General Relativity in terms of Dirac Eigenvalues

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The eigenvalues of the Dirac operator on a curved spacetime are diffeomorphism-invariant functions of the geometry. They form an infinite set of "observables" for general relativity. Recent work of Chamseddine and Connes suggests that they can be taken as variables for an invariant description of the gravitational field's dynamics. We compute the Poisson brackets of these eigenvalues and find them in terms of the energy-momentum of the eigenspinors and the propagator of the linearized Einstein equations. We show that the eigenspinors' energy-momentum is the Jacobian matrix of the change of coordinates from metric to eigenvalues. We also consider a minor modification of the spectral action, which eliminates the disturbing huge cosmological term and derive its equations of motion. These are satisfied if the energy momentum of the trans Planckian eigenspinors scale linearly with the eigenvalue; we argue that this requirement approximates the Einstein equations.

I. INTRODUCTION

One of the important lessons that we learn from general relativity is that fundamental physics is invariant under diffeomorphisms: there is no fixed nondynamical structure with respect to which location or motion could be defined. A fully diffeomorphism-invariant description of the geometry has consequently long been searched in general relativity; but so far without much success. As emphasized by many authors, such a description would be precious for quantum gravity [1].

Recently, Alain Connes' intriguing attempt of using the particle physics standard model for unraveling a microscopic noncommutative structure of spacetime [2–4], has generated –in a sense as a side product– the remarkable idea that the curved-spacetime Dirac operator D codes the full information about spacetime geometry in a way that can be used for describing the dynamics of the latter. Indeed, not only the geometry can be reconstructed from the (normed) algebra generated by (the inverse of) D and the smooth functions on the manifold, but the Einstein-Hilbert action itself is approximated by the trace of a simple function of D [5]. In its simplest reading, this result suggests the possibility of taking the eigenvalues λ^n of the Dirac operator as "dynamical variables" for general relativity. Indeed, these form an infinite family of fully four-dimensional diffeomorphism invariant observables, precisely the kind of object that was long searched in relativity. This approach has limitations. The most serious of these are that so far it works for Euclidean general relativity only (see [6] for some attempts to overcome this problem), and somewhat unrealistic predictions for the bare couplings [7]. However, it definitely opens a new window on the study of the dynamics of spacetime.

In order to use these ideas in the classical or in the quantum theory, one must translate structures from the metric variables to the λ^n variables. In particular, one needs information on the constraints that the λ^n's satisfy if they correspond to a smooth geometry, and on their Poisson brackets. The difficulty is that the dependence of the λ^n's on the geometry is defined in a very implicit manner, and these quantities seem too hard to compute.

In this letter, we show that these difficulties can be circumvented. Following some earlier results in [8] (valid only for the non 4d-invariant eigenvalues of the fixed-time Weyl operator), we derive here an expression for the Poisson brackets of the Dirac eigenvalues. Rather surprisingly, this expression turns out to be given in terms of the energy-momentum tensors of the Dirac eigenspinors. These tensors form the matrix elements of the Jacobian matrix of the change of coordinates between metric and eigenvalues. The Poisson brackets are quadratic in these tensors, with a kernel given by the propagator of the linearized Einstein equations. Thus, the energy-momentum tensors of the Dirac eigenspinors turn out to be the key tool for analyzing the representation of spacetime geometry in terms of Dirac eigenvalues.

We study also the Chamseddine-Connes spectral action. In the form presented in [5] this is quite unrealistic as a pure gravity action, because of a huge cosmological constant term that implies that the geometries for which the action approximates the Einstein-Hilbert action are *not* solutions of the theory. However, a very small modification of the action eliminates the cosmological constant term. We derive the equations of motion directly from the (modified) spectral action. We argue that they amount to the requirement that the energy momenta of the high mass eigenspinors scale linearly with the mass, and that this requirement approximates the vacuum Einstein equations.

