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Stimulated particle emission – a simple example.

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Abstract: We find evidence of stimulated photon emission in a simple model of the atom plus radiation system, namely a two state atom interacting with a quantised scalar field in the rotating wave approximation. We use an initial state where the atom is excited and there is one photon present, and the photon energy is close to the energy difference between the two atomic states. By taking the long-time, weak-coupling limit, we compute in closed form the probability that the atom remains in its excited state as a function of the rescaled time. Our results indicate that stimulated emission does occur for certain initial photon states, but only at quadratic order in the rescaled time. We also find partial recurrence of the excited state.

1. Introduction.

The broad subject of this paper is the interaction of matter and radiation, in particular the absorption and emission of photons from a one-electron atom. The literature on this subject is vast – we refer the reader to [CDG], [D], [St], [WW] for the basic physics. Recently there has been considerable progress in the mathematical analysis of the Hamiltonian which describes this interaction ([BFS1], [BFS2]). In particular it has been shown how an excited state of the atom turns into a resonance for the atom-radiation system. The evolution of initial states in a quantum mechanical model is strongly influenced by the existence of such resonances, as they generally determine the long-time asymptotics of solutions of Schrödinger's equation. In the atom-radiation model they lead to decay of excited states, along with emission of photons. For additional background on the mathematics of quantum radiation, resonances and decay of states, we refer the reader to [A1], [A2], [Da], [DE], [Fro], [HSp], [Ho], [Hu], [JP], [K1], [OY], [Sp2].

The influence of such a resonance on the time evolution operator can be studied in a simplified model where the atom is replaced by a two-state system, the radiation field is replaced by a massless scalar field and the rotating wave approximation is used in the interaction. This model and its generalizations have been widely studied – some examples are to be found in [Da], [EY], [Fri], [GT], [JC], [K1], [K2], [Pf], [Sp1], [W]. In particular the issue of spontaneous emission of a photon has been examined; here one follows the time evolution of the initial state Ψ_0 where the atom is excited and there are no photons present. In this paper we will consider initial states Ψ_1 for this same model where the atom is excited and there is *one* photon present. Our goal is to find evidence for "stimulated emission", namely an increase in the rate at which the atom relaxes to its ground state, due to the presence of the photon.

The precise formulation of such a result requires some choices. We follow the lead of Davies [Da] and explore a long-time, weak-coupling limit which can be used to define the rate of spontaneous emission. Namely we introduce a parameter β which governs the size of the coupling between atom and field, and we rescale the time coordinate t by defining $\tau = \beta^2 t$. We then take the limit $\beta \to 0$, $t \to \infty$ with τ fixed. As Davies showed ([Da]), when the initial state is Ψ_0 the probability that the atom remains in its excited state can be computed exactly in this limit. The result is $\exp[-2\Gamma\tau]$ where Γ is the usual decay rate. As mentioned above, in this paper we choose an initial state where the atom is excited, and there is one photon present. In order to obtain an interesting result (that is, something other than $\exp[-2\Gamma\tau]$), we must choose the photon state to depend on the parameter β, in such a way that its energy is localised to within $O(\beta^2)$ of the atom's energy gap. Let $p(\beta,t)$ be the resulting probability that the atom remains excited at time t, with this initial state. It turns out that we can explicitly compute $\alpha(\tau) = \lim_{\beta\to 0} p(\beta, \beta^{-2}\tau)$ as a function of τ (see equations () – () in section 2.5). It shows interesting behavior, depending on the precise form of the initial one-photon state. Initially $\alpha(\tau)$ decreases more rapidly than the corresponding probability with no initial photon, and goes to zero exponentially as $\tau \to \infty$. However it behaves as $1 - 2\Gamma\tau$ for small τ, which means that the *linear* decay rate is unaffected by the initial one-photon state. In addition, $\alpha(\tau)$ may actually increase at some later times. Presumably this records some re-excitation of the atom by the initial one-photon state. Such a "rebound" or recurrence effect is well-known in models with discrete spectrum, for example the Jaynes-Cummings model ([JC]), but occurs less often in models with only continuous spectrum, as is the case here.

We prove the result by using a resummed perturbation theory, based on a resolvent expansion. Specifically we separate the interaction into its direct and exchange parts, and we resum the direct part. The technical part of our work is concerned with obtaining bounds for the remainder terms in this expansion, and one of the chief tools here is dilation analyticity (for background on this topic, see [AC], [BC], [Hu], [Si]). It is worth noting that our results cannot be obtained using the standard formulation of time-dependent perturbation theory for this model. Indeed repeating the calculation which leads to Fermi's Golden Rule for this model shows that the rate of decay is independent of the initial one-photon state. The reason is that time-dependent perturbation theory yields a useful expansion for times which are less than $O(\beta^{-1})$, while our results show that the effects of stimulated emission occur at times which are $O(\beta^{-2})$.

The paper is organised as follows. In section 2 we define our model and state the main results in and . In section 3 we use spectral theory to derive our resummed perturbation theory, and prove . Section 4 contains the results about stimulated emission, and the proof of

. In section 5 we prove convergence of our resummed perturbation theory, and the technical estimates needed in sections 3 and 4.

Concerning notation, we are cavalier about numerical factors in the estimates throughout the paper. In particular we use "C" to denote a positive number which is independent of t, β and any variables. Its value may change from line to line.

2. Definitions and Results.

2.1 The model.

First we define the model. The state space is $\mathbf{C}^2 \otimes \mathcal{F}$, where $\mathcal{F}$ is the symmetric Fock space over $L^2(\mathbf{R})$ (see [RS], sections II.4 and VIII.10 for relevant definitions). For any $g \in L^2(\mathbf{R})$, let $a(g)$ and $a^\dagger(g)$ denote respectively the annihilation and creation operators acting on $\mathcal{F}$. They are defined by their action on $\mathcal{F}_n$, the n-particle subspace of $\mathcal{F}$. Let $\phi_n(k_1, \ldots, k_n)$ be a state in $\mathcal{F}_n$; then the state $a^\dagger(g)\phi_n$ lies in $\mathcal{F}_{n+1}$, and is given by

$$\big(a^\dagger(g)\phi_n\big)(k_1, \ldots, k_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} \phi_n(k_1, \ldots, \hat{k}_j, \ldots, k_{n+1}) g(k_j) \qquad ()$$

where the argument $\hat{k}_j$ on the right side is omitted. Similarly, for $n \geq 1$, $a(g)\phi_n$ lies in $\mathcal{F}_{n-1}$, and is given by

$$\big(a(g)\phi_n\big)(k_1, \ldots, k_{n-1}) = \sqrt{n} \int \phi_n(k_1, \ldots, k_n) g(k_n) dk_n \qquad ()$$

The subspace $\mathcal{F}_0 \simeq \mathbf{C}$ is spanned by the Fock vacuum Ω, and $a(g)\Omega = 0$.

Let h denote the second quantised Hamiltonian for the free scalar field on $\mathcal{F}$ defined by the single particle dispersion relation $\omega(k) = |k|$. Since h preserves each subspace $\mathcal{F}_n$, it is defined by its restriction to $\mathcal{F}_n$, which we call h_n; for $\phi_n(k_1, \ldots, k_n)$ in $\mathcal{F}_n$,

$$h_n \phi_n(k_1, \ldots, k_n) = \sum_{j=1}^{n} |k_j| \phi_n(k_1, \ldots, k_n) \qquad ()$$

The full Hamiltonian $H(\beta)$ for our model is most conveniently written as a 2×2 matrix, with respect to the decomposition $\mathbf{C}^2 \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F}$:

$$H(\beta) = \begin{pmatrix} h + E & \beta a(\bar{f}) \\ \beta a^\dagger(f) & h \end{pmatrix} \qquad ()$$

Here $E > 0$ is the energy gap between the ground state and excited state of the atom, β is the (real) coupling parameter, and $f(k)$ is the atomic form factor. We shall make strong and explicit regularity assumptions for f below. Note that for all $n \geq 0$, the subspace

$$\mathcal{H}_n = \mathcal{F}_n \oplus \mathcal{F}_{n+1} \qquad ()$$

is invariant under $H(\beta)$. Hence under the decomposition $\mathbf{C}^2 \otimes \mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{H}_n$, $H(\beta)$ has block diagonal form. We let $H_n(\beta)$ denote its restriction to $\mathcal{H}_n$.

