Renormalization Group Pathologies and the Definition of Gibbs States

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Dedicated to the memory of Roland L. Dobrushin

Abstract

We show that the so-called Renormalization Group pathologies in low temperature Ising models are due to the fact that the renormalized Hamiltonian is defined only almost everywhere (with respect to the renormalized Gibbs measures). We construct this renormalized Hamiltonian using a Renormalization Group method developed for random systems and we show that the pathologies are analogous to Griffiths' singularities.

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1 Introduction.

The Renormalization Group (RG) has been one of the most useful tools of theoretical physics during the past decades. It has led to an understanding of universality in the theory of critical phenomena and of the divergences in quantum field theories. It has also provided a nonperturbative calculational framework as well as the basis of a rigorous mathematical understanding of these theories.

Even though the RG was primarily devised for the study of (approximatively) scale invariant situations such as statistical mechanical models at the critical point, it was found useful in the mathematical analysis of problems that were not critical but that nevertheless were "multiscale": for example, first order phase transitions in regular [8] and disordered [2] spin systems. The spin variables give an appropriate representation of these systems at and above the critical point; however, at low temperatures, these models are most naturally expressed in terms of contours (domain walls) that separate the different ground states. To apply the RG method, one inductively sums over the small scale contours, producing an effective theory for the larger scale contours.

However, the real power of the RG both theoretically and in most applications has been to realize it as a map between Hamiltonians, and the latter are usually expressed in terms of the spin variables. So, one would like to be define rigorously such a map, but this program has met some difficulties. It was observed in simulations [15] that the RG transformation seems, in some sense, "discontinuous" as a map between spin Hamiltonians. These observations led subsequently to a rather extensive discussion of the so-called "pathologies" of Renormalization Group Transformations (RGT): van Enter, Fernandez and Sokal have shown [30, 31] that, first of all, the RG transformation is not really discontinuous. But they also show, using results of Griffiths and Pearce [12, 13] and of Israel [16], that, roughly speaking, there does not exist a renormalized Hamiltonian for many RGT applied to Ising-like models at low temperatures and in some cases even at high temperatures (in particular in a large external field, see [28, 29]).

More precisely, van Enter, Fernandez and Sokal consider various real-space RGT (block spin, majority vote, decimation) that can be easily and rigorously defined as maps acting on measures (i.e. on probability distributions of the infinite volume spin system). The problem occurs when one tries to rewrite this map in terms of Hamiltonians. Hamiltonians are usually expressed as sums of (n-body) interactions of the spins that have sufficient decay properties so as to define infinite volume Gibbs measures. If we start with a Gibbs measure μ corresponding to a given Hamiltonian H, one can easily define the renormalized measure μ'. The problem then is to reconstruct a renormalized Hamiltonian H' (i.e. a set of interactions) for which μ' is a Gibbs measure. Although this is trivial in finite volume, it is not so in the thermodynamic limit, and it is shown in [31] that, in many cases at low temperatures, even if H contains only nearest-neighbour interactions, there is no absolutely summable interaction (defined in (2.7) below) giving rise to a Hamiltonian H' for which μ' is a Gibbs measure.

What is one supposed to think about these pathologies? From a practical point of view, we understand the reason why there *might* be problems: one is using the wrong variables, i.e. the spin variables rather than the contours variables. The fact that the usefulness of the RG method depends crucially on choosing the right variables has been known for a long time. The "good" variables should be such that a single RG transformation, which can be interpreted as solving the statistical mechanics of the small scale variables with the large ones kept fixed, should be "noncritical" i.e. should be away from the parameter regions where phase transitions occur. This is true in particular in the low temperature region if one uses the contour variables. In all the cases where pathologies were found, they were due to the fact that a single RG transformation involves a system that has a phase transition for some fixed values of the large scale variables.

However, from the theoretical point of view, we believe that it is interesting to see just how pathological the renormalized Hamiltonians are. This question is related to another one, of independent interest: when is a measure Gibbsian for some Hamiltonian? For example, Schonmann showed [27] that, when one projects a Gibbs measure (at low temperatures) to the spins attached to a lattice of lower dimension, the resulting measure is not, in general, Gibbsian.

There has been an extensive investigation of this problem of pathologies and Gibbsianness. Martinelli and Olivieri [25, 26] have shown that, in a non-zero external field, the pathologies disappear after sufficiently many decimations . Fernandez and Pfister [7] study the set of configurations that are responsible for those pathologies. They give criteria which hold in particular in a non-zero external field, and which imply that this set is of zero measure with respect to the renormalized measures. Following the work of Kennedy [17], several authors [14, 1, 3] analyze the absence of pathologies near the critical point.

Our goal in this paper is to further clarify the situation: following an idea of Dobrushin [5], we prove that, for several examples considered in [31], the renormalized Hamiltonian actually exists, but the corresponding interaction satisfies a weaker summability condition than the one used in [31]. Our condition ((2.11) below) is however sufficient to define Gibbs measures, in a way that is similar to the one used before for "unbounded spins". Thus in a sense, the pathologies are not there in the end, and the renormalized measures are Gibbsian. However, it turns out that the renormalized Hamiltonian is only defined almost everywhere with respect to the renormalized measure: it becomes "pathological" on a set of measure zero that in particular includes the configurations used in [31] to exhibit the pathologies. Our result is similar to the one of Maes and Vande Velde [24] on the Schonmann example of the measures projected on a lattice of lower dimension, except that, in our case, we show that the two renormalized states are Gibbsian with respect to the same Hamiltonian (while this question is left open in [24]). However, as pointed out to us by A. Sokal, with our definition of Gibbs states, the interaction defining the Hamiltonian is not unique, while it is essentially unique (up to physical equivalence) with the usual definition.

Of course, we do not claim to justify the RG method in general, and, besides, our results do not hold near the critical point. We do not even prove that, upon iteration, the RGT drives the Hamiltonians to a trivial fixed point, although this can probably be done, in some of the examples discussed below. But we do clarify the nature of the "pathologies".

Our proof is based on the following idea: we consider the spins distributed by the renormalized measures as random external fields acting on the original system, and we apply the methods of [2] to construct the renormalized Hamiltonian. So, we use the Renormalization Group in order to justify the Renormalization Group. As in all random systems, there is a set of measure zero of "bad" configurations of the random fields, for which "typical" results (e.g. decay of correlations) do not hold, and which are responsible for the Griffiths' singularities ([11]). We shall see below that the pathologies used in [31] are actually due to those bad configurations. But, once one excludes this set of measure zero, the renormalized system has a nice Hamiltonian, with rapidly decaying interactions.

2 Results.

We consider the nearest-neighbour Ising model on $\mathbf{Z}^d$, $d \geq 2$, at β large, for simplicity. To each $i \in \mathbf{Z}^d$, we associate a variable $\sigma_i \in \{-1, +1\}$, and the (formal) Hamiltonian is

$$-\beta H = \beta \sum_{\langle ij \rangle} (\sigma_i \sigma_j - 1) \tag{1}$$

where $\langle ij \rangle$ denotes a nearest-neighbour pair and β is the inverse temperature. At low temperatures, there are two extremal translation invariant Gibbs measures corresponding to (1), μ_+ and μ_-. To define our RGT, let $\mathcal{L} = (b\mathbf{Z})^d$, $b \in \mathbf{N}$, $b \geq 2$ and cover $\mathbf{Z}^d$ with disjoint *b-boxes* $B_x = B_0 + x$, $x \in \mathcal{L}$ where B_0 is a box of size b centered around 0. Associate to each $x \in \mathcal{L}$ a variable $s_x \in \{-1, +1\}$, denote by σ_A an element of $\{-1, +1\}^A$, for $A \subset \mathbf{Z}^d$, $|A| < \infty$, and introduce the probability kernels

$$T_x = T(\sigma_{B_x}, s_x)$$

for $x \in \mathcal{L}$, i.e. T_x satisfies

$$\begin{aligned} &1) \quad T(\sigma_{B_x}, s_x) \geq 0 \\ &2) \quad \sum_{s_x} T(\sigma_{B_x}, s_x) = 1 \end{aligned} \tag{2}$$

For any measure μ on $\{-1, +1\}^{\mathbf{Z}^d}$, we denote by $\mu(\sigma_A)$ the probability of the configuration σ_A.

Definition. Given a measure μ on $\{-1, +1\}^{\mathbf{Z}^d}$, the *renormalized measure* μ' on $\Omega = \{-1, +1\}^{\mathcal{L}}$ is defined by:

$$\mu'(s_A) = \sum_{\sigma_{B_A}} \mu(\sigma_{B_A}) \prod_{x \in A} T(\sigma_{B_x}, s_x) \tag{3}$$

where $B_A = \cup_{x \in A} B_x$, $A \subset \mathcal{L}$, $|A| < \infty$, and $s_A \in \Omega_A = \{-1, +1\}^A$.

It is easy to check, using 1) and 2), that μ' is a measure. We shall call the spins σ_i the *internal* spins and the spins s_x the *external* ones (they are also sometimes called the block spins).

We shall need two other conditions on T: we assume that T is symmetric:

$$T(\sigma_{B_x}, s_x) = T(-\sigma_{B_x}, -s_x) \tag{4}$$

and that

$$0 \leq T(\sigma_{B_x}, s_x) \leq e^{-\beta} \tag{5}$$

if $\sigma_i \neq s_x \quad \forall i \in B_x$.

The condition (5) means that there is a coupling which tends to align s_x and the spins in the block B_x. It would be more natural to have, instead of (5), $0 \leq T \leq \epsilon$ (with ϵ independent of β but small enough). However, assuming (5) simplifies the proofs. Note that (2, 4, 5) imply that

$$\overline{T} \equiv T(\{\sigma_i = +1\}_{i \in B_x}, +1) = T(\{\sigma_i = -1\}_{i \in B_x}, -1) \geq 1 - e^{-\beta} \geq \frac{1}{2} \tag{6}$$

for β large. The usual transformations, discussed in [31], sect. 3.1.2, like decimation or majority rule, obviously satisfy (4,5). The Kadanoff transformation satisfies (5) for p large. Our results could be extended to the block spin transformations (where s_x does not belong to $\{-1, +1\}$).

Let us now summarize the main result of [31]. Consider *interactions* $\Phi = (\Phi_X)$, which are families of functions

$$\Phi_X : \Omega_X \to \mathbf{R},$$

indexed by $X \subset \mathcal{L}$, $|X| < \infty$. Assume that Φ is

a) translation invariant:

$$\Phi_X = \Phi_{X+x} \;\; \forall x \in \mathcal{L}$$

b) uniformly absolutely summable:

$$\sum_{X \ni 0} \|\Phi_X\|_\infty < \infty \tag{7}$$

This set of interactions obviously forms a Banach space, with the norm (7) (note that our terminology differs slightly from the one of [31]: we add the word "uniformly" to underline the difference with respect to condition (11) below) .

Then, one defines, $\forall V \subset \mathcal{L}$, $|V| < \infty$, the Hamiltonian

$$H(s_V|\bar{s}_{V^c}) = -\sum_{X \cap V \neq \emptyset} \Phi_X(s_{X \cap V} \vee \bar{s}_{X \cap V^c}) \tag{8}$$

where $s_V \in \Omega_V$,$\bar{s}_{V^c}$ is the restriction to V^c of $\bar{s} \in \Omega$ and, for $X \cap Y = \emptyset$, $s_X \vee s_Y$ denotes the obvious configuration in $\Omega_{X \cup Y}$. The condition (7) implies that H is a *continuous* function of s_V and $\bar{s}_{V^c}$ (in the product topology), and that

$$\pi_V^\Phi(s_V|\bar{s}_{V^c}) = Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V|\bar{s}_{V^c})) \tag{9}$$

(where $Z^{-1}(\bar{s}_{V^c})$ is the obvious normalization factor) defines a quasilocal specification in the sense of [31].

Definition. μ is a *Gibbs measure* for Φ if the conditional probabilities satisfy, $\forall V \subset \mathcal{L}$, $|V|$ finite, $\forall s_V \in \Omega_V$,

$$\mu(s_V|\bar{s}_{V^c}) = \pi_V^{\Phi}(s_V|\bar{s}_{V^c}) \tag{10}$$

μ a.e.

Note that we left out the inverse temperature β. When we refer to β below, we mean the inverse temperature of the original Ising model (1), before acting with the RGT.

The main result of [31] is that, for a variety of RGT, there is no interaction satisfying a) and b) above for which μ'_+ or μ'_- are Gibbs measures.

However, we observe that, in order to define $H(s_V|\bar{s}_{V^c})$ and $\pi_V^{\Phi}(s_V|\bar{s}_{V^c})$, it is not necessary to assume (7); it is enough to assume the existence of a tail set $\overline{\Omega} \subset \Omega$ on which the following *pointwise* bounds hold:

b') $\overline{\Omega}$-pointwise absolutely summable:

$$\sum_{X \ni x} |\Phi_X(s_X)| < \infty \quad \forall x \in \mathcal{L}, \forall s \in \overline{\Omega}. \tag{11}$$

We shall therefore enlarge the class of "allowed" interactions by dropping the condition (7) and assuming (11) instead. Actually, a similar setup was already used in the theory of "unbounded spins" with infinite range interactions, see [9].

With this condition, we can define the specification $\pi^{\Phi,\overline{\Omega}}$ by

$$\pi_V^{\Phi,\overline{\Omega}}(s_V|\bar{s}_{V^c}) \;=\; \begin{cases} Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V|\bar{s}_{V^c})) & \text{for } \bar{s}_{V^c} \in \overline{\Omega} \\ 0 & \text{for } \bar{s}_{V^c} \notin \overline{\Omega} \end{cases} \tag{12}$$

and then define Gibbs measures for the pair $\Phi, \overline{\Omega}$ as follows:

Definition. Given a tail set $\overline{\Omega} \subset \Omega$, μ is a *Gibbs measure* for the pair $(\Phi, \overline{\Omega})$ if $\mu(\overline{\Omega}) = 1$, and there exists a version of the conditional probabilities that satisfy, $\forall V \subset \mathcal{L}$, $|V|$ finite, $\forall s_V \in \Omega_V$,

$$\mu(s_V|\bar{s}_{V^c}) = \pi_V^{\Phi,\overline{\Omega}}(s_V|\bar{s}_{V^c}) \tag{13}$$

$\forall \bar{s} \in \overline{\Omega}$.

