A FORMULA FOR THE ESSENTIAL SPECTRAL RADIUS OF RUELLE'S TRANSFER OPERATOR ON SMOOTH HÖLDER SPACES

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Abstract. We study the deterministic and random Ruelle transfer operator $\mathcal{L}$ induced by an expanding map f of a smooth n-dimensional manifold X and a bundle automorphism φ of an ℓ-dimensional vector bundle E. We prove the following exact formula for the essential spectral radius of $\mathcal{L}$ on the space $C^{\mathbf{r},\alpha}$ of $\mathbf{r}$-times continuously differentiable sections of E with α-Hölder $\mathbf{r}$-th derivative:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L};C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in\mathrm{Erg}}\{h_\nu + \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}\right).$$

Here Erg is the set of f-ergodic measures, h_ν is the entropy of f with respect to ν, λ_ν is the largest Lyapunov-Oseledets exponent of the cocycle $\varphi^k(x) = \varphi(f^{k-1}x)\cdot\ldots\cdot\varphi(x)$, and χ_ν is the smallest Lyapunov-Oseledets exponent of the differential $Df^k(x)$, $x\in X$, $k=1,2,\ldots$. A similar result holds for the random case.

1. Main Results

Ruelle's deterministic and random transfer operators play a significant role in dynamical systems and statistical physics. A study of their spectral properties is important in the theory of Gibbs measures [9, 27], hyperbolic dynamics [23], zeta-function [3, 23, 26, 28], thermodynamic formalism [25], piece-wise monotone transformations [4], Fredholm determinants [24], wavelets [12, Sections 3,5] and [16], and many other questions.

In the current paper we give exact formulas for the essential spectral radius of the deterministic and random matrix coefficient transfer operator $\mathcal{L}$ induced by an expanding map on the space $C^{\mathbf{r},\alpha}$ of differentiable Hölder vector-functions. For the scalar case the spectrum of the transfer operator $\mathcal{L}$ on the space C^0 of continuous functions is a disc centered at zero with the radius e^P, where P is the topological pressure (see [9, 23, 27, 29]). Since $\mathcal{L}$ preserves the cone of positive functions, $\mathcal{L}$ has a positive eigenfunction that corresponds to e^P, and there is a corresponding eigenvector (which is the Gibbs measure) for $\mathcal{L}^*$ (see [9]). When one passes from the space of continuous to the space of smooth Hölder functions, the essential spectrum of $\mathcal{L}$ "shrinks", showing finitely many isolated eigenvalues. There are quite a few reasons why one wants to obtain a formula for the radius of the essential spectrum of the Ruelle transfer operator on the space $C^{\mathbf{r},\alpha}$. We mention the following: (1) the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L};C^{\mathbf{r},\alpha})$ gives the exponential rate of decay of correlations for all but finitely many linearly independent test functions from $C^{\mathbf{r},\alpha}$, see [21] for the literature and recent advances on this topic; (2) the reciprocal to $\mathrm{r}_{\mathrm{ess}}(\mathcal{L};C^{\mathbf{r},\alpha})$ gives the radius of meromorphy for the weighted ζ-function associated with the system, see [3, 15, 23, 26].

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In this paper we consider the deterministic transfer operator in the following setting that is due to D. Ruelle [25]. Let f be an expanding map (small distances are increased by a factor $\theta > 1$) of a smooth compact n-dimensional manifold X. We assume that f is not one-to-one, that is $N = \mathrm{card}\{f^{-1}\{x\}\} > 1$. Let E be a smooth real ℓ-dimensional vector bundle over X, $E_x \simeq \mathbb{R}^\ell$ and φ be a smooth bundle automorphism over f, that is $\varphi(x) : E_x \to E_{fx}$ and $\det \varphi(x) \neq 0$. For $\mathbf{r} = 0, 1, \ldots$ and $\alpha \in [0,1]$ let $C^{\mathbf{r},0} = C^{\mathbf{r}}$ denote the space of $\mathbf{r}$-times continuously differentiable sections Φ of E, and $C^{\mathbf{r},\alpha}$, $0 < \alpha \leq 1$ denote the space of $\mathbf{r}$-times continuously differentiable sections Φ with $\mathbf{r}$-th derivative that satisfies a (global) Hölder condition with exponent α. On the space $C^{\mathbf{r},\alpha}$, $\alpha \in [0,1]$ consider the matrix coefficient Ruelle transfer operator, $\mathcal{L}$, defined as follows:

$$(\mathcal{L}\Phi)(x) = \sum_{y \in f^{-1}\{x\}} \varphi(y)\Phi(y), \quad x \in X, \quad \Phi \in C^{\mathbf{r},\alpha}.$$

Let $\mathrm{Erg} = \mathrm{Erg}(f, X)$ denote the set of all f-invariant ergodic Borel probability measures on X, and h_ν denote the entropy of f with respect to $\nu \in \mathrm{Erg}$.

D. Ruelle in his celebrated paper [25] proved, in particular, the following estimate for the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha})$ of the operator $\mathcal{L}$ on $C^{\mathbf{r},\alpha}$:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \leq \exp\left(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_X \log\|\varphi(x)\|\, d\nu - (\mathbf{r}+\alpha)\log\theta\}\right),$$

where $\|\cdot\|$ is the matrix norm, and θ denotes the corresponding expanding constant. This result was obtained by showing that $\log \mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \leq P(\log\|\varphi\|) - (\mathbf{r}+\alpha)\log\theta$ and using the variational principle [9, 29] for the topological pressure $P(\cdot)$.

