The Reversible Umbilic Bifurcation

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Abstract

Hamiltonian systems with several degrees of freedom regularly lead to the investigation of bifurcating equilibria in reduced one-degree-of-freedom systems. This paper concerns equilibria with vanishing linearization, a co-dimension two phenomenon in the reversible context. Under appropriate transversality conditions such equilibria have versal unfoldings related to the elliptic and hyperbolic umbilic catastrophes. This has applications to gyrostat motion and also helps to explain the dynamics defined by the normal form of the Hénon-Heiles system. The occurring unfoldings turn out to be versal even in the general reversible context of not necessarily Hamiltonian systems.

§1 Introduction

Given a Hamiltonian system with n degrees of freedom, it is very helpful if one can reduce it by means of a symmetry group. Often it is not the system itself, but some suitably chosen normal form that admits these symmetries, and if the symmetry group is sufficiently large this implies integrability. Such integrable approximations render the study of a wide number of problems more accessible, ranging from laser polarisation to galactic dynamics. Recall that the general (bounded) motion in a non-degenerate integrable Hamiltonian system is quasi-periodic with n frequencies. The corresponding n-parameter families of invariant n-tori in phase space are complemented by $(n-1)$-parameter families of invariant $(n-1)$-tori, ..., one-parameter families of periodic orbits and isolated equilibria. This makes the phase space a *ramified torus bundle*, the regular fibres being the maximal invariant tori. As the original system is approximated by the integrable normal form, the equilibria and periodic orbits survive by means of the implicit mapping theorem, while KAM-theory ensures this for most of the invariant n-tori. Under certain conditions on the *normal frequencies*, the eigenvalues of the (normal) linearization, this result extends to the lower dimensional tori, see [Broer,Huitema,Sevryuk;96] and references therein.

The invariant $(n-1)$-tori are particularly interesting because they determine the distribution of the maximal tori in phase space. Indeed, the n-parameter families of invariant n-tori are separated by the stable and unstable manifolds of normally hyperbolic $(n-1)$-tori and shrink down to normally elliptic $(n-1)$-tori. Reducing the $(n-1)$-torus symmetry of the normal form leads to a one-degree-of-freedom problem, with the $n-1$ actions conjugate to the toral angles acting as (distinguished) parameters. In this way we are led to study the equilibria, and generically bifurcations of co-dimension up to $n-1$ may occur.

Bifurcations in 1-parameter families have been thoroughly studied, cf. [Meyer;70,75], [Rimmer;78,83], [Lamb;94], [Lamb,Capel;95] and references therein. The only local co-dimension one bifurcation that occurs in (not necessarily reversible) Hamiltonian systems with one degree of freedom is the *centre-saddle bifurcation*. A standard family is $\frac{1}{2}p^2 + \frac{1}{3}q^3 + \lambda q$, under variation of the parameter λ a centre and a saddle meet and vanish. This particular family is also reversible with respect to the involution $p \mapsto -p$, *i.e.* under $(q,p,t) \mapsto (q,-p,-t)$ the equations of motion remain invariant. Correspondingly, the centre-saddle bifurcation is also a codimension one bifurcation in the reversible context. The standard families $\frac{1}{2}p^2 \pm \frac{1}{4}q^4 + \lambda q^2$ for the *Hamiltonian pitchfork bifurcation* are stable within families that are reversible with respect to $q \mapsto -q$. Depending on the sign of $\frac{1}{4}q^4$, under variation of λ either two centres meet in a saddle, leading to one centre, or a saddle splits into a centre and two saddles. Like in the centre-saddle bifurcation, the bifurcating equilibrium is *parabolic*, having linearization $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In this paper we treat equilibria with linearization $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ that occur in generic 2-parameter families of reversible Hamiltonian systems. For one-degree-of-freedom systems the integral curves are given by the level sets of the Hamilton function. In this way an unfolding of the bifurcating equilibrium is governed by the corresponding unfolding of the Hamilton function and singularity theory applies, cf. *e.g.* [Golubitsky,Guillemin;73], [Bröcker,Lander;75], [Poston,Stewart;78]. The present case turns out to be related to the elliptic and hyperbolic umbilic catastrophes, the versal unfoldings of the $D_4^{\pm}$ singularities.

In fact this already holds true in the generic Hamiltonian case, see [Broer,Chow,Kim, Vegter;93,95], where 3 parameters are needed. The reversibility assumption amounts to replacing singularity theory by $\mathbb{Z}_2$-equivariant singularity theory, cf. [Poènaru;76], thereby reducing the number of parameters from 3 to 2 . In [Lamb;94], [Lamb,Capel;95] a further (non-reversing) $\mathbb{Z}_2$-symmetry is used to obtain co-dimension one bifurcations.

On the other hand, the 2-parameter families we obtain are versal not only in the reversible Hamiltonian context, but also if the assumption that the system be Hamiltonian is dropped and only reversibility is required. A planar system is Hamiltonian if it is area preserving, and given a reversible unfolding of a $D_4^{\pm}$ singularity, it is possible to define a suitable invariant measure on $\mathbb{R}^2$. This allows to recover part of the results of [Teixeira;97].

This paper is organised as follows. In Section 2 we present the versal $\mathbb{Z}_2$-invariant unfoldings of the D_4^+ and the D_4^- singularity and discuss the corresponding Hamiltonian systems. Section 3 contains applications to the torque free gyrostat with two flywheels rotating about the stable principal axes of inertia and to the Hénon-Heiles system. The latter is invariant under the axial S^1-action, and normalising introduces a further S^1-symmetry that allows the reduction from three degrees to one degree of freedom. After a brief treatment of bifurcating equilibria in several degrees of freedom, we show in the final Section 4 that the families of Section 2 are also stable in the non-Hamiltonian reversible context. Furthermore we discuss open questions and possible generalizations.