Since conditional probabilities are defined almost everywhere, this definition is very similar to the usual one. However, when condition (7) holds, the conditional probabilities can be extended everywhere, and are continuous, which is not the case here. We can now state our main result:

Theorem 1 *Under assumptions (4, 5) on T, and for β large enough, there exist disjoint translation-invariant (hence, tail) sets $\overline{\Omega}_+, \overline{\Omega}_- \subset \Omega$ such that $\mu'_+(\overline{\Omega}_+) = \mu'_-(\overline{\Omega}_-) = 1$ and an interaction Φ satisfying a) and b') with $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ such that μ'_+ and μ'_- are Gibbs measures for the pair $(\Phi, \overline{\Omega})$.*

Remarks.

1. It should be emphasized that we have the *same* interaction for both μ'_+ and μ'_- (which, moreover, has stronger decay properties than (11), see (3.5) below). Thus, the result is different from the one of Maes and Vande Velde [23] on the projected Gibbs measure.

2. One can distinguish different type of models where "pathologies" occur. Our framework is the one where the pathologies are the weakest. In the case of [23], it is an open question whether one can take the *same* interaction for μ'_+ and μ'_-. But if one combines projection with enough decimation, as in [20], then one knows that each of the resulting states is Gibbsian (in the strongest sense, i.e. with interactions satisfying (7)), but for different interactions. This in turn implies that non-trivial convex combinations of these states are not quasilocal everywhere, see [32], where other examples of "robust" non-Gibbsianness can be found.

3. Note that in the theory of "unbounded spins" with long range interactions, a set $\overline{\Omega}$ of "allowed" configurations has to be introduced, where a bound like (11) holds [9]. But here, of course, contrary to the unbounded spins models, each $\|\Phi_X\|_\infty$ is finite. We even have the bound $\|\Phi_X\|_\infty \leq C\beta|X|$, see (3.4) below.

4. The set $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ is not "nice" topologically: e.g. it has an empty interior (in the usual product topology). Besides, our effective potentials do not belong to a natural Banach space like the one defined by (7). However, this underlines the fact that the concept of Gibbs measure is a measure - theoretic notion and the latter often do not match with topological notions.

5. We can regard $\{s_x\}$ as a set of quenched external fields coupled to the spins σ by the probability kernels T. The distribution of $\{s_x\}$ is given by μ'_+ or μ'_-. Then, as in most disordered systems, there is a set of "good" configurations of the random fields ($\overline{\Omega}$ here) for which the system with the spins σ has good clustering properties. And, implicitely, we shall use the latter to construct our Hamiltonian. This is why part of the proof below uses the techniques of [2, 8]. Of course, there are also "bad" configurations of the random fields for which the σ spins do not have good clustering properties, and those are essentially the ones used in [31] to prove thatthere is no absolutely summable potential for which μ'_+, μ'_- are Gibbsian.

6. To illustrate the role of the set $\overline{\Omega}$, consider the (trivial) case, where $b = 1$, and $T = \delta(\sigma_i - s_x)$ with $i = x$, i.e. the "renormalized" system is identical to the original one (this example was suggested to us by A. Sokal). Then $\overline{\Omega}$, as constructed in our proof, will be the set of configurations such that all the (usual) Ising contours are finite and each site is surrounded by at most a finite number of contours. When $X =$ a contour γ (considered as a suitable set of sites), we let

$$\Phi_X(s_X) = 2\beta|\gamma| \tag{14}$$

for $s_X =$ a configuration making γ a contour, and $\Phi_X(s_X) = 0$ otherwise. Obviously, this Φ satisfies (11) but not (7). One can write $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$, according to the values of the spins in the infinite connected component of the complement of the contours. It is easy to see that μ^+, μ^- are indeed, at low temperatures, Gibbs measures (in the sense considered here) for this new interaction: a Peierls argument shows that $\mu^+(\overline{\Omega}_+) = \mu^-(\overline{\Omega}_-) = 1$, and for $s \in \overline{\Omega}$ the (formal) Hamiltonian (2.1) is $\beta H = 2\beta \sum_\gamma |\gamma|$. Actually, we shall prove the Theorem by using a kind of perturbative analysis around this example. Of course, in this example one could alternatively take $\overline{\Omega} = \Omega$ and $\Phi =$ the original nearest-neighbor interaction; this shows the *nonuniqueness* of the pair $(\Phi, \overline{\Omega})$ in our generalized Gibbs-measure framework.

3 Outline of the proof.

3.1 The main Propositions.

We shall construct the interaction Φ inductively. We shall now give the strategy and indicate the different steps of the proof.

Consider μ'_+; using (2.3), one sees that the conditional probabilities $\mu'_+(s_V|\bar{s}_{V^c})$ can be obtained through the following limit, if it exists:

$$\mu'_+(s_V|\bar{s}_{V^c}) = \lim_{\Lambda_2 \uparrow \mathcal{L}} \lim_{\Lambda_1 \uparrow \mathbf{Z}^d} \frac{Z^+_{\Lambda_1}(s_V \vee \bar{s}_{V^c \cap \Lambda_2})}{\sum_{\tilde{s}_V} Z^+_{\Lambda_1}(\tilde{s}_V \vee \bar{s}_{V^c \cap \Lambda_2})}. \tag{1}$$

Where

$$Z^+_{\Lambda_1}(s_{\Lambda_2}) = \sum_{\sigma_{\Lambda_1}} \prod_{x \in \Lambda_2} T_x e^{-\beta H(\sigma_{\Lambda_1}|+)} \tag{2}$$

where $H(\sigma_{\Lambda_1}|+)$ is defined as in (2.1), but with the sum restricted to $i \in \Lambda_1$, with $\sigma_j = +1$, $\forall j \in \Lambda_1^c$. The conditional probabilities $\mu'_-(s_V|\bar{s}_{V^c})$ can be obtained by similar formulas, with $+$ replaced by $-$.

We shall prove

Proposition 1 *Under assumptions (2.4, 2.5), there exists, for all $\eta > 0$, a $\bar{\beta} < \infty$, such that, $\forall \beta \geq \bar{\beta}$ in (2.1), (2.5), there exists a set $\overline{\Omega} \subset \Omega$, $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$, and an interaction Φ, such that, $\forall s^1, s^2 \in \overline{\Omega}_+$, $\forall V \subset \mathcal{L}$, $|V| < \infty$*

$$\lim_{\Lambda_2 \uparrow \mathcal{L}} \lim_{\Lambda_1 \uparrow \mathbf{Z}^d} \frac{Z^+_{\Lambda_1}(s^1_{\Lambda_2})}{Z^+_{\Lambda_1}(s^2_{\Lambda_2})} = \exp\Big(\sum_{X \cap V \neq \emptyset} \Big(\Phi_X(s^1_X) - \Phi_X(s^2_X)\Big)\Big) \tag{3}$$

if $s^1_x = s^2_x$ $\forall x \notin V$. The functions Φ_X satisfy:

$$\|\Phi_X\|_\infty \leq c\beta|X| \tag{4}$$

for some $c < \infty$, and, $\forall x \in \mathcal{L}$, $\forall s \in \overline{\Omega}$,

$$\sum_{X \ni x} |\Phi_X(s_X)| \exp(d(X)^{1-\eta}) = C(x,s) < \infty \tag{5}$$

where $d(X) = \mathrm{diam}(X)$. A formula similar to (3) holds with $+$ replaced by $-$ for all s^1, $s^2 \in \overline{\Omega}_-$.

Remark.

The factor $\exp(d(X)^{1-\eta})$ is not optimal; we could replace it by $\exp(d(X)^{1-\eta}+|X|^{1-\eta})$; but we expect $|\Phi_X(s_X)|$ to decay as $\exp(-d(X)-|X|)$.

Proposition 2 $\mu'_+(\overline{\Omega}_+) = \mu'_-(\overline{\Omega}_-) = 1$.

Clearly, (1) and these two Propositions imply Theorem 1.

Remark. The set $\overline{\Omega}$ will be of measure zero for Gibbs measures which are not convex combinations of μ'_+ and μ'_-, such as the non-translation-invariant Gibbs measures with interfaces that exist for $d \geq 3$ [4]. An open question is to find another set $\tilde{\Omega}$, of μ' measure one, for the renormalized measure corresponding to a non-translation invariant Gibbs measure μ (for $d \geq 3$), and an interaction with respect to which μ' is Gibbsian.

3.2 The contour representation.

To prove these propositions, we shall use a "contour", or "polymer" representation of $Z^+_{\Lambda_1}(s_{\Lambda_2})$. But, since we regard the external spins as random fields acting on the internal ones, we shall first define the sets where the external spins are "bad", namely where they change sign and exert opposite influences on the internal spins. Let

$$D(s) = \bigcup \{B_x | x \in \mathcal{L},\ \exists y \in \mathcal{L}, |x-y| = b, s_x \neq s_y\} \tag{6}$$

where we use throughout the paper:

$$|x| = \max_{i=1,\cdots,d} |x_i|. \tag{7}$$

$D(s)$ is determined by the set of (ordinary) contours of the configuration s. Define also

$$D_{\Lambda_2} \equiv D^+(s_{\Lambda_2}) \tag{8}$$

with

$$D^+(s_{\Lambda_2}) = \bigcup \{B_x | x \in \Lambda_2,\ \exists\, y \in \Lambda_2, |x-y| = b, s_x \neq s_y\} \bigcup \{B_x | x \in \partial\Lambda_2, s_x = -1\} \tag{9}$$

where

$$\partial\Lambda_2 = \{x \in \Lambda_2, d(x, \Lambda_2^c) = b\}. \tag{10}$$

(d is the distance corresponding to (7)). $D^-(s_{\Lambda_2})$ is defined similarly, with $s_x = +1$ instead of $s_x = -1$.

Now, we introduce the "contours" of the internal spins: let, for each term in (2)

$$\underline{\Gamma}(\sigma_{\Lambda_1}|+) = \bigcup \{B_x \subset \Lambda_1 | \forall i \in B_x,\ \sigma_i \neq s_x \text{ or } \exists \langle ij \rangle, i \in B_x\ \sigma_i \neq \sigma_j\} \bigcup D_{\Lambda_2} \tag{11}$$

where $\sigma_i = +1$ for $i \notin \Lambda_1$.

So $\underline{\Gamma}$ includes the boxes B_x where all the internal spins differ from s_x, and the boxes intersected by the usual contours of the configuration σ_{Λ_1}, plus all the sites in D_{Λ_2}. Of course, these sets are not disjoint: in a box B_x belonging to D_{Λ_2}, $x \notin \partial\Lambda_2$, either all internal spins in B_x differ from s_x or all internal spins differ from $s_y \neq s_x$ in some box B_y with $|x-y| = b$ or there is a pair $\langle ij \rangle$ with $\sigma_i \neq \sigma_j$ in $B_x \cup B_y$. We include D_{Λ_2} in the contours, and we coarse-grain them into b-boxes for convenience.

Note that in all boxes B_x not contained in $\underline{\Gamma}$, the internal spins are constant (and, in Λ_2, are equal to the external ones). So, one may decompose $\underline{\Gamma} = \bigcup \underline{\gamma}$ into connected components (a subset Y of $\mathbf{Z}^d$ is *connected* if any two points of Y can be joined by a path (i_α), with $|i_\alpha - i_{\alpha+1}| = 1$), and one may define contours as pairs $\gamma = (\underline{\gamma}, \sigma(\gamma))$ where $\underline{\gamma}$ is the *support* of the contour, and $\sigma(\gamma)$ is a configuration $\{\sigma_i(\gamma)\}_{i \in \underline{\gamma}^c}$, $\sigma_i(\gamma) = +1$ or -1, defined on the complement of $\underline{\gamma}$, which is constant on the connected components of $\underline{\gamma}^c$ (this notion of contour will be slightly generalized below).

Definition. A family of contours Γ is *compatible* if the supports of the contours are mutually disjoint: $\underline{\gamma}_1 \cap \underline{\gamma}_2 = \emptyset$, and if their signs match and agree with the boundary conditions on Λ_1.

So, the notion of compatibility is as for the usual Ising contours, and if Γ is compatible, $\sigma_i(\Gamma)$ is unambiguously defined, $\forall i \in \underline{\Gamma}^c$.

Definition. A family of contours Γ is *s-compatible* if Γ is compatible and, moreover,

$$\sigma_x(\Gamma) = s_x \quad \forall x \in (\Lambda_2 \cap \mathcal{L}) \backslash \underline{\Gamma}$$

The notion of s-compatibility imposes a constraint due to the external spins. For example, if all the external spins have value $+1$, a single (Ising) contour surrounding Λ_2, with $+$ spins outside and $-$ spins inside, is compatible but is not s-compatible.