These results suggest that –even independently from its application to the standard model– the Chamseddine-Connes *gravitational* theory can be viewed as a manageable gravitational theory by itself (see also [9,10]), possibly with powerful applications to classical and quantum gravity. It reproduces general relativity at low energies; it is formulated in terms of fully diffeomorphism invariant variables; and, of course, it prompts fascinating extensions of the very notion of geometry.

II. GR IN TERMS OF EIGENVALUES

Consider Euclidean general relativity (GR) on a compact 4d (spin-) manifold without boundary. We work in terms of the tetrad field $E^I_\mu(x)$. Here $\mu = 1\ldots4$ are spacetime indices and $I = 1\ldots4$ are internal Euclidean indices, raised and lowered by the Euclidean metric δ_{IJ}. The metric field is given by $g_{\mu\nu}(x) = E^I_\mu(x)E_{\nu I}(x)$, and is used to raise and lower spacetime indices. The spin connection $\omega^I_{\mu J}$ is defined by the equation $\partial_{[\mu}E^I_{\nu]} = \omega^I_{[\mu J}E^J_{\nu]}$. The dynamics is defined by the action $S[E] = \frac{1}{16\pi G}\int d^4x\sqrt{g}R$, where g and R are the determinant and the Ricci scalar of the metric.

In spite of a curiously widespread popular belief of the contrary, phase space is a covariant notion: the covariant definition of the phase space is as the space of the solutions of the equations of motion, modulo gauge transformations [11]. In the theory considered, the gauge transformations are given by 4d diffeomorphisms and by the local internal rotations of the tetrad field. Thus, the phase space Γ of GR is the space of the tetrad fields E that solve the equation of motion (Einstein equations), modulo internal rotations and diffeomorphisms. Γ can be identified with the space of the gauge orbits on the constraint surface and with the space of the Ricci flat 4-geometries.

We shall use the following notation. We denote the space of smooth tetrad fields as $\mathcal{E}$; the space of the orbits of the gauge groups –diffeomorphisms and local orthogonal tetrad rotations– in $\mathcal{E}$ as $\mathcal{G}$ (these are "4-geometries"); and the space of the orbits that satisfy the Einstein equation as Γ (these are the Ricci-flat 4-geometries, which form the phase space of GR). We call functions on Γ "observables". Observables correspond to functions on the constraint surface that commute with *all* the constraints.

We now define an infinite family of such observables. Consider spinor fields ψ on the manifold and the curved Dirac operator

$$D = \imath\gamma^I E^\mu_I \left(\partial_\mu + \omega_{\mu\, JK}\gamma^J\gamma^K\right), \tag{1}$$

which acts on them. Here γ are the (Euclidean) hermitian Dirac matrices. For each given field E, the Dirac operator is a self-adjoint operator on the Hilbert space of spinor fields with scalar product

$$(\psi, \phi) = \int d^4x\sqrt{g}\ \overline{\psi(x)}\phi(x). \tag{2}$$

where the bar indicates complex conjugation, and the scalar product in spinor space is the natural one in C^4. Therefore, D admits a complete set of real eigenvalues and eigenfunctions ("eigenspinors"). Since the manifold is compact, the spectrum is discrete. We write

$$D\psi^n = \lambda^n\psi^n. \tag{3}$$

Here and below, $n = 0, 1, 2...$ is an index, not an exponent. We convene to label the eigenvalues so that λ^n is non decreasing in n, namely $\lambda^n \leq \lambda^{n+1}$ (each eigenvalue is repeated according to its multiplicity). In order to emphasize the dependence of Dirac operator, eigenvalues and eigenspinors on the tetrad field, we use also the notation

$$D[E]\ \psi^n[E] = \lambda^n[E]\ \psi^n[E] \tag{4}$$

where the dependence on the tetrad is indicated explicitly.