§2 The families in one degree of freedom

We are given a Hamilton function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $DH(0) = 0 = D^2H(0)$. In the corresponding Hamiltonian system, with equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

the origin is an equilibrium, the linearization of which has 0 as eigenvalue with geometric multiplicity two. From [Broer,Chow,Kim,Vegter;93,95] we know that the least degenerate such equilibria are given by the singularities $D_4^\pm$, with Hamilton functions

$$H^+(q,p) = p^2 q - \frac{1}{3}q^3 \quad , \quad H^-(q,p) = p^2 q + \frac{1}{3}q^3$$

and that the versal unfoldings of these are provided by the elliptic and hyperbolic umbilic catastrophes

$$H^\pm_{\lambda,\mu,\nu}(q,p) \;=\; p^2 q \mp \frac{1}{3}q^3 + \lambda\,(p^2 \pm q^2) + \mu\, q + \nu\, p \quad . \tag{1}$$

We further assume that our Hamiltonian systems are reversible with respect to the involution $(q,p) \mapsto (q,-p)$. Dropping the last term in $H^\pm_{\lambda,\mu,\nu}$ we are led to the unfoldings

$$H^\pm_{\lambda,\mu}(q,p) \;=\; p^2 q \mp \frac{1}{3}q^3 + \lambda\,(p^2 \pm q^2) + \mu\, q \tag{2}$$

of $D_4^\pm$. For later reference we call these the *reversible umbilics*. These unfoldings turn out to be versal. Let us recall the precise meaning of this notion.

Definition 2.1

Two dynamical systems are C^k-equivalent if there is a C^k-isomorphism η that sends orbits to orbits, preserving the direction of time. An unfolding $(\varphi_\lambda)_{\lambda\in\Lambda}$ of a dynamical system φ_{λ_0} is C^k-versal if, given another unfolding ψ_ν of φ_{λ_0}, there is a C^k-reparametrisation $\nu \mapsto \lambda(\nu)$ such that for all parameters ν the systems ψ_ν and $\varphi_{\lambda(\nu)}$ are C^k-equivalent.

In this paper we only encounter the cases $k = 0$ and $k = \infty$.

Theorem 2.2

The 2-parameter family $H^\epsilon_{\lambda,\mu}$ given by (2) defines a C^∞-versal unfolding of the Hamiltonian system X_{H^ϵ} ; $\epsilon = \pm 1$.

Proof :

We have to show that, given a reversible unfolding X_{K_ν} of X_{H^ϵ} , there is a C^∞-reparametrisation $(\lambda, \mu) : \mathbb{R}^n \longrightarrow \mathbb{R}^2$ and a smooth parameter-dependent equivalence $\eta_\nu : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ between X_{K_ν} and $X_{H^\epsilon_{\lambda(\nu),\mu(\nu)}}$. The latter amounts to

$$K_\nu \;=\; H^\epsilon_{\lambda(\nu),\mu(\nu)} \circ \eta_\nu \qquad , \qquad \det \eta_\nu \;>\; 0$$

and η_ν is called a *right equivalence* between the functions K_ν and $H^\epsilon_{\lambda(\nu),\mu(\nu)}$, cf. [Broer, Chow,Kim,Vegter;93,95]. Now (2) defines a universal $\mathbb{Z}_2$-invariant unfolding of H^ϵ , cf. [Poènaru;76], *i.e.* given a smooth family

$$\begin{array}{rccc} K: & \mathbb{R}^2 \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & ((q,p),\nu) & \mapsto & K_\nu(q,p) \end{array}$$

of $\mathbb{Z}_2$-invariant (Hamilton) functions with $K_0 = H^\epsilon$, there is a unique smooth $\mathbb{Z}_2$-equivariant mapping

$$\begin{array}{rccc} (\eta,\lambda,\mu): & \mathbb{R}^2 \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^2 \times \mathbb{R}^2 \\ & (q,p,\nu) & \mapsto & (\eta_\nu(q,p), \lambda(\nu), \mu(\nu)) \end{array}$$

providing both the right equivalences and the reparametrisations **;** **q.e.d.**

Since the singularities $D_4^\pm$ are 3-determined, cf. [Bröcker,Lander;75], [Poston,Stewart;78], it is sufficient to study the 3-jet of a given family K_ν . This is important for applications.

