PROOF OF PERSISTENCE OF INVARIANT TORI IN NONHAMILTONIAN PERTURBATIONS OF INTEGRABLE SYSTEMS

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1. Introduction 0

We study here the dynamics of the system

$$\begin{cases} \dot{I} = \varepsilon\ f(I,\phi,\varepsilon) \\ \dot{\phi} = \omega_0(I) + \varepsilon\ g(I,\phi,\varepsilon) \end{cases} \tag{1}1$$

where $I \in \mathcal{G} \subseteq \mathbb{R}^n$ (with $\mathcal{G}$ open) are the slow variables, $\phi \in \mathbf{T}^m$ are the fast angular variables, and ε is a (small) real parameter. We assume that all functions appearing here are analytic.

We will prove that, provided the average of f over the angles has a hyperbolic attractive zero at a point $I_* \in \mathcal{G}$, and the corresponding frequency $\omega_0(I_*)$ is sufficiently nonresonant, then there exists a normally hyperbolic attractive invariant torus of the system (1) which is close to the torus $I_* \times \mathbf{T}^m$.

We introduce the set $\Gamma(\tau,\gamma,K)$ of variables which are "τ,γ diophanitine up to order K", namely we define

$$\Gamma(\tau,\gamma,K) := \left\{ I \in \mathcal{G}\ :\ |\omega_0(I)\cdot k| \geq \frac{\gamma}{|k|^\tau}\ ,\quad \forall k \in \mathbf{Z}\setminus\{0\}\ \text{such that}\ |k| := \sum_{i=1}^m |k_i| \leq K \right\}\ .$$

We will denote by $\langle f\rangle$ the average of the main part of f with respect to the angles, namely

$$\langle f\rangle(I) := \frac{1}{(2\pi)^m}\int_{\mathbf{T}^m} f(I,\phi,0)d^m\phi$$

(similarly we will use the notation $\langle g\rangle$).

Consider $\langle f\rangle$: we assume that there exists an attractive zero I_* of $\langle f\rangle$, namely a zero such that all the eigenvalues of $d\langle f\rangle(I_*)$ are strictly negative

Theorem 1.1m. *Assume that there exist constants τ,γ,K such that $I_* \in \Gamma(\tau, gamma, K)$, thne there exist constants ϵ_*, K_*, C_1, such that, if $K > K_* \left|\ln\varepsilon_*\right|^\tau$, and $\exp\frac{K}{K_*} < \varepsilon < \varepsilon_*$, then (1) has a stable invariant torus close to the torus $\mathbf{T}_* := I_* \times \mathbf{T}^m$, precisely, one has*

$$\min_{(I,\phi)\in\mathbf{T}_*} \|I - I_*\| \leq C_1\varepsilon\left|\ln\varepsilon\right|^{2\tau+1} \tag{2}\mathtt{dis}$$

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2. The normal form lemma nor

We first give a quantitative form to our smoothness assumptions. Let $D \subset \mathcal{G}$ (strictly) be open. We introduce the complex open set

$$D_\rho := \bigcup_{I \in \mathcal{D}} B_\rho(I) \ ,$$

where $B_\rho(I) \subset \mathbf{C}^n$ is the closed ball of radius ρ centered at I. We also introduce the complexification of the m–dimensional torus

$$(\mathbf{T} + i\sigma)^m := \{(\phi_i) \in (\mathbf{T} + i\,\mathrm{I\!R})^m \ : \ |\mathrm{Im}\,\phi_i| \le \sigma\} \ .$$

Then, there exist some $\bar\rho, \bar\sigma, \eta > 0$ such that the functions f, g, and ω_0 can be extended to complex analytic functions on

$$\mathcal{D}_{\bar\rho, 2\bar\sigma, \eta} := D_{\bar\rho} \times (\mathbf{T}^m + i2\bar\sigma) \times \mathcal{B}_\eta(0) \ ,$$

where $\mathcal{B}_\eta(0) := \{\epsilon \in \mathbf{C} \ : \ |\varepsilon| \le \eta\}$, which moreover are bounded on such set; moreover, there exist constants F, G, and Ω such that

$$\sup_{\mathcal{D}_{\bar\rho,2\bar\sigma,\eta}} \|f(I,\phi,\varepsilon)\| \le F$$
$$\sup_{\mathcal{D}_{\bar\rho,2\bar\sigma,\eta}} \|g(I,\phi,\varepsilon)\| \le G$$
$$\sup_{\mathcal{D}_{\bar\rho,2\bar\sigma,\eta}} \|d\omega_0(I)\| \le \Omega \ .$$

We will also consider the complex extension

$$\Gamma_\rho(\tau,\gamma,K) := \bigcup_{I \in \Gamma(\tau,\gamma,K)} B_\rho(I) \ .$$

Here we will prove the following

Lemma 2.1nf.1. *Fix $\tau \ge m$ and $\gamma > 0$, then, provided ϵ is small enough there exist constants $C_2, ..., C_8$ and an analytic coordinate transformation*

$$\begin{aligned} I &= J + \varepsilon A(J,\psi,\varepsilon) \ , \\ \phi &= \psi + \varepsilon B(J,\psi,\varepsilon) \ , \end{aligned} \tag{3}$$
c.c

defined on $\Gamma_{\rho/2}(\tau,\gamma,C_2\,|\ln\varepsilon|^\tau) \times (\mathbf{T} + i\bar\sigma/2)^m$, where

$$\rho := \frac{C_3}{|\ln\varepsilon|^{\tau+1}} \ ; \tag{4}$$
rho

which transforms (1) into the system

$$\begin{aligned} \dot J &= \varepsilon\langle f\rangle(J) + \varepsilon^2 Z(J,\varepsilon) + R(J,\psi,\varepsilon) \\ \dot\psi &= \omega_0(J) + \varepsilon\langle g\rangle(J) + R_1(J,\psi,\varepsilon) \ . \end{aligned} \tag{5}$$
sys

Moreover, the vector valued functions A, B, Z, R, R_1 are analytic on $\Gamma_{\rho/2}(\tau,\gamma,C_2\,|\ln\varepsilon|^\tau) \times (\mathbf{T}+i\bar\sigma/2)^m$, and, satisfy the estimates

$$\sup_{\Gamma_{\rho/2}(\tau,\gamma,C_2|\ln\varepsilon|^\tau)\times(\mathbf{T}+i\bar\sigma/2)^m} \|A\| \le C_4\,|\ln\varepsilon|^\tau$$
$$\sup_{\Gamma_{\rho/2}(\tau,\gamma,C_2|\ln\varepsilon|^\tau)\times(\mathbf{T}+i\bar\sigma/2)^m} \|B\| \le C_5\,|\ln\varepsilon|^{2\tau}$$
$$\sup_{\Gamma_{\rho/2}(\tau,\gamma,C_2|\ln\varepsilon|^\tau)\times(\mathbf{T}+i\bar\sigma/2)^m} \|Z\| \le C_6\,|\ln\varepsilon|^{2\tau+1}$$
$$\sup_{\Gamma_{\rho/2}(\tau,\gamma,C_2|\ln\varepsilon|^\tau)\times(\mathbf{T}+i\bar\sigma/2)^m} \|R\| \le C_7\varepsilon^3\,|\ln\varepsilon|^{4\tau+2}$$
$$\sup_{\Gamma_{\rho/2}(\tau,\gamma,C_2|\ln\varepsilon|^\tau)\times(\mathbf{T}+i\bar\sigma/2)^m} \|R_1\| \le C_8\varepsilon^2\,|\ln\varepsilon|^{3\tau+1}$$