One may write:

$$Z^+_{\Lambda_1}(s_{\Lambda_2}) = (\overline{T})^{|\Lambda_2|} \sum_{\Gamma \supset D_{\Lambda_2}} \rho(\Gamma) \tag{12}$$

where the sum runs over s-compatible families of contours with $\underline{\Gamma} \subset \Lambda_1$, $\rho(\Gamma) = \prod_{\gamma \in \Gamma} \rho(\gamma)$ with $\rho(\gamma) = 0$ if $\underline{\gamma}$ does not contain the connected components of D_{Λ_2} that it intersects and if $\sigma_x(\Gamma) \neq s_x$, for some x with B_x adjacent to $\underline{\gamma}$; it equals

$$\rho(\gamma) = \sum_{\sigma_{\underline{\gamma}}}^{\star} \prod_{x \in \underline{\gamma} \cap \mathcal{L}} \frac{T_x}{\overline{T}} \exp(-\beta H(\sigma_{\underline{\gamma}} | \sigma(\gamma)) \tag{13}$$

otherwise. Here $H(\sigma_{\underline{\gamma}}|\sigma(\gamma))$ is defined in the same way as (2.8) but with the Hamiltonian (2.1); $\sigma_i(\gamma)$, $i \notin \underline{\gamma}$, is fixed by the signs associated to the complement of $\underline{\gamma}$, and the sum $\sum^\star$ runs over spin configurations $\sigma_{\underline{\gamma}}$ such that γ is a contour of the configuration $\sigma_{\underline{\gamma}} \vee \sigma(\gamma)$. (and $\rho(\gamma) = 0$ if the sum is empty); $\rho(\gamma)$ is a function of the external spins $\{s_x | x \in \Lambda_2, d(x, \underline{\gamma} \cap \mathcal{L}) \leq 2b\}$, since it vanishes unless $\underline{\gamma}$ contains the connected components of D_{Λ_2} that it intersects (observe that the property, for a set D_i, to be a connected component of D_{Λ_2} depends on $\{s_x | x \in \Lambda_2, d(x, D_i \cap \mathcal{L}) \leq 2b\}$). The fact that we sum in (12) over s-compatible families introduces a global constraint on the set of contours which will be characterized explicitly in Lemma 4.1 below.

It is easy to see that:

$$0 \leq \rho(\gamma) \leq \exp(-\beta_0|\underline{\gamma}\backslash(D_{\Lambda_2} \cap \partial\Lambda_2)|) \tag{14}$$

where $\beta_0 = \beta_0(b, \beta)$ depends on the choice of b in the definition of $\mathcal{L}$ and goes to infinity as β in (2.1) goes to infinity. To prove (14), observe that one gets a factor $e^{-2\beta}$ from the Hamiltonian (2.1) for each pair $\langle ij \rangle$ with $\sigma_i \neq \sigma_j$, and a factor $e^{-\beta}$ for each box B_x such that $\forall i \in B_x$, $\sigma_i \neq s_x$, from our assumption (2.5) on the probability kernels T_x; for the boxes B_x in D_{Λ_2}, but with $x \notin \partial\Lambda_2$, we use the observation made above that in or near each such box, either there is a pair $\langle ij \rangle$ with $\sigma_i \neq \sigma_j$ or all the internal spins differs from the external one. Finally, using the lower bound (2.6) on $\overline{T}$, we get (14) for

$$\beta_0 = c\beta, \tag{15}$$

with some $c > 0$ (which has to be taken small enough because, in the above argument, we implicitely assigned the same factor $e^{-2\beta}$ or $e^{-\beta}$ to different sites of $\underline{\gamma}$).

3.3 Renormalization.

Let us now introduce the coarse-grained description of the system on which our inductive scheme is based. Let $L > b$ be some odd integer (which will be taken large enough below). Divide $\mathbf{Z}^d$ into disjoint $L-$ boxes $\{i| \, |i - Lx| < \frac{L}{2}\}$ where $x \in \mathbf{Z}^d$, i.e. each $i \in \mathbf{Z}^d$ can be written as $i = Lx + j$ with $x \in \mathbf{Z}^d$ and $|j_\mu| < \frac{L}{2}, \mu = 1, \cdots, d$ (here and below, we use the letters x, y to denote sites in the new lattices $\mathbf{Z}^d$). We define $[L^{-1}i] = x$ and the $L-$box of sites i such that $[L^{-1}i] = x$ is denoted by Lx. Also for a set $Y \subset \mathbf{Z}^d, [\frac{Y}{L}] = \{[L^{-1}i]|i \in Y\}$ while $LY = \cup\{Lx|x \in Y\}$. We use a similar notation for all scales $L^n, n = 1, 2, \cdots$.

We shall now describe D_{Λ_2} and $D(s)$ on these different coarse-grained scales. Let us introduce the random variables $N_x^n = N_x^n(s)$, $n = 0, 1, 2 \cdots$, defined inductively as follows:

$$\begin{aligned} N_x^0 &= 2d \quad \text{if} \quad x \in D(s) \\ N_x^0 &= 0 \quad \text{otherwise.} \end{aligned} \tag{16}$$

$$N_{x'}^{n+1} = L^{-1+\eta} \sum_{y \in Lx' \backslash \mathcal{D}^n(s)} N_y^n \tag{17}$$

where

$$\mathcal{D}^n(s) = \{D_i | D_i \text{ is a connected component of } D^n(s), \ |D_i| \leq L^\alpha, N^n(D_i) \leq L^{-3\alpha}\} \tag{18}$$

with η as in Proposition 1, $\alpha = \frac{\eta}{4}$, $D^0(s) = D(s)$;

$$N^n(Y) = \sum_{x \in Y} N_x^n, \tag{19}$$

for $Y \subset \mathbf{Z}^d$, and

$$D^{n+1}(s) = \overline{[L^{-1}(D^n(s) \backslash \mathcal{D}^n(s))]}, \tag{20}$$

where, for a set $Y \subset \mathbf{Z}^d$, we write:

$$\overline{Y} = \{i | d(i, Y) \leq 1\}. \tag{21}$$

It is easy to see inductively that

$$N_x^n = 0 \text{ if } x \notin D^n(s) \tag{22}$$

and

$$D^n(s) \subset \overline{\{x | N_x^n \neq 0\}} \tag{23}$$

We define also variables N_{x,Λ_2}^n and sets $D_{\Lambda_2}^n$, $\mathcal{D}_{\Lambda_2}^n$ $n = 2, \cdots$ by the same formulas (17, 20), but starting with D_{Λ_2} instead of $D(s)$ in (16).

Then, we define, $\forall x \in \mathbf{Z}^d$,

$$\Omega_x = \{s \in \Omega | \exists n(x) \;\; \forall n \geq n(x), N_x^n = 0\}$$

and

$$\overline{\Omega} = \bigcap_{x \in \mathbf{Z}^d} \Omega_x \tag{24}$$

To understand intuitively the meaning of $\overline{\Omega}$, observe that iterating the operation (20) removes, at each step, the "small" connected components of $D(s)$, and "glues" or "blocks" together the "large" ones that are not too far from each other. Then, the configurations in $\overline{\Omega}$ are those for which this operation ends, after finitely many steps, for each sequence of L^n-boxes labelled by a given site of $\mathbf{Z}^d$. Note that in the configurations used in [31] to construct "counterexamples", $D(s)$ covers an infinite connected subset of the lattice (and obviously $N_0^n \neq 0$, for all large enough n's).

Now we can formulate the inductive representation for the partition function which will be used to prove Proposition 1. We shall need a somewhat more general notion of contour: here and below, a *contour* on scale n will be a triple $\gamma = (\underline{\gamma}, \hat{\gamma}, \sigma(\gamma))$ where $\underline{\gamma}$ is a connected subset of $\mathbf{Z}^d$, $\hat{\gamma} \subset \underline{\gamma}$ ($\hat{\gamma}$ is not necessarily connected) and $\sigma(\gamma)$ is a collection of signs $\sigma_x(\gamma)$, $x \in \mathbf{Z}^d$ on the complement of $\underline{\gamma}$ which are constant on the connected components of the complement of $\underline{\gamma}$. However, when $\underline{\gamma} = D_i$, for D_i a connected component of $D_{\Lambda_2}^n$, we shall have $\underline{\gamma} = \hat{\gamma}$, and we shall simply denote γ by D_i. For those contours, on scale $n = 0$, the signs $\sigma(D_i)$ coincide with the values of the external spins in D_i^c. We shall define below (at the end of the proof of Proposition 3) "renormalized" values s_x^n of the external spins, for each $x \in [L^{-n}\Lambda_2] \backslash D_{\Lambda_2}^n$. On scale n, the signs $\sigma(D_i)$ will also coincide with the values s^n of the external spins in D_i^c. As before, a set of contours Γ is *compatible* if $\underline{\gamma}_1 \cap \underline{\gamma}_2 = \emptyset$, $\forall \gamma_1, \; \gamma_2 \in \Gamma$, and if the signs match (among the contours and with the boundary conditions on Λ_1). It is *s-compatible*, on scale n, if, moreover, $\sigma_x(\Gamma) = s_x^n$, $\forall x \in \mathbf{Z}^d \backslash \underline{\Gamma}$.

We shall derive inductively the following representation for the partition function (12):

$$Z_{\Lambda_1}^{+}(s_{\Lambda_2}) = e^{f_{+,\Lambda_1}^n(s_{\Lambda_2})} \exp\Big(\sum_{X \subset \Lambda_2} \Phi_X^n(s_X)\Big) \widetilde{Z}_{\Lambda_1}^{+}(s_{\Lambda_2}) \tag{25}$$

where $f^n_{+,\Lambda_1}(s_{\Lambda_2})$ corresponds to a "bulk" free energy. Since in Proposition 1, we study a ratio of partition functions, we need only to bound the difference between two $f^n_{+,\Lambda_1}(s_{\Lambda_2})$, for different s_{Λ_2}'s, and this is done in (32) below. Φ^n_X will converge, as $n \to \infty$, to the interactions Φ_X, while

$$\widetilde{Z}^+_{\Lambda_1}(s_{\Lambda_2}) = \sum_{\Gamma} \rho^n(\Gamma) \exp(W^n(\Gamma)) \tag{26}$$

where the sum runs over s^n-compatible sets of contours, with $\underline{\Gamma} \equiv \cup_{\gamma \in \Gamma} \underline{\gamma} \subset [L^{-n}\Lambda_1]$ and the constraint $\widehat{\Gamma} \equiv \cup_{\gamma\in\Gamma}\widehat{\gamma} \supset D^n_{\Lambda_2}$;

$$\rho^n(\Gamma) = \prod \rho^n(\gamma). \tag{27}$$

$\rho^n(\gamma)$, is, for $n \geq 1$, a function of $\{s_x | x \in L^n\overline{\underline{\gamma}} \cap \Lambda_2\}$ while

$$W^n(\Gamma) = \sum_{Y \subset [L^{-n}\Lambda_1]} \Psi^n(Y, \Gamma), \tag{28}$$

where $\Psi^n(Y,\Gamma)$ is, for $n \geq 1$, a function of $\{s_x | x \in L^n \overline{Y} \cap \Lambda_2\}$. We shall prove that, eventually, $\widetilde{Z}^+_{\Lambda_1}(s_{\Lambda_2}) \to 1$ as $\Lambda_2 \uparrow \mathcal{L}$. Proposition 1 will then follow easily from such a representation.

In the Proposition below, we collect the bounds satisfied by $\rho^n(\gamma)$, $\Psi^n(Y,\Gamma)$ and $f^n_{+,\Lambda_1}(s_{\Lambda_2})$:

Proposition 3 *Under the hypotheses of Proposition 1, $\forall s_{\Lambda_2} \in \Omega_{\Lambda_2}$, and for n such that $|[L^{-n}\Lambda_2]| \geq L^d$, (25, 26) hold, where Φ^n_X satisfies (3.4, 3.5) uniformly in n,*

$$0 \leq \rho^n(\gamma) \leq \exp(\beta_n k_n N^n_{\Lambda_2}(\underline{\gamma}) - \beta_n |\underline{\gamma} \backslash D^n_{\Lambda_2}|) \tag{29}$$

and, for each connected component D_i of $D^n_{\Lambda_2}$,

$$\rho^n(D_i) \geq \exp(-\beta_n k_n N^n_{\Lambda_2}(D_i)). \tag{30}$$

Moreover,

$$|\Psi^n(Y,\Gamma)| \leq e^{-\beta_n |Y|}; \tag{31}$$

$\Psi^n(Y,\Gamma)$ depends on Γ only through $\Gamma \cap Y \equiv \{\gamma | \underline{\gamma} \cap Y \neq \emptyset\}$ and $\Psi^n(Y,\Gamma) = 0$ unless Y is connected and $\underline{\Gamma} \cap Y \neq \emptyset$.

Finally, $\forall s^1_{\Lambda_2}, s^2_{\Lambda_2} \in \Omega_{\Lambda_2}$, with $s^1_x = s^2_x$, $\forall x \in V$, and for n such that $d([L^{-n}V],[L^{-n}\Lambda_2]^c) \geq L$,

$$|f^n_{+,\Lambda_1}(s^1_{\Lambda_2}) - f^n_{+,\Lambda_1}(s^2_{\Lambda_2})| \leq C|V| \exp(-(d(V,\Lambda^c_2))^{1-2\eta}) \tag{32}$$

for some $C < \infty$, and

$$\begin{aligned} \beta_n &= L^{(1-\eta)n}\beta_0 \\ k_n &= k_0 - L^{-n} \end{aligned} \tag{33}$$

where $k_0 < \infty$, $\beta_0 = c\beta$ and $\eta = 4\alpha$.

Similar formulas and bounds hold for $Z^-_{\Lambda_1}$, with a function f^n_{-,Λ_1} satisfying (32), and D_{Λ_2} ($=D^+(s_{\Lambda_2})$) replaced by $D^-(s_{\Lambda_2})$.

Remarks.

1. We shall see in the proof (eq.(4.18) below) that, at each scale, there are two contributions to Φ^n_X. One is given by $\ln \rho^n(D_i)$, and is similar to the contour energy in (2.14). Theother contribution comes from the sum over the internal spins that introduces "interactions" between the contours of the external spins, and gives rise to the last term in (4.18).

2. The restriction on n in the Proposition is technical: for larger n's, all the statements would remain true, except that β_n would no longer increase as in (33). However, in the limit $\Lambda_2 \uparrow \mathcal{L}$, the largest value of n to which the Proposition applies increases to infinity.

3. In the proofs, we shall denote by c or C a generic constant that depends only on the lattice dimension or on b, but not on the choice of L. This constant may vary from place to place. We shall use $C(L)$ to denote a generic constant that may depend also on L. We shall assume that L is chosen large enough (given $\eta = 4\alpha$ in Proposition 1) so that inequalities like $C \leq L^\alpha$ can be used. Besides, we shall assume that $\beta_0 = c\beta$ (see (3.15)) is large enough, so that inequalities like $C(L) \leq \beta_0$ can be used.