In the current paper we prove the following sharp formula for the essential spectral radius of the matrix transfer operator. For each $\nu \in \mathrm{Erg}$, let λ_ν denote the largest Lyapunov-Oseledets exponent of the cocycle $\varphi^k(x) = \varphi(f^{k-1}x) \cdot \ldots \cdot \varphi(x)$, and χ_ν denote the smallest Lyapunov-Oseledets exponent of the differential $Df^k(x)$, $x \in X$, $k = 1, 2, \ldots$.

Theorem 1.1. *Assume f is expanding. Then*

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}\right). \tag{1}$$

The proof of Theorem 1.1 involves a refinement of Ruelle's technique from [25] as well as a further development of an idea due to J. Mather [22] on localization along trajectories of f of "almost-eigenfunctions" for an associated evolution operator, see [10] and [11]. We sketch the proof in the next section.

Theorem 1.1 gives a new formula even for the scalar case $\ell = 1$, $\varphi : X \to \mathbb{R}$:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_X \log|\varphi(x)|\, d\nu - (\mathbf{r}+\alpha)\chi_\nu\}\right),$$

and for the one-dimensional case $n = 1$, $X = S^1$:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_X \log|\varphi(x)|\, d\nu - (\mathbf{r}+\alpha)\int_X \log|f'(x)|\, d\nu\}\right). \tag{2}$$

A formula similar to (1) holds for the evolution operator

$$(T\Phi)(x) = \varphi(f^{-1}x)\Phi(f^{-1}x), \quad \Phi \in C^{\mathbf{r},\alpha}, \quad f \quad \text{is a diffeomorphism,}$$

that is,

$$\mathrm{r}_{\mathrm{ess}}(T; C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in\mathrm{Erg}}\{\lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}\right). \tag{3}$$

For $\mathbf{r} = \alpha = 0$, that is, on the space of continuous functions C^0, and for the *spectral radius* $\mathrm{r}_{\mathrm{sp}}(\cdot)$ the formula

$$\mathrm{r}_{\mathrm{sp}}(\mathcal{L}, C^0) = \exp\left(\sup_{\nu\in\mathrm{Erg}}\{h_\nu + \lambda_\nu\}\right) \tag{4}$$

was obtained in [10, Theorem 3] (see also [20, Theorem 4.15], where another type of transfer operators was considered). For $\alpha = 0$, that is, for the space $C^{\mathbf{r}}$, the inequality "$\leq$" in (1) for the essential spectral radius was proved in [10, Theorem 2]; also, see [28] for a study of the transfer operator on $C^{\mathbf{r}}$. Among other results, an inequality "$\leq$", similar to (2), was obtained in [13] for $\ell = n = 1$. Formula (3) completes the line of research in [1, 18, 20] where various formulas for the spectral radius of T were given; in particular, it was proved in [20] that $\mathrm{r}_{\mathrm{sp}}(T, C^0) = \exp\left(\sup_{\nu\in\mathrm{Erg}}\{\lambda_\nu\}\right)$.

We also consider the *random* Ruelle transfer operator in the following setting. Let $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ be an invertible ergodic abstract dynamical system as a model for noise, and f be a non-degenerate expanding bundle random dynamical system [2, 8] on $\mathcal{Y} \subset \Omega \times X$ over ϑ consisting of random compact n-manifolds $\mathcal{Y}_\omega \subset X$, where X is a topological space with Borel σ-algebra $\mathcal{B}$. In particular f is a smooth cocycle over ϑ with generator $f(\omega) : \mathcal{Y}_\omega \to \mathcal{Y}_{\vartheta\omega}$, $\omega \in \Omega$, as it satisfies $f^{k+m}(\omega) = f^k(\vartheta^m\omega) \circ f^m(\omega)$ for all $m, k \in \mathbb{N}_+$, $\omega \in \Omega$. It induces a skew-product transformation Θ on $\mathcal{Y}$ by $\Theta(\omega, x) = (\vartheta\omega, f(\omega)x)$. We assume that f admits a symbolic description (cf. [6]), which need not be true for general expanding random systems (see [17]). Moreover we consider a smooth random ℓ-dimensional real vector bundle E over $\mathcal{Y}$, i.e. the fibres $E_{\omega,x}$ depend measurably on $(\omega, x) \in \mathcal{Y}$ and, for fixed $\omega \in \Omega$, $C^{\mathbf{r}}$-smoothly on $x \in \mathcal{Y}_\omega$ and they satisfy $E_{\omega,x} \simeq E_{\Theta(\omega,x)} \simeq \mathbb{R}^\ell$. We consider smooth random bundle maps φ on E, i.e. $\varphi(\omega, x) : E_{\omega,x} \to E_{\Theta(\omega,x)}$ shows the same dependence on $(\omega, x) \in \mathcal{Y}$ as $E_{\omega,x}$. We denote the space of such $C^{\mathbf{r},\alpha}$-smooth random bundle maps by $\mathbb{L}^{0,\mathbf{r},\alpha}(\mathcal{Y}, \mathcal{L}(E))$. We are only interested in those φ satisfying $\det \varphi(\omega, x) \neq 0$ and