Remark 2.21.2. *The dependence of A, B, Z, R, R_1 on ε is in general discontinuous (see remark 2.4 below).*

`0) Preliminaries`

We will first decompose f and g in parts of different order in ε. So we write

$$f(I,\phi,\varepsilon) = \tilde f^0(I,\phi) + \varepsilon \tilde f^1(I,\phi) + \varepsilon^2 \tilde f^2(I,\phi,\varepsilon)$$
$$g(I,\phi,\varepsilon) = \tilde g^0(I,\phi) + \varepsilon \tilde g^1(I,\phi,\varepsilon) \ ,$$

where

$$\tilde f^0(I,\phi) := f(I,\phi,0) \ , \quad \tilde f^1 := \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} \ , \quad \tilde f^2 := \frac{1}{\varepsilon^2}\left[f - \tilde f^0 - \varepsilon \tilde f^1\right] \ ,$$
$$\tilde g^0(I,\phi) := g(I,\phi,0) \ , \quad \tilde g^1 := \frac{1}{\varepsilon}\left[g - \tilde g^0\right] \ .$$

Then we perform the so called "ultraviolet cutoff", so we fix a positive K (which will be related to ε in a while), and define

$$\begin{aligned} f^0(I,\phi) &:= \sum_{|k|\le K} \tilde f^0_k(I) e^{ik\phi} \\ f^1(I,\phi) &:= \frac{1}{\varepsilon}\sum_{K<|k|\le 2K} \tilde f^0_k(I) e^{ik\phi} + \sum_{|k|\le 2K} \tilde f^1_k(I) e^{ik\phi} \\ f^2(I,\phi,\varepsilon) &:= \tilde f^2(I,\phi,\varepsilon) + \frac{1}{\varepsilon^2}\sum_{|k|>2K} \tilde f^0_k(I) e^{ik\phi} + \frac{1}{\varepsilon}\sum_{|k|>2K} \tilde f^1_k(I) e^{ik\phi} \\ g^0(I,\phi) &:= \sum_{|k|\le K} \tilde g^0_k(I) e^{ik\phi} \\ g^1(I,\phi,\varepsilon) &:= \tilde g^1(I,\phi,\varepsilon) + \frac{1}{\varepsilon}\sum_{|k|>K} \tilde g^0_k(I) e^{ik\phi} \ , \end{aligned} \tag{6)Ta}$$

where $\tilde f^i_k$ is the $k-th$ fourier coefficient of $\tilde f^i$, and similarly for $\tilde g^i_k$ (and for the Fourier coefficients of other functions incountered below).

Lemma 2.3uc. *Define*

$$\begin{aligned} F_0 &:= F\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \\ F_1 &:= F\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \frac{e^{-\sigma K/2}}{\varepsilon} + \frac{F}{\eta}\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \\ F_2 &:= \frac{8F}{\eta} + F\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \left(\frac{e^{-\sigma K/2}}{\varepsilon}\right)^2 + \frac{F}{\eta}\frac{e^{-\sigma K/2}}{\varepsilon} \\ G_0 &:= G\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \\ G_1 &:= \frac{2G}{\eta} + G\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^m \frac{e^{-\sigma K/2}}{\varepsilon} \ . \end{aligned}$$

Then one has

$$
\begin{aligned}
\sup_{\mathcal{D}_{\bar\rho,\bar\sigma,\eta/2}} \left\| f^i \right\| &\le \sum_{|k|\le (i+1)K} \sup_{I\in D_\rho} \left\| f^i_k(I) \right\| e^{\bar\sigma |k|} \le F_i \ , \quad i=0,1 \\
\sup_{\mathcal{D}_{\bar\rho,\bar\sigma,\eta/2}} \left\| g^0 \right\| &\le \sum_{|k|\le K} \sup_{I\in D_\rho} \left\| g^0_k(I) \right\| e^{\bar\sigma |k|} \le G_0 \ , \\
\sup_{\mathcal{D}_{\bar\rho,\bar\sigma,\eta/2}} \left\| f^2 \right\| &\le F_2 \ , \quad \sup_{\mathcal{D}_{\bar\rho,\bar\sigma,\eta/2}} \left\| g^1 \right\| \le G_1
\end{aligned}
\qquad (7)\mathtt{A.2}
$$

Proof. Just use Cauchy inequality to estimate $\tilde f^1$ and $\tilde g^1$, and (following [1], lemma 8) the exponential decay of Fourier coefficients to get the thesis. □

In order to obtain that the constants F_i and G_i do not depend on ε we choose

$$
K := \frac{2}{\sigma} |\ln \varepsilon| \ . \qquad (8)(\mathtt{k})
$$

We are thus reduced to the system

$$
\begin{aligned}
\dot I =& \varepsilon f^0(I,\phi) + \varepsilon^2 f^1(I,\phi) + \varepsilon^3 f^2(I,\phi,\varepsilon) \ , \\
\dot \phi =& \omega_0(I) + \varepsilon g^0(I,\phi) + \varepsilon^2 g^1(I,\phi,\varepsilon) \ ,
\end{aligned}
\qquad (9)\mathtt{v}
$$

Remark 2.41.1. *The functions f^0, f^1, g^0 (and therefore also the functions f^2, g^1) depend on ϵ through K (cf. (6) and (8)), and such a dependence is in general discontinuous.*

`1) Formal theory`

We look for a formal coordinate transformation

$$
\begin{aligned}
I =& J + \epsilon A_1(J,\psi) + \epsilon^2 A_2(J,\psi,\varepsilon) = J + \varepsilon A(J,\psi,\varepsilon) \\
\phi =& \psi + \epsilon B(J,\psi) \ ,
\end{aligned}
\qquad (10)\mathtt{t}
$$

which reduces the system (9) to the form (5). We substitute the first of (10) in the first of (9) and determine A_1 in order to eliminate at first order the dependence of f on the angles. We have

$$
\dot J = \varepsilon \left[f^0(J,\psi) - d_\psi A_1(J,\psi)\omega_0(J) \right] + o(\varepsilon) \qquad (11)\mathtt{A1}
$$

(here and in the following, the notation $d_\psi A \omega_0$ means the application of the differential with respect to the angles to ω_0, in coordinates it has the form $\omega_0^i \partial A/\partial \psi^i$, and similarly for equivalent notations to be used below).