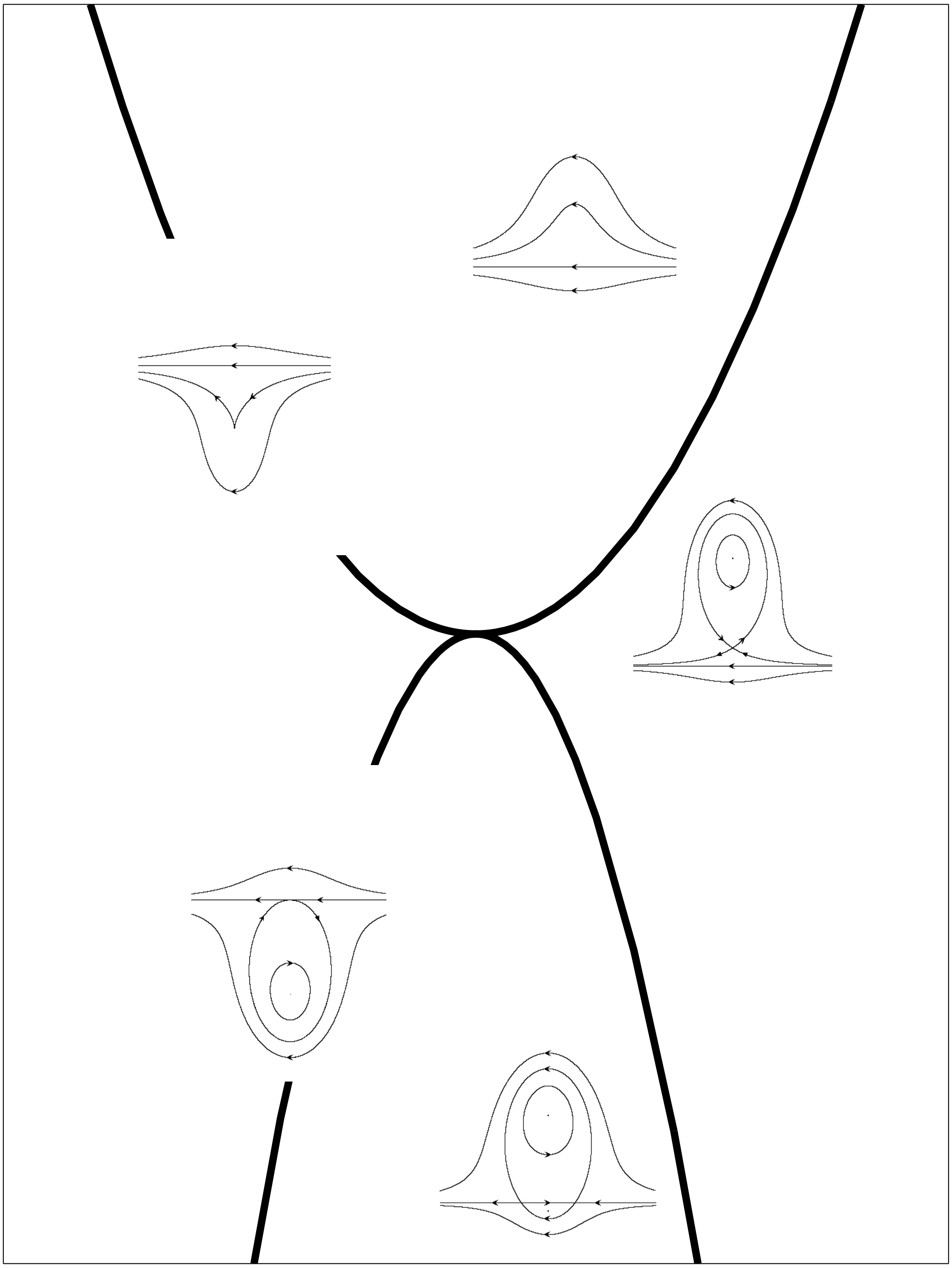

Figure 2.1: Bifurcation diagram of the hyperbolic reversible umbilic, with axes λ and μ .

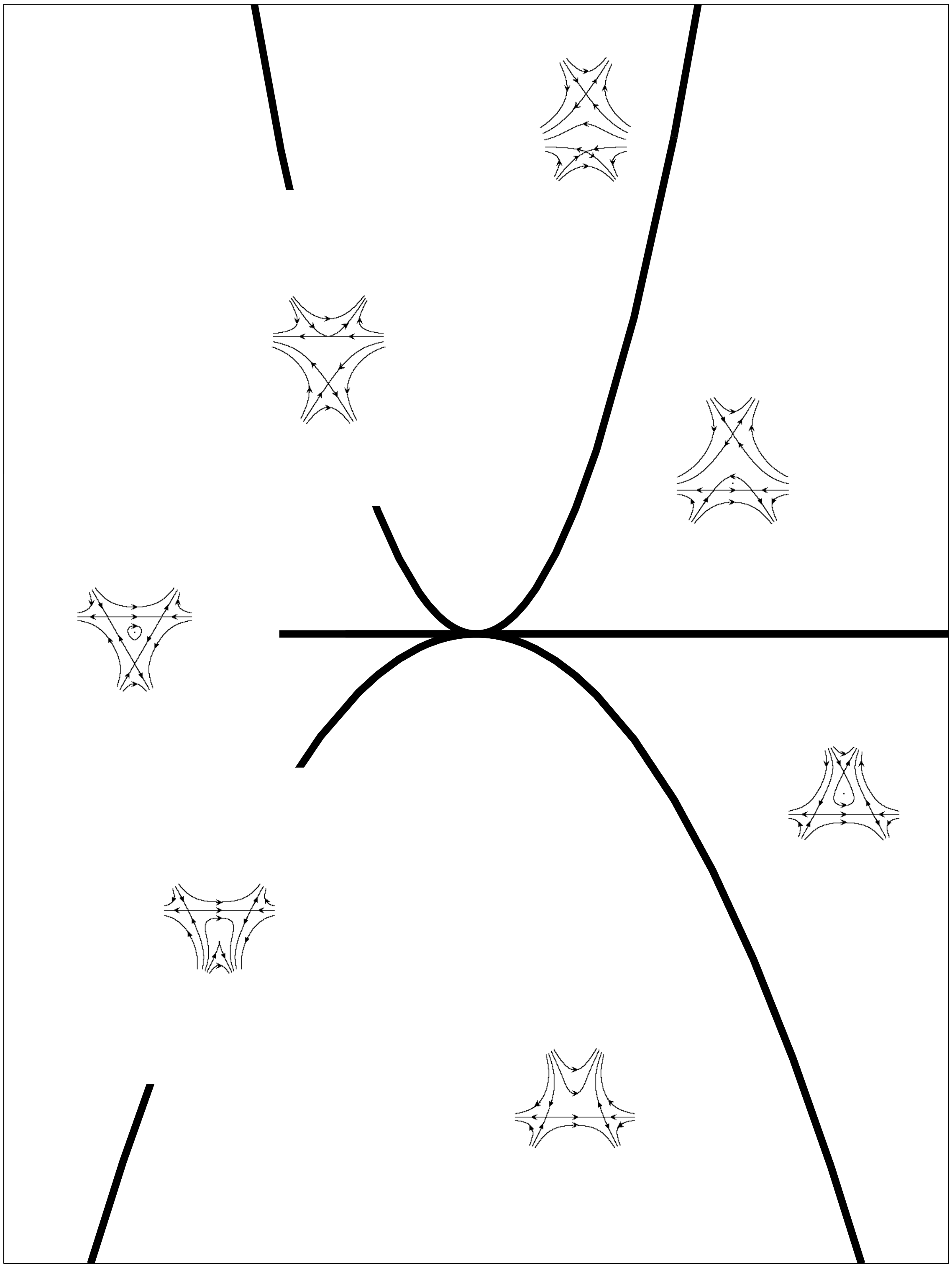

Figure 2.2: Bifurcation diagram of the elliptic reversible umbilic, with axes λ and μ .

To conclude this section we present the bifurcation diagrams of the systems defined by (2). Let us start with the hyperbolic reversible umbilic.