We shall be concerned with the dynamics generated by $H_1(\beta)$ on $\mathcal{H}_1$. Explicitly,

$$H_1(\beta) = \begin{pmatrix} h_1 + E & \beta V \\ \beta V^\dagger & h_2 \end{pmatrix} \qquad ()$$

where the operators $h_1, h_2, V, V^\dagger$ act as follows: let $\phi_1 \in \mathcal{F}_1 = L^2(R)$ and $\phi_2 \in \mathcal{F}_2 = \left(L^2(R) \otimes L^2(R)\right)_{\mathrm{sym}}$;

$$\begin{aligned} (h_1\phi_1)(k) &= |k|\phi_1(k) \\ (h_2\phi_2)(k_1, k_2) &= (|k_1| + |k_2|)\phi_2(k_1, k_2) \\ (V^\dagger\phi_1)(k_1, k_2) &= \frac{1}{\sqrt{2}}[\phi_1(k_1)f(k_2) + \phi_1(k_2)f(k_1)] \\ (V\phi_2)(k_1) &= \sqrt{2}\int \bar{f}(k_2)\phi_2(k_1, k_2)dk_2 \end{aligned} \qquad ()$$

2.2 Dilation analyticity.

We say that a function $g(k)$ is $\mathcal{F}_1$-dilation analytic in the set $\mathcal{S} \subset \mathbf{C}$, if g_θ is analytic as an $\mathcal{F}_1$-valued function of θ for all $\theta \in \mathcal{S}$, where

$$g_\theta(k) = e^{-i\theta/2} g(e^{-i\theta}k) \qquad ()$$

Henceforth the statement "$g(k)$ is dilation analytic" will mean that g is $\mathcal{F}_1$-dilation analytic in a set $\mathcal{S} \subset \mathbf{C}$ which contains some strip $\{\theta \mid -a \le \mathrm{Re}\theta \le b\}$, for some $a, b > 0$. Note in this case it is also true that the function $\bar{g}(k)$ is dilation analytic, and $(\bar{g})_\theta(k) = \overline{g_{-\bar{\theta}}(k)}$. Furthermore the product of dilation analytic functions is dilation analytic, as is the inverse of a dilation analytic function which is bounded away from zero.

Suppose $f(k)$ and $g(k)$ are both dilation analytic in some strip containing $\mathrm{Re}\theta = 0$. Then the inner product $\langle f_{-\bar{\theta}}, g_\theta \rangle$ is independent of θ. This follows because it is analytic in θ, and when θ is pure imaginary the θ-dependence can be removed by a change of variables. Next, recall the multiplication operator h_1 defined in (). Let $h_{1,\theta}$ be the following multiplication operator:

$$(h_{1,\theta}\phi_1)(k) = e^{-i\theta}|k|\phi_1(k). \qquad ()$$

Suppose that $\mathrm{Im}z > 0$. Then the inner product $\langle f_{-\bar{\theta}}, (h_{1,\theta} - z)^{-1} g_\theta \rangle$ is independent of θ in the region $0 \le \mathrm{Re}\theta \le a$ for some $a > 0$. Again this follows by noting that the inner product is analytic as a function of θ in some open set containing the strip $0 \le \mathrm{Re}\theta \le a$, and that the θ-dependence can be removed by a change of variables when $\mathrm{Re}\theta = 0$. Similarly if $\mathrm{Im}z < 0$, $\langle f_{-\bar{\theta}}, (h_{1,\theta} - z)^{-1} g_\theta \rangle$ is independent of θ in the region $-a < \mathrm{Re}\theta \le 0$.

Finally we say that a bounded operator A on $\mathcal{F}_1$ is dilation analytic if $A\psi$ is dilation analytic for every ψ which is dilation analytic, and if there exists A_θ bounded on $\mathcal{F}_1$ such that

$$(A\psi)_\theta = A_\theta \psi_\theta. \qquad ()$$

2.3 Assumptions on $f(k)$.

For comparison we recall some results for the spin-boson Hamiltonian. This Hamiltonian has the same form as (), with the off-diagonal terms both replaced by $\beta\big(a(\bar{f}) + a^\dagger(f)\big)$. For this model, it is known that a sufficient condition for essential self-adjointness and existence of a unique ground state is $\int |f(k)|^2(1+|k|^{-2})dk < \infty$ ([Sp1], [HSp]). We now state our assumptions on the form factor.

i) $f(k)$ is real, and $f(k) = f(-k)$ for all k.
ii) $f(E) \neq 0$.
iii) $f(k)$ is $\mathcal{F}_1$-dilation analytic in some strip $-\theta_0 \le \mathrm{Re}\theta \le \theta_0$, with $\theta_0 > 0$.
iv) For some $\delta > 0$, $C > 0$, $|f_\theta(k)|^2 \le C|k|^{1+\delta}$ for all $|k| \le 1$, and for all $-\theta_0 \le \mathrm{Re}\theta \le \theta_0$.
v) $|f(k)|^2 \le C\,(1+k^2)^{-1}$ for all k.

Some comments on these assumptions are in order: (i) is unessential, and is chosen for convenience; (ii) is essential – otherwise the transition is forbidden in leading order; (iii) is essential for our method to work; (iv) is necessary for our estimates – it is slightly stronger than the conditions stated above for the spin-boson model; (v) gives a useful shortcut to our result, and is physically reasonable.

2.4 Special initial states.

In order to find evidence of stimulated emission, we consider a family of initial states which depend on β. Specifically, we choose states which are localised at energy E to within

O(β^2). For Im$w \neq 0$ we define

$$\psi^{(w)}(k) = \frac{\kappa(w)}{|k| - E + w}, \qquad \Psi^{(w)} = \begin{pmatrix} \psi^{(w)} \\ 0 \end{pmatrix} \tag{}$$

where $\kappa(w)$ is chosen so that $||\psi^{(w)}||_2 = 1$. Then our special initial states will be $\Psi^{(\beta^2 w)}$, and we will take Im$w > 0$ for definiteness. Note that

$$\lim_{\beta \to 0} \beta^{-2} |\kappa(\beta^2 w)|^2 = \frac{|\mathrm{Im} w|}{2\pi} \tag{}$$

2.5 Results.

We define

$$u = \lim_{\eta \to 0} \langle f, (h_1 - E - i\eta)^{-1} f \rangle \stackrel{\mathrm{def}}{=} E_1 + i\Gamma \tag{}$$

It follows from section 2.3 that

$$\Gamma = 2\pi |f(E)|^2 > 0. \tag{}$$

Note that $2\beta^2\Gamma$ is the rate of decay of the excited state which follows by applying the Fermi Golden Rule to this model.

We write P_1 and P_2 for the orthogonal projections onto the 1- and 2-particle subspaces of $\mathcal{H}_1$ respectively:

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{}$$

Let $\psi \in \mathcal{F}_1$, and $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$. We are interested in the time evolution of the initial state Ψ, projected onto the 1-particle subspace. Accordingly let

$$p_1(\beta, t; \psi) = ||P_1\big(e^{-iH_1(\beta)t}\Psi\big)||^2 \tag{}$$

We examine the bahavior of this probability as a function of the rescaled time $\tau = \beta^2 t$, in the limit $\beta \to 0$.