$$\varphi \in \mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, \mathcal{L}(E)) := \{h \in \mathbb{L}^{0,\mathbf{r},\alpha}(\mathcal{Y}, \mathcal{L}(E)) : \int \|h(\omega, .)\|_{C^{\mathbf{r},\alpha}} d\mathbb{P}(\omega) < \infty\},$$

where we used $\|.\|_{C^{\mathbf{r},\alpha}}$ for the $C^{\mathbf{r},\alpha}$ operator norm. Such a φ gives rise to a cocycle over Θ, namely $\varphi^k(\omega, x) = \varphi(\Theta^{k-1}(\omega, x)) \cdot \ldots \cdot \varphi(\omega, x)$. Analogous to the spaces of smooth random bundle maps $\mathbb{L}^{i,\mathbf{r},\alpha}(\mathcal{Y}, \mathcal{L}(E))$, $i = 0, 1$, one can define the spaces of smooth random sections of E. We denote them by $\mathbb{L}^{i,\mathbf{r},\alpha}(\mathcal{Y}, E)$ and introduce for $\varphi \in \mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, \mathcal{L}(E))$ random matrix coefficient transfer operators $\mathcal{L} = \mathcal{L}(\varphi)$ on $\mathbb{L}^{0,k,\alpha}(\mathcal{Y}, E)$ which act ω-wise as $\mathcal{L}(\omega)$, $\omega \in \Omega$ by

$$(\mathcal{L}(\omega)\Phi)(x) = \sum_{y\in f(\omega)^{-1}\{x\}} \varphi(\omega, y)\Phi(y),$$

where Φ is a smooth section of E_ω. Here we used $f(\omega)^{-1}$ to specify the preimages of $f(\omega)$. Note that such a transfer operator $\mathcal{L}(\omega)$ maps $C^{\mathbf{r},\alpha}$ sections of E_ω to $C^{\mathbf{r},\alpha}$ sections of $E_{\vartheta\omega}$, as long as $\varphi(\omega)$ is at least $C^{\mathbf{r},\alpha}$. Moreover, it generates a cocycle $\mathcal{L}^n(\omega) = \mathcal{L}(\vartheta^{n-1}\omega) \cdot \ldots \cdot \mathcal{L}(\omega)$ for $n \in \mathbb{N}_+$. It gives rise to a mapping on $\mathbb{L}^{0,\mathbf{r},\alpha}(\mathcal{Y}, E)$ by

$$(\mathcal{L}\Phi)(\omega, x) = (\mathcal{L}(\vartheta^{-1}\omega)\Phi(\vartheta^{-1}\omega))(x).$$

Note that further integrability conditions are needed to guarantee that this operator is also acting on $\mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)$. If, for example, N denotes the random variable counting the number of preimages under f and $N(\omega)\|\varphi(\omega, .)\|_{C^{\mathbf{r},\alpha}}$ is in $\mathbb{L}^\infty$, then it is clear that $\mathcal{L}$ acts on $\mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)$. In the following we will make this assumption.

The problem arising in the investigation of the random Ruelle operator lies in the point spectrum. It is known by the Ruelle-Perron-Frobenius Theorem (see [23]) that in the deterministic case there exists a positive eigenvalue of largest absolute value, while this is in general not true for the random case ([7]). Nevertheless this special case can be achieved by renormalization such that 1 becomes the dominating eigenvalue of $\mathcal{L}$ ([14]). In general one should look at the limits of the cocycles $\mathcal{L}^n(\omega)$ to keep the theory - the formalism and the interpretations - alive. Thus ω-wise asymptotic growth rates have to be considered. In general these do not coincide with the asymptotic growth rates for the operator $\mathcal{L}$ on $\mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)$, but they play the same role in the random theory as the spectral quantities in the deterministic one. In particular the role of the essential spectral radius is taken over by the exponential of the index of compactness (see, e.g. [6]) which is denoted by $\mathrm{ind_c}$.

Theorem 1.2. *Assume that f defines an expanding bundle random dynamical system that allows a symbolic description and guarantees that $\mathcal{L}$ defines an operator on $\mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)$ provided $\varphi \in \mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)$. If the additional integrability condition (5) is satisfied, then*

$$\mathrm{ind_c}(\mathcal{L}; \mathbb{L}^{1,\mathbf{r},\alpha}(\mathcal{Y}, E)) = \sup_{\nu \in \mathrm{Erg}_{\mathbb{P}}} \{h_\nu + \lambda_\nu - (\mathbf{r} + \alpha)\chi_\nu\}.$$

In this theorem $\mathrm{Erg}_{\mathbb{P}} = \mathrm{Erg}_{\mathbb{P}}(f, \mathcal{Y})$ denotes the set of all Θ-ergodic probability measures on $(\mathcal{Y}, \mathcal{Y} \cap \mathcal{F} \otimes \mathcal{B})$ with marginal $\mathbb{P}$ on Ω, λ_ν the largest Lyapunov-Oseledets exponent of the cocycle φ^k, χ_ν the smallest Lyapunov-Oseledets exponent of the cocycle induced by the differential of f^k with respect to x, and h_ν the (fibre-) entropy of the random mapping f (see [8]).