$\lambda^n[E]$ defines a discrete family of real functions on the space $\mathcal{E}$ of the tetrad fields

$$\begin{aligned} \lambda^n :\ & \mathcal{E} \longrightarrow R \\ & E \longmapsto \lambda^n[E]. \end{aligned} \tag{5}$$

Equivalently, they define a function λ from $\mathcal{E}$ into the space of infinite sequences R^∞

$$\begin{aligned} \lambda :\ & \mathcal{E} \longrightarrow R^\infty \\ & E \longmapsto \{\lambda^n[E]\}. \end{aligned} \tag{6}$$

Since we have chosen to order the λ^n's in non-decreasing order, the image of $\mathcal{E}$ under this map, which we denote as $\lambda(\mathcal{E})$ is entirely contained in the cone $\lambda^n \leq \lambda^{n+1}$ of R^∞.

Now, for every n, the function λ^n is invariant under 4d diffeomorphisms and under internal rotations of the tetrad field. Therefore the functions λ^n are well defined functions on $\mathcal{G}$. In particular, they are well defined on Γ: they are observables of GR.

Two metric fields with the same λ^n's are called "isospectral". Isometric (that is, gauge equivalent) E fields are isospectral, but the converse might fail to be true [12,8]. Therefore λ may not be injective even if restricted to $\mathcal{G}$. The λ^n's may fail to coordinatize $\mathcal{G}$. They may also fail to coordinatize Γ. However, they presumably "almost do it". In the following, we explore the idea of analyzing GR in terms of the set of observables $\lambda^n[E]$.

III. SYMPLECTIC STRUCTURE

The phase space Γ is a symplectic manifold and a Poisson brackets structure is defined on the physical observables. We now study the Poisson brackets $\{\lambda^n, \lambda^m\}$.

To this purpose, we first construct the symplectic structure on Γ. This can be written in covariant form following Ref [13]. A vector field X on Γ can be written as a differential operator

$$X = \int d^4x\ X^I_\mu(x)[E]\ \frac{\delta}{\delta E^I_\mu(x)} \tag{7}$$

where $X^I_\mu(x)[E]$ is any solution of the Einstein equations *linearized* over the background E. The symplectic two-form Ω of GR is given by

$$\Omega(X,Y) = \frac{1}{32\pi G}\int_\Sigma d^3\sigma\ n_\rho[X^I_\mu \overleftrightarrow{\nabla}_\tau Y^J_\nu]\epsilon^\tau_{IJv}\epsilon^{v\rho\mu\nu} \tag{8}$$

where $[X^I_\mu \overleftrightarrow{\nabla}_\tau Y^J_\nu] \equiv [X^I_\mu \nabla_\tau Y^J_\nu - Y^I_\mu \nabla_\tau X^J_\nu]$; from now on we put $32\pi G = 1$. Both sides of (8) are functions of E, namely scalar functions on Γ; this E is used to transform internal indices into spacetime indices. Here $\Sigma : \sigma \longmapsto x(\sigma)$ is an arbitrary three-dimensional "ADM" surface, and n_ρ is the normal one-form to this surface. The coefficients of the symplectic form can be written as

$$\Omega^{\mu\nu}_{IJ}(x,y) = \int_\Sigma d^3\sigma\ n_\rho[\delta(x,x(\sigma))\overleftrightarrow{\nabla}_\tau\delta(y,x(\sigma))]\epsilon^\tau_{IJv}\epsilon^{v\rho\mu\nu}. \tag{9}$$

The Poisson bracket between two functions f and g on Γ is given by

$$\{f,g\} = \int d^4x \int d^4y\ \ P^{IJ}_{\mu\nu}(x,y)\ \frac{\delta f}{\delta E^I_\mu(x)}\ \frac{\delta g}{\delta E^I_\mu(y)}. \tag{10}$$

where $P^{IJ}_{\mu\nu}(x,y)$ is the inverse of the symplectic form matrix. It is defined by

$$\int d^4y\ P^{IJ}_{\mu\nu}(x,y)\ \Omega^{\nu\rho}_{JK}(y,z) = \delta(x,z)\ \delta^\rho_\mu\ \delta^I_K. \tag{11}$$