Theorem : *Let $\psi(k)$ be dilation analytic. Then for $\tau \geq 0$,*

$$\lim_{\beta \to 0} p_1(\beta, \beta^{-2}\tau; \psi) = e^{-2\Gamma\tau} \tag{}$$

In order to state the next result we need some notation. Define

$$K(u,w;\tau,\alpha;v)=\frac{e^{i\tau u}}{v+w}\Big((\alpha+1)e^{-i\tau v}-e^{i\tau w}\Big)-\frac{e^{i\tau w}}{v+u}\Big(e^{-i\tau v}-e^{i\tau u}\Big)\qquad ()$$

Recall the definition of $\psi^{(w)}$ in (), and of u in ().

Theorem : *For* $\mathrm{Im}w>0$,

$$\lim_{\beta\to 0}\,p_1(\beta,\beta^{-2}\tau;\psi^{(\beta^2 w)})=\frac{4\Gamma^2\mathrm{Im}w}{\pi|u-w|^2}\int_{-\infty}^{\infty}\left|K(u,w;\tau,\frac{u-w}{2i\Gamma};v)\right|^2 dv.\qquad ()$$

The integral in () can be computed using Cauchy's theorem; we record the result in section 4.6. One case of interest is when $\mathrm{Re}(w)=E_1=\mathrm{Re}(u)$, so we state this as a separate result below. We rescale time by setting $s=\Gamma\tau$, and we write $w=E_1+ic\Gamma$.

Corollary : *For any* $c>0$,

$$\begin{aligned}\lim_{\beta\to 0}\,p_1(\beta,\beta^{-2}\Gamma^{-1}s;\psi^{\left(\beta^2(E_1+ic\Gamma)\right)})=&\Big(\frac{3-c}{1-c}\Big)^2e^{-2s}+\frac{4c}{(1-c)^2}e^{-2sc}\\&-\frac{8c(3-c)}{(1-c)^2(1+c)}e^{-s-sc}-\frac{8}{1+c}e^{-2s-2sc}\end{aligned}\qquad ()$$

As one would expect, the right side of () approaches the right side of () as $c\to\infty$. In fact this also happens in the limit $c\to 0$. So the interesting behavior occurs for $0<c<\infty$. Perhaps the most interesting case is when $w=u$. We present this as another special case below.

Corollary :

$$\lim_{\beta\to 0}\,p_1(\beta,\beta^{-2}\Gamma^{-1}s;\psi^{(\beta^2 u)})=[5-8s+4s^2]e^{-2s}-4e^{-4s}\qquad ()$$

2.6 Discussion.

We present and its corollaries as evidence for stimulated emission in this model. The initial one photon state $\psi^{(\beta^2 w)}$ is designed to mimic the spontaneously emitted photon state, so we expect that it should produce an enhanced decay rate for the atom. In position space it is centered at the atom at time zero. As the results show, enhancement does occur at some values of the rescaled time $\beta^{-2}t$. Notice that for τ small, the right sides of () and () are both approximately $1-2\Gamma\tau$, so they have the same linear decay rate. Therefore the effects of stimulated emission show up at quadratic order in $\tau = \beta^2 t$, which gives another way of seeing that application of standard time dependent perturbation theory does not yield an enhanced decay rate.

As noted in the introduction, the probability that the atom remains excited may increase at later times. This is a little unexpected in our model, and we take this interesting feature as motivation for the continued study of this most 'elementary' quantum field theory model.

3. Stone's formula and the resolvent expansion.

3.1 The setup.

To prove our results we use a resolvent expansion which is essentially a resummation of the usual perturbative expansion. We separate the interaction between atom and field into a direct part and an exchange part, and we resum the direct part.

For any $\Psi \in \mathcal{H}_1$, Stone's formula [RS] gives

$$e^{-iH_1(\beta)t}\Psi = \lim_{\eta\downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-itx} \Big[\big(H_1(\beta) - x - i\eta\big)^{-1}\Psi - \big(H_1(\beta) - x + i\eta\big)^{-1}\Psi \Big] dx \qquad ()$$

The resolvent $\big(H_1(\beta) - z\big)^{-1}$ can be expressed as a 2×2 matrix with respect to the direct sum decomposition $\mathcal{H}_1 = \mathcal{F}_1 \oplus \mathcal{F}_2$ (see ()); for $\mathrm{Im} z \neq 0$,

$$\big(H_1(\beta) - z\big)^{-1} = \begin{pmatrix} G(\beta,z) & -\beta G(\beta,z) V (h_2 - z)^{-1} \\ -\beta (h_1 - z)^{-1} V^{\dagger} G(\beta,z) & M(\beta,z) \end{pmatrix} \qquad ()$$

where

$$G(\beta,z) = \big(h_1 + E - z - \beta^2 V (h_2 - z)^{-1} V^{\dagger}\big)^{-1} \qquad ()$$

$$M(\beta,z) = \big(h_2 - z - \beta^2 V^{\dagger} (h_1 + E - z)^{-1} V\big)^{-1} \qquad ()$$

For our purposes it will be enough to apply Stone's formula to a state $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, and to project the result onto the first component. Therefore we will study

$$P_1(e^{-iH_1(\beta)t}\Psi) = \lim_{\eta\downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-itx} \Big[G(\beta, x + i\eta)\psi - G(\beta, x - i\eta)\psi \Big] dx \qquad ()$$

Our resummed expansion is obtained by splitting the term $V(h_2 - z)^{-1} V^{\dagger}$ in () into two factors, corresponding to the direct and exchange parts of the interaction. For $\mathrm{Im} z \neq 0$, we define the following operators on $\mathcal{F}_1$:

$$(S(z)\psi)(k) = \left(\int_{-\infty}^{\infty} \frac{|f(p)|^2}{|p| + |k| - z} dp \right) \psi(k) \qquad ()$$

$$(T(z)\psi)(k) = \left(\int_{-\infty}^{\infty} \frac{\bar{f}(p)\psi(p)}{|p| + |k| - z} dp \right) f(k) \qquad ()$$

Then for all $\psi \in \mathcal{F}_1$,

$$V(h_2 - z)^{-1} V^{\dagger} \psi = S(z)\psi + T(z)\psi. \qquad ()$$

We also define

$$G_0(\beta, z) = \big(h_1 + E - z - \beta^2 S(z)\big)^{-1} \qquad ()$$

$$G_1(\beta, z) = \beta^2 G_0(\beta, z) T(z) G_0(\beta, z) \qquad ()$$

$$G_R(\beta, z) = \beta^4 G_0(\beta, z) T(z) G(\beta, z) T(z) G_0(\beta, z) \qquad ()$$

Using (), and applying the second resolvent formula to (), we obtain

$$G(\beta, z) = G_0(\beta, z) + G_1(\beta, z) + G_R(\beta, z) \qquad ()$$

3.2 Leading order parts and remainder.

On substituting the right side of () into Stone's formula, we obtain an expansion for the time evolved state. Define

$$T_\sigma(\psi, t) = \lim_{\eta \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-itx} \big[G_\sigma(\beta, x + i\eta)\psi - G_\sigma(\beta, x - i\eta)\psi\big]\, dx \qquad ()$$

where $\sigma \in \{0, 1, R\}$. Then we obtain

$$P_1\left(e^{-iH_1(\beta)t} \begin{pmatrix} \psi \\ 0 \end{pmatrix}\right) = T_0(\psi, t) + T_1(\psi, t) + T_R(\psi, t) \qquad ()$$

The following lemmas give bounds for the three terms on the right side of (). The first result, , will be proved in section 4.5. Recall the definition of u in ().