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2. Sketch of Proofs

The main idea of the proof of Theorem 1.1 is to relate to $\mathcal{L}$ an another transfer operator, $\mathcal{K}$, induced by a certain "projectivization" F of the map f and by a cocycle $\{\psi^k\}$ that depends on $\{\varphi^k\}$, $\mathbf{r}$ and α; see Theorem 2.1 below and [19] for the case $N = \ell = 1$. The map F acts on an extension $\mathcal{X}$ of the manifold X that also depends on $\mathbf{r}$ and α. The operator $\mathcal{K}$ is considered on the space of *continuous* sections of a bundle $\mathcal{E}$ over $\mathcal{X}$. We prove that the *essential* spectral radius of $\mathcal{L}$ on the space of *smooth Hölder* continuous sections of E coincides with the *spectral* radius of $\mathcal{K}$ on the space of *continuous* sections of $\mathcal{E}$. We apply formula (4) to compute the spectral radius of $\mathcal{K}$. Then we relate the entropy of F and the largest Lyapunov-Oseledets exponents of the cocycle $\{\psi^k\}$ to that of f and the cocycles $\{\varphi^k\}$ and $\{Df^k\}$ to get (1). A similar strategy works for the random Ruelle transfer operator under the additional assumptions which are mainly used to apply the Pointwise and the Subadditive Ergodic Theorem for investigations of convergence properties.

We will describe the constructions mentioned above in some more details. For the sake of simplicity we restrict ourself to the deterministic case and remark where additional work has to be done in the random case. Also, we "pretend" that X is a subset of $\mathbb{R}^n$, and E is a subset of $X \times \mathbb{R}^\ell$. Since our constructions are local, a generalization for the case of nontrivial bundles is obvious.

For $\mathbf{r} = 0, 1, \dots$ and $\alpha \in [0,1]$ we define, locally, $\mathcal{X} = X \times S^{n-1} \times \dots \times S^{n-1}$, where we take $\mathbf{r}$ copies of S^{n-1} if $\alpha = 0$ and $\mathbf{r}+1$ copies if $\alpha > 0$. We denote $\mathbf{v} = (v_1, \dots, v_\mathbf{r})$ if $\alpha = 0$ and $\mathbf{v} = (v_0, v_1, \dots, v_\mathbf{r})$ if $\alpha > 0$, $\|v_j\| = 1$, and let $(x, \mathbf{v})$ denote points in $\mathcal{X}$. Globally, one might want to think of $\mathcal{X}$ as a bundle over X whose fibers are direct products of $\mathbf{r}$ or $\mathbf{r}+1$ copies of the fibers of the unit tangent bundle at the corresponding points of X. Using the differential $Df(\cdot)$, let us define a map $F : \mathcal{X} \to \mathcal{X}$ as

$$F(x, \mathbf{v}) = \left(fx, \frac{Df(x)v_0}{\|Df(x)v_0\|}, \frac{Df(x)v_1}{\|Df(x)v_1\|}, \dots, \frac{Df(x)v_\mathbf{r}}{\|Df(x)v_\mathbf{r}\|} \right)$$

for $\alpha > 0$ and similarly (that is, with no term $Df(x)v_0/\|Df(x)v_0\|$) for $\alpha = 0$. It can be shown that F is expanding. Let $\mathcal{E}$ be the ℓ-dimensional bundle over $\mathcal{X}$ with fibers $\mathcal{E}_{(x,\mathbf{v})} = E_x$. Define a bundle automorphism ψ of $\mathcal{E}$ over F as

$$\psi(x, \mathbf{v}) = \frac{\varphi(x)}{\|Df(x)v_0\|^\alpha \cdot \prod_{j=1}^{\mathbf{r}} \|Df(x)v_j\|}$$

for $\alpha > 0$ and similarly (that is, with no term $\|Df(x)v_0\|^\alpha$) for $\alpha = 0$. Let $\mathcal{C}^0$ denote the space of *continuous* sections Ψ of $\mathcal{E}$. On $\mathcal{C}^0$ consider the following Ruelle transfer operator $\mathcal{K}$ induced by F and ψ:

$$(\mathcal{K}\Psi)(x, \mathbf{v}) = \sum_{(y,\mathbf{u}) \in F^{-1}\{(x,\mathbf{v})\}} \psi(y, \mathbf{u})\Psi(y, \mathbf{u}), \quad (x, \mathbf{v}) \in \mathcal{X}.$$

Let us remark that in the random case this operator becomes

$$(\mathcal{K}(\omega)\Psi)(x, \mathbf{v}) = \sum_{(y,\mathbf{u}) \in F(\omega)^{-1}\{(x,\mathbf{v})\}} \psi(\omega, y, \mathbf{u})\Psi(y, \mathbf{u}), \quad (x, \mathbf{v}) \in \mathcal{X}(\omega),$$

where the ω-versions of the notions in use are obvious. The integrability condition

$$\log \|\mathcal{K}(\omega)\|_\infty \in \mathbb{L}^1(\Omega) \tag{5}$$

is needed in order to estimate spectral properties with the help of the Subadditive Ergodic Theorem.