The catastrophe set of $H^-_{\lambda,\mu}$ is the union of the parabolas $\{\mu = \lambda^2\}$ and $\{\mu = -3\lambda^2\}$, see *Figure 2.1.* For $\mu > \lambda^2$ the system $X_{H^-_{\lambda,\mu}}$ has no equilibria. At $\mu = \lambda^2$ a centre-saddle bifurcation takes place, giving rise to a centre $(q,p) = (\lambda + \mathrm{sgn}\,(\lambda)\sqrt{\lambda^2-\mu}, 0)$ and a saddle $(q,p) = (\lambda - \mathrm{sgn}\,(\lambda)\sqrt{\lambda^2-\mu}, 0)$. The latter undergoes at $\mu = -3\lambda^2$ a Hamiltonian pitchfork bifurcation, thereby turning into a centre and giving rise to two saddles $(q,p) = (-\lambda, \pm\sqrt{3\lambda^2+\mu})$. Since there are no further bifurcations, the bifurcation set of $X_{H^-_{\lambda,\mu}}$ coincides with the catastrophe set of $H^-_{\lambda,\mu}$. Due to the reversibility the two saddles $(q,p) = (-\lambda, \sqrt{3\lambda^2+\mu})$ and $(q,p) = (-\lambda, -\sqrt{3\lambda^2+\mu})$ have the same energy and are connected by heteroclinic orbits. In the generic context of (1) this is reflected by $\{\, (\lambda,\mu,\nu) \mid \mu \leq -3\lambda^2, \nu = 0 \,\}$ being part of the bifurcation set, see [Broer,Chow,Kim,Vegter;93,95].

We now turn to the elliptic reversible umbilic. The catastrophe set of $H^+_{\lambda,\mu}$ is the union of the parabolas $\{\mu = 3\lambda^2\}$ and $\{\mu = -\lambda^2\}$, see *Figure 2.2.* For $\mu > 3\lambda^2$ the system $X_{H^+_{\lambda,\mu}}$ has two saddles $(q,p) = (\lambda + \mathrm{sgn}\,(\lambda)\sqrt{\lambda^2+\mu}, 0)$ and $(q,p) = (\lambda - \mathrm{sgn}\,(\lambda)\sqrt{\lambda^2+\mu}, 0)$. The latter undergoes at $\mu = 3\lambda^2$ a Hamiltonian pitchfork bifurcation, thereby turning into a centre and giving rise to two further saddles $(q,p) = (-\lambda, \pm\sqrt{3\lambda^2-\mu})$. Again, the heteroclinic connections between these saddles are due to reversibility. However, at $\mu = 0$ all three saddles have the same energy and a heteroclinic or *connection bifurcation* takes place. Here the bifurcation set of $X_{H^+_{\lambda,\mu}}$ extends the catastrophe set of $H^+_{\lambda,\mu}$. For $\mu < 0$ the saddle $(q,p) = (\lambda + \mathrm{sgn}\,(\lambda)\sqrt{\lambda^2+\mu}, 0)$ has a homoclinic orbit around the centre, and at $\mu = -\lambda^2$ these two equilibria meet in a centre-saddle bifurcation and vanish.

§3 Applications

A *gyrostat* consists of a rigid body, to which one or several flywheels are attached in such a way that the individual rotation of the latter does not influence the mass distribution of the gyrostat, see [Leimanis;65], [Elipe;96], [Elipe,Arribas,Riaguas;97]. We concentrate on a gyrostat with two flywheels rotating about the two stable principal axes of inertia. In [Elipe;96], [Elipe,Arribas,Riaguas;97] the Hamilton function of the system is expressed as

$$H_{f,h}(g_1,g_2,g_3) \;=\; \frac{1}{2}\sum_{i=1}^{3} a_i g_i^2 \;-\; a_1 f g_1 \;-\; a_3 h g_3 \tag{3}$$

where $a_1 > a_2 > a_3 > 0$ are the inverse of the principal moments of inertia, g_i are the components of the angular momentum of the system in the body representation, and

f, h denote the moments of the (relative) rotation of the two flywheels. In the absence of external forces or torques the latter, as well as the total angular momentum $G = \sqrt{g_1^2 + g_2^2 + g_3^2}$ are conserved quantities. Thus we are left with a one-degree-of-freedom problem on the sphere $\{ g_1^2 + g_2^2 + g_3^2 = G^2 \}$ with Poisson bracket $\{g_i, g_j\} = -\varepsilon_{ijk} g_k$ where the alternating Levi-Civita symbol ε_{ijk} denotes the sign of the permutation $\left(\begin{smallmatrix} i & j & k \\ 1 & 2 & 3 \end{smallmatrix} \right)$. Note that the system is reversible with respect to the reflection $g_2 \mapsto -g_2$.

Following [Elipe;96], [Elipe,Arribas,Riaguas;97] we substract the Casimir $a_2 G^2$, rescale time by $(a_1 - a_2)G^2$ and introduce dimensionless variables $u = \frac{g_1}{G}$, $v = \frac{g_3}{G}$ and $w = \frac{g_2}{G}$, thereby 'relabelling' the last two variables. This leads to the Hamilton function

$$H(u, v, w) \;=\; \frac{1}{2} u^2 \;+\; \frac{P}{2} v^2 \;+\; Q u \;+\; R v \tag{4}$$

on the unit sphere (with Poisson bracket $\{u, v\} = w$ and cyclic permutations) where

$$P = \frac{a_3 - a_2}{a_1 - a_2} < 0 \quad , \quad Q = \frac{-a_1}{a_1 - a_2} \frac{f}{G} \quad , \quad R = \frac{-a_3}{a_1 - a_2} \frac{h}{G} \qquad .$$

The 3-parameter family (4) has been studied in [Lanchares,Iñarrea,Salas,Sierra,Elipe;95]. For $P < 0$ the bifurcation diagram is shown to consist of a surface of centre-saddle bifurcations and a surface of Hamiltonian pitchfork bifurcations, touching along the four lines

$$Q_P \;=\; \frac{\pm 1}{\sqrt{1 - P}} \quad , \quad R_P \;=\; \frac{\pm P \sqrt{-P}}{\sqrt{1 - P}}$$

parametrised by $P \in \,]-\infty, 0[$. The bifurcating equilibrium has co-ordinates

$$u = -Q_P \quad , \quad v = -\frac{R_P}{P} \quad , \quad w = 0 \qquad .$$

We claim that a hyperbolic reversible umbilic bifurcation takes place.