4 Proofs

Proof of Proposition 3

The proof will be made for the $+$ boundary conditions. The proof for the $-$ boundary condition is similar, but we shall indicate which quantities may depend on the boundary conditions. Although the proof is rather technical, the main idea is quite simple: we cannot take directly the logarithm of $Z^+_{\Lambda_1}(s_{\Lambda_2})$, wherever s_{Λ_2} is not constant, because the change of signs of s_{Λ_2} in D_{Λ_2} introduces constraints in that partition function, i.e. it forces the presence of contours. We define a sort of local partition function, $\rho(\mathcal{D}_{\Lambda_2})$ (see (1) below), containing the contours that are constrained only by the "small" connected components of D_{Λ_2}, and factor it out of the sum. Then the sum over the contours that do not intersect $(D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})$ can be exponentiated via the usual polymer formalism. The exponent is divided into three parts: the terms that are not inside Λ_2 contribute to $f^n_{+,\Lambda_1}(s_{\Lambda_2})$ (see (20)), those that are inside Λ_2 but do not depend on the remaining contours (i.e. those that intersect $(D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})$) contribute to $\Phi^n(X)$ (see (18)), and finally those that depend on the remaining contours contribute to $\Psi^n(Y, \Gamma)$ (see (19, 14)). Then, we "block" the remaining contours and iterate the operation. By definition of $\overline{\Omega}$, eventually D_{Λ_2} becomes empty, there are no constraints left and $\tilde{Z}^+_{\Lambda_1}(s_{\Lambda_2})$ converges to 1 (see (34)).

Turning to the proof, we see that, for $n = 0$, (3.25) and (3.26) follow from (3.12) with $\widehat{\Gamma} = \underline{\Gamma}$, $\Phi^0_X = 0$, $\Psi^0(Y, \Gamma) = 0$, $f^0_{+,\Lambda_1}(s_{\Lambda_2}) = |\Lambda_2| \ln \overline{T}$ (which is independent of s_{Λ_2}, so that (3.32) holds trivially) and $\rho^0(\gamma) = \rho(\gamma)$. The bounds on $\rho(\gamma)$ will be discussed later (see proof of Lemma 5).

Now assume that the Proposition holds for n, and let us prove it for $n+1$. We shall delete the indices n, $n+1$ and denote by a prime the scale $n+1$.

Let

$$\rho(\mathcal{D}_{\Lambda_2}) = \prod_{D_i \in \mathcal{D}_{\Lambda_2}} \rho(D_i) \tag{1}$$

and write (3.26) as:

$$\widetilde{Z} = \rho(\mathcal{D}_{\Lambda_2}) \sum_{\Gamma} \prod_{\gamma \in \Gamma} \overline{\rho}(\gamma) \exp(W(\Gamma)) \tag{2}$$

$W(\Gamma)$ was defined in (3.28), and

$$\overline{\rho}(\gamma) = \frac{\rho(\gamma)}{\prod_i \rho(D_i)} \tag{3}$$

where the product runs over $D_i \subset \underline{\gamma}$, $D_i \in \mathcal{D}_{\Lambda_2}$. In (2), we use the fact that $\underline{\Gamma} \supset D_{\Lambda_2} \supset \mathcal{D}_{\Lambda_2}$.

A contour γ is *small* if,

$$V(\gamma) \not\supset [L^{-n}\Lambda_2] \tag{4}$$

and

$$\underline{\gamma} \cap (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) = \emptyset \tag{5}$$

where $V(\gamma)$ is the complement of the infinite connected component of $\mathbf{Z}^d \backslash \underline{\gamma}$. A contour is *large* otherwise. Note that, unlike in [2], [8], the notion of small contour does not refer to the size of $\underline{\gamma}$, but, basically, to the subset of D_{Λ_2} intersected by $\underline{\gamma}$. It is convenient to include in the large contours those for which $V(\gamma) \supset [L^{-n}\Lambda_2]$. Indeed, as we shall see in Lemma 1 below, the global constraints on families of contours due to the fact that they have to be s-compatible can be expressed entirely in terms of those contours. This inclusion is, however, what limits the values of n in Proposition 3: when $|[L^{-n}\Lambda_2]|$ becomes too small, all the contours are "large" and the iteration stops.

As we shall see, the bounds (3.29, 3.30) are sufficient to control the sum over the small contours (see Lemma 3 below). So, we rewrite the sum in (2) as

$$\sum_{\widehat{\Gamma}_1 \supset D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}}^{\ell} \overline{\rho}(\Gamma_1) \exp(W(\Gamma_1)) \sum_{\Gamma_2}^{s} \overline{\rho}(\Gamma_2) \exp(W(\Gamma_1, \Gamma_2)) \tag{6}$$

$\sum_{\Gamma_1}$ runs over all families of large contours such that $\widehat{\Gamma}_1 \supset (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})$ and $\sum_{\Gamma_2}$ runs over the set $C_s(\Gamma_1)$ of families of small contours Γ_2 such that $\Gamma_1 \cup \Gamma_2$ is s-compatible and such that $\widehat{\Gamma}_1 \cup \widehat{\Gamma}_2 \supset D_{\Lambda_2}$. If $C_s(\Gamma_1) = \emptyset$, then $\sum_{\Gamma_2} = 0$. Finally,

$$W(\Gamma_1, \Gamma_2) = W(\Gamma_1 \cup \Gamma_2) - W(\Gamma_1) = \sum_Y \Psi(Y, \Gamma_1 \cup \Gamma_2) - \sum_Y \Psi(Y, \Gamma_1) \equiv \sum_Y \Psi(Y, \Gamma_1, \Gamma_2) \tag{7}$$

where the sums run over $Y \subset [L^{-n}\Lambda_1]$, and the last sum runs over $Y \cap \underline{\Gamma}_2 \neq \emptyset$ because $\Psi(Y, \Gamma)$ depends on Γ only through $\Gamma \cap Y$.

Let us first characterize explicitely the constraint that the families of contours have to be s-compatible. For that, we define

$$Out(\Gamma) = \{\gamma \in \Gamma | V(\gamma) \supset [L^{-n}\Lambda_2]\}.$$

Lemma 1 *A compatible family of contours Γ such that $\widehat{\Gamma} \supset D_{\Lambda_2}$ is s-compatible if and only if $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible.*

The proof of this Lemma and of the other ones is given in the Appendix. Using this Lemma, one may characterize the families of contours that enter the sum $\sum_{\Gamma_2}^{s}$:

Lemma 2 *If Γ_1 is a family of contours such that $\widehat{\Gamma}_1 \supset D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}$ and $C_s(\Gamma_1) \neq \emptyset$, then $C_s(\Gamma_1)$ is the set of families of small contours Γ_2 such that:*

$$\begin{aligned} &1) \quad \underline{\Gamma}_2 \cap \underline{\Gamma}_1 = \emptyset && (8)\\ &2) \quad \textit{The signs of the contours in } \Gamma_2 \cup \Gamma_1 \textit{ match} && (9)\\ &\quad\;\; \textit{among themselves and with the boundary conditions on } \Lambda_1 && (10)\\ &3) \quad \widehat{\Gamma}_2 \supset (\mathcal{D}_{\Lambda_2} \backslash \widehat{\Gamma}_1). && (11) \end{aligned}$$

We shall show that the sum $\sum_{\Gamma_2}^{s}$ can be exponentiated using this Lemma, the bounds (3.29, 3.30) and the standard polymer formalism:

Lemma 3

$$\sum_{\Gamma_2}^{s} \overline{\rho}(\Gamma_2) \exp(W(\Gamma_1, \Gamma_2)) = \exp\Big(\sum_{Y \subset [L^{-n}\Lambda_1]} \varphi^+(Y, \Gamma_1)\Big) \tag{12}$$

where $\varphi^+(Y, \Gamma_1)$ is a function of $\{s_x, x \in L^n\overline{Y} \cap \Lambda_2\}$, for $n \geq 1$, and of $\{s_x | x \in \Lambda_2, d(x, Y) \leq 2b\}$ for $n = 0$; $\varphi^+(Y, \Gamma_1)$ depends on Γ_1 only through $\Gamma_1 \cap Y$. In particular, $\varphi^+(Y, \Gamma_1) = \varphi^+(Y, \emptyset)$ if $\underline{\Gamma}_1 \cap Y = \emptyset$, and we denote it by $\varphi^+(Y)$ in that case. Moreover, $\varphi^+(Y, \Gamma_1)$ satisfies the bound:

$$|\varphi^+(Y, \Gamma_1)| \leq \exp(-\beta L^{-2\alpha}|Y|) \tag{13}$$

and $\varphi^+(Y, \Gamma_1) = 0$ unless Y is connected. Finally, one may define $\varphi^-(Y, \Gamma_1)$ with $-$ boundary conditions, and we have $\varphi^+(Y, \Gamma_1) = \varphi^-(Y, \Gamma_1)$, if $L^n\overline{Y} \subset \Lambda_2$.

Now, insert (12) in (6). We write

$$\begin{aligned} \sum_{Y \subset [L^{-n}\Lambda_1]} \varphi^+(Y, \Gamma_1) &= \sum_{Y \subset [L^{-n}\Lambda_1]} \varphi^+(Y) + \sum_{Y \subset [L^{-n}\Lambda_1]} (\varphi^+(Y, \Gamma_1) - \varphi^+(Y))\chi(Y \cap \underline{\Gamma}_1 \neq \emptyset) \\ &= \sum_{Y \subset [L^{-n}\Lambda_1]} \varphi^+(Y) + \sum_{Y \subset [L^{-n}\Lambda_1]} \tilde{\varphi}^+(Y, \Gamma_1) \end{aligned}$$

where

$$\tilde{\varphi}^+(Y, \Gamma_1) = (\varphi^+(Y, \Gamma_1) - \varphi^+(Y))\chi(Y \cap \underline{\Gamma}_1 \neq \emptyset). \tag{14}$$

We get, using (2, 6) and writing Γ for Γ_1,

$$\tilde{Z} = \rho(\mathcal{D}_{\Lambda_2}) \exp\Big(\sum_{Y \subset [L^{-n}\Lambda_1]} \varphi^+(Y)\Big) \sum_{\widehat{\Gamma} \supset D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}}^{\ell} \overline{\rho}(\Gamma) \exp\Big(W(\Gamma) + \sum_{Y \subset [L^{-n}\Lambda_1]} \tilde{\varphi}^+(Y, \Gamma)\Big) \tag{15}$$

Write:

$$\sum_{Y\subset[L^{-n}\Lambda_1]}\varphi^+(Y) = \sum_{X\subset\Lambda_2}\sum_{Y}\varphi^+(Y)\chi(L^n\overline{Y}=X) + \sum_{Y\subset[L^{-n}\Lambda_1]}\varphi^+(Y)\chi(L^n\overline{Y}\cap\Lambda_2^c\neq\emptyset) \tag{16}$$

and a similar formula for $\ln\rho(\mathcal{D}_{\Lambda_2})$. Then, using (3.25), we get:

$$Z^+_{\Lambda_1}(s_{\Lambda_2}) = e^{f'_{+,\Lambda_1}(s_{\Lambda_2})}\exp(\sum_{X\subset\Lambda_2}\Phi'_X(s_X))\sum^{\ell}_{\widehat{\Gamma}\supset D_{\Lambda_2}\setminus\mathcal{D}_{\Lambda_2}}\overline{\rho}(\Gamma)\exp(\widetilde{W}(\Gamma)) \tag{17}$$

where, for $n\geq 1$, and $X\subset\Lambda_2$,

$$\Phi'_X = \Phi_X + \sum_{D_i\in\mathcal{D}_{\Lambda_2}}\ln\rho(D_i)\chi(L^n\overline{D}_i=X) + \sum_Y\varphi^+(Y)\chi(L^n\overline{Y}=X) \tag{18}$$

and, using (3.28),

$$\widetilde{W}(\Gamma) = \sum_{Y\subset[L^{-n}\Lambda_1]}\Big(\Psi(Y,\Gamma)+\tilde{\varphi}^+(Y,\Gamma)\Big) \tag{19}$$

while

$$\begin{aligned} f'_{+,\Lambda_1}(s_{\Lambda_2}) &= f_{+,\Lambda_1}(s_{\Lambda_2}) + \sum_{Y\subset[L^{-n}\Lambda_1]}\varphi^+(Y)\chi(L^n\overline{Y}\cap\Lambda_2^c\neq\emptyset) \\ &\quad + \sum_{D_i\in\mathcal{D}_{\Lambda_2}}\ln\rho(D_i)\chi(L^n\overline{D}_i\cap\Lambda_2^c\neq\emptyset) \end{aligned} \tag{20}$$

For $n=0$, one modifies (18, 20) by replacing $L^n\overline{D}_i$ by $\{x|d(x,D_i)\leq 2b\}$, and $L^n\overline{Y}$ by $\{x|d(x,Y)\leq 2b\}$. Note that all the terms in (19) vanish unless $Y\cap\underline{\Gamma}\neq\emptyset$ (using Proposition 3 and (14)).