Since

$$
f^0(J,\psi) = \sum_{|k|\le K} f^0_k(J) e^{ik\cdot\psi} \ ,
$$

we define (as usual)

$$
A_1(J,\psi) := \sum_{0\ne|k|\le K} \frac{f^0_k(J)}{i\omega_0(J)\cdot k} e^{ik\cdot\psi} \ , \qquad (12)\mathtt{A.1}
$$

which makes sense provided $\omega_0(J)\cdot k \ne 0 \ \forall k$ with $|k|\le K$ and $\forall J$ in the domain we are interested in; with this, the square bracket in (11) reduces to $\langle f \rangle$. We consider now the equation for ψ. Substituting (10) in it we obtain

$$
\dot\psi = \omega_0(J) + \varepsilon \left[g^0(J,\psi) + d_J\omega_0(J) A_1(J,\psi) - d_\psi B(J,\psi)\omega_0(J) \right] + o(\varepsilon) \ . \qquad (13)\mathtt{psip}
$$

We choose B as (using an obvious notation)

$$B(J,\psi) = \sum_{0\neq|k|\leq K} \frac{1}{i\omega_0(J)\cdot k}\left(g_k^0(J) - d_J\omega_0(J)\frac{f_k^0(J)}{i\omega_0(J)\cdot k}\right)e^{ik\cdot\psi} \ , \qquad (14)\texttt{B1}$$

which reduces the square bracket of (13) to $\langle g\rangle$. We consider again the equation for J in order to eliminate at second order the dependence on the angles.

Lemma 2.577. *Define Ψ by*

$$\Psi := f^1 + d_\psi f^0 B - d_\psi A_1\langle g\rangle + d_J f^0 A_1 - d_J A_1\langle f\rangle \ . \qquad (15)\texttt{psi}$$

and

$$A_2(J,\psi) := \sum_{0\neq|k|\leq 2K} \frac{\Psi_k(J)}{i\omega_0(J)\cdot k}\, e^{ik\cdot\psi} \ ,$$

then the formal coordinate transformation (10) reduces (9) to the form (5) with Z the average of Ψ over the angles,

$$\begin{aligned} R := &\left(\mathcal{L}^{-1} - \mathbb{1} + \varepsilon d_J A_1\right)\varepsilon\langle f\rangle + \varepsilon\mathcal{L}^{-1}d_\psi A_1\left(\omega_0 + \varepsilon\langle g\rangle - \dot\psi\right) + \varepsilon^2\left(\mathcal{L}^{-1} - \mathbb{1}\right)\left(Z + d_J A_1\langle f\rangle\right) \\ &-\mathcal{L}^{-1}\left[\varepsilon\Delta f^0 + \varepsilon^2(d_J A_2\dot J - \Delta f^1) + \varepsilon^2 d_\psi A_2(\dot\psi - \omega_0) - \varepsilon^3 f^2(I,\phi) - \varepsilon^3 d_J f^0 A_2\right] \end{aligned}$$

and

$$R_1 := \Delta\omega_0 + \varepsilon\Delta g_0 + \varepsilon^2 g^1 + \varepsilon d_\psi B(\omega_0 - \dot\psi) - \varepsilon d_J B\dot J \ . \qquad (16)\texttt{r1}$$

We also introduced the notations

$$\begin{aligned} \mathcal{L} :=\ & \mathbb{1} + \varepsilon d_J A_1 \\ \Delta f^0 =& f^0(I,\phi) - f^0(J,\psi) - \varepsilon\left[d_J f^0(J,\psi)A_1(J,\psi) + d_\psi f^0(J,\psi)B(J,\psi)\right] \ , \\ \Delta f^1 =& f^1(I,\phi) - f^1(J,\psi) \ , \\ \Delta g_0 =& g^0(I,\phi) - g^0(J,\psi) \ , \\ \Delta\omega_0 =& \omega_0(I) - \omega_0(J) - \varepsilon d_J\omega_0(J)A_1(J,\psi) \ . \end{aligned} \qquad (17)\texttt{delta}$$

Proof. Consider the equation for J. We have

$$\begin{aligned} &\mathcal{L}\dot J + \varepsilon^2 d_J A_2\dot J + \varepsilon d_\psi A_1\dot\psi + \varepsilon^2 d_\psi A_2\dot\psi = \\ &\quad\varepsilon\left[f^0 + d_J f^0\varepsilon A_1 + d_\psi f^0\varepsilon B + \Delta f^0\right] + \varepsilon^2\left[f^1 + \Delta f^1\right] + \varepsilon^3 f^2(I,\phi) \end{aligned}$$

here the functions are evalueted in J,ψ unless otherwise specified. Thus we can rewrite the above formula as

$$\begin{aligned} \mathcal{L}\dot J = &\ \varepsilon\left[f^0 - d_\psi A_1\omega_0(J)\right] + \varepsilon^2\left\{f_1 + d_\psi f^0 B + d_J f^0 A_1 - d_\psi A_2\dot\psi\right\} \\ &- \left(\varepsilon\Delta f^0 + \varepsilon^2(\Delta f^1 - d_J A_2\dot J + \varepsilon^3 f^2(I,\phi) + d_J f^0 A_2)\right) \ , \end{aligned} \qquad (18)\texttt{pp}$$

Using the fact that A_1 satisfies the homological equation, the square bracket in (18) is nothing else than $\langle f\rangle$ we rewrite (18) as

$$\mathcal{L}\dot J = \varepsilon\langle f\rangle + \varepsilon^2\left\{f_1 + d_\psi f^0 B + d_J f^0 A_1 - d_\psi A_2\omega_0\right\} + \mathcal{F}_3 \ , \qquad (19)\texttt{15}$$

where

$$\begin{aligned} &\mathcal{F}_3 := -\varepsilon\left[d_\psi A_1(\dot\psi - \omega_0(J) - \varepsilon\langle g\rangle)\right] \\ -\varepsilon^2 d_J A_2\dot J + \varepsilon\Delta f^0 + &\varepsilon^2\Delta f^1 + \varepsilon^3 f^2(I,\phi) - \varepsilon^2 d_\psi A_2(\dot\psi - \omega_0(J)) \ . \end{aligned}$$

Multiplying by $\mathcal{L}^{-1}$, and taking into account the definition of A_2 we get

$$\dot{J} = \varepsilon\langle f\rangle - \varepsilon^2 d_J A_1\langle f\rangle + \varepsilon\left(\mathcal{L}^{-1} - \mathbb{1} + \varepsilon d_J A_1\right)\langle f\rangle \\ +\varepsilon^2\{...\} + \mathcal{L}^{-1}\mathcal{F}_3 + \varepsilon^2\left(\mathcal{L}^{-1} - \mathbb{1}\right)\{...\} \ .$$

where the curly bracket coincides with that of (19). Taking into account that

$$\{...\} + d_J A_1\langle f\rangle = \Psi + d_\psi A_2\omega_0$$

and the definition of A_2 we get the expression of R. □

2) Quantitative estimates

We begin by determininig the domain. To this end the following lemma is usefull

Lemma 2.64.1. *For K large enough (so that $\rho \le \bar{\rho}$) define*

$$\rho = \frac{\gamma}{\Omega 2^{\tau+2}K^{\tau+1}} \ , \tag{20}$$

4.lem1

then, one has

$$\Gamma_\rho(\tau,\gamma,2K) \subset \Gamma(\tau,\frac{\gamma}{2},2K) \ .$$

Proof. Just use Lagrange theorem; for details see [2] lemma 4.1. □

In what follows we will use the notation

$$\alpha_K := \frac{\gamma}{2K^\tau} \ . \tag{21}$$

14.1

¿From now on we assume that ρ satifies (20).