Though the definition of F, ψ and $\mathcal{K}$ are rather cumbersome, their appearance in the current context is quite natural. Indeed, assume $\alpha = 0$ and consider the operator "of differentiation" $\mathcal{D} : C^\mathbf{r} \to \mathcal{C}^0$ defined as $(\mathcal{D}\Phi)(x, \mathbf{v}) = D\Phi(x)(\mathbf{v})$, where $D\Phi$ is the differential. The following fact was proved in [10, Lemma 3] (see also [19] for the case $N = \ell = 1$):

$$\mathcal{D}\mathcal{L}\Phi = \mathcal{K}\mathcal{D}\Phi + R\Phi, \quad \Phi \in C^\mathbf{r}, \tag{6}$$

where R is a compact operator from $C^\mathbf{r}$ to $\mathcal{C}^0$. To see this fact, one just differentiates $\mathcal{L}\Phi$, and uses product rule and chain rule. In fact, $R\Phi$ contains all terms with low order derivatives. For example, for $\mathbf{r} = 1$ one has:

$$\begin{aligned}
(\mathcal{D}\mathcal{L}\Phi)(x, v) &= \sum_{y \in f^{-1}\{x\}} \phi(y)(D\Phi)(y)[(Df(y))^{-1}(v)] + (R\Phi)(x, v) \\
&= \sum_{y \in f^{-1}(x)} \phi(y) \cdot \|(Df(y))^{-1}(v)\| \cdot (D\Phi)(y) \left[\frac{(Df(y))^{-1}(v)}{\|(Df(y))^{-1}(v)\|} \right] + (R\Phi)(x, v),
\end{aligned}$$

which gives the desired result (6).

Theorem 2.1. *Assume f is expanding, $\mathbf{r}=0,1,\dots$ and $\alpha\in[0,1]$. Then*

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L},C^{\mathbf{r},\alpha})=\mathrm{r}_{\mathrm{sp}}(\mathcal{K},\mathcal{C}^0). \tag{7}$$

Sketch of the proof. For $\alpha=0$ the proof of the estimate "$\leq$" in (7) follows (see [10]) from the intertwining (6) and general facts from operator theory. Indeed, $\mathcal{D}$ has finite dimensional kernel and closed range. Hence, $\mathcal{D}$ has a left regularizer $\mathcal{D}^{-1}$, that is, an operator from $\mathcal{C}^0$ to $C^{\mathbf{r}}$ such that $\mathcal{D}^{-1}\mathcal{D}-I$ is a compact operator. Thus, $\mathcal{L}-\mathcal{D}^{-1}\mathcal{K}\mathcal{D}$ is compact, and the general formula

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L})=\lim_{m\to\infty}\inf\{\|\mathcal{L}^m-R\|^{1/m}: R\quad\text{is compact}\} \tag{8}$$

for the essential spectral radius does the job. This argument, essentially, was used in [10].

Unfortunately, for $\alpha\in(0,1]$ we do not see such a simple argument. To prove the estimate "$\leq$" in (7) for any $\alpha\in[0,1]$ we adopted the technique used by D. Ruelle in [25]. The main idea of the proof, as in [25, p. 249] is to estimate the norm of $\mathcal{L}^m-K_{(m)}$, where $K_{(m)}$ are operators of finite rank related to the Taylor expansion of order $\mathbf{r}$.

We use symbolic dynamics and adopt notations from [25]. In particular, fix open sets U_i (see [25, Proposition 2.3]) and let $ix\in U_i\cap f^{-1}\{x\}$ denote the the i-th preimage of x under f. Similarly, we denote $i_1\dots i_m x\in U_{i_1\dots i_m}$ a preimage of $x\in X$ under f^m that corresponds to an admissible m-tuple $i^{(m)}:=i_1\dots i_m$.

Define an operator $\mathcal{M}^{(m)}\approx\mathcal{L}^m$ as in formula (4.4) of [25]. Note, that $\mathcal{M}^{(m)}$ is "almost equivalent" to the transfer operator $\mathcal{L}^m$, that is, $\mathrm{r}_{\mathrm{ess}}(\mathcal{L},C^{\mathbf{r},\alpha})$ can be computed as on the right-hand side of (8) with $\mathcal{L}^m$ replaced by $\mathcal{M}^{(m)}$. Choose $\bar{x}(i_0\dots i_m)\in U(i_0\dots i_m)$, define $K_{(m)}=\mathcal{M}^{(m)}\mathcal{F}_{\mathbf{r}}$, where $\mathcal{F}_{\mathbf{r}}\Phi$ is the Taylor expansion of Φ of order $\mathbf{r}$ at $\bar{x}(i_0\dots i_m)$, see [25, p. 249]. If $\mathcal{R}$ denotes the remainder of the Taylor expansion, we need to estimate (cf. formula (4.7) in [25]) the $C^{\mathbf{r},\alpha}$-norm of the function

$$(\mathcal{M}^{(m)}-K_{(m)})\Phi(x)=\sum_{i_0\dots i_{m-1}}\varphi^m(i_0\dots i_{m-1}x)\mathcal{R}(i_0\dots i_{m-1}x),\quad x\in U_{i_m}. \tag{9}$$

We note that the function on the right-hand side of (9) is, in fact, $\mathcal{L}^m\mathcal{R}$. We will estimate the norm of this function in terms of $\mathrm{r}_{\mathrm{sp}}(\mathcal{K},\mathcal{C}^0)$.

