A REMARK ON INVARIANT MANIFOLDS: THE OPTIMAL GAP CONDITION.

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ABSTRACT. We give optimal gap conditions on Lipschitz constant of the nonlinear term and growth bounds of the linear term that imply the existence of Lipschitz invariant manifolds for semilinear equations on Banach spaces. This result improves recent theorems by C. Foias and by N. Castañeda and R. Rosa.

1. MAIN RESULTS

From its inception in now classic works, [?, ?, ?, ?, ?], the invariant manifold theory plays an important role in the modern theory of infinite dimensional dynamical systems. We refer the reader to [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] and to the literature cited therein. Recently, several advances in the invariant manifold theory have been made to obtain optimal gap conditions, see [?, ?, ?].

In this paper we consider semilinear autonomous differential equations on Banach spaces with globally Lipschitz nonlinearities. We give new sharp gap conditions on the spectrum of the linear part and Lipschitz constants for the nonlinear part under which the equation has Lipschitz invariant manifolds. Descriptively, our result says the following. Suppose the real part of the spectrum of the linear term of the equation has several disjoint spectral segments. Fix a spectral segment. Add the Lipschitz constant of the nonlinear term that correspond to this segment and the Lipschitz constant of the nonlinear term that corresponds to all segments to the right (respectively, to the left) of this fixed segment. Lipschitz invariant manifolds exist provided this *sum* is strictly less then the length of the spectral gap that lies to the right (respectively, to the left) of the segment.

First, we give a precise formulation of this result for the case of two spectral segments. Let Z be a Banach space, and $X, Y \subset Z$ are subspaces of Z such that $Z = X \oplus Y$. We equip Z with the norm $|z|_Z = |x + y|_Z = \max\{|x|_X, |y|_Y\}$. Consider a system of differential equations on Z of the form

$$\dot{x} = Ax + F(x + y), \tag{1}$$

$$\dot{y} = By + G(x + y). \tag{2}$$

Let us assume that the operator A is an infinitesimal generator of a strongly continuous semigroup on X, the operator $-B$ is a generator of a strongly continuous semigroup on Y, the maps $F : Z \to X$ and $G : Z \to Y$ are globally Lipschitz, and $F(0) = G(0) = 0$. Further, let us assume there exist constants $a < b$ such that

$$\|e^{tA}\| \le e^{ta}, \quad t \ge 0, \quad \|e^{tB}\| \le e^{tb}, \quad t \le 0. \tag{3}$$

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We stress that (3) implies the inequalities $\mathrm{Re}\,\sigma(A) \le a$ and $\mathrm{Re}\,\sigma(B) \ge b$ for the spectra of A and B. Denote $\mathrm{Lip}(F) = \sup\{|F(z_1) - F(z_2)|/|z_1 - z_2| : z_1 \ne z_2\}$, and let $\mathrm{Lip}_1^0(X, Y) := \{\varphi : X \to Y \,|\, \varphi(0) = 0,\ \mathrm{Lip}(\varphi) \le 1\}$.

We prove the following sharp results on the existence of "stable"- and "unstable"-like invariant manifolds for (1)–(2) . Assume gap condition

$$\mathrm{Lip}(F) + \mathrm{Lip}(G) < b - a. \tag{4}$$

Theorem 1. *There exists a function $\varphi : X \to Y$, $\varphi \in \mathrm{Lip}_1^0(x, Y)$ such that its graph $\mathcal{M}_\varphi = \{x + \varphi(x) | x \in X\}$ is a forward invariant manifold for (1)–(2) and*

$$\mathrm{Lip}(\varphi) \le \frac{\mathrm{Lip}(G)}{b - a - \mathrm{Lip}(F)}. \tag{5}$$

If, in addition, A generates a strongly continuous group on X, then $\mathcal{M}_\varphi$ is also backward invariant.

By reversing time, one obtains:

Corollary 2. *There exists a function $\psi : Y \to X$, $\psi \in \mathrm{Lip}_1^0(Y, X)$ such that its graph $\mathcal{M}_\psi = \{\psi(y) + y | y \in Y\}$ is a backward invariant manifold for (1)–(2) and*

$$\mathrm{Lip}(\psi) \le \frac{\mathrm{Lip}(F)}{b - a - \mathrm{Lip}(G)}.$$

If, in addition, B generates a strongly continuous group on Y, then $\mathcal{M}_\psi$ is also forward invariant.

By renorming, conditions (3) can be relaxed as follows. Assume, that there exist M_A, $M_B > 1$ such that

$$\|e^{tA}\| \le M_A e^{ta}, \quad t \le 0, \quad \|e^{tB}\| \le M_B e^{tb}, \quad t < 0.$$

Assume the gap condition

$$M_A \,\mathrm{Lip}(F) + M_B \,\mathrm{Lip}(G) < b - a.$$

Corollary 3. *There exists $\varphi : X \to Y$ with a forward invariant graph $\mathcal{M}_\varphi$ such that*

$$\mathrm{Lip}(\varphi) \le M_A \frac{M_B \,\mathrm{Lip}(G)}{b - a - M_A \,\mathrm{Lip}(F)}.$$

Similar fact holds for $\psi : Y \to X$.

The proofs of these results are given in Section 2. The main ingredients are: a connection of Lyapunov-Perron operators on spaces of exponentially bounded functions on axis, and Lyapunov-Perron operators on spaces of Lipschitz maps; a contraction mapping theorem for the Lyapunov-Perron operator; Gronwall's inequality.

In fact, our method is just a refinement of the technique developed in [**?**, **?**], where the case of *groups* generators A and B was considered. We note, that in [**?**] the gap condition

$$4 \max\{\mathrm{Lip}(F)\,, \mathrm{Lip}(G)\} < b - a$$

was used instead of (4). In [**?**] this condition was improved to state

$$2 \max\{\mathrm{Lip}(F)\,, \mathrm{Lip}(G)\} < b - a, \tag{6}$$