Lemma 4 *The functions Φ^n_X, defined inductively by (18), are functions of $\{s_x\}_{x\in X}$, are independent of the boundary conditions on Λ_1, and satisfy the bounds in Proposition 1 uniformly in n. The limit $\lim_{n\to\infty}\Phi^n_X = \Phi_X$ exists. Moreover, $f^n_{+,\Lambda_1}(s_{\Lambda_2})$, defined inductively by (20), and $f^n_{-,\Lambda_1}(s_{\Lambda_2})$ defined similarly, satisfy (3.32).*

Now we shall "block" the terms in

$$\sum^{\ell}_{\widehat{\Gamma}\supset D_{\Lambda_2}\setminus\mathcal{D}_{\Lambda_2}}\overline{\rho}(\Gamma)\exp(\widetilde{W}(\Gamma)) \tag{21}$$

in order to obtain the representation (3.26) for $\tilde{Z}$ on the next scale. Let, for each term in (21), $\widehat{\Gamma}' = [L^{-1}\underline{\Gamma}]\cup D'_{\Lambda_2}$ (note that D'_{Λ_2} is not necessarily included in $[L^{-1}\underline{\Gamma}]$, because of the bar in (3.20)) and decompose $\widehat{\Gamma}'$ into connected components: $\widehat{\Gamma}' = \bigcup_i \hat{\gamma}_i'$. Write also

$$\widetilde{W}(\Gamma) = \sum_{Y \subset [L^{-n}\Lambda_1]}^{\ell} (U(Y,\Gamma) + \tilde{\Psi}(Y,\widehat{\Gamma}')) + \sum_i E(\hat{\gamma}_i', \underline{\Gamma} \cap \hat{\gamma}_i') \tag{22}$$

where, in $\sum^{\ell}$, we sum only over Y with $d(Y) \geq \frac{L}{4}$, and we define

$$U(Y,\Gamma) = \Psi(Y,\Gamma) + \tilde{\varphi}^+(Y,\Gamma) - \min_{\Gamma}(\Psi(Y,\Gamma) + \tilde{\varphi}^+(Y,\Gamma)), \tag{23}$$

and

$$\tilde{\Psi}(Y,\widehat{\Gamma}') = \min_{\Gamma}(\Psi(Y,\Gamma) + \tilde{\varphi}^+(Y,\Gamma)) \tag{24}$$

where $\min_\Gamma$ is taken over all Γ such that $[L^{-1}\underline{\Gamma}] \bigcup D'_{\Lambda_2} = \widehat{\Gamma}'$. Finally,

$$E(\hat{\gamma}', \underline{\Gamma} \cap \hat{\gamma}') = \sum_{Y \cap L\hat{\gamma}' \neq \emptyset} (\Psi(Y,\Gamma) + \tilde{\varphi}^+(Y,\Gamma)) \tag{25}$$

where all the terms satisfy $d(Y) < \frac{L}{4}$. Note that those Y's can intersect $L\hat{\gamma}'$ for at most one $\hat{\gamma}'$, since, for disconnected sets Y_1, Y_2 in $\mathbf{Z}^d$, $d(LY_1, LY_2) \geq L$.

Let, for $Y' \subset [L^{-(n+1)}\Lambda_1]$,

$$\Psi'(Y',\widehat{\Gamma}') = \sum_{[L^{-1}Y]=Y'}^{\ell} \tilde{\Psi}(Y,\widehat{\Gamma}') \tag{26}$$

We get

$$(21) = \sum_{\widehat{\Gamma}' \supset D'_{\Lambda_2}} \exp(\sum_{Y' \subset [L^{-(n+1)}\Lambda_1]} \Psi'(Y',\widehat{\Gamma}')) \sum_{[L^{-1}\underline{\Gamma}] \cup D'_{\Lambda_2} = \widehat{\Gamma}'}^{\ell} \overline{\rho}(\Gamma) \exp(\sum_i E(\hat{\gamma}_i', \underline{\Gamma} \cap \hat{\gamma}_i') + \sum_Y^{\ell} U(Y,\Gamma)). \tag{27}$$

We need to do a Mayer expansion on $\exp(\sum_Y^{\ell} U(Y,\Gamma))$ in order to factorize the sum over Γ in (27). We note, for further use, that, by (23), $U(Y,\Gamma) \geq 0$. We write

$$\exp(\sum_Y^{\ell} U(Y,\Gamma)) = \prod_Y (e^{U(Y,\Gamma)} - 1 + 1) = \sum_{\mathcal{Y}} \prod_{Y \in \mathcal{Y}} (e^{U(Y,\Gamma)} - 1) \equiv \sum_{\mathcal{Y}} V(\mathcal{Y},\Gamma), \tag{28}$$

where $V(\mathcal{Y},\Gamma) \geq 0$.

Insert (28) in (27). We can write the first $\sum^{\ell}$ in (27) as:

$$\sum_{[L^{-1}\underline{\Gamma}] \cup D'_{\Lambda_2} = \widehat{\Gamma}'}^{\ell} \overline{\rho}(\Gamma) \exp(\sum_i E(\hat{\gamma}_i', \underline{\Gamma} \cap \hat{\gamma}_i')) \sum_{\mathcal{Y}} V(\mathcal{Y},\Gamma) = \sum_{\underline{\Gamma}' \supset \widehat{\Gamma}'} \prod_{\gamma'} \rho'(\gamma') \tag{29}$$

where $\underline{\Gamma}' = \widehat{\Gamma}' \cup [L^{-1}\mathcal{Y}] = \cup \underline{\gamma}'$ is decomposed into connected components and

$$\rho'(\gamma') = \sum_{(\Gamma,\mathcal{Y})}^{\ell} \overline{\rho}(\Gamma) V(\mathcal{Y},\Gamma) \exp(\sum_i E(\hat{\gamma}_i', \underline{\Gamma} \cap \hat{\gamma}_i')) \tag{30}$$

where the sum runs over $(\Gamma, \mathcal{Y})$ such that

$$[L^{-1}\underline{\Gamma}] \cup (D'_{\Lambda_2} \cap \underline{\gamma}') = \widehat{\Gamma}' \cap \underline{\gamma}', \tag{31}$$

$$[L^{-1}(\underline{\Gamma} \cup \mathcal{Y})] \cup (D'_{\Lambda_2} \cap \underline{\gamma}') = \underline{\gamma}', \tag{32}$$

and such that the signs $\{\sigma(\gamma)\}_{\gamma\in\Gamma}$ are the same for all the terms in the sum. Note that all the terms in (30) are positive by (23).

The factorization of the sum in (29) holds because $U(Y, \Gamma)$ depends on Γ only through $\Gamma \cap Y$, and all the terms in (25) intersect only one $\tilde{\gamma}'$. We define $\gamma' = (\underline{\gamma}', \hat{\Gamma}' \cap \underline{\gamma}', \sigma(\gamma'))$, where $\sigma(\gamma')$ is determined by the common signs of $\{\sigma(\gamma)\}_{\gamma\in\Gamma}$, and

$$\Psi'(Y', \Gamma') = \Psi'(Y', \widehat{\Gamma}'). \tag{33}$$

Finally, we define $s'_{x'}$ for each $x' \in [L^{-n-1}\Lambda_2]\backslash D'_{\Lambda_2}$, as the (constant) value of s_y, for $y \in Lx'\backslash(\cup_{D\subset\mathcal{D}_{\Lambda_2}} V(D))$. To see that s_y is constant, observe that $Lx' \cap (D_{\Lambda_2}\backslash \mathcal{D}_{\Lambda_2}) = \emptyset$ since $x' \notin D'_{\Lambda_2}$, and to see that the set $Lx'\backslash(\cup_{D\subset\mathcal{D}_{\Lambda_2}} V(D))$ is not empty, notice that $|V(D)| \leq cL^{d\alpha} < |Lx| = L^d$, for $D \subset \mathcal{D}^n_{\Lambda_2}$ and α small. So, $s'_{x'}$ is well-defined.

With this definition of s' and of $\sigma(\gamma')$, we see that the sum (29) runs over s'-compatible families of contours. Inserting (29) in (27), and combining (17, 27) we get (3.25, 3.26) on the next scale and the proof of Proposition 3 is finished with:

Lemma 5 $\Psi'(Y', \Gamma')$ *defined by (26, 33) and* $\rho'(\gamma')$ *defined by (30), satisfy the claims of Proposition 3, for* $\beta' = L^{1-4\alpha}\beta$.

Remark Before proving Proposition 1, let us characterize $\overline{\Omega}_+$ and $\overline{\Omega}_-$. It is easy to see that, if $s \in \overline{\Omega}$, each connected component D_i of $D(s)$ is finite. Moreover, for each $x \in \mathcal{L}$, there are at most a finite number of D_i's with $x \in V(D_i)$, (where $V(D_i) =$ the complement of the infinite connected component of $\mathbf{Z}^d\backslash D_i$). Indeed, otherwise, $\forall n$, there exists $m \geq n$ and a connected component D_i of $D^m(s)$ such that $x \in V(D_i)$, $|D_i| \geq L^\alpha$, $D_i \cap L^2\{x\} \neq \emptyset$, where $L^2\{x\}$ is the cube of size L^2 centered at x (indeed, if this last condition is not satisfied for some m, it will hold for a larger m, because of the "blocking" in (3.20)). This would imply (by 3.23) that, $\forall n$, there exists $m \geq n$ and $y \in \mathbf{Z}^d$, with $|x - y| \leq L^2 + 1$, and $N^m_y \neq 0$; but since there is a finite number of such y's, this in turn means that $s \notin \overline{\Omega}$. So, combining these two facts, we see that, for each $s \in \overline{\Omega}$, there exists a unique infinite b-connected set (a subset Y of $\mathcal{L}$ is *b-connected* if any two points of Y can be joined by a path (x_i), $x_i \in \mathcal{L}$ with $|x_i - x_{i+1}| = b$) where s_x is of a given sign and this sign defines $\overline{\Omega}_+$ and $\overline{\Omega}_-$.

Now we can give the

Proof of Proposition 1

Let us apply Proposition 3 up to the largest n such that $|[L^{-n}\Lambda_2]| \geq L^d$. For that n, $[L^{-n}\Lambda_2] \subset L^2\{0\}$, where $L^2\{0\}$ is the cube of size L^2 centered at 0. Since $\Lambda_2 \uparrow \mathcal{L}$, $n \to \infty$. So, we have $[L^{-n}V] = \{0\}$, for n (i.e. Λ_2) large, since V is fixed. Therefore, we have, for n as above, $d([L^{-n}V],[L^{-n}\Lambda_2]^c)\geq L$ and we can use (3.32). Then, given the bounds on Φ^n_X and (3.32), it is enough to show that

$$\widetilde{Z}^+_{\Lambda_1}(s_{\Lambda_2})=1+\mathcal{O}(e^{-c\beta_n}). \tag{34}$$

for $s\in\overline{\Omega}_+$. We claim that, in that case, for $s\in\overline{\Omega}_+$ and n large enough,

$$D^n_{\Lambda_2}=\emptyset \tag{35}$$

(which implies, by (3.22), $N^n_{\Lambda_2,x}=0$, $\forall x$). Postponing the proof of (35), and using the representation (3.26), where now $\Gamma=\emptyset$, $\rho(\emptyset)=1$, $W(\emptyset)=0$, enters the sum since $D^n_{\Lambda_2}=\emptyset$, we get:

$$\widetilde{Z}^+_{\Lambda_1}(s_{\Lambda_2})=1+\sum_{\Gamma\neq\emptyset}\rho^n(\Gamma)\exp(W^n(\Gamma)) \tag{36}$$

where, for each γ in the sum, $V(\gamma)\cap[L^{-n}\Lambda_2]\neq\emptyset$ (all other γ's were small on the first scale, see (4, 5)).

Now, we use the bound (3.29), with $D^n_{\Lambda_2}=\emptyset$, and $N^n_{\Lambda_2,x}=0$, $\forall x$. We use also (3.28) and

$$|W^n(\Gamma)|\leq ce^{-\beta_n}|\underline{\Gamma}|,$$

which follows from (3.31), and the fact that $\Psi^n(Y,\Gamma)=0$ unless Y is connected and $Y\cap\underline{\Gamma}\neq\emptyset$, to get

$$\sum_{\Gamma\neq\emptyset}\rho^n(\Gamma)\exp(W^n(\Gamma))\leq\sum_{\Gamma\neq\emptyset}\exp(-\beta_n|\underline{\Gamma}|/2). \tag{37}$$

Since for each γ in the previous sum, $V(\gamma)\cap[L^{-n}\Lambda_2]\neq\emptyset$, and since $|[L^{-n}\Lambda_2]|\leq L^{2d}$, for n as above ($[L^{-n}\Lambda_2]\subset L^2\{0\}$), we have,

$$(37)\leq\sum_{k\geq1}\Big(\sum_{V(\gamma)\cap[L^{-n}\Lambda_2]\neq\emptyset}\exp(-\beta_n|\underline{\gamma}|/2)\Big)^k\leq\sum_{k\geq1}(e^{-c\beta_n}L^{2d})^k, \tag{38}$$

which proves (34).

We are left with the proof of (35). Now observe that (on scale $n=0$), if D_i is a connected component of $D_{\Lambda_2}=D^+(s_{\Lambda_2})$ (or of $D^-(s_{\Lambda_2})$) such that $D_i\cap\partial\Lambda_2=\emptyset$, then $D_i\subset D(s)$. Besides, if $s\in\overline{\Omega}_+$ and $s_x=-1$, $x\in\partial\Lambda_2$ then $x\in V(D_i)$ for some connected $D_i\subset D(s)$.

Hence, if $s\in\overline{\Omega}_+$,

$$D^+(s_{\Lambda_2})\subset\bigcup\{V(D_i)|D_i\subset D(s),V(D_i)\cap\Lambda_2\neq\emptyset\}.$$

This implies inductively that

$$D^n_{\Lambda_2}\subset\bigcup\{V(D_i)|D_i\subset D^n(s),V(D_i)\cap[L^{-n}\Lambda_2]\neq\emptyset\}. \tag{39}$$

On the other hand, if $s\in\overline{\Omega}_+$, there exists n such that, $\forall m\geq n$,

$$D^m(s)\cap L^2\{0\}=\emptyset. \tag{40}$$

This implies that

$$V(D^n(s)) \cap L^2\{0\} = \cup_{D_i \subset D^n(s)} V(D_i) \cap L^2\{0\} = \emptyset. \tag{41}$$

Indeed, if (41) does not hold, and $D^n(s) \cap L^2\{0\} = \emptyset$, then, because of the blocking in (3.20), $D^m(s) \cap L^2\{0\} \neq \emptyset$, for some $m \geq n$. By taking Λ_2 large, we may assume that this n is the one chosen at the beginning of the proof, in particular that $[L^{-n}\Lambda_2] \subset L^2\{0\}$. Combining this fact and (39, 41), we get $D^n_{\Lambda_2} = \emptyset$ i.e. (35).