We fix now τ and γ (while K is determined by (8)). For a function h analytic in $\Gamma_{\alpha\rho}(\tau,\gamma,2K)\times\mathbf{T}^m + i\beta\sigma$, where $\alpha,\beta\in(0,1]$ we define

$$\|h\|_{\alpha\rho,\beta\sigma} := \sup_{B_{\alpha\rho}(I_*)\times\mathbf{T}^m+i\beta\sigma}\|h(I,\phi)\| \ .$$

We come to the estimates.

Clearly, from (12), (7), (14), we have

$$\|A_1\|_{\rho,\sigma} \le \frac{F_0}{\alpha_K} \ , \quad \|B\|_{\rho,\sigma} \le \left(\frac{\Omega F_0}{\alpha_K} + G_0\right)\frac{1}{\alpha_K} \le \frac{2\Omega F_0}{\alpha_K^2} \ ,$$

where we used the fact that $\alpha_K \to 0$ as $\varepsilon \to 0$ cf. (8), (21) to identify (and retain) only the main term.

Then, using the Cauchy inequalities it is easy to see that

$$\|\Psi\|_{3\rho/4,3\sigma/4} \le F_1 + \frac{4F_0^2}{\alpha_K}\left(\frac{2}{\rho} + \frac{\Omega}{e\sigma\alpha_K} + \frac{G_0}{e\sigma}\right) \le \frac{16F_0^2}{\alpha_K\rho} \ ,$$

where we used $\rho = o(\alpha_K)$ (cf. (20), (21)) to identify the leading term. ¿From this we obtain

$$\|A_2\|_{3\rho/4,3\sigma/4} \le \frac{16F_0^2}{\alpha_K\alpha_{2K}\rho}$$
$$\|A\|_{3\rho/4,3\sigma/4} \le \frac{2F_0}{\alpha_K}$$

(provided ε is small enough).

Before estimating the remainders R, R_1 we need some preliminary estimates.

Lemma 2.7ln.1. *One has*

$$\left\|\dot{J}\right\|_{3\rho/4,3\sigma/4} \le 2\varepsilon F_0 \ , \quad \left\|\dot{\psi}-\omega_0\right\|_{3\rho/4,3\sigma/4} \le 2\varepsilon\frac{\Omega F_0}{\alpha_K}$$
$$\|\Delta\omega_0\|_{\rho/2,\sigma/2} \le \varepsilon^2\frac{24\Omega F_0^2}{\rho\alpha_K\alpha_{2K}} \ , \quad \|\Delta g_0\|_{\rho/2,\sigma/2} \le \varepsilon\frac{8G_0F_0}{\rho\alpha_K}$$
$$\|\Delta f_0\|_{\rho/2,\sigma/2} \le \varepsilon^2\frac{2^7F_0^3}{\rho^2\alpha_K\alpha_{2K}} \ , \quad \|\Delta f_1\|_{\rho/2,\sigma/2} \le \varepsilon\frac{8F_1F_0}{\rho\alpha_K}$$
$$\|\mathcal{L}\|_{\rho/2,\sigma/2} \le 2 \ , \quad \|\mathcal{L}-\mathbb{1}\|_{\rho/2,\sigma/2} \le \frac{4\varepsilon}{\rho\alpha_K}F_0 \ , \quad \left\|\mathcal{L}-\mathbb{1}+\varepsilon\frac{\partial A_1}{\partial J}\right\|_{\rho/2,\sigma/2} \le \frac{8\varepsilon^2F_0^2}{\rho^2\alpha_K^2} \ .$$

Proof. We begin by $\Delta\omega_0$: one has

$$\begin{aligned}\|\Delta\omega_0\| &\le \left\|\omega_0(I)-[\omega_0(J)+\varepsilon\frac{\partial\omega_0}{\partial J}A]\right\| + \varepsilon^2\left\|\frac{\partial\omega_0}{\partial J}\right\|\|A_2\| \\ &\le \left\|\frac{\partial^2\omega_0}{\partial J^2}\right\|\varepsilon^2\|A_1+\varepsilon A_2\|^2 + \varepsilon^2\left\|\frac{\partial\omega_0}{\partial J}\right\|\|A_2\| \ ,\end{aligned}$$

from which

$$\|\Delta\omega_0\|_{\rho/2,\sigma/2} \le \frac{4}{\rho}\Omega\varepsilon^2\left(\frac{F_0}{\alpha_K}+\varepsilon\frac{16F_0^2}{\alpha_K\alpha_{2K}\rho}\right)^2 + \varepsilon^2\Omega\frac{16F_0^2}{\alpha_K\alpha_{2K}\rho} \le \frac{24\Omega F_0^2\varepsilon^2}{\rho\alpha_K\alpha_{2K}} \ .$$

Concerning Δg_0 we have

$$\|\Delta g_0\|_{\rho/2,\sigma/2} \le \left\|\frac{\partial g_0}{\partial I}\right\|\left\|\varepsilon A_1+\varepsilon^2A_2\right\| + \left\|\frac{\partial g_0}{\partial\psi}\right\|\varepsilon\|B\| \ ,$$

which can be easily estimated giving the claimed result.

We come to $\dot{J}$ and $\dot{\psi}-\omega_0$. It is easy to obtain the following equations

$$\begin{aligned}(\mathbb{1}+\varepsilon d_JA)\dot{J} &= \varepsilon\langle f\rangle - \varepsilon d_\psi A(\dot{\psi}-\omega_0) - \varepsilon^2 d_\psi A_2\omega_0 + \varepsilon(f^0(I,\phi)-f^0(J,\psi)) + \varepsilon^2 f^1 + \varepsilon^3 f^2 \\ (\mathbb{1}+\varepsilon d_\psi B)(\dot{\psi}-\omega_0) &= [\omega_0(I)-\omega_0(J)] + \varepsilon\langle g\rangle(J) + \varepsilon\Delta g^0 + \varepsilon^2 g^1(I,\phi) - \varepsilon d_J B\dot{J} \ .\end{aligned}$$

Solving the second with respect to $\dot{\psi}-\omega_0$, substituing in the first, and using the definition of A_2 to estimate the last term of the so obtained inequality, one gets the wonted estimates for $\dot{J}$ and $\dot{\psi}-\omega_0$.