One can prove (see [10]) that

$$\begin{aligned}\mathrm{r}_{\mathrm{sp}}(\mathcal{K},\mathcal{C}^0) &= \lim_{m\to\infty}\max_{(x,\mathbf{v})\in\mathcal{X}}\left(\sum_{(y,\mathbf{u})\in F^{-m}\{(x,\mathbf{v})\}}\|\psi^m(y,\mathbf{u})\|\right)^{1/m}\\ &= \lim_{m\to\infty}\max_{(x,\mathbf{v})\in\mathcal{X}}\left(\sum_{i^{(m)}}\|\varphi^m(i^{(m)}x)\|\prod_j\|[Df^m(i^{(m)}x)]^{-1}v_j\|\right)^{1/m}.\end{aligned}$$

The main technical estimate that we need is as follows (we skip its proof).

Lemma 2.2. *Fix $k=0,1,\dots,\mathbf{r}$. For each $\epsilon>$ there exists $C=C(\epsilon)$ such that for all $m\in\mathbb{N}$ and $x_1\neq x_2$ one has:*

$$\begin{aligned}\max_{(v_1,\dots,v_k),\|v_j\|=1}\sum_{i^{(m)}}\|\varphi^m(i^{(m)}x_1)\|\prod_{j=1}^{k}\|[Df^m(i^{(m)}x_1)]^{-1}v_j\|\cdot\left[\frac{\mathrm{dist}(i^{(m)}x_1,i^{(m)}x_2)}{\mathrm{dist}(x_1,x_2)}\right]^{\mathbf{r}-k+\alpha}\\ \leq C\left(\mathrm{r}_{\mathrm{sp}}(\mathcal{K},\mathcal{C}^0)+\epsilon\right)^m,\end{aligned}$$

where the sum is taken over all admissible m-tuples.

An argument, based on (6) and Lemma 2.2, gives the estimate

$$\|\mathcal{L}^m\mathcal{R}\|_{C^{\mathbf{r},\alpha}} \le p(m)\,\mathrm{const}\,\|\Phi\|_{C^{\mathbf{r},\alpha}}\left(\mathrm{r_{sp}}(\mathcal{K},\mathcal{C}^0)+\epsilon\right)^m, \tag{10}$$

where $p(m)$ is a polynomial of degree m (cf. formula (4.8) in [25]). Indeed, for $k = 0, 1, \dots, \mathbf{r}$ the intertwining (6) gives $D^k\mathcal{L}^m\mathcal{R} = \mathcal{K}^m D^k\mathcal{R} + \mathcal{R}_k$, where the expression $\mathcal{R}_k$ contains a variety of terms with the derivatives of $\mathcal{R}$ up to the order $k-1$, but does not contain any k-derivatives of $\mathcal{R}$. Now, using Lemma 2.2, one can show that

$$\|\mathcal{K}^m D^k\mathcal{R}\|_\infty \le \mathrm{const}\,\|\Phi\|_{C^{\mathbf{r},\alpha}}\left(\mathrm{r_{sp}}(\mathcal{K},\mathcal{C}^0)+\epsilon\right)^m.$$

Since $\mathcal{R}_k$ contains $p(m)$ terms that satisfy similar estimates, one has (10) for $C^{\mathbf{r}}$-norm. Similarly for the Hölder norm, and the inequality "$\le$" in (7) follows.

The main difficulty in the proof of Theorem 2.1 is the inequality "$\ge$" in (7). To prove this inequality, we developed an idea of J. Mather [22] on localization of "almost-eigenfunctions" of the evolution operator T along trajectories of f, see also the proof of Theorem 4 in [10]. Note, that the approximate point spectrum $\sigma_{\mathrm{ap}}(\mathcal{K},\mathcal{C}^0)$ of the operator $\mathcal{K}$ on $\mathcal{C}^0$ contains the whole circle with the radius $\mathrm{r_{sp}}(\mathcal{K},\mathcal{C}^0)$ (see [10, Theorem 4]). By a simple rescaling, one could assume that $\mathrm{r_{sp}}(\mathcal{K},\mathcal{C}^0) = 1$ and $1 \in \sigma_{\mathrm{ap}}(\mathcal{K},\mathcal{C}^0)$. Thus, for any small $\epsilon > 0$ and big $N \in \mathbb{N}$ there exists $\Psi \in \mathcal{C}^0$ of unit norm such that $\|\mathcal{K}^N\Psi - \Psi\|_{\mathcal{C}^0} \le \epsilon$. Using this Ψ, we will construct an "almost-eigenfunction" g_ϵ for $\mathcal{L}$ on $C^{\mathbf{r},\alpha}$, supported in a finite piece of the f-trajectory of a small ball in X. That is, we will construct $g_\epsilon \in C^{\mathbf{r},\alpha}$ such that

$$\|\mathcal{L}g_\epsilon - g_\epsilon\|_{C^{\mathbf{r},\alpha}} \le \mathrm{const}\cdot\epsilon\cdot\|g_\epsilon\|_{C^{\mathbf{r},\alpha}}. \tag{11}$$

This shows that $1 \in \sigma_{\mathrm{ap}}(\mathcal{L}, C^{\mathbf{r},\alpha})$, and the inequality "$\ge$" in (7) follows.

For simplicity of the exposition, we give here the construction of g_ϵ only for $\mathbf{r} = 0$, that is, for the Hölder space $C^{0,\alpha}$; the case $\mathbf{r} = 1, 2, \dots$ requires more technicalities.