To prove Proposition 2 and Lemma 5, we need the following Lemma.

Lemma 6 *Let D_i be a connected component of D^n. Then*

$$\sqrt{\beta_0} L^{\alpha n/2} |D_i| \leq \beta_n N^n(D_i) \leq 2d\beta_0 L^{nd} |D_i|. \tag{42}$$

A similar bound holds for $D^n_{\Lambda_2}$, $N^n_{\Lambda_2}(D_i)$.

Now we shall prove that the configurations in $\overline{\Omega}_+$, $\overline{\Omega}_-$ are typical for μ'_+, μ'_-:

Proof of Proposition 2

It is enough to show, see (3.24), that, for any $x \in \mathbf{Z}^d$,

$$\mu'_+(\Omega^c_x) = \mu'_-(\Omega^c_x) = 0, \tag{43}$$

because, by (3.14) and a simple Peierls argument, one sees that the probability with respect to μ'_+ (resp. μ'_-) of an infinite b-connected set of $-$ (resp. $+$) spins is zero, hence $\mu'_+(\overline{\Omega}_-) = \mu'_-(\overline{\Omega}_+) = 0$. We shall prove that, $\forall A \subset \mathbf{Z}^d$, and $\forall \{N^n_x \in L^{-(1-\eta)n}\mathbf{N}\}_{x\in A}$,

$$\mu'_+(\prod_{x\in A} \chi_{N^n_x} \chi_A) \leq \exp(-\widetilde{\beta}_n N^n(A)) \tag{44}$$

where $\chi_{N^n_x}$ means that the random variable $N^n_x(s)$ defined by (3.16, 3.17) takes the value N^n_x (by (3.17), $N^n_x \in L^{-(1-\eta)n}\mathbf{N}$), χ_A is the indicator function of the event: "A is a union of connected components of $D^n(s)$", and

$$\widetilde{\beta}_n = cL^{-\alpha n/4}\beta_n. \tag{45}$$

Then (43) follows from

$$\mu'_+(\Omega^c_x) \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu'_+(N^n_x \neq 0)$$
$$\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \sum_{A\ni x}^{c} \exp(-c\sqrt{\beta_0} L^{\alpha n/4}|A|) \leq \lim_{N\to\infty} C \sum_{n=N}^{\infty} \exp(-c\sqrt{\beta_0}L^{\alpha n/4}) = 0 \tag{46}$$

where $\sum^c$ runs over connected sets, and, in the second inequality, we use (44) and the lower bound in (42). For μ'_-, we can use the symmetry $\mu'_+(\Omega^c_x) = \mu'_-(\Omega^c_x)$, which follows from (2.4).

To prove (44), consider first $n=0$. Using (3.16), it is enough to show:

$$\mu'_+(A \subset D(s)) \leq \exp(-c\beta_0|A|) \tag{47}$$

We have, by definition (3.6) of $D(s)$,

$$\mu'_+(A \subset D(s)) = \lim_{\Lambda_2\uparrow\mathcal{L}} \lim_{\Lambda_1\uparrow\mathbf{Z}^d} \sum{}^{A} \frac{Z^+_{\Lambda_1}(\{s_x|d(x,A)\leq b\})}{Z^+_{\Lambda_1}} \tag{48}$$

where the sum runs over $\{s_x|d(x,A)\leq b\}$ such that, $\forall x\in A$, $\exists y\in\mathcal{L}$, $|x-y|=b$ and $s_x\neq s_y$. Using the representation (3.12) in the numerator of (48), we get:

$$(\overline{T})^{-|\Lambda_2|} \sum{}^{A} Z^+_{\Lambda_1}(\{s_x|d(x,A)\leq b\}) \leq \sum_{\underline{\Gamma}\supset B_A} \rho(\Gamma) = \sum_{\underline{\Gamma}_1\supset B_A}^{1} \rho(\Gamma_1) \sum_{\Gamma_2}^{\Gamma_1} \rho(\Gamma_2) \tag{49}$$

where the first sum runs over Γ_1 such that $\underline{\gamma}\cap B_A\neq\emptyset$, $\forall\gamma\in\Gamma_1$, with $B_A=\cup_{x\in A}B_x$, and the second sum runs over $\underline{\Gamma}_2\cap B_A=\emptyset$, $\underline{\Gamma}_2\cap\underline{\Gamma}_1=\emptyset$ Now, for any Γ_1 in (49), $(\overline{T})^{-|\Lambda_2|}Z^+_{\Lambda_1}\geq\sum^{\Gamma_1}_{\Gamma_2}\rho(\Gamma_2)$. Inserting this and (49) in (48), and using the bound (3.14), where we can put $\exp(-\beta_0|\underline{\gamma}|)$ in the RHS (by taking Λ_2 large enough), we get

$$\begin{aligned}
\mu'_+(A \subset D(s)) &\leq \sum_{\underline{\Gamma}_1\supset B_A}^{1} \exp(-\beta_0|\underline{\Gamma}_1|) \\
&\leq \exp(-\beta_0|A|/2) \sum_{\underline{\Gamma}_1\supset B_A}^{1} \exp(-\beta_0|\underline{\Gamma}_1|/2) \\
&\leq \exp(-\beta_0|A|/2)\exp(ce^{-\beta_0/2}|A|) \\
&\leq \exp(-c\beta_0|A|)
\end{aligned} \tag{50}$$

for β_0 large enough, using $\underline{\Gamma}_1\supset B_A$ in the first inequality, and the fact that each (connected) $\underline{\gamma}$ in $\underline{\Gamma}_1$ contains a box in B_A in the second inequality, which implies

$$\sum_{\underline{\Gamma}_1\supset B_A}^{1} \exp(-\beta_0|\underline{\Gamma}_1|/2) \leq \prod_{i\in B_A}(1+\sum_{\underline{\gamma}\ni i} e^{-\beta_0|\underline{\gamma}|/2}) \leq \exp(c'e^{-\beta_0/2}|B_A|) \leq \exp(ce^{-\beta_0/2}|A|) \tag{51}$$

with $c=c'b^d$. This proves (47), i.e. (44), for $n=0$.

Then we proceed inductively: we have, by (3.17, 3.20),

$$\mu'_+(\prod_{x'\in A'}\chi_{N'_{x'}}\chi_{A'}) \leq \sum_{\overline{[L^{-1}A]=A',\{N_y\}}} \mu'_+(\prod_{y\in A}\chi_{N_y}\chi_A) \prod_{x'\in A'} \chi(L^{-1+\eta}\sum_{y\in Lx'\cap A} N_y = N'_{x'}) \tag{52}$$

Since A is a union of connected components of $D^n(s)$, using (44) on scale n and Lemma 6, we have:

$$(52) \leq \sum_{A,\{N_y\}}^{*} e^{-\tilde{\beta}N(A)} \prod_{x'\in A'} \chi(L^{-1+\eta}\sum_{y\in Lx'\cap A} N_y = N'_{x'}) \tag{53}$$

where $\sum^*$ runs over $A, \{N_y\}$ such that $\tilde{\beta}N(A) \geq c\sqrt{\beta_0}L^{\alpha n/4}|A|$ (see (42, 45)). Since $N_y \in L^{-(1-\eta)n}\mathbf{N}$, the sum over $N_y, y \in Lx' \cap A$ has at most $L^{(1-\eta)(n+1)}N'_{x'}$ terms. Note also that, by (3.17, 3.33), $\beta N(A) = \beta' N'(A')$, i.e., again by (45), $\tilde{\beta}N(A) \geq \tilde{\beta}' L^{\alpha/4}N'(A')$.

So, using $\tilde{\beta}N(A) \geq \frac{\tilde{\beta}'}{2}L^{\alpha/4}N'(A') + \sqrt{\beta_0}L^{\frac{\alpha n}{4}}\frac{|A|}{2}$, we have:

$$(53) \leq \exp\Big(-\frac{\tilde{\beta}'}{2}L^{\alpha/4}N'(A')\Big)\Big(\prod_{x'\in A'} N'_{x'}\Big)^{L^d} \sum_{[L^{-1}A]=A'} L^{(1-\eta)(n+1)|A|}\exp\Big(-\sqrt{\beta_0}L^{\frac{\alpha n}{4}}\frac{|A|}{2}\Big); \tag{54}$$

for β_0 and L large, the sum over A is bounded by $\mathcal{O}(1)$, and we get:

$$(54) \leq \exp(-\tilde{\beta}' N'(A')), \tag{55}$$

which proves (44) on scale $n+1$.

Appendix: Proof of the Lemmas

Proof of Lemma 1

First, note that a compatible family Γ that contains an s-compatible family $\tilde{\Gamma}$ is trivially s-compatible: each $x \in \underline{\Gamma}^c \cap \mathcal{L}$ belongs also to $\underline{\tilde{\Gamma}}^c$ and we have $\sigma_x(\tilde{\Gamma}) = s_x$ (because $\tilde{\Gamma}$ is s-compatible) and $\sigma_x(\tilde{\Gamma}) = \sigma_x(\Gamma)$ (because Γ is compatible and $\tilde{\Gamma} \subset \Gamma$).

Now, letting $\tilde{\Gamma} \equiv Out(\Gamma) \cup D_{\Lambda_2}$ this shows that, if $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible, then, Γ is s-compatible.

To prove the converse, we proceed inductively. Let us consider first $n = 0$. Assume that Γ is an s-compatible family of contours containing D_{Λ_2}. We shall show that $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible. First of all, note that $Out(\Gamma)$ can be written as $(\gamma_1, \ldots, \gamma_n)$ with $\underline{\gamma}_{i+1} \subset Int(\gamma_i)$ and $\Lambda_2 \subset V(\gamma_n)$. Obviously, since Γ is compatible, and since $Out(\Gamma)$ contains all the contours in Γ such that $\Lambda_2 \subset V(\gamma)$, the sign in $(V(\gamma_{i+1}))^c$ must match the one in the component of $\underline{\gamma}_i^c$ containing $\underline{\gamma}_{i+1}$; therefore, $Out(\Gamma)$ is compatible.

To show that $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible, we consider the following cases:

a) $\underline{\gamma}_n \cap \Lambda_2 \neq \emptyset$. In that case the signs of $\sigma(\gamma_n)$ must coincide with the signs of s_x for all B_x adjacent to $\underline{\gamma}_n$ (because γ_n belongs to Γ, which is s-compatible). But since the signs of s_x are constant outside D_{Λ_2}, this implies that $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible.

b) $\underline{\gamma}_n \cap \Lambda_2 = \emptyset$, which means that $\underline{\gamma} \cap \Lambda_2 = \emptyset$, for all $\gamma \in Out(\Gamma)$ (γ_n is the innermost contour in $Out(\Gamma)$). It also means that $s_x = 1$ for some $x \in \partial\Lambda_2$: indeed, otherwise, $\partial\Lambda_2$ would entirely belong to D_{Λ_2}, and $\partial\Lambda_2$ would be a part of a contour $\gamma \in \Gamma$ such that $\Lambda_2 \subset V(\gamma)$; hence this γ would belong to $Out(\Gamma)$, but obviously $\underline{\gamma} \cap \Lambda_2 \neq \emptyset$, contradicting our assumption on γ_n.

We shall show that the (constant) sign given by $\sigma(\gamma_n)$ to Λ_2 must be $+1$. Assuming this result, we see that this sign is compatible with those $s_x = 1$ for $x \in \partial\Lambda_2$, and again, since the signs of s_x are constant outside D_{Λ_2}, this implies that $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible.

To show that this sign must be $+1$, assume that it is -1. This means that all $x \in \partial\Lambda_2$ with $s_x = 1$ must be in $V(\gamma)$ for some contour $\gamma \in \Gamma \backslash Out(\Gamma)$, since Γ is s-compatible. But since all $x \in \partial\Lambda_2$ with $s_x = -1$ belong to $\Gamma \supset D_{\Lambda_2}$, by definition, and since $\partial\Lambda_2$ is connected, $\partial\Lambda_2$ must be in $V(\gamma)$. But then $\Lambda_2 \subset V(\gamma)$, which contradicts the fact that $\gamma \in \Gamma \backslash Out(\Gamma)$.

Let us now proceed inductively: if Γ' is an s'-compatible family of contours on scale $n+1$, then there is an s-compatible family $\Gamma \subset L\Gamma'$. Moreover, $Out(\Gamma) \cup (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) \subset L(Out(\Gamma') \cup D'_{\Lambda_2})$. But since, by assumption, $Out(\Gamma) \cup D_{\Lambda_2}$ is s-compatible, $Out(\Gamma') \cup D'_{\Lambda_2}$ is s'-compatible because of the way s' and $\sigma(\Gamma')$ were inductively defined, at the end of the proof of Proposition 3.

Proof of Lemma 2

Since the subset Γ_1 of large contours of an s-compatible family Γ always satisfies $Out(\Gamma_1) = Out(\Gamma)$, $C_s(\Gamma_1) \neq \emptyset$ means, by Lemma 1, that $\Gamma_1 \cup D_{\Lambda_2}$ is s-compatible. Thus, any compatible family $\Gamma \supset \Gamma_1 \cup D_{\Lambda_2}$ is s-compatible, by the argument given at the beginning of the previous proof. So, $C_s(\Gamma_1)$ consists of all the families of contours Γ_2 so that $\Gamma_1 \cup \Gamma_2$ is compatible and contains D_{Λ_2}, which is equivalent to the statements in the Lemma.