Lemma : *Let $|\beta|$ be sufficiently small and let ψ be dilation analytic. There is $z_0 \in \mathbf{C}$, with $|z_0 - E + \beta^2 u| = \mathrm{O}(\beta^4)$, such that for all $t \geq 0$,*

$$||T_1(\psi, t)|| \leq C\,\beta^2, \qquad ||T_0(\psi, t) - e^{-it(h_1 + z_0)}\psi|| \leq C\,\beta^2 \qquad ()$$

The number z_0 appearing in () will be defined below in (). The next lemma gives a bound for the remaining term on the right side of (); the proof is in section 5. Recall that δ was defined in section 2.3. It will be convenient to use the following additional notation:

$$\beta^{\{2,\delta\}} = \begin{cases} \beta^{2\delta} & \text{if } 0 < \delta < 1 \\ \ln(\beta^{-2})\beta^2 & \text{if } \delta = 1 \\ \beta^2 & \text{if } \delta > 1 \end{cases} \qquad ()$$

Lemma : *For $|\beta|$ sufficiently small, for all $t \geq 0$ and all $\psi \in \mathcal{F}_1$,*

$$||T_R(\psi, t)|| \leq C\,\beta^{\{2,\delta\}}\,||\psi|| \qquad ()$$

3.3 Proof of .

Combining the results of and we see that for $|\beta|$ small enough, for any dilation analytic state ψ with $||\psi|| = 1$ there is C such that

$$||P_1\left(e^{-iH_1(\beta)t}\begin{pmatrix}\psi\\0\end{pmatrix}\right) - e^{-it(h_1+z_0)}\psi|| \leq C\,\beta^{\{2,\delta\}} \qquad ()$$

for all $t \geq 0$. Since

$$||e^{-it(h_1+z_0)}\psi|| = e^{-t(\beta^2\Gamma + O(\beta^4))}, \qquad ()$$

the result follows immediately. ∎

3.4 Propagators.

We require some detailed properties of the propagators $G_0(\beta, z)$ and $G_1(\beta, z)$. To this end, we define the following functions for $\mathrm{Im} z \neq 0$:

$$\Pi(z) = \langle f, (h_1 - z)^{-1} f\rangle = \int_{-\infty}^{\infty} \frac{|f(p)|^2}{|p| - z}\,dp \qquad ()$$

$$g(z) = E - z - \beta^2\Pi(z). \qquad ()$$

It follows that $S(z)\psi(k) = \Pi(z - |k|)\psi(k)$, so we have

$$\big(G_0(\beta, z)\psi\big)(k) = g(z - |k|)^{-1}\psi(k). \qquad ()$$

Following the comments in Section 2.2, for Im$z > 0$ and $0 \leq \mathrm{Re}\theta \leq \theta_0$, and for Im$z < 0$ and $0 \geq \mathrm{Re}\theta \geq -\theta_0$, we have

$$\Pi(z) = \langle f_{-\bar{\theta}}, (h_{1,\theta} - z)^{-1} f_\theta \rangle = \int_{-\infty}^{\infty} \overline{f_{-\bar{\theta}}(p)} f_\theta(p) (e^{-i\theta}|p| - z)^{-1} dp \qquad ()$$

Define the two subsets of the complex plane:

$$\begin{aligned} \mathcal{D}_+ &= \{z \,:\, |z| > 0, -\theta_0 < \mathrm{Arg} z < \pi\} \\ \mathcal{D}_- &= \{z \,:\, |z| > 0, -\pi < \mathrm{Arg} z < \theta_0\} \end{aligned} \qquad ()$$

Then letting $\theta = \pm\theta_0$ (see section 2.3) () provides the analytic continuations of $\Pi(z)$ from the region Im$z > 0$ to $\mathcal{D}_+$, and from the region Im$z < 0$ to $\mathcal{D}_-$. We denote the former by $\Pi_+(z)$ and the latter $\Pi_-(z)$. These in turn define the corresponding analytic continuations of $g(z)$ to $\mathcal{D}_+, \mathcal{D}_-$ which we denote by $g_+(z)$ and $g_-(z)$.

We list some properties of $\Pi_+(z)$:

(i) $\mathrm{Im}(z) \geq 0 \quad \Rightarrow \quad \mathrm{Im}\big(\Pi_+(z)\big) \geq 0$

(ii) $\lim_{\eta \downarrow 0} \Pi_+(i\eta) = \int |p|^{-1} |f(p)|^2 dp < \infty$.

(iii) for all $z \in \mathcal{D}_\pm$,

$$|\Pi_\pm(z)| \leq \int_{-\infty}^{\infty} |f_{\mp\theta_0}(p)|\, |f_{\pm\theta_0}(p)|\, |p|^{-1}\, dp \,<\, \infty$$

(iv) $\Pi_+(E) = u = E_1 + i\Gamma$.

Since by assumption $\mathrm{Im}(\Pi_+(E)) = \Gamma > 0$, it follows that there is $\epsilon > 0$ such that $\mathrm{Im}\big(\Pi_+(z)\big) > 0$ for all z satisfying $|z - E| \leq \epsilon$.

For $|\beta|$ sufficiently small, $g_+(z)$ has a unique zero in $\mathcal{D}_+$, which we denote z_0. It satisfies the following:

$$g_+(z_0) = 0, \quad g_+'(z_0) = -1 + O(\beta^2), \quad z_0 = E - \beta^2 u + O(\beta^4) \qquad ()$$

The following bound is easily verified; for $|\beta|$ sufficiently small and for all $z \in \mathcal{D}_+$,

$$|g_+(z)| \geq \frac{|z - z_0|}{2} \qquad ()$$

Together with $f(E) \neq 0$ this in turn implies that there is $C > 0$ such that for all z with Im$z \geq 0$,

$$|g_+(z)| \geq C(|z - E|^2 + \beta^4)^{1/2} \qquad ()$$

Similar bounds hold for $g_-(z)$: for all $z \in \mathcal{D}_-$,

$$|g_-(z)| \geq \frac{|z - \bar{z}_0|}{2}; \qquad ()$$

for all z with $\mathrm{Im} z \leq 0$,

$$|g_-(z)| \geq C(|z - E|^2 + \beta^4)^{1/2}. \qquad ()$$

Returning to () we deduce that for any function $\psi(k) \in \mathcal{F}_1$, and any z with $\mathrm{Im} z \neq 0$,

$$|(G_0(\beta, z)\psi)(k)| \leq C\big(|z - |k| - E|^2 + \beta^4\big)^{-1/2} |\psi(k)|. \qquad ()$$

We shall also need some estimates for the dilated version of the operator G_0 (see section 2.2). Since the operator G_0 is multiplication by a function of $|k|$, its dilation is obtained by replacing $|k|$ everywhere by $e^{-i\theta}|k|$. So we define

$$(G_{0,\theta}(\beta, z)\psi)(k) = g(z - e^{-i\theta}|k|)^{-1}\psi(k). \qquad ()$$

For $\mathrm{Im} z > 0$, $g(z - e^{-i\theta}|k|)$ is analytic in θ for θ in some open set containing the strip $0 \leq \mathrm{Re}\theta < \theta_0$. By considering separately the regions $|e^{-i\theta}|k| + E - z| \leq \epsilon$ and $|e^{-i\theta}|k| + E - z| > \epsilon$ for ϵ sufficently small, we get the bounds

$$\begin{aligned} |g(z - e^{-i\theta}|k|)| =& \big|e^{-i\theta}|k| + E - z - \beta^2 \Pi(z - e^{-i\theta}|k|)\big| \\ \geq& C \max(\beta^2\Gamma, \mathrm{Re}(\theta)|k| + \mathrm{Im} z) \end{aligned} \qquad ()$$

This shows that the operator $G_{0,\theta}(\beta, z)$ is analytic in θ and z in the region $0 \leq \mathrm{Re}\theta \leq \theta_0$ and $\mathrm{Im} z > 0$, and satisfies the bound

$$|(G_{0,\theta}(\beta, z)\psi)(k)| \leq C \min\{\frac{1}{\beta^2}, \frac{1}{|k||\mathrm{Re}\theta|}, \frac{1}{|\mathrm{Im} z|}\} \qquad ()$$

By similar reasoning the same bound holds for $\mathrm{Im} z < 0$ and $-\theta_0 \leq \mathrm{Re}\theta \leq 0$.