Fix a small $\epsilon > 0$. Choose $N = N(\epsilon)$ big enough. Note, that by Lemma 2.2 with $\mathrm{r_{sp}}(\mathcal{K},\mathcal{C}^0) = 1$ there exists $C = C(\epsilon^2)$ such that for any $x_1 \ne x_2$ one has:

$$\sum_{i^{(2N+1)}} \|\varphi^{2N+1}(i^{(2N+1)}x_1)\| \left[\frac{\mathrm{dist}(i^{(2N+1)}x_1, i^{(2N+1)}x_2)}{\mathrm{dist}(x_1,x_2)}\right]^\alpha \le C(\epsilon^2)(1+\epsilon^2)^{2N+1}. \tag{12}$$

Choose Ψ as described above and small $\delta = \delta(N,\epsilon)$. Fix $(x_0, v_0) \in \mathcal{X}$ such that $\|\Psi(x_0,v_0)\| \ge 1/2$. Choose a ball $B \ni x_0$ with radius d so small that the components of $f^{k-N}(B)$, $k = 0, \dots, 2N+1$, are disjoint. Choose a bump function $\beta : X \to [0,\delta]$ such that $\mathrm{supp}(\beta) \subset \varphi^{-N}(B)$, $\|\beta\|_{C^{0,\alpha}} \le 1$, $\beta(y_0) = 0$ for all $y_0 = i^{(N)}x_0 \in f^{-N}(x_0)$ and, in addition,

$$\lim_{t\to 0} t^{-\alpha}\beta\circ f^{-N}_{i^{(N)}}(x_0 + tv_0) = \|[Df^N(y_0)]^{-1}v_0\|^\alpha$$

for each branch $f^{-N}_{i^{(N)}}$ of the inverse map for f^N. Denote

$$h(y) = \beta(y)\Psi(y_0,u_0), \quad y \in f^{-N}_{i^{(N)}}(B), \quad y_0 = i^{(N)}x_0, \quad u_0 = [Df^N(i^{(N)}x_0)]^{-1}v_0.$$

We define the desired "almost-eigenfunction" for $\mathcal{L}$ as follows (cf. [19] and [10]):

$$g_\epsilon(x) = (1-\epsilon)^{|k-N|}(\mathcal{L}^k h)(x), \quad x \in \varphi^{k-N}(B), \quad k = 0, \dots, 2N+1.$$

Let us remark that this construction also works in the random case due to the assumption of the existence of symbolic dynamics for the expanding random system.

We claim that (11) holds for this g_ϵ. Indeed, the choice of g_ϵ leads to the estimate

$$\|g_\epsilon\|_{C^{0,\alpha}} \ge \sup_{0\le t\le d} t^{-\alpha}\|g_\epsilon(x_0+tv_0)\| \ge 1/4.$$

A calculation shows that

$$\|g_\epsilon - \mathcal{L}g_\epsilon\|_{C^{0,\alpha}} \le (1-\epsilon)^{N+1}\|h\|_{C^{0,\alpha}} + 2\epsilon\|g_\epsilon\|_{C^{0,\alpha}} + (1-\epsilon)^N\|\mathcal{L}^{2N+1}h\|_{C^{0,\alpha}}.$$

We note that

$$\|h\|_{C^{0,\alpha}} \le \|\beta\|_{C^{0,\alpha}}\|\Psi(y_0,v_0)\| \le 1 \le 4\|g_\epsilon\|_{C^{0,\alpha}}.$$

To give an estimate for $\|\mathcal{L}^{2N+1}h\|_{C^{0,\alpha}}$, we recall that $\|\mathcal{L}^{2N+1}h\|_{C^0} \le \delta\|\mathcal{L}^{2N+1}\|$ due to the choice of h and $\|\beta\|_{C^0} \le \delta$. To estimate the Hölder seminorm

$$\sup\{\|\mathcal{L}^{2N+1}h(x_1) - \mathcal{L}^{2N+1}h(x_2)\| \cdot [\mathrm{dist}(x_1,x_2)]^{-\alpha} : x_1 \ne x_2\},$$

we use (12). After some calculations one has

$$\|\mathcal{L}^{2N+1}h\|_{C^{0,\alpha}} \le \delta(C + \|\mathcal{L}^{2N+1}\|) + C(\epsilon^2)(1+\epsilon^2)^{2N+1}$$

with a constant C, which is known as soon as ϵ and N are fixed. Therefore, for sufficiently small δ, one has

$$\|g_\epsilon - \mathcal{L}g_\epsilon\|_{C^{0,\alpha}} \le 4(1-\epsilon)^{N+1}\|g_\epsilon\|_{C^{0,\alpha}} + 2\epsilon\|g_\epsilon\|_{C^{0,\alpha}} + 2(1-\epsilon)^N C(\epsilon^2)(1+\epsilon^2)^{2N+1}\cdot 4\|g_\epsilon\|_{C^{0,\alpha}}.$$

This gives the desired result (11) provided $N = N(\epsilon)$ was chosen large enough. □