All the other estimates can be obtained almost in the same way. □

To obtain the proof of lemma 2.1 we proceed now as follows: first use the above lemma to obtain the estimate

$$\|R_1\|_{\rho/2,\sigma/2} \le \varepsilon^2\frac{48\Omega F_0}{\rho\alpha_K\alpha_{2K}} \ . \qquad (22)\text{dpsi}$$

Then, notice that the r.h.s. of (22) gives also an estimate of $\dot{\psi}-\omega_0-\varepsilon\langle g\rangle$. Using such an estimate it is easy to obtain the following estimate for R:

$$\|R\|_{\rho/2,\sigma/2} \le \varepsilon^3\frac{2^{11}F_0^3}{\rho^2\alpha_K\alpha_{2K}} \ .$$

This gives Lemma 2.1 with the claimed estimates.

3. Persistence of stable hyperbolic tori hyp

We consider here a system of the form (5). We will assume that $\langle f\rangle$ has an attractive zero which is suffitiently nonresonant, and we will prove that the correspnding invariant torus of the averaged system can be continued to an invariant tous of system (5)

We assume that there exists an attractive zero J_* of $\langle f\rangle$, namely a zero such that all the eigenvalues of $d\langle f\rangle(J_*)$ are strictly negative We first continue it to a zero of $\langle f\rangle+\varepsilon Z$. This requires some care since Z is defined only on nonresonant sets, and moreover such function depends discontinuously on ε.

Lemma 3.1h.1. *Assume that there exist τ,γ,K such that the zero J_* of $\langle f\rangle$ belongs to $\Gamma(\tau,\gamma,K)$, then there exist ε_*,C_9 such that the following holds true: if*

$$K\geq C_2|\ln\varepsilon_*|^{\tau} \tag{23}$$ s

then for any ε satisfying

$$e^{-K/C_2}\leq\varepsilon<\varepsilon_* \tag{24}$$ *

the function $\langle f\rangle+\varepsilon Z$ has an attractive zero $J_0(\varepsilon)$ close to J_*, namely such that

$$\|J_0(\varepsilon)-J_*\|\leq C_9\varepsilon|\ln\varepsilon|^{2\tau+1}\ ; \tag{25}$$ dj0

moreover there exists $\lambda>0$ such that all the eigenvalues $\lambda_i=\lambda_i(\varepsilon)$ of $d\langle f\rangle(J_0)+\varepsilon d_J Z(J_0)$ satisfy

$$\inf_{e^{-K/C_2}\leq\varepsilon<\varepsilon_*}|\lambda_1(\varepsilon)|\geq\lambda\ . \tag{26}$$ la

Proof. We fix ε and consider the function

$$F(J,\mu):=\langle f\rangle(J)+\frac{\mu}{\varepsilon|\ln\varepsilon|^{2\tau+1}}Z(J,\varepsilon)\ , \tag{27}$$ F

to which we can apply the implicit function theorem. So we obtain that there exists μ_* such that provided $\mu<\mu_*$ such function has a zero $J_1(\mu)$ smoothly (analitically) depending on μ. Define now ε_* by $\varepsilon_*|\ln\varepsilon_*|^{2\tau+1}=\mu_*$. Provided condition (24) is satisfyed it is possible to choose ε so small that the fraction in (27) is one, and the function J_1 is defined. We then put $J_0(\varepsilon):=J_1(\varepsilon|\ln\varepsilon|^{2\tau+1})$.

We come to the estimates (25) and (26). Remark that $\frac{\partial J_1}{\partial\mu}$ (which can be computed explicitly) is bounded uniformly on ε, and therefore (25) is a consequence of the smoothness of $J_1(\mu)$. The eigenvalues of $d_J F(J_1(\mu),\mu)$ depend smoothly on μ, and therefore they are close to the eigenvalues of $d\langle f\rangle(J_*)$, so that also condition (26) holds. □

In all the rest of the paper we will denote by ρ the quantity (4).

Remark 3.2h.2. *By (25) one has*

$$B_{\rho/4}(J_0(\varepsilon))\subset\Gamma_{\rho/2}(\tau\gamma,K)\ ,$$

so that $B_{\rho/4}(J_0(\varepsilon))$ is contained in the domain of definition of (5).

Remark 3.3h.3. *Under the hypotheses of lemma 3.1 the system*

$$\begin{aligned}\dot J&=\varepsilon\langle f\rangle+\varepsilon^2 Z\\ \dot\psi&=\omega+\varepsilon\langle g\rangle\end{aligned} \tag{28}$$ sys.1

has an attractive torus $J_0(\varepsilon) \times \mathbf{T}^m$ close to the torus $J_* \times \mathbf{T}^m$.

We are going to continue such a normally hyprebolic torus to a torus of system (5).

We will follow almost literaly the proof of persistence of normally hyperbolic manifolds given by Fenichel, just adding quantitative estimates on the size of the threshold. Such estimates are needed in our case (in which the Lyapunov exponents of the invariant manifold and the size of the perturbation are related).

In all the rest of the paper we will assume that ε is small enough and that J_* is suffitiently nonresonant. By this we mean that $J_* \in \Gamma(\tau, \gamma, K)$ with $K > C_2 |\ln \varepsilon|^\tau$.

We give now a suitable form to system (5). First we make an ε dependent translation in J space so that the attractive zero $J_0(\varepsilon)$ coincides with 0, namely so that $\langle f \rangle(0) + \varepsilon Z(0, \varepsilon) = 0$. Then we write our system in the form

$$\begin{aligned} \dot{J} &= A_\varepsilon J + \mathcal{N}(J, \varepsilon) + \varepsilon^{2+p} \mathcal{R}(J, \psi, \varepsilon) \\ \dot{\psi} &= \omega(J, \varepsilon) + \varepsilon^{1+p} \mathcal{R}_1 \ , \end{aligned} \tag{29}$$ h.s

where p is any real number satisfying $0 < p < 1$, and

$$\begin{aligned} A_\varepsilon &:= d_J \langle f \rangle(0) + \varepsilon d_J Z(0, \varepsilon) \ , \quad \mathcal{N}(J, \varepsilon) := \langle f \rangle(J) + \varepsilon Z(J, \varepsilon) - A_\varepsilon J \\ \omega(J, \varepsilon) &:= \omega_0(J) + \varepsilon \langle g \rangle(J) \ , \\ \mathcal{R} &= \frac{R}{\varepsilon^{2+p}} \ , \quad \mathcal{R}_1 = \frac{R_1}{\varepsilon^{1+p}} \ . \end{aligned}$$

In what follows we will systematically omit the ε dependence of the various functions. This is not dangerous by virtue of the following

Lemma 3.4h.9. *For any integer q the C^q norms of $\mathcal{N}, \mathcal{R}, \mathcal{R}_1, \omega$ are bounded on $B_{\rho/4}(0) \times (\mathbf{T} + i\bar{\sigma}/4)^m$, uniformly in ε; moreover there exists C_N such that*

$$\|\mathcal{N}(J)\| \le C_N \|J\|^2 \ , \quad \|d\mathcal{N}(J)\| \le 2 C_N \|J\| \ . \tag{30}$$ s.n