Since the symplectic form is degenerate on the space of the fields (it is non-degenerate only when restricted to the space of equivalent classes of gauge-equivalent fields), we can only invert it on this space by fixing a gauge. Let us assume this has been done. More precisely, integrating the last equation against a vector field $F^K_\rho(z)$ that satisfies the linearized Einstein equations over E, we have

$$\begin{aligned}&\textstyle\int d^4y \int d^4z\ P^{IJ}_{\mu\nu}(x,y)\ \Omega^{\nu\rho}_{JK}(y,z)\ F^K_\rho(z) = \\ &\qquad = \textstyle\int d^4z\ \delta(x,z)\ \delta^\rho_\mu\ \delta^I_K\ \ F^K_\rho(z),\end{aligned} \tag{12}$$

Integrating over the delta functions, and using (9), we have

$$\begin{aligned}&\textstyle\int_\Sigma d^3\sigma\ n_\rho[P^{IJ}_{\mu\nu}(x,x(\sigma))\overleftrightarrow{\nabla}_\rho F^K_\tau(x(\sigma))]\epsilon^\rho_{JKv}\epsilon^{v\nu\tau\sigma} = \\ &\qquad = F^I_\mu(x).\end{aligned} \tag{13}$$

This equation, where F is any solution of the linearized equations, defines P. But this equation is precisely the definition of the propagator of the linearized Einstein equations over the background E (in the chosen gauge). For instance, let us chose the surface Σ as $x^4 = 0$ and fix the gauge with

$$X^4_4 = 1,\quad X^4_a = 0,\quad X^i_4 = 1,\quad X^i_a = 0. \tag{14}$$

where $a = 1,2,3$ and $i = 1,2,3$. Then equation (13) becomes

$$F^i_a(\vec{x},t) = \int d^3\vec{y}\ (P^{ib}_{aj}(\vec{x},t;\vec{y},0)\overleftrightarrow{\nabla}_0 F^j_b(\vec{y},0)), \tag{15}$$

where we have used the notation $\vec{x} = (x^1,x^2,x^3)$ and $t = x^4$, and the propagator can be easily recognized.

Next, we need the functional derivative of the eigenvalues with respect to the metric. The variation of λ^n for a variation of E can be computed using standard techniques for first order variations of eigenvalues; (standard time-independent quantum-mechanics perturbation theory). For a self-adjoint operator D depending on a parameter v and whose eigenvalues are nondegenerate, we have

$$\frac{d\lambda^n}{dv} = (\psi^n|\frac{d}{dv}D(v)|\psi^n). \tag{16}$$

In our situation, for generic metrics with nondegenerate eigenvalues we have that

$$\begin{aligned}\frac{\delta\lambda^n}{\delta E^I_\mu(x)} &= (\psi^n|\frac{\delta}{\delta E^I_\mu(x)}D|\psi^n) \qquad (17)\\ &= \int \sqrt{g}\ \bar{\psi}^n \frac{\delta}{\delta E^I_\mu(x)}D\psi^n \\ &= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ \bar{\psi}^n D\psi^n - \int\frac{\delta\sqrt{g}}{\delta E^I_\mu(x)}\bar{\psi}^n D\psi^n \\ &= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ \bar{\psi}^n D\psi^n - \int\frac{\delta\sqrt{g}}{\delta E^I_\mu(x)}\bar{\psi}^n\lambda^n\psi^n \\ &= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ (\bar{\psi}^n D\psi^n - \lambda^n\bar{\psi}^n\psi^n) \\ &= T^{n}{}^\mu_I(x). \qquad (18)\end{aligned}$$

Now, $T^n{}^\mu_I(x)$ is nothing but the usual energy-momentum tensor of the spinor field ψ^n in tetrad notation (see for instance [14]). Indeed, the usual Dirac energy-momentum tensor is defined by

$$T^\mu_I(x) \equiv \frac{\delta}{\delta E^I_\mu(x)}S_{\text{Dirac}}, \tag{19}$$

where $S_{\text{Dirac}} = \int\sqrt{g}\ (\bar{\psi}D\psi - \lambda\bar{\psi}\psi)$ is the usual Dirac action of a spinor with mass λ.