4. Computation of the leading order terms.

4.1 The setup.

As shown by in section 3.2, the leading order contribution to the time evolved state is provided by the two terms $T_0(\psi,t)$ and $T_1(\psi,t)$. In this section we will extract and compute the most important parts of these terms when the state ψ is a special initial state as defined in (), and estimate the error terms.

4.2 Leading order part of T_0.

The leading order part of $T_0(\psi,t)$ can be extracted for any state ψ. For any $\psi \in \mathcal{F}_1$, we have

$$T_0(\psi,t)(k) = \frac{1}{2\pi i}\lim_{\eta\to 0}\int_{-\infty}^{\infty} e^{-itx}\left[g(x+i\eta-|k|)^{-1}\psi(k) - g(x-i\eta-|k|)^{-1}\psi(k)\right]dx \qquad ()$$

Recall the definition of z_0 in (). Define:

$$L_0(\psi,\beta,t;k) = e^{-it(|k|+z_0)}\psi(k). \qquad ()$$

Lemma : *For all $t \geq 0$, and ψ with $||\psi|| = 1$,*

$$\left(\int_{-\infty}^{\infty}\left|T_0(\psi,t)(k) - L_0(\psi,\beta,t;k)\right|^2\right)^{1/2} \leq C\beta^2 \qquad ()$$

proof: First, it is sufficient to prove that for all k,

$$\left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-itx}\left[g_+(x-|k|)^{-1} - g_-(x-|k|)^{-1}\right]dx - e^{-it(|k|+z_0)}\right| \leq C\beta^2 \qquad ()$$

Second, the bounds (), () imply that for $|x-|k|-E| \geq E/2$, we have

$$|g_\pm(x-|k|)|^{-1} \leq C|x-|k|-E|^{-1}. \qquad ()$$

Combining this with the formula

$$g_+(z)^{-1} - g_-(z)^{-1} = g_+(z)^{-1}g_-(z)^{-1}[-\beta^2\Pi_-(z) + \beta^2\Pi_+(z)], \qquad ()$$

and recalling that $\Pi_\pm(z)$ is uniformly bounded, it is sufficient to show that

$$\left|\frac{1}{2\pi i}\int_{|k|+E/2}^{|k|+3E/2} e^{-itx}\left[g_+(x-|k|)^{-1} - g_-(x-|k|)^{-1}\right]dx - e^{-it(|k|+z_0)}\right| \leq C\beta^2 \qquad ()$$

Let S be the semicircular arc in the lower half plane which begins at $|k|+E/2$ and ends at $|k|+3E/2$. For $|\beta|$ sufficiently small the pole of $g_+(z-|k|)^{-1}$ lies above S. Therefore by deforming the contours of integration we get

$$\begin{aligned} \frac{1}{2\pi i}\int_{|k|+E/2}^{|k|+3E/2} & e^{-itx}\left[g_+(x-|k|)^{-1}-g_-(x-|k|)^{-1}\right]dx \\ &= \frac{1}{2\pi i}\int_S e^{-itz}\left[g_+(z-|k|)^{-1}-g_-(z-|k|)^{-1}\right]dz \\ &\qquad\qquad - \mathrm{Res}\left[e^{-itz}g_+(z-|k|)^{-1}; z=z_0+|k|\right] \end{aligned} \qquad ()$$

where $\mathrm{Res}(F(z); z=w)$ denotes the residue of the meromorphic function $F(z)$ at $z=w$. Using () again, and the bounds (), (), we deduce that the integral along S is $\mathrm{O}(\beta^2)$. Finally the evaluation of the residue and the estimate $g_+'(z_0)=-1+\mathrm{O}(\beta^2)$ yield the stated result.

∎

4.3 Leading order part of T_1.

Unlike the estimate in section 4.2 for T_0, we must now use the particular form of the initial state. Recall that the special initial state is

$$\psi(k)=\psi^{(\beta^2 w)}(k)=\frac{\kappa(\beta^2 w)}{|k|-E+\beta^2 w}$$

where $\mathrm{Im}(w)>0$, and κ is chosen so that $||\psi^{(\beta^2 w)}||=1$. Define for $\mathrm{Im} z\neq 0$,

$$B(z,w,|k|)=\int_{-\infty}^{\infty}\frac{\bar{f}(p)}{|p|+|k|-z}g(z-|p|)^{-1}\psi^{(\beta^2 w)}(p)dp \qquad ()$$

Then we can write

$$\begin{aligned} T_1(\psi^{(\beta^2 w)},t)(k)=\frac{\beta^2 f(k)}{2\pi i}\lim_{\eta\to 0}\int_{-\infty}^{\infty} & e^{-itx}\left[g(x+i\eta-|k|)^{-1}B(x+i\eta,w,|k|)\right. \\ & \left. - g(x-i\eta-|k|)^{-1}B(x-i\eta,w,|k|)\right]dx \end{aligned} \qquad ()$$

Using $f(p)=f(-p)$ real we can rewrite () as

$$B(z,w,|k|)=2\int_0^{\infty}\frac{f(p)}{p+|k|-z}g(z-p)^{-1}\frac{\kappa(\beta^2 w)}{p-E+\beta^2 w}dp \qquad ()$$

For $\mathrm{Im}(z)>0$, the integrand in () has only one pole below the real axis, namely $p=E-\beta^2 w$. Its residue is

$$b(z,w,|k|)=-2\pi i\kappa(\beta^2 w)\mathrm{Res}\left(\frac{2f(p)}{p+|k|-z}g(z-p)^{-1}; p=E-\beta^2 w\right) \qquad ()$$

where f now denotes the analytic continuation off the real axis. To estimate the contribution of () to the integral (), we will rotate the contour of integration. Accordingly we define

$$B_\theta(z,w,|k|) = 2\int_0^\infty \frac{f(pe^{-i\theta})}{pe^{-i\theta}+|k|-z} g(z-pe^{-i\theta})^{-1} \frac{\kappa(\beta^2 w)}{pe^{-i\theta}-E+\beta^2 w} e^{-i\theta} dp \qquad ()$$

Lemma : *There exists C such that for all $\eta > 0$, all θ with $\theta_0/2 \le \mathrm{Re}\theta \le \theta_0$,*

$$\int_{-\infty}^{\infty} \left|B_{\pm\theta}(x \pm i\eta, w, |k|)\right|^2 dx \le C\beta^2 \qquad ()$$

proof: Let $\theta \ge \theta_0/2$ and $\eta > 0$. We use the bounds (recall ()):

$$\begin{aligned} |g(x+i\eta-pe^{-i\theta})| &\ge \frac{1}{2}|x+i\eta-pe^{-i\theta}-z_0| \ge C[p^2\sin^2\theta + (x-p\cos\theta-\mathrm{Re}z_0)^2]^{1/2} \\ |pe^{-i\theta}+|k|-x-i\eta| &\ge C[p^2\sin^2\theta+(x-p\cos\theta-|k|)^2]^{1/2} \\ |pe^{-i\theta}-E+\beta^2 w| &\ge [p^2\sin^2\theta+(E-p\cos\theta)^2]^{1/2} \ge C \end{aligned} \qquad ()$$