Proof. Consider first $\mathcal{R}$. By Cauchy inequality its C^q norm is bounded on $B_{\rho/4}(0) \times (\mathbf{T} + i\bar{\sigma}/4)^m$ by a constant multiplied by

$$\frac{\varepsilon^3 |\ln \varepsilon|^{4\tau+2} |\ln \varepsilon|^{q(\tau+1)}}{\varepsilon^{2+p}} \ ,$$

which is bounded provided $p < 1$. Similarly for $\mathcal{R}_1$ and $\mathcal{N}$. The estimate of ω is easily obtained by remarking that such function is analytic and bounded on $D_{\bar{\rho}}$, with $\bar{\rho}$ defined at the beginning of the paper, and independent of ε. The estimate (30) is a simple consequence of the twice differentiability of $\langle f \rangle + \varepsilon Z$ with respect to J. □

We fix now $0 < p < 1$, and $q \ge 1$. We introduce some notations. We denote by $\bar{\mathcal{R}}, \bar{\mathcal{R}}_1$ the constants bounding the C^q norm of $\mathcal{R}$ and $\mathcal{R}_1$ respectively. The C^{q-1} norm of $d\omega$ is bounded by 2Ω.

We denote by $F^t(J, \psi)$ the solution of (29) starting at the point (J, ψ). With an obvious notation we write $(J(t), \psi(t)) = F^t(J, \psi)$. We will also denote

$$H_t(J, \psi) := J(t) \ , \quad \Phi_t(J, \psi) = \psi(t) \ , \tag{31}$$ ev

and

$$\mathcal{U}_{\rho_1} := (D \cap B_{\rho_1}(0)) \times \mathbf{T}^m \ , \quad \omega_* := \omega(J_0) \ .$$

We will often use the formula of variation of constants, which gives

$$H_t(J,\psi) = e^{\varepsilon At}J + \varepsilon\int_0^t e^{\varepsilon A(t-s)}\left(\mathcal{N}(H_s(J,\psi)) + \varepsilon^{1+p}\mathcal{R}(H_s(J,\psi),\Phi_s(J,\psi))\right)ds\ , \tag{32}$$

and the formula

$$\Phi_t(J,\psi) = \psi + \int_0^t \left[\omega(H_s(J,\psi)) + \varepsilon^{1+p}\mathcal{R}_1(H_s(J,\psi),\Phi_s(J,\psi))\right]ds\ . \tag{33}$$

We recall that Fenichel's method consists in looking for an invariant manifold of the map F^T, where T is a fixed positive time.

We fix

$$T := \varepsilon^{-7p/8}\ . \tag{34}$$

Notice that this establishes a relation between the time T and the smallest Lyapunov exponent $\varepsilon\lambda$ of the unperturbed invariant torus $J = 0$, in particular one has $\varepsilon\lambda T \ll 1$, but $\varepsilon \ll \varepsilon T$ (this is the key trick for the proof).

We have the following

Lemma 3.5inv. *For any ρ_1 satisfying $\rho \geq \rho_1$ and $\lambda\rho_1 \geq 2\varepsilon^{1+p}\bar{\mathcal{R}}$, and $t \leq T$ one has*

$$F^t(\mathcal{U}_{\rho_1}) \subset \mathcal{U}_{\rho_1}\ .$$

Proof. Using (32) and (30) we have

$$\begin{aligned}\|J(t)\| &\leq e^{-\varepsilon\lambda t}\rho_1 + \varepsilon\int_0^t e^{\lambda\varepsilon(t-s)}[C_N\rho_1^2 + \varepsilon^{1+p}\bar{\mathcal{R}}]ds \\ &\leq e^{-\varepsilon\lambda t}\rho_1 + \varepsilon t(C_N\rho_1^2 + \varepsilon^{1+p}\bar{\mathcal{R}}) \\ &= (1-\lambda\varepsilon t)\rho_1 + \varepsilon t(C_N\rho_1^2 + \varepsilon^{1+p}\bar{\mathcal{R}}) + o(\varepsilon t)\rho_1\ ,\end{aligned}$$

and this is less than ρ_1 provided ρ_1 is small, εt is small, and $\varepsilon^{1+p}\bar{\mathcal{R}} + o(\varepsilon t)\rho_1 \leq \lambda\rho_1$. □

In particular it follows from this lemma that the evolution up to time T leaves $\mathcal{U}_{\rho_1}$ invariant; in other words, the evolution does not push (up to time T) an initial datum in $\mathcal{U}_{\rho_1}$ out of the domain of definition of the system (29). We fix now

$$\rho_1 := \frac{2\bar{\mathcal{R}}}{\lambda}\varepsilon^{1+p}\ .$$

We recall that, following Fenichel, the invariant manifold is found as the graph of a section $\sigma : \mathbf{T}^m \to \mathbb{R}^n$ ($J = \sigma(\psi)$). Moreover, σ is characterized as the unique fixed point of the "graph transform" G. Coming to the definition of such a graph transform, let us fix a section σ, and consider the map defined by

$$\psi \mapsto \xi = \xi(\psi) := \Phi_T(\sigma(\psi - \omega_* T), \psi - \omega_* T) \tag{35}$$

(where we used the notation of (31)); it will be needed to invert such a map.

It is usefull to introduce the function

$$\mathcal{S}(\psi) := \xi(\psi) - \psi\ ,$$

so that $\xi(\psi) = \psi + \mathcal{S}(\psi)$.

Lemma 3.6k.1. *Let $\sigma : \mathbf{T}^m \to B_{\rho_1}(0)$, then*

$$\|\mathcal{S}(\psi)\| \leq (4\lambda\bar{\mathcal{R}} + \bar{\mathcal{R}}_1)\varepsilon^{1+p}T \ll 1 \ .$$

Proof. By (33) one has

$$\mathcal{S}(\psi) = \int_0^t \left[\omega(H_s(J,\psi)) + \varepsilon^{1+p}\mathcal{R}_1(H_s(J,\psi), \Phi_s(J,\psi))\right] ds - \omega_* T$$
$$\int_0^t \left[\omega(H_s(J,\psi)) - \omega_*\right] ds + \varepsilon^{1+p} \int_0^t \left[\mathcal{R}_1(H_s(J,\psi), \Phi_s(J,\psi))\right] ds \ .$$

But, taking into account lemma 3.5 one has $\|H_s\| \leq \rho_1$, and therefore the argument of the first integral is the difference between the value of the function ω at points ρ_1 close, and moreover $\|\mathcal{R}_1\|$ is bounded. So we have

$$\|\mathcal{S}(\psi)\| \leq 2\Omega\rho_1 T + \varepsilon^{1+p}\bar{\mathcal{R}}_1 T \ ,$$

which, taking into account the definition of ρ_1 gives (23). □

By the implicit function theorem it follows that $\xi(\psi)$ is invertible.