Proof of Lemma 3

We write

$$\exp(W(\Gamma_1, \Gamma_2)) = \prod_Y (e^{\Psi(Y, \Gamma_1, \Gamma_2)} - 1 + 1) = \sum_{\mathcal{Y}} \tau(\mathcal{Y}, \Gamma_1, \Gamma_2) \tag{1}$$

where the product over Y runs over $Y \cap \underline{\Gamma}_2 \neq \emptyset$, see (4.7), and

$$\tau(\mathcal{Y}, \Gamma_1, \Gamma_2) = \prod_{Y \in \mathcal{Y}} (e^{\Psi(Y, \Gamma_1, \Gamma_2)} - 1). \tag{2}$$

From (3.31), we have

$$|e^{\Psi(Y, \Gamma_1, \Gamma_2)} - 1| \leq C e^{-\beta |Y|} \tag{3}$$

and, using (4.3), (3.29, 3.30),(3.18) and (3.22) we have:

$$\overline{\rho}(\gamma) \leq \exp(2\beta k L^{-3\alpha} |\underline{\gamma} \cap \mathcal{D}_{\Lambda_2}| - \beta |\underline{\gamma} \backslash \mathcal{D}_{\Lambda_2}|) \tag{4}$$

because γ is small, which implies $\underline{\gamma} \cap (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) = \emptyset$.

On the other hand,

$$|\underline{\gamma} \backslash \mathcal{D}_{\Lambda_2}| \geq \frac{1}{2d} L^{-\alpha} |\underline{\gamma}|, \quad \text{if} \quad \gamma \not\subset \mathcal{D}_{\Lambda_2}, \tag{5}$$

which follows from the fact that $\underline{\gamma}$ is connected while the connected components of $\mathcal{D}_{\Lambda_2}$ contain at most L^α points, so that, for each connected component of $\mathcal{D}_{\Lambda_2}$, there exists at least one site in $\underline{\gamma} \backslash \mathcal{D}_{\Lambda_2}$ which is adjacent to that connected component, and the same site can be adjacent to a fixed number of such components (the worst case is when $\underline{\gamma}\backslash\mathcal{D}_{\Lambda_2}$ contains only one site separating different components of $\mathcal{D}_{\Lambda_2}$ of size L^α). Using the fact that we also have $|\underline{\gamma}\cap\mathcal{D}_{\Lambda_2}] \le |\underline{\gamma}| \le 2dL^\alpha|\underline{\gamma}\backslash\mathcal{D}_{\Lambda_2}|$, we get, for $\gamma \notin \mathcal{D}_{\Lambda_2}$,

$$\overline{\rho}(\gamma) \le \exp(-c\beta L^{-\alpha}|\underline{\gamma}|). \tag{6}$$

and

$$\overline{\rho}(\gamma) = 1 \tag{7}$$

for $\gamma \in \mathcal{D}_{\Lambda_2}$. Now inserting (1) in (4.12), we get

$$\sum_{\Gamma_2}^{s} \overline{\rho}(\Gamma_2)\exp(W(\Gamma_1,\Gamma_2)) = \sum_{\mathcal{Z}}\prod_{Z\in\mathcal{Z}} A(Z,\Gamma_1) \tag{8}$$

where $\mathcal{Z} = \underline{\Gamma}_2 \cup \mathcal{Y}$, and $Z = (\Gamma_2, Y)$ with $\underline{\Gamma}_2 \cup Y$ connected;

$$A(Z,\Gamma_1) = \overline{\rho}(\Gamma_2)\tau(\mathcal{Y},\Gamma_1,\Gamma_2) \tag{9}$$

with $\underline{\Gamma}_2 \cup Y = Z$. $A(Z,\Gamma_1)$ depends on Γ_1 only through $\Gamma_1 \cap Z$ and is a function of $\{s_x|x \in L^n\overline{Z}\cap\Lambda_2\}$ for $n \ge 1$ or of $\{s_x|x \in \Lambda_2, d(x,Z) \le 2b\}$ for $n = 0$, because it is a product of factors with these properties. Combining (3) and (6, 7), we get

$$A(Z,\Gamma_1) = 1 \tag{10}$$

if $Z = (D_i, \emptyset)$, with $D_i \in \mathcal{D}_{\Lambda_2}$, and

$$|A(Z,\Gamma_1)| \le \exp(-c\beta L^{-\alpha}|Z|) \tag{11}$$

otherwise. We shall now see how to apply the polymer formalism (see e.g. [18]). There are constraints on Γ_2 coming from (1,2,3) in Lemma 2: to deal with constraint (3), define a polymer as $\tilde{Z} = ((\underline{\Gamma}_2\backslash\mathcal{D}_{\Lambda_2})\cup Y$ with $A(\tilde{Z},\Gamma_1) = A(Z,\Gamma_1)$ for the corresponding Z, and define $\tilde{Z}$ to be "connected" if $\underline{\Gamma}_2 \cup Y$ is connected. Since $A(Z,\Gamma_1) = 1$, if $Z = (D_i,\emptyset)$, with $D_i \in \mathcal{D}_{\Lambda_2}$, we have $A(\tilde{Z},\Gamma_1) = 1$ if $\tilde{Z} = \emptyset$ and the bound (11) allows us to control the sum over "connected" polymers. The remaining constraints on $\tilde{Z}$ come from (1,2) in Lemma 2. The constraint (1) gives rise to the usual hard-core constraint between polymers. To deal with constraint (2), observe that, since the contours in Γ_2 are small, if $\gamma \in \Gamma_2$, and if a finite connected component V_i of $\underline{\gamma}^c$ intersects $[L^{-n}\Lambda_2]$, it must be adjacent to $\underline{\gamma}\cap\Lambda_2$, and the constraint (2) in Lemma 2 is automatically satisfied for that component of $\underline{\gamma}^c$, since the signs $\sigma(\gamma)$ must agree (by definition of a contour) with those of the internal spins in the blocks adjacent to$\underline{\gamma}$. On the other hand, if a connected component V_i does not intersect Λ_2, then the s-compatibility reduces there to compatibility, as for the usual Ising contours and the constraint can be dealt with as in that case.

The claims of the Lemma follow then from the polymer formalism ([18]) applied to (8), for β_0 large; $\varphi^+(Y,\Gamma_1)$ is a sum of products of $A(Z,\Gamma_1)$, with $Z \subset Y$ so its dependence on $\{s_x\}$ and on Γ_1 follows from the one of the $A(Z,\Gamma_1)$'s mentioned above. We put $L^{-2\alpha}$ in (4.13), instead of $L^{-\alpha}$, in order to control constants.

To prove that $\varphi^+(Y,\Gamma_1) = \varphi^-(Y,\Gamma_1)$, if $L^n\overline{Y} \subset \Lambda_2$, observe that, if $\underline{\gamma} \subset \Lambda_2$, and $\underline{\gamma} \cap \partial\Lambda_2 = \emptyset$, the values of $\sigma_i(\gamma)$, $i \notin \underline{\gamma}$ are determined by the external spins outside $\underline{\gamma}$, since, by definition, the internal spins coincide with the external ones on the boxes adjacent to $\underline{\gamma}$. Hence, the value of $\rho(\gamma)$, see (3.13), is independent of the boundary conditions on Λ_1. It is then easy to see inductively that, on scale n, if $L^n\overline{Y} \subset \Lambda_2$, $\varphi^+(Y,\Gamma_1)$ is independent of the boundary conditions.

Proof of Lemma 4

First, observe that Φ^n_X is indeed a function of $\{s_x\}_{x\in X}$, since, by Proposition 3 and Lemma 3, $\rho(D_i)$, for $L^n\overline{D}_i = X$, and $\varphi^+(Y)$, for $L^n\overline{Y} = X$, are functions of $\{s_x\}_{x\in X}$. Besides, Φ^n_X is independent of the boundary conditions on Λ_1, by the argument given at the end of the proof of Lemma 3.

Let us prove the bound (3.5). We consider separately each contribution to (4.18), and sum them over n. From (3.29, 3.30, 3.33), we have:

$$|\ln \rho^n(D_i)| \le k_0\beta_n N^n_{\Lambda_2}(D_i) \le k_0 c\beta_0 L^{nd}|D_i| \le C(L)\beta_0 L^{nd} \tag{12}$$

where, in the second inequality, we use Lemma 6, and in the third, we use $|D_i| \le L^\alpha$ since D_i belongs to $\mathcal{D}_{\Lambda_2}$. Besides,

$$d(X) \le C(L)L^n \tag{13}$$

if $X = L^n\overline{D}_i$ and D_i belongs to $\mathcal{D}_{\Lambda_2}$.

Let, for $x \in \mathcal{L}$,

$$n(x,s) = \max\{n | [L^{-n}x] \in \overline{D^n_{\Lambda_2}}(s)\}. \tag{14}$$

If $s \in \overline{\Omega}$, $n(x,s) < \infty$, because $[L^{-n}x] \in \overline{D^n_{\Lambda_2}}(s)$ means, by (3.23), that there exists a y, with $d(y,[L^{-n}x]) \le 2$, and $N^n_y \ne 0$. However, for any fixed x, $N^n_y = 0$ for all such $y's$, for n large enough, and $s \in \overline{\Omega}$. So we get the bound

$$\begin{aligned}
&\sum_n \sum_{X\ni x} \sum_{D\in\mathcal{D}^n_{\Lambda_2}} |\ln \rho^n(D)| \chi(L^n\overline{D} = X) \exp(d(X)^{1-\eta}) \\
\le\ & \sum_n \sum_{D\in\mathcal{D}^n_{\Lambda_2}} |\ln \rho^n(D)| \chi([L^{-n}x] \in \overline{D}) \exp(C(L)L^{n(1-\eta)}) \\
\le\ & C(L)\beta_0 \sum_{n=0}^{n(x,s)} L^{nd} \exp(C(L)L^{n(1-\eta)}) \equiv C_1(x,s) < \infty,
\end{aligned} \tag{15}$$

since, for each n, $[L^{-n}x] \in \overline{D}$ for at most one $D \in \mathcal{D}^n_{\Lambda_2}$.

On the other hand,

$$d(X) \le C(L)L^n d(Y) \le C(L)L^n|Y| \tag{16}$$

if $X = L^n\overline{Y}$ and Y is connected.

So, with $\eta = 4\alpha$,

$$\begin{aligned}
&\sum_n \sum_{X \ni x} \sum_Y |\varphi^+(Y)| \chi(L^n \overline{Y} = X) \exp(d(X)^{(1-4\alpha)}) \\
\leq\ & \sum_n \sum_{Y \ni [L^{-n}x]} |\varphi^+(Y)| \exp(C(L) L^{n(1-4\alpha)} |Y|) \\
\leq\ & \sum_n \sum_{Y \ni [L^{-n}x]} |\varphi^+(Y)| \exp(\frac{\beta_n L^{-2\alpha}|Y|}{2}), \qquad (17)
\end{aligned}$$

where we used

$$C(L) L^{n(1-4\alpha)} \leq \frac{\beta_n L^{-2\alpha}}{2}, \qquad (18)$$

which holds by (3.33) for all $n \geq 0$, provided β_0 is large enough. Then, the previous sum is bounded by

$$\sum_n \sum_{Y \ni [L^{-n}x]}^{c} \exp(-\frac{\beta_n L^{-2\alpha}|Y|}{2}) \leq C \sum_n \exp(-\beta_n \frac{L^{-2\alpha}}{2}) \equiv C_2 \qquad (19)$$

using the bound (4.13) and the fact that the sum $\sum_Y^c$ runs over connected sets, since $\varphi^+(Y) = 0$ unless Y is connected. Now take $C(x,s) = C_1(x,s) + C_2$ to get (3.5).

Turning to the proof of (3.4),

$$\|\Phi_X^n\|_\infty \leq c\beta_0 |X|,$$

follows, for the second term in (4.18), from (12) (where $L^{nd}|D_i| \leq |L^n \overline{D_i}] = |X|$), since a given X can equal $L^n \overline{D}_i$, for $D_i \in \mathcal{D}$ for at most one n. The last term in (4.18) is controlled by (4.13, 19). The existence of the limit $n \to \infty$ follows from the bounds (15, 17, 19).

Turning to the proof of (3.32), observe that the last term in (4.20) is independent of s_V, as long as $d([L^{-n}V], [L^{-n}\Lambda_2]^c) \geq L$, since each $\rho(D_i)$ is a function of $\{s_x | x \in L^n \overline{D}_i\}$, $L^n \overline{D}_i \cap \Lambda_2^c \neq \emptyset$, and $|D_i| \leq L^\alpha$. The contribution to (3.32) of the previous term in (4.20) is bounded (on scale n), using (4.13) and the fact that $\varphi(Y)$ is a function of s_V only for $\overline{Y} \cap [L^{-n}V] \neq \emptyset$, by:

$$|[L^{-n}V]| \exp(-c\beta_n L^{-2\alpha} d([L^{-n}V], [L^{-n}\Lambda_2]^c)) \qquad (20)$$

Now, use (3.33), which implies $c\beta_n L^{-2\alpha} d([L^{-n}V], [L^{-n}\Lambda_2]^c) \geq L^{\eta n}(d(V, \Lambda_2^c))^{1-2\eta}$ (for β_0 large) and $|[L^{-n}V]| \leq cL^{-nd}|V|$. The factor $L^{\eta n}$ controls the sum over n, and we get (3.32).

Proof of Lemma 5

Consider first $\Psi'(Y', \Gamma')$. From (4.14), (3.31) and (4.13) we have

$$|\tilde{\Psi}(Y, \hat{\Gamma}')| \leq 3 \exp(-\beta L^{-2\alpha}|Y|). \qquad (21)$$

Since Y is connected and all the terms in (4.26) have $d(Y) \geq \frac{L}{4}$, we have

$$|Y| \geq cL|Y'|. \tag{22}$$

since $[L^{-1}Y] = Y'$. The number of terms in the sum in (4.26) is at most $2^{L^d|Y'|}$, since $Y \subset LY'$, so, using $cL\beta L^{-2\alpha} - L^d \ln 2 \geq \beta L^{1-4\alpha}$, for β_0 large, we get (see (4.33))

$$|\Psi'(Y',\Gamma')| \leq \exp(-\beta L^{1-4\alpha}|Y'|) \tag{23}$$

for β_0 large.