Hence an application of Cauchy-Schwarz shows that

$$\begin{aligned} \left|B_\theta(x+i\eta,w,|k|)\right|^2 \le & C|\kappa(\beta^2 w)|^2 \left(\int_0^\infty |f(pe^{-i\theta})|[p^2\sin^2\theta+(x-p\cos\theta-\mathrm{Re}z_0)^2]^{-1}dp\right) \\ & \left(\int_0^\infty |f(qe^{-i\theta})|[q^2\sin^2\theta+(x-q\cos\theta-|k|)^2]^{-1}dq\right) \end{aligned} \qquad ()$$

We now integrate over x. We interchange the x integral with the integrals over p and q, and use Cauchy-Schwarz to estimate the integral over x. The result is

$$\int_{-\infty}^{\infty} \left|B_\theta(x+i\eta,w,|k|)\right|^2 dx \le C|\kappa(\beta^2 w)|^2 \left(\int_0^\infty |f(pe^{-i\theta})|p^{-3/2}dp\right)^2 \qquad ()$$

By assumption the integral is finite (see section 2.3, (iv)), and $|\kappa(\beta^2 w)|^2 = \mathrm{O}(\beta^2)$, and hence the result follows. The analysis for $B_{-\theta}(x-i\eta,w,|k|)$ is identical. ∎

We now observe that for $\mathrm{Im}z < 0$, $B_\theta(z,w,|k|)$ is independent of $\theta \le 0$, since in this case the integrand on the right side of () has no poles above the real axis. For $\mathrm{Im}z > 0$, and $\theta \ge \theta_0/2$, an application of Cauchy's theorem gives

$$B(z,w,|k|) = b(z,w,|k|) + B_\theta(z,w,|k|) \qquad ()$$

Hence for $\theta_0/2 \leq \theta \leq \theta_0$,

$$\begin{aligned}T_1(\psi^{(\beta^2 w)},t)(k) = &\frac{\beta^2 f(k)}{2\pi i}\lim_{\eta\downarrow 0}\int_{-\infty}^{\infty} e^{-itx} g(x+i\eta-|k|)^{-1} b(x+i\eta,w,|k|)dx \\ &+\frac{\beta^2 f(k)}{2\pi i}\lim_{\eta\downarrow 0}\int_{-\infty}^{\infty} e^{-itx}\big[g(x+i\eta-|k|)^{-1}B_\theta(x+i\eta,w,|k|) \\ &\qquad - g(x-i\eta-|k|)^{-1}B_{-\theta}(x-i\eta,w,|k|)\big]dx\end{aligned} \qquad ()$$

We now use (), () to deduce

$$\int_{-\infty}^{\infty}\big|g(x\pm i\eta-|k|)\big|^{-2}dx \leq C\beta^{-2}. \qquad ()$$

It follows by Cauchy-Schwarz and () that

$$\left|\int_{-\infty}^{\infty} e^{-itx} g(x\pm i\eta-|k|)^{-1}B_{\pm\theta}(x\pm i\eta,w,|k|)dx\right| \leq C, \qquad ()$$

and hence the second term on the right side of () is bounded by $C\beta^2|f(k)|$. Therefore the dominant contribution to $T_1(\psi^{(\beta^2 w)},t)$ comes from the first term on the right side of (). This integral has three poles, all located at distance $\mathrm{O}(\beta^2)$ below the real axis (see () below for the evaluation of this residue): these are

$$\begin{aligned}x_1 =&|k|+z_0\\ x_2 =&E+|k|-\beta^2 w\\ x_3 =&E+z_0-\beta^2 w\end{aligned} \qquad ()$$

Given a,b,c with nonzero imaginary parts, we define

$$J(a,b,c;t)=\int_{-\infty}^{\infty} e^{-ity}(y-a)^{-1}(y-b)^{-1}(y-c)^{-1}dy \qquad ()$$

The next lemma shows that the leading order part of () is the following expression (recall () $u=\Pi_+(E)$):

$$L_1(w,\beta,t;k)=2\beta^2\kappa(\beta^2 w)f(E)f(k)J(|k|+E-\beta^2 u,|k|+E-\beta^2 w,2E-\beta^2 u-\beta^2 w;t) \quad ()$$

Lemma :

$$\left(\int_{-\infty}^{\infty}\left|T_1(\psi^{(\beta^2 w)},t)(k)-L_1(w,\beta,t;k)\right|^2 dk\right)^{1/2} \leq C|\beta|. \qquad ()$$

proof: It is enough to establish the bound $\left|T_1(\psi^{(\beta^2 w)},t)(k) - L_1(w,\beta,t;k)\right| \leq C|\beta||f(k)|$ for each k. Also using the estimate (), it is sufficient to establish such a bound for the difference between the first term on the right side of () and the function $L_1(w,\beta,t;k)$. The evaluation of the residue b gives

$$b(x+i\eta,w,|k|) = -4\pi i\kappa(\beta^2 w)f(E-\beta^2 w)(E+|k|-x-i\eta-\beta^2 w)^{-1}g(x+i\eta-E+\beta^2 w)^{-1} \quad ()$$

where f again denotes the analytic continuation. Inserting this in () we see that the integrand has three poles at the points x_1, x_2, x_3 defined in (). In the resulting expression we want to replace $g_+(x-|k|)^{-1}$ by $(x-x_1)^{-1}$ and $g_+(x-E+\beta^2 w)^{-1}$ by $(x-x_3)^{-1}$, and replace $f(E-\beta^2 w)$ by $f(E)$. This can be accomplished by noting that (i) only the neighborhoods of the poles at x_1, x_2, x_3 contribute to the integral – the integral over the remaining intervals is bounded by $C\beta^2|f(k)|$; (ii) in a neighborhood of $\mathrm{Re}x_1$, we have $g_+(x-|k|)^{-1} = (x-x_1)^{-1}(-1+\mathrm{O}(\beta^2))$ for real x (see ()), and in a neighborhood of $\mathrm{Re}x_3$ we have $g_+(x-E+\beta^2 w)^{-1} = (x-x_3)^{-1}(-1+\mathrm{O}(\beta^2))$ for real x; (iii) $f(E-\beta^2 w) = f(E) + \mathrm{O}(\beta^2)$. Now the following crude estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} |x-x_1|^{-1}|x-x_2|^{-1}|x-x_3|^{-1}dx \\
&\quad \leq \left(\int |x-x_1|^{-2}dx\right)^{1/2}\left(\int |x-x_2|^{-2}|x-x_3|^{-2}dx\right)^{1/2} \\
&\quad \leq \left(\int |x-x_1|^{-2}dx\right)^{1/2}\left(\int |x-x_2|^{-4}dx\right)^{1/4}\left(\int |x-x_3|^{-4}dx\right)^{1/4} \\
&\quad \leq C\beta^{-4}
\end{aligned} \quad ()$$

shows that

$$\left|\frac{\beta^2 f(k)}{2\pi i}\int_{-\infty}^{\infty} e^{-itx}g_+(x-|k|)^{-1}b(x,w,|k|)dx\right| \leq C\beta^{-1}|f(k)|. \quad ()$$

Therefore if we replace $g_+(x-|k|)^{-1}$ by $(x-x_1)^{-1}$ as discussed above, the error is bounded by $C|\beta||f(k)|$. For the same reason we can replace $g_+(x-E+\beta^2 w)^{-1}$ by $(x-x_3)^{-1}$, and $f(E-\beta^2 w)$ by $f(E)$, obtaining the same error. Putting it together, we have shown that

$$\left|T_1(\psi^{(\beta^2 w)},t)(k) - 2\beta^2\kappa(\beta^2 w)f(E)f(k)J(x_1,x_2,x_3;t)\right| \leq C\,|\beta|\,|f(k)| \quad ()$$

Now we use the estimate () again to deduce that for any $\epsilon_1, \epsilon_2, \epsilon_3$,

$$\left|J(x_1,x_2,x_3;t) - J(x_1+\beta^4\epsilon_1, x_2+\beta^4\epsilon_2, x_3+\beta^4\epsilon_3;t)\right| = \mathrm{O}(\beta^{-2}) \quad ()$$

Since $z_0 = E - \beta^2 u + \mathrm{O}(\beta^4)$, () implies that we can replace $J(x_1, x_2, x_3; t)$ in () with $J(|k| + E - \beta^2 u, |k| + E - \beta^2 w, 2E - \beta^2 u - \beta^2 w; t)$, with error $C|\beta||f(k)|$, thus proving the result.