We define the graph transform writing

$$G\sigma(\xi) := H_T(\sigma(\psi - \omega T), \psi - \omega T) \ ,$$

where ψ, and ξ are related by (35).

Define now the space S_d of the Lipschitz functions h from $\mathbf{T}^m$ to $\mathcal{U}_{\rho_1}$ with Lipschitz constants $\mathrm{Lip}(h)$ smaller than d. We choose

$$d = \varepsilon^{1+p/2} \ .$$

Following again Fenichel, we will prove that $G : S_d \to S_d$ is a contraction, so that it has a unique fixed point; this is the invariant torus. We split the proof in some lemmas.

Lemma 3.7l.1. *For $(J,\psi) \in \mathcal{U}_{\rho_1}$, the following relations hold:*

$$\begin{aligned}
&\sup_{J,\psi} \|d_J H_T(J,\psi)\| \leq 1 - \varepsilon T\lambda + \text{h.o.t.} \ , \\
&\sup_{J,\psi,s\leq t} \|d_J \Phi_s(J,\psi)\| \leq t2\Omega + \text{h.o.t.} \ , \\
&\sup_{J,\psi,s\leq t} \|d_\psi H_s(J,\psi)\| \leq \varepsilon^{2+p} t\bar{\mathcal{R}} + \text{h.o.t.} \\
&\sup_{J,\psi,s\leq t} \|d_\psi \Phi_s(J,\psi)\| \leq 1 + 2\Omega t^2 \varepsilon^{2+p}\bar{\mathcal{R}} + \text{h.o.t.} \ .
\end{aligned} \tag{36}$$

1.2

Proof. Let us write

$$\alpha_1 := \sup_{t\in[0,T]} \sup_{J,\psi} \|d_J H_t(J,\psi)\| \ , \quad \alpha_2(s) := \sup_{t\in[0,s]} \sup_{J,\psi} \|d_\psi H_t(J,\psi)\| \ ,$$
$$\alpha_3(s) := \sup_{t\in[0,s]} \sup_{J,\psi} \|d_J \Phi_t(J,\psi)\| \ , \quad \alpha_4(s) := \sup_{t\in[0,s]} \sup_{J,\psi} \|d_\psi \Phi_t(J,\psi)\| \ .$$

Differentiating eqs. (32) and (33) with respect to J, one has

$$D_J H_t = e^{\varepsilon A t} + \varepsilon \int_0^t e^{\varepsilon A(t-s)} \left[d\mathcal{N} d_J H_s + \mu \left(d_J \mathcal{R}_1 d_J H_s + d_\psi \mathcal{R}_1 d_J \Phi_s \right) \right] ds \ ,$$

from which one obtains the following inequalities

$$\begin{aligned} \|d_J H_t\| \leq & e^{-\lambda t \varepsilon} + \varepsilon t \left[2 C_N \rho_1 \alpha_1 + \varepsilon^{1+p} \left(\alpha_1 + \alpha_3 \right) \bar{\mathcal{R}} \right] \ , \\ \|d_J \Phi_t\| \leq & t \left[2\Omega \alpha_1 + \varepsilon^{1+p} \left(\alpha_1 + \alpha_3 \right) \bar{\mathcal{R}}_1 \right] \ . \end{aligned} \tag{37alpha}$$

Now, it is clear that $\alpha_1 = 1$, while the r.h.s. of the second of (37) is a function increasing with time; so we have that there exists $\tilde{\Phi} \geq \alpha_3$ such that

$$\tilde{\Phi} \leq t\Omega + \varepsilon^{1+p} t \left(\bar{\mathcal{R}}_1 + \tilde{\Phi} \bar{\mathcal{R}}_1 \right) \ ;$$

neglecting higher order terms this gives $\|d_J \Phi_t\| \leq \tilde{\Phi} \leq t\Omega$. Substituting in the first of (37) we get the first two of (36) (recall that $\varepsilon T \ll 1$). Differentiating (32) and (33) with respect to ψ, making some standard estimates we get a system of inequalities for α_2, α_4 which gives the remaining of (36). □

Lemma 3.81.2. *For any $\sigma \in S_d$, one has*

$$\mathrm{Lip}(G\sigma) \leq \left[1 + (2\Omega T d - \varepsilon \lambda T) \right] d + \text{h.o.t.} \ ;$$

hence $\mathrm{Lip}(G\sigma) < d$, *and G maps S_d into itself.*

Proof. Let ξ_1, ξ_2 be related to ψ_1, ψ_2 as in (35). We have to estimate

$$\|G\sigma(\xi_1) - G\sigma(\xi_2)\| = \|H_T\left(\sigma(\psi_1 - \omega_* T), \psi_1 - \omega_* T\right) - H_T\left(\sigma(\psi_2 - \omega_* T), \psi_2 - \omega_* T\right)\| \ .$$

Using Lagrange theorem one has that that the above quantity is less than

$$\mathrm{Lip}\left[H_T(\sigma(.) - \omega_* T, . - \omega_*) \right] \|\psi_1 - \psi_2\| \ .$$

Such a Lipschitz constant is estimated using lemma 3.7 by

$$\|d_J H_T\| \, d + \|d_\psi H_T\| \leq \left[(1 - \varepsilon \lambda T) d + \varepsilon^{2+p} T \bar{\mathcal{R}} \right] \ ,$$

where we neglected higher order terms. So, we need an estimate of $\|\psi_1 - \psi_2\|$ in terms of $\|\xi_1 - \xi_2\|$. To obtain it we estimate the Lipschitz constant of $\mathcal{S}$. We split the estimate in two parts, corresponding to the two parts of (33). We begin by

$$\begin{aligned} & \int_0^T \|\omega\left(H_s\left(\sigma(\psi_1 - \omega_* T), \psi_1 - \omega_* T\right)\right) - \omega\left(H_s\left(\sigma(\psi_2 - \omega_* T), \psi_2 - \omega_* T\right)\right)\| \, ds \\ & \leq T 2\Omega \left[d \|\psi_1 - \psi_2\| + \varepsilon^{2+p} T \bar{\mathcal{R}} \|\psi_1 - \psi_2\| \right] \simeq T 2\Omega d \|\psi_1 - \psi_2\| \ , \end{aligned}$$

where we neglected higher order terms, and used again lemma 3.7. This gives the estimate of the Lipschitz constant of such a term. Proceeding analogously for the other term one obtains that its Lipschitz constant is smaller than

$$\varepsilon^{1+p} T \bar{\mathcal{R}}_1 (d + \varepsilon^{2+p} T \bar{\mathcal{R}} + 2\Omega T d + 1 + 2\Omega T^2 \varepsilon^{2+p} \bar{\mathcal{R}}) \simeq \varepsilon^{1+p} T \bar{\mathcal{R}}_1 \ ,$$

so that $\mathrm{Lip}(\mathcal{S}) \leq 2\Omega T d + h.o.t.$. But one has

$$\begin{aligned} \|\psi_1 - \psi_2\| &= \|\xi_1 - \xi_2 - \mathcal{S}(\psi_1) + \mathcal{S}(\psi_2)\| \\ &\leq \|\xi_1 - \xi_2\| + \|\mathcal{S}(\psi_1) - \mathcal{S}(\psi_2)\| \leq \|\xi_1 - \xi_2\| + [\mathrm{Lip}\mathcal{S}] \, \|\psi_1 - \psi_2\| \ , \end{aligned}$$

so that (at first order)

$$\|\psi_1 - \psi_2\| \le \frac{\|\xi_1 - \xi_2\|}{1 - 2\Omega T d} \simeq (1 + 2\Omega T d)\,\|\xi_1 - \xi_2\| \ . \qquad (38)33$$