Note that the value of P is fixed by the design of the gyrostat, while Q and R depend on the initial conditions. For this reason the latter have to be treated as internal or *distinguished parameters* in that we only allow reparametrisations of the form

$$(P, Q, R) \;\;\mapsto\;\; (\tilde{P}(P), \tilde{Q}(P, Q, R), \tilde{R}(P, Q, R))$$

to ensure that the new $\tilde{Q}, \tilde{R}$ can still be interpreted as values of phase space variables, cf. [Wassermann;75], [Broer,Chow,Kim,Vegter;93]. As we are interested in a neighbourhood of (P, Q_P, R_P) we introduce

$$\beta \;=\; Q - Q_P \quad , \quad \gamma \;=\; R - R_P \qquad .$$

We choose an (x, y)-chart on S^2 around the bifurcating equilibrium according to

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\stackrel{!}{=}\; \sqrt{1 - x^2 - y^2} \begin{pmatrix} -Q_P \\ -\frac{R_P}{P} \\ 0 \end{pmatrix} \;+\; x \begin{pmatrix} \frac{R_P}{P} \\ -Q_P \\ 0 \end{pmatrix} \;+\; y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad .$$

Geometrically this corresponds to a projection along the equilibrium vector. In particular x and y are not symplectic co-ordinates, having Poisson bracket $\{x, y\} = \sqrt{1 - x^2 - y^2}$. The reversing involution amounts in these co-ordinates to $y \mapsto -y$ and the 3-jet of the Hamilton function reads

$$H^{\text{3-jet}}_{\beta,\gamma}(x, y) \;=\; \frac{-\epsilon\sqrt{-P}}{2}(x^3 + xy^2) \;+\; (\beta Q_P + \gamma \frac{R_P}{P})\frac{x^2 + y^2}{2} \;+\; (\beta \frac{R_P}{P} - \gamma Q_P)x \qquad (5)$$

where ϵ denotes the sign of the product $Q_P R_P$. To put (5) into the form (2), with standard symplectic structure $\mathrm{d}q \wedge \mathrm{d}p$, we apply the co-ordinate transformation

$$\begin{aligned} x \;&\mapsto\; q(x) \;=\; \sqrt[4]{3}\, x \;-\; \frac{2\epsilon}{\sqrt[4]{27P^2}}(\beta Q_P + \gamma \frac{R_P}{P}) \\ y \;&\mapsto\; p(x, y) \;=\; \frac{-\epsilon}{\sqrt[4]{3}} \arcsin \frac{y}{\sqrt{1 - x^2}} \qquad , \end{aligned}$$

rescale time by $-\frac{\epsilon}{2}\sqrt[4]{3P^2}$ and use the reparametrisation

$$\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \;\mapsto\; \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \;=\; \frac{-\epsilon}{3\sqrt{-P}} \begin{pmatrix} 0 & \sqrt[4]{3} \\ 2\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{R_P}{P} & -Q_P \\ Q_P & \frac{R_P}{P} \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

which combines scaling and relabelling with a rotation. This proves the following result.

Theorem 3.1

Let H be the Hamilton function (3) *of a gyrostat with moments of inertia* $0 < \frac{1}{a_1} < \frac{1}{a_2} < \frac{1}{a_3}$ *and two flywheels rotating about the first and third principal axes of inertia, with moments f and h . Then the system undergoes at*

$$f \;=\; \pm\sqrt{\tfrac{(a_1 - a_2)^3}{a_1^3 - a_1^2 a_3}}\; G \quad , \quad h \;=\; \pm\sqrt{\tfrac{(a_2 - a_3)^3}{a_1 a_3^2 - a_3^3}}\; G$$

a hyperbolic reversible umbilic bifurcation, where G denotes the total angular momentum.

Keeping f and h fixed and varying only G we recover the 'double teardrop bifurcation' reported in [Lanchares,Iñarrea,Salas,Sierra,Elipe;95].

It is instructive to break the reversibility of the system by introducing a third flywheel rotating about the unstable principal axis of inertia, with momentum k . Repeating the above considerations shows that one merely has to add the term νy in (5), with $\nu = \frac{-a_2}{a_1 - a_2}\frac{k}{G}$. In this way the hyperbolic reversible umbilic bifurcation is embedded in a (general) hyperbolic umbilic bifurcation of the gyrostat with three flywheels.

We now turn to the normalised Hénon-Heiles system. The Hénon-Heiles system serves as a model equation for the motion in axially symmetric galaxies in the central region of a few core radii, see [Hénon,Heiles;64]. The invariance under rotations about the third spatial axis allows to reduce to a two-degrees-of-freedom problem, parametrised by the value a of the third component N of the angular momentum. Most studies in the literature are restricted to the subsystem $a = 0$, see *e.g.* [Hénon,Heiles;64], [Gustavson;66], [Verhulst;79], [Lichtenberg,Lieberman;83], and the influence of non-zero values of N on the dynamics has been investigated only recently in [Ferrer,Palacián,San Juan,Viartola,Yanguas;97], [Ferrer,Lara,Palacián,San Juan,Viartola,Yanguas;97a,b]. Near the origin a scaling by ε turns the system into

$$H_\varepsilon(x,p) \;=\; \frac{1}{2}\sum_{i=1}^{3} p_i^2 \;+\; \frac{\omega^2}{2}\sum_{i=1}^{3} x_i^2 \;+\; \varepsilon\,(\frac{x_3^3}{3} - (x_1^2+x_2^2)x_3)$$

and reveals it to be a perturbation of the isotropic harmonic oscillator $H_0 = \omega L$, $L = \frac{1}{2\omega}\sum p_i^2 + \frac{\omega}{2}\sum x_i^2$. The unperturbed system has a periodic flow with frequency $\omega > 0$, and a normal form procedure makes this flow a formal S^1-symmetry of H_ε.

In [Ferrer,Lara,Palacián,San Juan,Viartola,Yanguas;97b] the fourth order normal form $\bar{H}_\varepsilon$ is analysed. The two S^1-actions defined by the Hamiltonian vector fields X_N and X_L commute and induce a 2-torus action, which allows to reduce to a family $\bar{H}^{a,b}$ of one-degree-of-freedom systems, parametrised by the values (a,b) of the momentum mapping (N,L). We remark that the corresponding family defined by the second order normal form turns out to be degenerate, while the first and third order normal form do not contribute extra terms as polynomials of uneven order vanish under normalization. Note that the dependence on the external parameter ε is suppressed in the notation. Indeed, the internal parameters a and b act as *distinguished parameters* in that we only consider reparametrisations of the form

$$(a,b,\varepsilon) \;\mapsto\; (\tilde{a}(a,b,\varepsilon), \tilde{b}(a,b,\varepsilon), \tilde{\varepsilon}(\varepsilon))$$

to ensure that the new $\tilde{a}, \tilde{b}$ can still be interpreted as values of phase space variables.