■

4.4 Proof of .

Recall (). With some abuse of notation, we write

$$L_0(w, \beta, t; k) = L_0(\psi^{(\beta^2 w)}, \beta, t; k) \qquad ()$$

Then by the results of the previous sections, we have

$$||P_1(e^{-iH_1(\beta)t}\Psi^{(\beta^2 w)})||^2 = \int_{-\infty}^{\infty} |L_0(w, \beta, t; k) + L_1(w, \beta, t; k)|^2 dk + \mathrm{O}(|\beta|) \qquad ()$$

We make the change of variables $|k| = E + \beta^2 v$, and recall the definition $u = \Pi_+(E)$. Then the integral can be written

$$\int_{-\infty}^{\infty} |L_0(w, \beta, t; k) + L_1(w, \beta, t; k)|^2 dk = 2\beta^2 \int_{-E/\beta^2}^{\infty} |F(w, \beta, t; v)|^2 dv \qquad ()$$

where

$$\begin{aligned} F(w, \beta, t; v) =& e^{-it(z_0 - E + \beta^2 v)} \frac{\kappa(\beta^2 w)}{\beta^2 v + \beta^2 w} \\ &+ 2\beta^2 \kappa(\beta^2 w) f(E) f(E + \beta^2 v) J(-\beta^2 u + \beta^2 v, \beta^2 v - \beta^2 w, -\beta^2 u - \beta^2 w; t), \end{aligned} \qquad ()$$

and we have removed an overall phase factor e^{-2iEt}. By rescaling the integration variable in the definition of J, we can rewrite this as

$$F(w, \beta, t; v) = \frac{\kappa(\beta^2 w)}{\beta^2} \left[\frac{e^{-it(\beta^2 v + z_0 - E)}}{v + w} + 2f(E) f(E + \beta^2 v) J(-u + v, v - w, -u - w; \beta^2 t) \right] \quad ()$$

Let $\tau = \beta^2 t$; evaluating J gives

$$\frac{-1}{2\pi i} J(-u+v, v-w, -u-w; \tau) = \frac{e^{-i\tau(v-u)}}{(-u+w)(v+w)} + \frac{e^{-i\tau(v-w)}}{(u-w)(v+u)} + \frac{e^{-i\tau(-u-w)}}{(-w-v)(-u-v)} \quad ()$$

Substituting this in () we see that the integral over v is dominated by the contributions from the neighborhoods of the poles. Therefore we can replace $f(E + \beta^2 v)$ by $f(E)$ with an error $\mathrm{O}(\beta^2)$, and we can replace the lower limit of integration $-E/\beta^2$ by $-\infty$ with the same error. Finally, replacing $e^{-it(\beta^2 v + z_0 - E)}$ by $e^{-i\beta^2 t(v-u)}$ produces an error which is $\mathrm{O}(\beta^4 t) = \mathrm{O}(\beta^2 \tau)$.

Therefore since $\Gamma = 2\pi|f(E)|^2$ and using $\lim_{\beta\to 0}\beta^{-2}|\kappa(\beta^2 w)|^2 = \mathrm{Im}(w)/2\pi$, we get

$$\lim_{\beta\to 0} ||P_1\big(e^{-iH_1(\beta)(\beta^{-2}\tau)}\Psi^{(\beta^2 w)}\big)||^2$$

$$= \lim_{\beta\to 0} 2\beta^2 \int_{-E/\beta^2}^{\infty} |F(w,\beta,\beta^{-2}\tau;v)|^2 dv$$

$$= 2\Big(\frac{\mathrm{Im} w}{2\pi}\Big)\int_{-\infty}^{\infty}\left|\frac{e^{-i\tau(v-u)}}{v+w} + \frac{\Gamma}{\pi}J(-u+v, v-w, -u-w;\tau)\right|^2 dv \qquad ()$$

After some rearrangement this produces (). ■

4.5 Proof of .

Suppose that ψ in (), () is dilation analytic. Then the first term on the right side of () is absent, and hence () immediately implies the bound

$$\left|T_1(\psi^{(\beta^2 w)}, t)(k)\right| \leq C\beta^2|f(k)|, \qquad ()$$

which together with () gives (). ■

4.6 Evaluation of limit.

Here we write down the explicit expression for the integral on the right side of (). For convenience, define

$$\alpha = \frac{u-w}{2i\Gamma} \qquad ()$$

Then Cauchy's theorem gives the following:

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left|K(u,w;\tau,\alpha;v)\right|^2 dv = &\frac{e^{i\tau(u-\bar{u})}}{w-\bar{w}}\big[|1+\alpha|^2 - e^{i\tau(w-\bar{w})}(1+\alpha+\bar{\alpha})\big]\\ &+\frac{e^{i\tau(w-\bar{w})}}{u-\bar{u}}\big[1-e^{i\tau(u-\bar{u})}\big]\\ &-\frac{(1+\bar{\alpha})e^{i\tau(w-\bar{u})}}{u-\bar{w}}\big[1-e^{i\tau(u-\bar{w})}\big]\\ &-\frac{(1+\alpha)e^{i\tau(u-\bar{w})}}{w-\bar{u}}\big[1-e^{i\tau(w-\bar{u})}\big]\end{aligned} \qquad ()$$

5. Convergence of the expansion.

5.1 Proof of .

Our first goal is to prove which shows that $T_R(\psi, t)$ in () is uniformly bounded in t. Using Stone's formula () this follows immediately from the following estimate (recall the definition of G_R in (), and the definition of δ in section 2.3 (iv)).

Lemma : *For $|\beta|$ sufficiently small, and for all z with* $\mathrm{Im}z \neq 0$*,*

$$||G_R(\beta, z)|| \leq \frac{C\beta^{\{2,\delta\}}}{(\mathrm{Re}z - E)^2 + 1} \qquad ()$$

We will deduce this result from below. For $\mathrm{Im}z \neq 0$, and p, q real, define

$$\mathcal{A}(z; p, q) = \langle f, (h_1 + |p| - z)^{-1} G(\beta, z)(h_1 + |q| - z)^{-1} f\rangle \qquad ()$$

Then the integral kernel of the operator $T(z)G(\beta, z)T(z)$ is given as follows:

$$\big(T(z)G(\beta, z)T(z)\big)(p, q) = f(p)\mathcal{A}(z; p, q)f(q) \qquad ()$$

Therefore the integral kernel of the operator $G_R(\beta, z)$ in () is the following:

$$(G_R(\beta, z))(p, q) = \beta^4 g(z - |p|)^{-1} f(p)\mathcal{A}(z; p, q)g(z - |q|)^{-1} f(q) \qquad ()$$

Our main technical result is the following.