This completes the proof. □

Lemma 3.91.4. *The graph transform $G\big|_{S_d}$ is Lipschitz and its Lipschitz constant is less than*

$$1 + d4\Omega T - \varepsilon T \lambda \ . \qquad (39)\texttt{lipg}$$

Proof. We have to estimate $\|G\sigma(\xi) - G\sigma'(\xi)\|$; this is equal to

$$\|H_T\left(\sigma(\psi - \omega_* T), \psi - \omega_* T\right) - H_T\left(\sigma'(\psi' - \omega_* T), \psi' - \omega_* T\right)\| \qquad (40)\texttt{sigma}$$

(where ψ' is related to ξ by (35), via the section σ'). This is estimated, using Lagrange theorem and lemma 3.7 by

$$\begin{aligned} (1 - \varepsilon\lambda T)\,\|\sigma(\psi - \omega_* T) - \sigma'(\psi' - \omega_* T)\| + \varepsilon^{2+p} T \bar{\mathcal{R}}\,\|\psi - \psi'\| \le \\ (1 - \varepsilon\lambda T)\left[\|\sigma(\psi - \omega_* T) - \sigma(\psi' - \omega_* T)\| + \|\sigma(\psi' - \omega_* T) - \sigma'(\psi' - \omega_* T)\|\right] + \varepsilon^{2+p} T \bar{\mathcal{R}}\,\|\psi - \psi'\| \ . \end{aligned} \qquad (41)34$$

To estimate this quantity we need an estimate of $\psi - \psi'$. Defining J and J' by

$$\begin{aligned} (J, \xi) &= F^T(\sigma(\psi - \omega T), \psi - \omega T) \\ (J', \xi) &= F^T(\sigma'(\psi' - \omega_* T), \psi' - \omega_* T) \end{aligned}$$

one has that $F^{-T}(J, \xi)$ and $F^{-T}(J', \xi')$ is in $\mathcal{U}_{\rho_1}$, and moreover

$$\psi = \Phi_{-T}(\xi, J) + \omega_* T \ , \quad \psi' = \Phi_{-T}(\xi, J') + \omega_* T \ .$$

¿From this, using (33) one has

$$\|\psi - \psi'\| \le T[4\omega + 2\varepsilon^{1+p}\bar{\mathcal{R}}_1] \ .$$

Inserting in (41) we get the thesis. □

So, G is a contraction and therefore has a unique fixed point which is the invariant manifold. Moreover, by construction the fixed point is a section $\sigma : \mathbf{T}^m \to \mathcal{U}_{\rho_1}$, and therefore the perturbed invariant torus is $\mathcal{O}(\rho_1)$ close to the the torus $J = 0$, which in turn is $\mathcal{O}(\varepsilon\,|\ln\varepsilon|^{2\tau+1})$ close to the invariant torus $J_* \times \mathbf{T}^m$ of the averaged system. Finally the change of variables (3) is $\varepsilon\,|\ln\varepsilon|^{\tau}$ close to the identity, and therefore the invariant torus satisfyies (2).

4. Extension to the general hyperbolic case `hypg`

In the discussion above, we have assumed that the unperturbed invariant torus is not only hyperbolic, but also attractive. We give now a sketch of the proof of persistence of the normally hyperbolic manifold in the general case, in which the unperturbed invariant manifold is not attractive (or repulsive).

Following Fenichel and many other authors we first look at the (local) unstable manifold of the torus $J = 0$, which is a conctractive manifold, and so we apply the theory of the previous section to prove its persistence.

Then we look at the stable manifold, inverting time this becomes an attractive manifold, so we obtain in the same way its persistence. The normally hyperbolic torus is obtained as the intersection of the two manifolds.

To obtain quantitative estimates in this case one needs some care. To explain this we introduce some notations. First we consider again the operator A, and we assume that it is diagonal. Let $-\lambda_1, ..., -\lambda_k$ be the negative eigenvalues of A, and $\mu_1, ..., \mu_l$ be the positive eigenvalues. Finally denote by $\mathcal{J}$ the coordinates along the attractive directions, and by $\mathcal{I}$ those in the expanding directions, i.e.

$$A \begin{pmatrix} \mathcal{J} \\ \mathcal{I} \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \mathcal{J} \\ \mathcal{I} \end{pmatrix}$$

with

$$(A_1 \mathcal{J})_i = -\lambda_i \mathcal{J}_i \quad , \quad (A_2 \mathcal{I})_i = \mu_i \mathcal{I}_i \ .$$

We then write the original system as

$$\begin{cases} \dot{\mathcal{J}} = \varepsilon A_1 \mathcal{J} + \varepsilon \mathcal{H}_{\mathcal{J}}(\mathcal{J}, \mathcal{I}) + \varepsilon^{2+p} \mathcal{R}_{\mathcal{J}}(\mathcal{J}, \mathcal{I}, \psi) \\ \dot{\mathcal{I}} = \varepsilon A_2 \mathcal{I} + \varepsilon \mathcal{H}_{\mathcal{I}}(\mathcal{J}, \mathcal{I}) + \varepsilon^{2+p} \mathcal{R}_{\mathcal{I}}(\mathcal{J}, \mathcal{I}, \psi) \\ \dot{\psi} = \omega(\mathcal{I}, \mathcal{J}) + \varepsilon^{1+p} \mathcal{R}_1(\mathcal{I}, \mathcal{J}, \psi) \end{cases}$$

where the terms $\mathcal{H}_{\mathcal{J}}$ and $\mathcal{H}_{\mathcal{I}}$ are the different components of $\mathcal{F} - A$. Since $\mathcal{Z}$ and $\langle f \rangle$ are C^2, it follows that $\mathcal{F}$ is also C^2, and therefore $\mathcal{H}_{\mathcal{I}}$ and $\mathcal{H}_{\mathcal{J}}$ are $\mathcal{O}(\rho_1)$, which implies that $\varepsilon \mathcal{H}_{\mathcal{J}}$ is of the same order as the remainder $\varepsilon^{2+p}\mathcal{R}$; thus the theory of the previous section applies with minor changes. In particular, one should again obtain the appropriate estimates, in a way adapted to the new form; we mention that, in particular, the estimates corresponding to lemma 3.7 take a slightly worse form.