu\,(g - g_A) + \{\chi_A, H\}$, we can write :

$$\begin{aligned}||H_A - H||_{(1-8\delta)K} \leq & \varepsilon\mu\,||g_A - g||_{(1-8\delta)K} + ||\{\chi_A, H\}||_{(1-8\delta)K} \\ \leq & \left[3\left(1+\frac{2}{\delta}\right)+1\right]||\{\chi_A, H\}||_{(1-8\delta)K} \\ \leq & \frac{38\sigma E_1^2\varepsilon\mu}{m_*^3\gamma\delta\,(\delta K_*)^{2(n+1)}} \Longrightarrow ||H_A||_{(1-8\delta)K} \leq 3E_1 \text{ (using the assumption } A_1).\end{aligned}$$

and we find $\left|\left|\widetilde{d}(\varphi, s, u) - d_A(\varphi, s, u)\right|\right|_{(1-9\delta)K} \leq \frac{2016\sigma^3 E_1^3\mu}{m_*^6\gamma^2\delta^2(\delta K_*)^{4(n+2)}}$

$$\Longrightarrow \left|\left|\widetilde{d}(\varphi, s, u) - d(\varphi, s, u)\right|\right|_{(1-9\delta)K} \leq \frac{(13\sigma E_1)^3\,\mu}{m_*^6\gamma^2\delta^2\,(\delta K_*)^{4(n+2)}}$$

Hence, for all $\mathbf{V} \in \mathbf{R}^n$, we have

$$\begin{aligned}\left|\left|\begin{pmatrix} 2\varepsilon\overline{\widetilde{d}}(0,0) & \widetilde{\omega} \\ \widetilde{\omega} & 0\end{pmatrix}\mathbf{V}\right|\right| \geq & \left|\left|\begin{pmatrix} 2\varepsilon\overline{d}(0,0) & \omega \\ \omega & 0\end{pmatrix}\mathbf{V}\right|\right| - \left|\left|\begin{pmatrix} 2\varepsilon\overline{\widetilde{d} - d}(0,0) & \varepsilon\mu\eta_A\omega \\ \varepsilon\mu\eta_A\omega & 0\end{pmatrix}\mathbf{V}\right|\right| \\ \geq & \left[\varepsilon\mathrm{p} - 2\left(\varepsilon\left|\left|\left(\widetilde{d} - d\right)(\varphi, 0, 0)\right|\right| + \varepsilon\mu\,|\eta_A|\,|1+\eta|\,||\omega_0||\right)\right]||\mathbf{V}||\end{aligned}$$

then,using $\nu_0 \leq E_1$ and $1 + \eta < 3/2$, we find the estimate for $\widetilde{\mathrm{p}}$ given in the assumption (A_2).

In the same way, we have $\left|\left|\widetilde{d}(\varphi, 0, 0)\right|\right|_{(1-9\delta)K} \leq \mathrm{p}^{-1} + \frac{(13\sigma E_1)^3\mu}{m_*^6\gamma^2\delta^2(\delta K_*)^{4(n+2)}} \leq \widetilde{\mathrm{p}}^{-1}$ and this gives the second part of the assumption (A_2).

There remains to estimate the variations on the main part of the Hamiltonian, we have previously derived :

$$||H_A - H||_{(1-8\delta)K} \leq \frac{38\sigma E_1^2\varepsilon\mu}{m_*^3\gamma\delta\,(\delta K_*)^{2(n+1)}} \text{ and } ||H_A||_{(1-8\delta)K} \leq 3E_1$$

in the same way $\widetilde{H} - H_A = \varepsilon\mu g_A + \{\chi_B, H_A\}$ allows to write :

$$\begin{aligned}\left|\left|\widetilde{H} - H_A\right|\right|_{(1-10\delta)K} \leq & \varepsilon\mu\,||g_A||_{(1-10\delta)K} + ||\{\chi_B, H_A\}||_{(1-10\delta)K} \\ \leq & \frac{113\sigma^3 E_1^4\varepsilon\mu}{m_*^6\gamma^2\delta^4\,(\delta K_*)^{4n+6}}\end{aligned}$$

and we find the bound given in the lemma A.5.

Appendix B : Results on the isoenergetic nondegeneracy condition

This second appendix is composed of three basic lemmas of linear algebra.

Lemma B.0.