Lemma : *For $|\beta|$ sufficiently small, for all z with* $\mathrm{Im}z \neq 0$*, and for all p, q,*

$$|\mathcal{A}(z; p, q)| \leq C\beta^{-2}\beta^{\{2,\delta\}} \qquad ()$$

Proof of .

We use the Hibert-Schmidt norm to bound the operator norm:

$$||G_R(\beta, z)||^2 \leq \int\int |(G_R(\beta, z))(p, q)|^2 \, dpdq \qquad ()$$

Using and also the bounds () we deduce that for $\mathrm{Im}z \neq 0$

$$\begin{aligned} &||G_R(\beta, z)||^2 \\ &\leq C\beta^4(\beta^{\{2,\delta\}})^2 \int\int |f(p)|^2|f(q)|^2 \big(|z - |p| - E|^2 + \beta^4\big)^{-1}\big(|z - |q| - E|^2 + \beta^4\big)^{-1} dpdq \end{aligned} \qquad ()$$

Recall from section 2.3(v) that $|f(k)|^2 \leq C(k^2+1)^{-1}$; combined with the estimate

$$\int_{-\infty}^{\infty} \left((|z-|p|-E)^2+\beta^4\right)^{-1}(p^2+1)^{-1}dp \leq \frac{C\beta^{-2}}{(\mathrm{Re}z-E)^2+1} \qquad ()$$

$$|f(k)|^2 \leq C(k^2+1)^{-1}, \qquad ()$$

we deduce the result. ■

So the proof reduces to for the kernel $\mathcal{A}$. Our proof of this uses dilation analyticity in an essential way. First we dilate the operator $T(z)$ defined in (), using the prescription (). We assume below that $\mathrm{Im}z > 0$; the proof for $\mathrm{Im}z < 0$ is the same. If $\mathrm{Im}z > 0$, then the operator $T(z)$ is dilation analytic for θ satisfying $0 \leq \mathrm{Re}\theta \leq \theta_0$, and for these values

$$(T_\theta(z)\psi)(k) = f_\theta(k)\langle f_{-\bar\theta}, (h_{1,\theta}+e^{-i\theta}|k|-z)^{-1}\psi\rangle. \qquad ()$$

Furthermore since the integral kernel of this operator is

$$(T_\theta(z))(p,q) = \frac{f_\theta(p)\overline{f_{-\bar\theta}(q)}}{e^{-i\theta}|p|+e^{-i\theta}|q|-z} \qquad ()$$

we deduce that for $\mathrm{Im}z > 0$ and $\theta_0 \geq \mathrm{Re}\theta \geq 0$,

$$|(T_\theta(z))(p,q)| \leq C|f_\theta(p)||f_{-\bar\theta}(q)|\mathrm{min}\Big(\frac{1}{\mathrm{Im}z}, \frac{1}{\mathrm{Re}(\theta)\sqrt{|p|\,|q|}}\Big) \qquad ()$$

Combining this with () produces the bound:

$$\left|\left(G_{0,\theta}^{1/2}T_\theta(z)G_{0,\theta}^{1/2}\right)(p,q)\right| \leq C\,|f_\theta(p)||f_{-\bar\theta}(q)|\mathrm{min}\Big(\frac{1}{(\mathrm{Im}z)^2}, \frac{1}{(\mathrm{Re}(\theta))^2|p|\,|q|}\Big) \qquad ()$$

Using the Hilbert-Schmidt norm to bound the operator norm, and using the fact that $|k|^{-r}|f_\theta(k)|^2$ is integrable for $0 \leq r \leq 2$, we get

$$\left\|\left(G_{0,\theta}^{1/2}T_\theta(z)G_{0,\theta}^{1/2}\right)\right\| \leq C\mathrm{min}\Big(\frac{1}{(\mathrm{Im}z)^2}, \frac{1}{(\mathrm{Re}(\theta))^2}\Big) \qquad ()$$

We now define the dilated version of the operator $G(\beta,z)$, by the following series:

$$G_\theta(\beta,z) = G_{0,\theta}(\beta,z)\sum_{N=0}^{\infty}\left(\beta^2T_\theta(z)G_{0,\theta}(\beta,z)\right)^N \qquad ()$$

For $|\beta|$ sufficiently small, the bound () shows that (i) this series converges uniformly in z and θ for $\mathrm{Im}z > 1$ and $0 \leq \mathrm{Re}\theta < \theta_0$, and (ii) if $\mathrm{Re}\theta$ is bounded away from zero, then this series converges uniformly in z for $\mathrm{Im}z > 0$. Hence $G_\theta(\beta, z)$ defined by () is analytic in θ, z in both of these regions.

Returning to $\mathcal{A}(z)$ in (), it follows that for $\mathrm{Im}z > 1$ and for $0 \leq \mathrm{Re}\theta \leq \theta_0$,

$$\mathcal{A}(z;p,q) = \langle f_{-\bar{\theta}}, (h_{1,\theta} + |p| - z)^{-1} G_\theta(\beta, z)(h_{1,\theta} + |q| - z)^{-1} f_\theta \rangle \qquad ()$$

The independence of θ follows as usual by noting that the right side is analytic in θ, and is independent of θ when $\mathrm{Re}\theta = 0$.

Now fix $\theta = \theta_0$. Since $G(\beta, z)$ is a projection of the resolvent of a self-adjoint operator, $\mathcal{A}(z;p,q)$ is an analytic function of z for all $\mathrm{Im}z > 0$. As noted above, the right side of () is also analytic for $\mathrm{Im}z > 0$. Thus by the uniqueness of analytic continuation,

$$\mathcal{A}(z;p,q) = \langle f_{-\theta_0}, (h_{1,\theta_0} + |p| - z)^{-1} G_{\theta_0}(\beta, z)(h_{1,\theta_0} + |q| - z)^{-1} f_{\theta_0} \rangle \qquad ()$$

for all z with $\mathrm{Im}z > 0$.

We now bound $\mathcal{A}(z)$. ¿From () and () it follows that

$$\begin{aligned} &|\mathcal{A}(z;p,q)| \\ &\quad \leq C \| \big(G_{0,-\theta_0}(\beta, \bar{z})\big)^{\frac{1}{2}} (h_{1,-\theta_0} + |p| - \bar{z})^{-1} f_{-\theta_0} \| \| \big(G_{0,\theta_0}(\beta, z)\big)^{\frac{1}{2}} (h_{1,\theta_0} + |q| - z)^{-1} f_{\theta_0} \|. \end{aligned} \qquad ()$$

Further, using () and section 2.3(iv),

$$\begin{aligned} &\| \big(G_{0,\theta_0}(\beta, z)\big)^{\frac{1}{2}} (h_{1,\theta_0} + |q| - z)^{-1} f_{\theta_0} \|^2 \\ &\qquad = \int \frac{|f_{\theta_0}(p)|^2}{\left|e^{-i\theta_0}|p| + |q| - z\right|^2 \left|e^{-i\theta_0}|p| + E - z - \beta^2 \Pi(z - e^{-i\theta_0}|p|)\right|} \, dp \\ &\qquad \leq C \int_0^1 \frac{|p|^{1+\delta}}{|p|^2 (p + \beta^2)} \, dp + C \\ &\qquad \leq C \beta^{-2} \beta^{\{2,\delta\}} \end{aligned} \qquad ()$$

The same estimate holds for $\| \big(G^{(1)}_{0,-\theta_0}(\beta, \bar{z})\big)^{\frac{1}{2}} (h_{1,-\theta_0} + |p| - \bar{z})^{-1} f_{-\theta_0} \|^2$. The bound claimed in () now follows. ∎

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