We use a special set of action-angle variables for the isotropic harmonic oscillator to write down the normalised Hamilton function, the nodal-Lissajous variables (ℓ, g, ν, L, G, N) proposed in [Ferrer,Gárate;96]. The actions N and L are completed by the total angular momentum G, while the conjugate angles ν, ℓ and g allow for an intuitive geometric interpretation in view of the orbital ellipses of the oscillator. Indeed, the angle ℓ describes the position of the point x on this ellipse in configuration space, measured from the semi-minor axis, g is the angle from the nodal line to this semi-minor axis and ν represents the angle of the ascending node of the orbital plane, the inclination of which is given by $\cos I = \frac{N}{G}$.

In particular, the above 2-torus action amounts to rotating along ν and ℓ, making these angles cyclic variables of the normal form. In this way we may use (g, G) as local co-ordinates on the reduced phase space, double-valued because (ℓ, g, ν, L, G, N) and $(\ell - \pi, g + \pi, \nu, L, G, N)$ represent the same point (x,p). The (single-valued) co-ordinates

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Figure 3.1: Part of the bifurcation diagram of $X_{\bar{H}^{a,b}}$, with axes b and $\frac{a}{b}$. The phase portraits to the small regions are depicted below.

$(2g, G)$ can be thought of as co-ordinates on a topological 2-sphere. While this sphere is smooth at the "north pole" $G = b$, the "south pole" $G = |a|$ is a conical singularity.

To simplify the formula for the family $\bar{H}^{a,b}$ we scale time to achieve $\omega = 1$. Furthermore we put $\varepsilon = 1$. Indeed, being close to the origin $(x, p) = (0, 0)$ may also be accomplished by restricting to small values of L. Up to constant terms the reduced Hamilton function then reads

$$\begin{aligned} \bar{H}^{a,b}(2g, G) \;\; = \;\; & \tfrac{84-7b}{144} G^2 + \tfrac{140a^4b+63a^2b^3}{54} G^{-2} - \tfrac{7a^6b+14a^4b^3}{3} G^{-4} + \tfrac{35a^6b^3}{9} G^{-6} \\ & - \left(\tfrac{7a^2}{27} G^4 - \tfrac{7a^4+14a^2b^2}{6} G^2 + \tfrac{35a^4b^2}{6} \right) \sqrt{b^2 - G^2} \, (G^2 - a^2) G^{-6} \cos 2g \\ & + \tfrac{7a^2b}{3} (b^2 - G^2)(G^2 - a^2)^2 G^{-6} \cos 4g \\ & - \tfrac{7}{18} (b^2 - G^2)^{\frac{3}{2}} (G^2 - a^2)^3 G^{-6} \cos 6g \qquad , \end{aligned} \tag{6}$$

see [Ferrer,Lara,Palacián,San Juan,Viartola,Yanguas;97b]. Being a "simple mechanical system" — the Hamilton function is the sum of the potential energy which only depends on the position x and the kinetic energy which is a quadratic form in the momentum p — the Hénon-Heiles system is reversible with respect to $(x, p) \mapsto (x, -p)$, and this is preserved under the normalisation process. In nodal-Lissajous variables the involution reads

$$(\ell, g, \nu, L, G, N) \;\; \mapsto \;\; (-\ell, -g, \nu, L, G, -N) \qquad ,$$

leading to the involution $(2g, G) \mapsto (-2g, G)$ on the reduced phase space which reveals $X_{\bar{H}^{a,b}}$ to be a family of reversible systems.

In [Ferrer,Lara,Palacián,San Juan,Viartola,Yanguas;97b] the bifurcation set of the family (6) is computed and the occurring phase portraits are given. Here we focus on the region

$$\{ \, (a, b) \in \mathbb{R}^2 \;\; \big| \;\; \tfrac{36}{39} \leq b \leq 2 \, , \, 0.23 \leq \frac{a}{b} \leq 0.35 \, \}$$

of the parameter space, cf. *Figure 3.1.* This part of the bifurcation diagram is organised by the three points $E_8 : (a, b) \approx (0.34, 1.12)$, $E_{10} : (a, b) \approx (0.43, 1.52)$ and $E_9 : (a, b) \approx (0.42, 1.25)$ where the bifurcation lines are tangent or form a cusp, respectively. We visualize the flow by 'painting the Hamilton function', *i.e.* using a colour code for the values of $\bar{H}^{a,b}$, see [Pascua,Rubio,Viartola,Ferrer;96]. This strongly suggests that two hyperbolic reversible umbilic bifurcations take place at E_8 and E_{10}. Furthermore, we expect $\bar{H}^{a,b}$ to undergo a cusp catastrophe at E_9, versally unfolding the singularity A_3^+ (cf. (8) below).

To prove these statements one has to check the necessary transversality conditions on the 3-jet of (6) in the bifurcating equilibrium. As already the corresponding parameter points E have unwieldy expressions, this computation is quite involved and will appear elsewhere.

§4 Extensions and open problems

Let us consider a family of reversible Hamiltonian systems X_{H_α} in several degrees of freedom around a bifurcating equilibrium, with a double eigenvalue zero and semi-simple linearization. Further hyperbolic eigenvalues may be dealt with by means of a centre manifold, so we concentrate on the case that all eigenvalues are purely imaginary. In suitable co-ordinates the 2-jet of the bifurcating Hamilton function reads

$$H_0^{\text{2-jet}}(x,y) \;=\; 0 \;+\; \frac{\omega_2}{2}(x_2^2+y_2^2) \;+\; \ldots \;+\; \frac{\omega_n}{2}(x_n^2+y_n^2) \;=\; \sum_{i=1}^{n} \omega_i I_i$$

where we abbreviated $I_i = \frac{1}{2}(x_i^2+y_i^2)$. Excluding strong resonances between the (normal) frequencies ω_i we can apply a Birkhoff normalisation to obtain

$$H_\alpha(x,y) \;=\; F_\alpha(x_1,y_1,I_2,\ldots,I_n) \;+\; \textit{higher order terms} \qquad ,$$

see [Broer,Chow,Kim,Vegter;93,95]. The reversing involution $(x,y) \mapsto (x,-y)$ acts as a reflection $\varphi_i \mapsto -\varphi_i$ on the angles conjugate to the actions I_i . The normal form F_α is cyclic in these angles, and reducing the corresponding $(n-1)$-torus action we are led to a family F_α^I of reversible one-degree-of-freedom systems. Here we have to distinguish the "normal parameters" I , *i.e.* only reparametrisations of the form $(I,\alpha) \mapsto (\tilde{I}(I,\alpha),\tilde{\alpha}(\alpha))$ are admitted to ensure that the new $\tilde{I}$ are again phase space variables. Furthermore we require our reparametrisations to respect the boundary $I=0$ by being of the form $\tilde{I}(I,\alpha) = I \cdot J(I,\alpha)$ with $J(0,\alpha) > 0 \ \forall_\alpha$.