We have shown that the energy-momentum tensor of the eigenspinors gives the Jacobian matrix of the transformation from E to λ; namely it gives the variation of the eigenvalues for a small change in the geometry. This fact suggests that we can study the map λ locally by studying the space of the eigenspinor's energy-momenta.

By combining (10,13) and (18) we obtain our main result:

$$\{\lambda^n,\lambda^m\} = \int d^4x\int d^4y\ T^{[n}{}^\mu_I(x)\ P^{IJ}_{\mu\nu}(x,y)\ T^{m]}{}^\nu_J(y) \tag{20}$$

which gives the Poisson bracket of two eigenvalues of the Dirac operator in terms of the energy-momentum tensor of the two corresponding eigenspinors and of the propagator of the linearized Einstein equations. The r.h.s. does not depend on the gauge chosen for P.

Finally, if the transformation between the "coordinates" $E^I_\mu(x)$ and the "coordinates" λ^n is locally invertible on the phase space Γ, we can write the symplectic form directly in terms of the λ^n's as

$$\Omega = \Omega_{mn}\ d\lambda^n \wedge d\lambda^m, \tag{21}$$

where a sum over indices is understood, and where Ω_{mn} is defined by

$$\Omega_{mn}\ T^{n}{}_I^{\mu}(x)\ T^{m}{}_J^{\nu}(y) = \Omega^{\mu\nu}_{IJ}(x,y). \tag{22}$$

Indeed, let $dE^I_\mu(x)$ be a (basis) one-form on Γ, namely the infinitesimal difference between two solutions of Einstein equations, namely a solution of the Einstein equations linearized over E. We have then

$$\begin{aligned}\Omega &= \int d^4x \int d^4y\ \Omega^{\mu\nu}_{IJ}(x,y)\ dE^I_\mu(x) \wedge dE^J_\nu(y)\\ &= \int d^4x \int d^4y \Omega_{mn}\ T^{n}{}_I^{\mu}(x)\ dE^I_\mu(x) \wedge T^{m}{}_J^{\nu}(y)\ dE^J_\nu(y)\\ &= \Omega_{mn}\ d\lambda^n \wedge d\lambda^m.\end{aligned} \tag{23}$$

An explicit evaluation of the matrix Ω_{nm} would be of great interest.

IV. ACTION

As shown in [5], the gravitational action can be expressed as

$$S = Tr[\chi(D)] \tag{24}$$

in natural units $\hbar = G = c = 1$. Here χ is a smooth monotonic function on R^+ such that

$$\begin{aligned}\chi(x) &= 1, \quad \text{for } x < 1-\delta,\\ \chi(x) &= 0, \quad \text{for } x > 1+\delta.\end{aligned} \tag{25}$$

where $\delta <<< 1$. Equivalently, S is the number of λ^n's smaller than the Planck mass, which is 1 in natural units.

The action (24) approximates the Einstein-Hilbert action with a large cosmological term for "large-scale" metrics with small curvature (with respect to the Planck scale). This can be seen as follows. Let $N[E]$ be the integer such that $\lambda^N[E]$ is the largest eigenvalue of $D[E]$ smaller than the Planck mass $M_P = \frac{1}{L_P} = 1$. A moment of reflection shows then that we have

$$S[E] = N[E] \tag{26}$$

by definition. For large n, the growth of the eigenvalues of the Dirac operator is given by the Weyl formula;

$$\lambda^n = V^{-\frac{1}{4}}\ n^{\frac{1}{4}} + \ldots \tag{27}$$

where V is the volume and we are neglecting factors of the order of unity. The next term in this asymptotic approximation can be obtained from the fact that the Dixmier trace (the logarithmic divergence of the trace) of D^{-2} is the Einstein-Hilbert action [2,5,4]; therefore

$$(\lambda_n)^{-2} = V^{\frac{1}{2}}\ n^{-\frac{1}{2}} + \int \sqrt{g}R\ n^{-1} + \ldots \tag{28}$$