Let Q be a matrix in $\mathcal{M}_{n+1}(\mathbf{R})$ which can be written :

$$Q = \begin{pmatrix} M & F \\ {}^tF & 0\end{pmatrix} \text{ where } M \in \mathcal{S}_n(\mathbf{R}) \text{ is a symmetric matrix and } F \in \mathbf{R}^n\backslash\{0\}.$$

We also assume that $||F|| \geq \tilde{p}$ where $\tilde{p} \in]0,1[$ and that $||M|| \leq k\tilde{p}$ ($k > 0$) for the operator norm induced by the Euclidean norm. Then, we have the following properties :

(i) If, for all $\tilde{v} \in < F >^{\perp} \subset \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, we have $||M\tilde{v} + \lambda F|| \geq \tilde{p}||\tilde{v}||$ then :

$$\text{for all } v \in \mathbf{R}^{n+1}, \text{ we can write } ||Qv|| \geq p||v|| \text{ where } p = \frac{\tilde{p}}{2(2+k)^2}.$$

(ii) Conversely, if for all $v \in \mathbf{R}^{n+1}$ we have $||Qv|| \geq p||v||$, then :

$$\text{for all } \tilde{v} \in < F >^{\perp} \subset \mathbf{R}^n \text{ and } \lambda \in \mathbf{R}, \text{ we can write } ||M\tilde{v} + \lambda F|| \geq p||\tilde{v}||.$$

Proof

(i) Let e_{n+1} be the last vector of the canonical basis and $\mathcal{F}$ the vectorial line generated by $\begin{pmatrix} F \\ 0 \end{pmatrix}$.

Then all $v \in \mathbf{R}^{n+1}$ can be decomposed in $v = v' + v'' + \alpha e_{n+1}$ with $v' \in \mathcal{F}^{\perp} \cap < e_{n+1} >^{\perp}$, $v'' \in \mathcal{F}$ and $\alpha \in \mathbf{R}$. We can write $Qv = X + Y$ with $X = M(v'+v'') + \alpha F$ and $Y = (v'+v''.F)e_{n+1}$, then $v' \in \mathcal{F}^{\perp}$ and $v'' \in \mathcal{F}$ imply $||v''|| = \frac{||Y||}{||F||} \leq \frac{||Y||}{\tilde{p}}$, our hypothesis (i) yields

$$||X|| = ||M(v'+v'') + \alpha F|| \geq \tilde{p}||v'|| - k\tilde{p}||v''|| \implies ||v'|| \leq \frac{||X||}{\tilde{p}} + k\frac{||Y||}{\tilde{p}},$$

finally $\alpha F = X - M(v'+v'')$ implies $|\alpha| \leq (1+k)\left[\frac{||X||}{\tilde{p}} + k\frac{||Y||}{\tilde{p}}\right]$. Then we find the majoration

$$||v||^2 = ||v'||^2 + ||v''||^2 + \alpha^2 \leq \frac{2}{\tilde{p}^2}(1+k)^2\left(1 + 2(1+k)^2\right)\left(||X||^2 + ||Y||^2\right)$$

and $||Qv||^2 = ||X||^2 + ||Y||^2$ gives the isoenergetic nondegeneracy condition with the value p computed in the lemma.

(ii) For $\tilde{v} \in < F >^{\perp} \subset \mathbf{R}^n$ and $\lambda \in \mathbf{R}$:

$$\left|\left|\begin{pmatrix} M & F \\ {}^tF & 0 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \lambda \end{pmatrix}\right|\right| = ||M\tilde{v} + \lambda F|| \geq p\sqrt{||\tilde{v}||^2 + \lambda^2} \geq p||\tilde{v}||.$$

Lemma B.1.
Let Q be a matrix in $\mathcal{M}_{n+1}(\mathbf{R})$ which can be written :

$$Q = \begin{pmatrix} \tilde{M} & A & F \\ {}^tA & a & 0 \\ {}^tF & 0 & 0 \end{pmatrix} \quad \textit{where } \tilde{M} \in \mathcal{S}_{n-1}(\mathbf{R}),\ A \in \mathbf{R}^{n-1},\ a \in \mathbf{R}\backslash\{0\} \textit{ and } F \in \mathbf{R}^{n-1}\backslash\{0\}.$$

We also assume that $||F|| \geq \tilde{p}$ where $\tilde{p} \in]0,1[$ and that $\left|\left|\begin{pmatrix} \tilde{M} & A \\ {}^tA & a \end{pmatrix}\right|\right| \leq k\tilde{p}$ $(k > 0)$.
Finally, the matrix Q is supposed isoenergetically non-degenerate, i.e. :

$$\textit{for all } \tilde{v} \in \mathcal{F}^{\perp} \subset \mathbf{R}^n \textit{ and } \lambda \in \mathbf{R}, \textit{ we have } \left|\left|\begin{pmatrix} \tilde{M} & A \\ {}^tA & a \end{pmatrix}\tilde{v} + \lambda\begin{pmatrix} F \\ 0 \end{pmatrix}\right|\right| \geq \tilde{p}||\tilde{v}||$$

where $\mathcal{F}$ is the vectorial line generated by $\begin{pmatrix} F \\ 0 \end{pmatrix}$.