Then observe that, by induction and Lemma 3, $\tilde{\Psi}(Y,\hat{\Gamma}') = 0$ unless Y is connected and $Y \cap \underline{\Gamma} \neq \emptyset$ (see(4.14)). Hence, since $[L^{-1}Y] = Y'$, $\Psi'(Y',\hat{\Gamma}') = \Psi'(Y',\Gamma') = 0$ unless Y' is connected and $Y' \cap \underline{\Gamma}' \neq \emptyset$. Also by induction and Lemma 3, $\Psi(Y,\Gamma)$, $\tilde{\varphi}^+(Y,\Gamma)$ depend on Γ only through $\Gamma \cap Y$; hence, since $\underline{\Gamma} \subset L\hat{\Gamma}'$, $\tilde{\Psi}(Y,\hat{\Gamma}')$ and $\Psi'(Y',\Gamma')$ depend on Γ' only through $\Gamma' \cap Y'$. Likewise, $\Psi(Y,\Gamma)$, $\tilde{\varphi}^+(Y,\Gamma)$ are functions (for $n \geq 1$) of $\{s_x | x \in L^n\overline{Y} \cap \Lambda_2\}$ but since $\overline{Y} \subset L\overline{Y}'$ when $[L^{-1}Y] = Y', \Psi'(Y',\Gamma')$ is a function of $\{s_x | x \in L^{n+1}\overline{Y}' \cap \Lambda_2\}$. The same conclusion holds for $n = 0$, because $\{x | d(x,Y) \leq 2b\} \subseteq L\overline{Y}'$ when $[L^{-1}Y] = Y'$, for L large enough.

Turning to $\rho'(\gamma')$, a similar argument shows that it is a function of $\{s_x | x \in L^{n+1}\overline{\underline{\gamma}'} \cap \Lambda_2\}$. Let us prove (3.29) inductively: observe that it holds trivially for $n = 0$ because of (3.14). To proceed inductively, we use (4.30):

$$\rho'(\gamma') = \sum_{(\Gamma,\mathcal{Y})}^{\ell} \bar{\rho}(\Gamma) V(\mathcal{Y},\Gamma) \exp(\sum_i E(\hat{\gamma}'_i, \underline{\Gamma} \cap \hat{\gamma}'_i)) \tag{24}$$

with the constraints (4.31) and (4.32). Let us bound the different parts of this expression; Using (4.25) and the bounds (3.31), (4.13), we have:

$$\sum_i |E(\hat{\gamma}'_i, \underline{\Gamma} \cap \hat{\gamma}'_i)| \leq C(L) e^{-\beta L^{-2\alpha}} \sum_i |\hat{\gamma}'_i| \leq e^{-\beta L^{-3\alpha}} |\underline{\gamma}'| \tag{25}$$

since the sum here runs over disjoint subsets $\hat{\gamma}'_i$of $\underline{\gamma}'$, and the sum in (4.25) runs over connected sets Y that intersect $L\hat{\gamma}'$. Next using (4.3), the bounds (3.29) and (3.30), we have:

$$\bar{\rho}(\Gamma) \leq \exp\Big(\beta k N_{\Lambda_2}((D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) \cap \underline{\Gamma}) + 2\beta k N_{\Lambda_2}(\mathcal{D}_{\Lambda_2} \cap \underline{\Gamma}) - \beta |\underline{\Gamma} \backslash D_{\Lambda_2}|\Big) \tag{26}$$

Since, by (3.22), $N_{\Lambda_2}(\underline{\Gamma}) = N_{\Lambda_2}(D_{\Lambda_2} \cap \underline{\Gamma}) = N_{\Lambda_2}\Big((D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) \cap \underline{\Gamma}\Big) + N_{\Lambda_2}(\mathcal{D}_{\Lambda_2} \cap \underline{\Gamma})$.

Now, we can write:

$$|\underline{\Gamma} \backslash D_{\Lambda_2}| \geq cL^{-\alpha} |\underline{\Gamma} \backslash (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})|. \tag{27}$$

because $\underline{\Gamma} \backslash D_{\Lambda_2}$ can be written as $\underline{\Gamma} \backslash (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2}) \backslash \mathcal{D}_{\Lambda_2}$ and, for each connected component of $\mathcal{D}_{\Lambda_2}$, there is always at least one site in $\underline{\Gamma} \backslash (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})$ which is adjacent to that component.

By definition of $\mathcal{D}_{\Lambda_2}$,

$$N_{\Lambda_2}(\mathcal{D}_{\Lambda_2} \cap \underline{\Gamma}) \leq L^{-3\alpha} |\mathcal{D}_{\Lambda_2} \cap \underline{\Gamma}| \leq L^{-3\alpha} |\underline{\Gamma} \backslash (D_{\Lambda_2} \backslash \mathcal{D}_{\Lambda_2})|. \tag{28}$$

(since $\mathcal{D}_{\Lambda_2} \cap \underline{\Gamma} \subset \underline{\Gamma}\backslash(D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2})$). So we get,

$$\overline{\rho}(\Gamma) \leq \exp\Big(\beta k N_{\Lambda_2}((D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2}) \cap \underline{\Gamma}) - c\beta L^{-\alpha}|\underline{\Gamma}\backslash(D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2})|\Big) \tag{29}$$

We have, by (4.28), (4.23), (3.31) and Lemma 3,

$$\Big|V(\mathcal{Y},\Gamma)\Big| \leq \prod_{Y\in\mathcal{Y}} \Big(Ce^{-\beta L^{-2\alpha}|Y|}\Big) \leq \exp\Big(-\beta L^{-3\alpha}|\mathcal{Y}|\Big) \tag{30}$$

where $|\mathcal{Y}| = \sum_{Y\in\mathcal{Y}} |Y|$, and we put $L^{-3\alpha}$ in order to eliminate the constant C.

Inserting (25, 29, 30) in (24), we get

$$\rho'(\gamma') \leq \exp\Big(e^{-\beta L^{-3\alpha}}|\underline{\gamma}'| + \beta' k N'_{\Lambda_2}(\gamma')\Big) \sum_{\Gamma,\mathcal{Y}} \exp\Big(-\beta L^{-3\alpha}|\mathcal{Y}| - c\beta L^{-\alpha}|\underline{\Gamma}\backslash(D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2})|\Big) \tag{31}$$

with the same constraints as before and we used, by definition (3.33) of β', and (3.17) of N'_{Λ_2},

$$\beta N_{\Lambda_2}\Big((D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2}) \cap \underline{\Gamma}\Big) = \beta' N'_{\Lambda_2}(\underline{\gamma}'). \tag{32}$$

In order to control the sum over $\underline{\Gamma}, \mathcal{Y}$, we decompose $\underline{\Gamma}\backslash(D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2}) = \bigcup_i A_i$ into connected components and call a component *long* if, either $A_i = \underline{\gamma}$, with $V(\gamma) \supset [L^{-n}\Lambda_2]$, or $A_i \cap L(\underline{\gamma}'\backslash D'_{\Lambda_2}) \neq \emptyset$ and we call A_i *short* otherwise.

Write $\mathcal{A}^\ell = \bigcup_\ell A_i$ and $\mathcal{A}^s = \bigcup_s A_i$, where $\bigcup_\ell(\bigcup_s)$ is the union over the long (short) components. We claim that a long component satisfies $d(A_i) \geq L$; to show that, consider two cases: either $A_i = \underline{\gamma}$, with $V(\gamma) \supset [L^{-n}\Lambda_2]$, and $d(A_i) \geq L$ because of our restriction on n in Proposition 3; or A_i intersects $L(\underline{\gamma}'\backslash D'_{\Lambda_2})$, and, since Γ contains only long contours, A_i must intersects $\overline{D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2}}$ (see (4.5), in which case it must cross the corridor of width L given by $LD'_{\Lambda_2}\backslash L([L^{-1}(D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2}])$, which implies $d(A_i) \geq L$;

Moreover, each box in $L(\underline{\gamma}'\backslash D'_{\Lambda_2})$ is intersected either by a long A_i or by a $Y \in \mathcal{Y}$ all of which have a diameter at least $\frac{L}{4}$; thus, we have

$$\sum_{A_i\in\mathcal{A}^\ell} |A_i| + |\mathcal{Y}| > cL|\underline{\gamma}'\backslash D'_{\Lambda_2}|. \tag{33}$$

We write then the sum over $\Gamma, \mathcal{Y}$ as follows

$$\sum_{\mathcal{A}^\ell,\mathcal{Y}} \exp\Big(-\beta(cL^{-\alpha}\sum_{A_i\in\mathcal{A}^\ell}|A_i| + L^{-3\alpha}|\mathcal{Y}|)\Big) \sum_{\mathcal{A}^s} \exp(-c\beta L^{-\alpha}\sum_{A_i\in\mathcal{A}^s}|A_i|) \tag{34}$$

using (33) and $\beta' = L^{1-4\alpha}\beta$, this sum can be bounded by $\exp(-2\beta'|\underline{\gamma}'\backslash D'_{\Lambda_2}|)\exp(e^{-\beta L^{-3\alpha}}C(L)|\underline{\gamma}'|)$, for L large enough, where the last factor bounds the sums in (34).

So, we get, going back to (31),

$$\rho'(\gamma') \leq \exp\Big(-\beta'|\underline{\gamma}'\backslash D'_{\Lambda_2}| + \beta' k N'_{\Lambda_2}(\gamma') + (C(L)+1)e^{-\beta L^{-3\alpha}}|D'_{\Lambda_2} \cap \underline{\gamma}'|\Big) \tag{35}$$

using $|\underline{\gamma}'| = |\underline{\gamma}'\backslash D'_{\Lambda_2}| + |D'_{\Lambda_2} \cap \underline{\gamma}'|$ and $(C(L)+1)e^{-\beta L^{-3\alpha}} \leq \beta'$.

By (3.22), we have $N'_{\Lambda_2}(\underline{\gamma}') = \sum_{D_i\subset\underline{\gamma}'} N'(D_i) = N'(D'_{\Lambda_2}\cap\underline{\gamma}')$, and, using Lemma 6, we can bound

$$\beta' k N'_{\Lambda_2}(\underline{\gamma}') + (C(L)+1)e^{-\beta L^{-3\alpha}}|D'_{\Lambda_2}\cap\underline{\gamma}'| \le \beta' k' N'_{\Lambda_2}(\underline{\gamma}') \tag{36}$$

because, by (3.33), $k'-k = L^{-n}(1-L^{-1})$ and $e^{-\beta_n L^{-3\alpha}} << L^{-n}$. This iterates (3.29).

To prove (3.30), observe that it follows for $n=0$ because, see (2.1),

$$|H_D(\sigma_D|\tilde{\sigma}_{D^c})| \le 4d|D| \tag{37}$$

and we obtain a lower bound, with $\beta_0 k_0 = 4d\beta$, by keeping in (3.12) only the term with $T_x = \bar{T}$. Inductively, we use

$$\rho'(D'_i) \ge \exp(E(D'_i))\rho\Big((D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2})\cap LD'_i\Big) \tag{38}$$

because all terms in (4.30) are positive. Then use (3.30) inductively, the fact, analogous to (32), that

$$\beta N_{\Lambda_2}\Big((D_{\Lambda_2}\backslash\mathcal{D}_{\Lambda_2})\cap LD'_i\Big) = \beta' N'_{\Lambda_2}(D'_i) \tag{39}$$

(25) and (36) with $\gamma' = D'_i$.

Proof of Lemma 6

The upper bound follows inductively from (3.16) (for $n=0$),

$$\beta N\Big((D(s)\backslash\mathcal{D}(s))\cap LD'_i\Big) = \beta' N'(D'_i)$$

and from

$$\sum_{D_j\in(D(s)\backslash\mathcal{D}(s))\cap LD'_i} |D_j| \le L^d|D'_i|$$

As for the lower bound, it is obvious for $n=0$ and β_0 large. As above, we have

$$\beta' N'(D'_i) = \sum_j \beta N(D_j) \tag{40}$$

where $\sum_j$ runs over $D_j \in (D(s)\backslash\mathcal{D}(s))\cap LD'_i$ and we have $D'_i = \overline{[L^{-1}(\cup_j D_j)]}$. Write the sum in (40) as $\sum^1 + \sum^2$ where in $\sum^1$, we have $|D_j| > L^\alpha$ and in $\sum^2$, $|D_j| \le L^\alpha$. In this latter case we must have $N(D_j) \ge L^{-3\alpha}|D_j|$ since $D_j \notin \mathcal{D}$ (see (3.18)). For $\sum^1$, we use Lemma 6 inductively

$$\sum_j{}^1 \beta N(D_j) \ge \sqrt{\beta_0} L^{\alpha n/2} \sum_j{}^1 |D_j| \tag{41}$$

and, since we have $|D_j| \ge L^\alpha$, we get

$$L^{\alpha n/2} \sum_j{}^1 |D_j| \ge L^{\alpha(n+1)/2}|D'_{i,1}| \tag{42}$$

with $D'_{i,1} = \overline{[L^{-1}(\bigcup_j^1 D_j)]}$. For $\sum^2$, we have, using the definition (3.33) of β_n and the fact that each box Lx' with $x' \in D'_{i,2} = \overline{[L^{-1}(\bigcup_j^2 D_j)]}$ must contain or be adjacent to a box intersected by some D_j in $\sum^2$:

$$\begin{aligned}\sum_j^2 \beta N(D_j) &\geq L^{-3\alpha}\beta \sum_j^2 |D_j| \\ &\geq \beta_0 L^{(1-4\alpha)n} L^{-3\alpha} c |D'_{i,2}| \\ &\geq \sqrt{\beta_0} L^{\alpha(n+1)/2} |D'_{i,2}| \end{aligned} \tag{43}$$

for α small enough and β_0 large enough. Obviously (40-43) imply the lower bound in (4.42) on scale $n+1$ since

$$|D'_{i,1}| + |D'_{i,2}| \geq |D'_i|. \tag{44}$$

The same arguments hold for $D^n_{\Lambda_2}$, $N^n_{\Lambda_2}(D_i)$.

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