The singularity we encounter in F_α^0 depends on the original system X_{H_α} . Generically we only expect singularities of co-dimension m to occur, where m is the number of parameters α . For $m=2$ these are the two singularities D_4^+ and D_4^- . Still assuming genericity, singularity theory yields a further parameter-dependent co-ordinate transformation $(x_1,y_1) \mapsto (q,p)$, this time not necessarily canonical, such that

$$F_\alpha^I(q,p) \;=\; p^2 q \mp \frac{1}{3}q^3 \;+\; \lambda(I,\alpha)\,(p^2 \pm q^2) \;+\; \mu(I,\alpha)\,q \qquad . \tag{7}$$

Our restricted class of reparametrisations does not enable us to simply obtain *e.g.* $\lambda(I,\alpha) = \tilde{I}_2$. As shown in [Broer,Chow,Kim,Vegter;93], one can generically achieve $\lambda(I,\alpha) = \tilde{\alpha}_1 \pm \tilde{I}_2$. If the number n of degrees of freedom of X_{H_α} is 3 or higher, we can similarly obtain $\mu(I,\alpha) = \tilde{\alpha}_2 \pm \tilde{I}_3$ and (7) is a universal unfolding. In two degrees of freedom the situation is different as (7) cannot be further simplified, see again [Broer,Chow,Kim,Vegter;93].

Starting from (7), one now can first reconstruct the dynamics of the integrable approximation F_α in n degrees of freedom and then consider the original Hamilton function H_α as a perturbation of F_α . Since the reversible Hamiltonian case can be handled completely analogous to the general Hamiltonian case, the reader is referred to [Broer,Chow,Kim,Vegter;93,95] for further details.

One last remark concerning the distinction of parameters. Given a Hamiltonian system in $n+m$ degrees of freedom that admits an m-torus symmetry, one can reduce to an m-parameter family in n degrees of freedom, like the one we started with. Here the internal parameters, the values of the actions conjugate to the toral angles, have to play the rôle of the α . In fact, in the presence of normal parameters I , internal parameters a and further external parameters γ one is led to the class of reparametrisations

$$(I, a, \gamma) \quad \mapsto \quad (\tilde{I}(I, a, \gamma), \tilde{a}(a, \gamma), \tilde{\gamma}(\gamma)) \qquad ,$$

defining a hierarchy of distinguished parameters.

For planar reversible Hamiltonian systems the approach of Section 2 can be immediately generalized. Bifurcating equilibria are determined by the singularity of the Hamilton function, and as in the proof of Theorem 2.2 the universal Z_2-invariant unfoldings provided by equivariant singularity theory yield versal unfoldings of such equilibria. Thus, there are four more local planar reversible Hamiltonian bifurcations of co-dimension two. The singularities $A_3^{\pm}$ have the unfolding

$$H^{\pm}_{\lambda,\mu} \;=\; \frac{1}{2}\,p^2 \;\pm\; \frac{1}{4}\,q^4 \;+\; \lambda\, q^2 \;+\; \mu\, q \tag{8}$$

if $(q,p) \mapsto (q,-p)$ is the reversing involution. In the case "$+\frac{1}{4}q^4$" this is the cusp catastrophe we already encountered in the normalised Hénon-Heiles system. The case "$-\frac{1}{4}q^4$" is the dual cusp catastrophe where for all non-zero unfolding parameters (λ, μ) at least one saddle is present. As all occurring equilibria lie on the fixed point set $\{p = 0\}$ of the reversing involution, the bifurcation set is the union of the catastrophe set $\{32\lambda^3 + 27\mu^2 = 0\}$ with the half line $\{\lambda > 0, \mu = 0\}$ where a connection bifurcation between the two saddles takes place.

In case the reversing involution is $(q,p) \mapsto (-q,p)$ the universal unfolding of the singularities $A_5^{\pm}$

$$H^{\pm}_{\lambda,\mu} \;=\; \frac{1}{2}\,p^2 \;\pm\; \frac{1}{6}\,q^6 \;+\; \lambda\, q^4 \;+\; \mu\, q^2$$

is related to the butterfly catastrophe. In the case "$+\frac{1}{6}q^6$" the catastrophe set $\{\mu = 0\} \cup \{\mu = 2\lambda^2, \lambda < 0\}$ again coincides with the bifurcation set, while the catastrophe set $\{\mu = 0\} \cup \{\mu = -2\lambda^2, \lambda > 0\}$ in the case "$-\frac{1}{6}q^6$" has to be completed by a connection bifurcation at $\{\mu = -\frac{3}{2}\lambda^2, \lambda > 0\}$ to yield the bifurcation set.

We conclude with a list of all local planar reversible Hamiltonian bifurcations of co-dimension three. The singularities of type A that occur with co-dimension three are A_4 and $A_7^{\pm}$. Their equivariant universal unfoldings are

$$\begin{array}{lll} \frac{1}{2}\,p^2 \;+\; \frac{1}{5}\,q^5 \;+\; \lambda\, q^3 \;+\; \mu\, q^2 \;+\; \nu\, q & \quad\text{under}\quad & (q,p) \mapsto (q,-p) \\ \frac{1}{2}\,p^2 \;\pm\; \frac{1}{8}\,q^8 \;+\; \lambda\, q^6 \;+\; \mu\, q^4 \;+\; \nu\, q^2 & \quad\text{under}\quad & (q,p) \mapsto (-q,p) \qquad , \end{array}$$

the former being the swallow tail catastrophe. For equilibria with vanishing linearization we always can arrange the reversing involution to be $(q,p) \mapsto (q,-p)$. The universal unfolding

$$p^2 q \;+\; \frac{1}{4} q^4 \;+\; \lambda\, p^2 \;+\; \mu\, q^2 \;+\; \nu\, q$$

of the singularity D_5 is related to the parabolic umbilic catastrophe, while the universal unfolding

$$\frac{1}{4} p^4 \;+\; \frac{1}{3} q^3 \;+\; \lambda\, p^2 q \;+\; \mu\, p^2 \;+\; \nu\, q$$

of the singularity E_6 is derived from the symbolic umbilic catastrophe. Note that the parabolic equilibria A_4 and $A_7^{\pm}$ only unfold to other singularities of type A , whereas the reversible parabolic and symbolic umbilic bifurcations diagrams also contain the reversible elliptic and hyperbolic umbilic in a subordinate way.