Then, the matrix $\begin{pmatrix} \tilde{M} - \frac{1}{a}A^tA & F \\ {}^tF & 0 \end{pmatrix}$ is also isoenergetically non-degenerate :

$$\textit{for all } v \in < F >^{\perp} \subset \mathbf{R}^{n-1} \textit{ and } l \in \mathbf{R}, \textit{ we have } \left|\left|(\tilde{M} - \frac{1}{a}A^tA)v + lF\right|\right| \geq \tilde{p}||v||$$

and $\left|\left|\tilde{M} - \frac{1}{a}A^tA\right|\right| \leq \frac{k^2\tilde{p}^2}{|a|}$.
Hence, the previous lemma B.0. shows that, for all $\bar{v} \in \mathbf{R}^n$, we have

$$\left|\left|\begin{pmatrix} \tilde{M} - \frac{1}{a}A^tA & F \\ {}^tF & 0 \end{pmatrix}\bar{v}\right|\right| \geq C_1|a|\tilde{p}||\bar{v}|| \textit{ where } C_1 \textit{ is a positive constant.}$$

Proof First, we consider $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbf{R}^n$ with $v_1.F = 0$ and $v_2 = -\frac{(A.v_1)}{|a|} \in \mathbf{R}$ then

$$\left|\left|(\tilde{M} - \frac{1}{a}A^tA)v_1 + lF\right|\right| == \left|\left|\begin{pmatrix} \tilde{M} & A \\ {}^tA & a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + l\begin{pmatrix} F \\ 0 \end{pmatrix}\right|\right| \geq \tilde{p}||v|| = \tilde{p}\sqrt{||v_1||^2 + v_2^2} \geq \tilde{p}||v_1||,$$

and, since $\sqrt{a^2 + ||A||^2} \leq k\tilde{p}$, we can write :

$$\left|\left|\left(\tilde{M} - \frac{1}{a}A^tA\right)v_1\right|\right| = \left|\left|\begin{pmatrix} \tilde{M} & A \\ {}^tA & a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|\right| \leq k\tilde{p}\sqrt{||v_1||^2 + v_2^2} \leq \frac{k^2\tilde{p}^2}{|a|}||v_1||.$$

Lemma B.2.
Let Q be a matrix in $\mathcal{M}_{n+1}(\mathbf{R})$ which can be written :

$$Q = \begin{pmatrix} M & F \\ {}^tF & 0 \end{pmatrix} \text{ where } M \in \mathcal{S}_n(\mathbf{R}) \text{ and } F \in \mathbf{R}^n \backslash \{0\},$$

and satisfies $||Mv + lF|| \geq \tilde{p}||v||$ for all $v \in < F >^\perp \subset \mathbf{R}^n$ and $l \in \mathbf{R}$.
Then, the matrix $Q_\varepsilon = \begin{pmatrix} \sqrt{\varepsilon}M & F \\ {}^tF & 0 \end{pmatrix}$ is also isoenergetically non-degenerate :

$$\text{for all } v \in < F >^\perp \subset \mathbf{R}^n \text{ and } l \in \mathbf{R}, \text{ we have } \left|\left|\sqrt{\varepsilon}Mv + lF\right|\right| \geq \sqrt{\varepsilon}\tilde{p}||v||.$$

Hence, the previous lemma B.0. shows that, for all $v \in \mathbf{R}^n$, we have

$$\left|\left|\begin{pmatrix} \sqrt{\varepsilon}M & F \\ {}^tF & 0 \end{pmatrix} v\right|\right| \geq C_2\sqrt{\varepsilon}\tilde{p}||v|| \text{ where } C_2 \text{ is a positive constant.}$$

Proof : For all $v \in < F >^\perp \subset \mathbf{R}^n$ and $l \in \mathbf{R}$, we have

$$\left|\left|\sqrt{\varepsilon}Mv + lF\right|\right| = \left|\left|M\left(\sqrt{\varepsilon}v\right) + lF\right|\right| \geq \tilde{p}||\sqrt{\varepsilon}v|| = \sqrt{\varepsilon}\tilde{p}||v||.$$

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