To derive Theorem 1.1 from Theorem 2.1, we apply formula (4) for the operator $\mathcal{K}$:

$$\mathrm{r_{sp}}(\mathcal{K}, \mathcal{C}^0) = \exp\left(\sup_{\tilde\nu\in\mathrm{Erg}(F,\mathcal{X})}\{h_{\tilde\nu}(F) + \tilde\lambda_{\tilde\nu}\}\right),$$

where $h_{\tilde\nu}(F)$ is the entropy of F and $\tilde\lambda_{\tilde\nu}$ is the largest Lyapunov-Oseledets exponent for the cocycle $\{\psi^k\}$ over F with respect to an F-ergodic measure $\tilde\nu$ on $\mathcal{X}$. Recall, that by the Multiplicative Ergodic Theorem, for each $\nu \in \mathrm{Erg}(f,X)$ there exists a full ν-measure subset X_ν of X such that for each point x from this subset there exists a filtration of the tangent space $T_xX = W_x^1 \supset \ldots \supset W_x^{n'} \supset W_x^{n'+1} = \{0\}$, $n' \le n$ such that the *exact* Lyapunov-Oseledets exponent $\chi(x,v) = \lim_{k\to\infty} k^{-1}\log\|Df^k(x)v\|$ for the cocycle $\{Df^k\}$ exists and is equal to χ_ν^i if and only if $v \in W_x^i \setminus W_x^{i+1}$. (Note that here we had to use the one-sided version of the Multiplicative Ergodic Theorem, as the linear cocycle $\{Df^k\}$ is defined only over a non-invertible skew-product transformation Θ.) Let $E_x^i = S^{n-1}\cap W_x^i$, $i = 1,\ldots,n'$ be the trace of the Oseledets subbundle W^i on the unit sphere in the tangent space T_xX.

To express the Lyapunov-Oseledets exponents $\tilde\lambda_{\tilde\nu}$ in terms of λ_ν and χ_ν, we will use a description of all measures $\tilde\nu$ on $\mathcal{X}$, that are F-ergodic for the projectivization F of f (see [2, Chapter 6]). This description says that every $\tilde\nu \in \mathrm{Erg}(F,\mathcal{X})$ has a form $\tilde\nu = \nu\times\mu_0\times\mu_1\times\ldots\times\mu_{\mathbf{r}}$ (for $\alpha > 0$, and with no term μ_0 for $\alpha = 0$) where μ_k are supported on the Oseledets subbundles E^i. We claim, that

$$\sup_{\tilde\nu\in\mathrm{Erg}(F,\mathcal{X})}\{h_{\tilde\nu}(F) + \tilde\lambda_{\tilde\nu}\} = \sup_{\nu\in\mathrm{Erg}(f,X)}\{h_\nu(f) + \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}, \tag{13}$$

where λ_ν is the largest Lyapunov-Oseledets exponent for the cocycle $\{\varphi^k\}$, and $\chi_\nu = \chi_\nu^{n'}$ is the smallest Lyapunov-Oseledets exponent for the cocycle $\{Df^k\}$.

To prove the inequality "$\le$" in (13), fix any $\tilde\nu \in \mathrm{Erg}(F,\mathcal{X})$. Note (see, e.g., [20], Lemma 4.14) that $h_\nu(f) = h_{\tilde\nu}(F)$ for $\nu = \mathrm{proj}\,\tilde\nu$, the projection of the measure $\tilde\nu$ on X. Pick a point $(x,\mathbf{v}) \in \mathcal{X}_{\tilde\nu}$ such that there exists the exact Lyapunov-Oseledets exponent $\tilde\lambda_{\tilde\nu} = \lim k^{-1}\log\|\psi^k(x,\mathbf{v})\|$ for the cocycle $\{\psi^k\}$, and, at the same time, $x \in X_\nu$, that is, there exist the exact Lyapunov-Oseledets exponents $\lambda_\nu = \lim k^{-1}\log\|\varphi^k(x)\|$ for $\{\varphi^k\}$ and $\chi_\nu(x, v_j)$ for $\{Df^k\}$. Then

$$\begin{aligned}
\tilde{\lambda}_{\tilde{\nu}} &= \lim_{k\to\infty} k^{-1}\log\|\varphi^k(x)\| - \sum_{j=1}^{\mathbf{r}} \lim_{k\to\infty} k^{-1}\log\|Df^k(x)v_j\| - \alpha \lim_{k\to\infty} k^{-1}\log\|Df^k(x)v_0\| \\
&\le \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu, \quad \mathbf{v} = (v_0, v_1, \dots, v_{\mathbf{r}}), \qquad (14)
\end{aligned}$$

since χ_ν is the *smallest* Lyapunov-Oseledets exponent for $\{Df^k\}$.

To prove the inequality "$\ge$" in (13), fix any $\nu \in \mathrm{Erg}(f, X)$. Pick a measure μ supported on $E^{n'}$, the trace of the Oseledets subbundle corresponding to the *smallest* exponent $\chi_\nu = \chi_\nu^{n'}$. Form $\tilde{\nu} = \nu \times \mu \times \ldots \times \mu$. Then $h_{\tilde{\nu}}(F) = h_\nu(f)$ as before. Also, for any $(x, \mathbf{v}) \in \mathcal{X}_{\tilde{\nu}}$, $\tilde{\nu}(\mathcal{X}_{\tilde{\nu}}) = 1$ where the exact limits $\tilde{\lambda}_{\tilde{\nu}}$, λ_ν, $\chi_\nu(x, v_j)$ exist, one has the *equality* in (14), since each of the vectors v_j is a vector from $E_x^{n'}$. This concludes the proof of Theorem 1.1.

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