Under our assumptions on the geometry, at the Planck scale we are in asymptotic regime: the first term dominates over the second, and the remaining terms are negligible. Writing the last equation for $n = N$ and using (26), we have

$$1 = V^{\frac{1}{2}}\ S^{-\frac{1}{2}} + \int \sqrt{g}R\ S^{-1} + \ldots \tag{29}$$

Solving for S we obtain

$$S = \frac{1}{L_P^4} V + \frac{1}{L_P^2} \int \sqrt{g}R + \ldots. \tag{30}$$

which shows that the action (24) is dominated by the Einstein-Hilbert action with a cosmological term. In the last equation, we have explicitly reinserted the Planck length L_P.

The presence of the huge Planck-mass cosmological term is a bit devastating because the solutions of the equations of motions have Planck-scale Ricci scalar, and therefore they are *all* out of the regime for which the approximation taken is valid!

However, the cosmological term can be canceled easily by replacing $\chi(x)$ with $\tilde\chi(x)$ defined by

$$\tilde\chi(x) = \chi(x) - \epsilon^4 \chi(\epsilon x) \tag{31}$$

for a small number ϵ. The additional term cancels exactly the cosmological term, leaving only the Einstein Hilbert action, with corrections which are small for low curvature geometries, which, *now, are* solutions of the theory. In fact

$$\begin{aligned}\tilde S &\equiv Tr(\tilde\chi(D))\\ &= Tr(\chi(D)) - \epsilon^4 Tr(\chi(\epsilon D))\\ &= \frac{V}{L_P^4} + \frac{1}{L_P^2}\int\sqrt{g}R\\ &\quad -\epsilon^4\left(\epsilon^{-4}\frac{V}{L_P^4} + \epsilon^{-2}\frac{1}{L_P^2}\int\sqrt{g}R\right) + \ldots.\\ &= \frac{1}{L_P^2}\int\sqrt{g}R + \ldots\end{aligned} \tag{32}$$

If we write S directly in terms of the observables λ^n, we have the following expression for the action

$$\tilde S[\lambda] = \sum_n\ \tilde\chi(\lambda^n). \tag{33}$$

One cannot vary the λ^n's in this expression to obtain (approximate) Einstein equations. These equations are obtained minimizing (33) on the surface $\lambda(\mathcal{E})$, not on the entire R^∞. In other words, the λ^n's are not independent: there are relations among them. These relations code the complexity of GR. The equations of motion are obtained by varying S with respect to the tetrad field. They can be written as

$$0 = \frac{\delta \tilde{S}}{\delta E^I_\mu(x)} = \sum_n \frac{\partial \tilde{S}}{\partial \lambda^n} \frac{\delta \lambda^n}{\delta E^I_\mu(x)} = \sum_n \frac{d\tilde{\chi}(\lambda^n)}{d\lambda^n} T^{n}{}^\mu_I(x). \tag{34}$$

Let $f(x) = \frac{d}{dx}\tilde{\chi}(x)$. The equations of motion of the theory are then

$$\sum_n f(\lambda^n)\; T^{n}{}^\mu_I(x) = 0. \tag{35}$$

We close by analyzing the content of these equations. $f(x)$ is a function that vanishes everywhere except on two narrow peaks. A positive peak (width δ and height $1/\delta$) around the Planck mass 1; and a negative peak (width $\delta\epsilon$ and height ϵ^5/δ) around the arbitrary large number $s = \frac{1}{\epsilon} >> 1$. The equation is therefore solved if above the Planck mass ($n >> N$), the energy momentum tensor scales as