As discussed in the introduction, a typical way to obtain a 2-parameter family of Hamiltonian systems in one degree of freedom is to start with a Hamiltonian system in three degrees of freedom and reduce a 2-torus action. Often this 2-torus action does not leave the original Hamilton function invariant, but some normal form that acts as an integrable approximation. This poses the problem whether the bifurcation scenario found in the reduced system (and reconstructed to a bifurcation of invariant 2-tori) persists the perturbation to the original system. In two degrees of freedom the corresponding question concerns periodic orbits and can be answered by means of the implicit mapping theorem.

To address the robustness of bifurcations of invariant tori one has to use KAM-techniques. In particular the resonant tori are not expected to survive the perturbation. The torus bifurcations of co-dimension one that occur in dissipative systems have been treated in [Braaksma,Broer,Huitema;90], while [Hanßmann;96] contains the simplest Hamiltonian torus bifurcation. In future research we plan to investigate the quasi-periodic counterpart of the umbilic bifurcations.

We finally consider reversible systems in two dimensions that are not necessarily Hamiltonian. Versal unfoldings of local bifurcations in such systems have been recently given in [Teixeira;97], and we will recover some of these results. The most general 2-jet of a reversible vector field in an equilibrium with vanishing linearization is given by

$$\begin{aligned} \dot{x} &= a\, x\, y \\ \dot{y} &= b\, y^2 \;+\; c\, x^2 \end{aligned} \tag{9}$$

where $(x, y) \mapsto (x, -y)$ is the reversing involution. By means of a scaling we may restrict to $b = -1$ and $c = \pm 1$. With these choices the two Hamiltonian cases are given by $a = 2$. This suggests the unfolding

$$\begin{aligned} \dot{x} &= a\, x\, y \;+\; \lambda\, y \\ \dot{y} &= -y^2 \;\pm\; x^2 \;\mp\; \lambda\, x \;+\; \mu \qquad . \end{aligned} \tag{10}$$

In fact, for $a > 0$ the flow defined by the subfamily $\lambda = 0$ of (10) is C^0-equivalent to the flow defined by the corresponding subfamily of (2) as the function

$$F(x, y) \;\;=\;\; \frac{a}{2}\, \mathrm{e}^{\frac{1}{a} \ln x^2} (y^2 \mp \frac{x^2}{1+a} - \mu)$$

provides a first integral that extends to $\{x = 0\}$, cf. [Guckenheimer,Holmes;86]. The proof of Theorem 4.1 will also show that this equivalence may be extended to $\lambda \neq 0$.

Note that, while we could use C^∞-equivalences in the (reversible) Hamiltonian context, we have to work for (non-Hamiltonian) reversible systems with the weaker notion of C^0-equivalences. Indeed, C^k-equivalences with $k \geq 1$ keep the ratio of eigenvalues of an equilibrium invariant, and while in a planar Hamiltonian system this ratio is always -1 , it may change under small perturbations for equilibria of a reversible system that lie off the symmetry axis.

Theorem 4.1

For $a > 0$ the 2-parameter family of vector fields defined by (10) is a C^0-versal unfolding of the reversible vector field $axy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} \pm x^2\frac{\partial}{\partial y}$.

Proof :

To lead the situation back to the Hamiltonian case dealt with in Theorem 2.2, we have to show that the flow defined by a given reversible unfolding leaves invariant a measure equivalent to the Lebesgue measure. This is decided by the recurrent part of the dynamics. As all equilibria off the fixed point set $\{y = 0\}$ of the reversing involution are saddles, there are no point attractors. In the neighbourhood of equilibria $(x, 0)$ the flow is already known to be equivalent to a Hamiltonian flow. Indeed, these are either centres, or parabolic of co-dimension one, or given by (10) with $a > 0$ and $\lambda = 0 = \mu$. In the situation at hand a periodic orbit has to encircle a centre and thus has to cross the fixed point set $\{y = 0\}$. This leads to whole 1-parameter families of periodic orbits that extend to homo- or heteroclinic orbits, revealing also this part of the dynamics to be equivalent to a Hamiltonian flow **;** **q.e.d.**

The technique in [Teixeira;97] is based on the treatment in [Višik;72], [Teixeira;77,79] of vector fields close to boundaries, see [Siersma;81] for the corresponding theory for functions. In [Teixeira;97] the two cases with $a > 0$ are called tangent saddles. The cases with $a < 0$, called tangent nodes in [Teixeira;97], are of essentially non-Hamiltonian nature. Indeed, already the flows for $\lambda = 0 = \mu$ do not preserve a measure equivalent to the Lebesgue measure. The bifurcating equilibria are further discriminated by $-1 < a < 0$ and $a < -1$, see [Guckenheimer,Holmes;86]. For the versal unfoldings (10) it is not necessary to distinguish between $-1 < a < 0$ and $a < -1$ as these unfoldings are C^0-equivalent off $\lambda = 0 = \mu$, cf. [Teixeira;97].

A different approach is pursued in [Teixeira,Jacquemard;94]. The bifurcating system as well as the linearizations of the systems in the unfolding are supposed to be reversible, while the higher order terms in the unfolding are allowed to break this symmetry. In further contrast to our approach the origin is always required to be an equilibrium.

The vector field (9) is furthermore symmetric under the reflection $(x, y) \mapsto (-x, y)$. With respect to reversible systems that also satisfy this Z_2-symmetry the co-dimension becomes 1 and correspondingly one restricts in (10) to $\lambda = 0$, see [Lamb;94], [Lamb,Capel;95].

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