$$\rho(1)\; T^{N}{}^\mu_I(x) = s^{-4}\; \rho(s)\; T^{N(s)}{}^\mu_I(x), \tag{36}$$

where $\rho(s)$ is the density of the eigenvalues at the scale s and $\lambda^{N(s)} = s$, because in this case the two terms from the two peaks cancel. But from (27) we have that the density of eigenvalues grows as N^3, and that $N(s) = s^4$. This yields

$$T^{n}{}^\mu_I(x) = \lambda^n \;\; T^{N}{}^\mu_I(x). \tag{37}$$

for $n >> N$. (Recall that $\lambda^N = 1$.) In other words: *the equations of motion for the geometry are solved when above the Planck mass the energy-momentum of the eigenspinors scales as the mass.*

To understand the meaning of this scaling requirement, notice that $T^{n}{}^I_\mu$ is formed by a term linear in the derivatives of the spinor field and a term independent from these, which is a function of $(\psi, E, \partial_\mu E)$ quadratic in ψ.

$$T^{n}{}^I_\mu = \bar{\psi}^n \gamma^I \overleftrightarrow{\partial}_\mu \psi^n + S^{n}{}^I_\mu[\psi, E, \partial E]. \tag{38}$$

If we expand the last term around a point of the manifold with local coordinates x, covariance and dimensional analysis require that

$$S^{n}{}^I_\mu = c_0 \lambda^n E^I_\mu + c_1\; R^I_\mu + c_2\; R E^I_\mu + O\left(\frac{1}{\lambda^n}\right). \tag{39}$$

for some fixed expansion coefficients c_0, c_1 and c_2. Here R^I_μ is the Ricci tensor. [To be convinced that terms of this form do appear, consider the following.

$$\begin{aligned} T^{n}{}^I_\mu &= \bar{\psi}^n \gamma^I D_\mu \psi^n + \ldots \\ &= (\lambda^n)^{-1}\; \bar{\psi}^n \gamma^I \gamma^\nu D_\mu D_\nu \psi^n + \ldots \\ &= (\lambda^n)^{-1}\; \bar{\psi}^n \gamma^I \gamma^\nu [D_\mu, D_\nu] \psi^n + \ldots \\ &= (\lambda^n)^{-1}\; \bar{\psi}^n \gamma^I \gamma^\nu R_{\mu\nu} \psi^n + \ldots \\ &= (\lambda^n)^{-1}\; \bar{\psi}^n \gamma^I \gamma^\nu R^{JK}_{\mu\nu} \gamma_J \gamma_K \psi^n + \ldots \\ &= Tr\;\; \gamma^I \gamma^\nu R^{JK}_{\mu\nu} \gamma_J \gamma_K + \ldots \\ &= R^I_\mu + \ldots\,] \end{aligned} \tag{40}$$

For sufficiently high n, the eigenspinors are locally plane waves in local cartesian coordinates: doubling the mass just doubles the frequency. If

$$\lambda^m = t\; \lambda^n \tag{41}$$

Then $\partial_\mu \psi^m = t\; \partial_\mu \psi^n$. It follows that in general the energy momentum scales as

$$\begin{aligned} T^{m}{}^I_\mu &= t\left[\bar{\psi}^n \gamma^I \partial_\mu \psi^n - \partial_\mu \bar{\psi}^n \gamma^I \psi^n + c_0 \lambda^n E^I_\mu\right] \\ &\quad + \left[c_1\; R^I_\mu + c_2\; R E^I_\mu\right] + O\left(\frac{1}{\lambda^n}\right). \end{aligned} \tag{42}$$

For large λ^n we can disregard the last term, and (37) requires that the second square bracket vanishes. Taking the trace we have $R = 0$, using which we conclude

$$R^I_\mu = 0 \tag{43}$$

which are the vacuum Einstein equations. Thus, the variation of the (modified) Chamseddine-Connes action implies a scaling requirement on the high mass eigenspinors energy momentum tensors, and this requirement, in turn, yields vacuum Einstein equations at low scale.

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