Eigenvalues and bands imbedded in the continuous spectrum for the system of resonators and waveguide: solvable model

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Abstract

Solvable model based on the operator extension theory is suggested for the description of trapped modes imbedded in the continuous spectrum. A system of resonators connected through small apertures with a waveguide is considered as for the case of Neumann and Dirichlet boundary conditions. That is, we study both acoustical and quantum waveguides. The existence of such modes is shown (corresponding sufficient condition is derived). An effective and simple algorithm for its determination is suggested. A system of acoustic (quantum) waveguide and periodic set of coupled cavities is studied in the framework of the model. The dispersion equation is obtained in an explicit form. The existence of bands imbedded in the continuous spectrum is proved, and an algorithm for its determination is described.

The problem of eigenvalues imbedded in the continuous spectrum is of great interest now both from mathematical and physical points of view. Mathematicians are studying it actively because there is no general theory allowing one to describe situation effectively, and to develop it they are trying to reveal general features by studying of concrete problems (see, for example, [1]). An important reason for physicists is that the problem in question is closely related with the description of transport properties of many physical systems in acoustics, quantum mechanics, fluid mechanics [2], [3], [4]. We shall deal with a system of resonators connected with a waveguide. Note that the problem of eigenvalues which are less then lower bound of the continuous spectrum is essentially more simple and has been studied better (see, for example, [5]).

Analogous (from mathematical point of view) systems were considered in fluid mechanics. Namely, the existence of trapped modes above a long submerged horisontal circular cylinder of sufficiently small radius, in deep water, was proved in [6]. The result was generalized in [7]. Recently, there has been a revival of interest in predicting those situations in which trap modes, or acoustic resonances, might occur, because of their importance in, for example, the design of turbomashinery [3]. A series of recent papers has been concerned with both existence proofs and numerical algorithms for the computation of trapped modes for different geometries [8], [4]. The proofs are based on using of variational inequalities. This limitation allows one to look for the effect in the case of Neumann boundary condition only. As a result, important case of Dirichlet boundary condition has not been considered. But it is Dirichlet condition that takes place for quantum waveguide. There is no results about trapped modes imbedded in continuous spectrum for quantum waveguides.

We use another approach. To describe complicated problem for system of resonators coupled with a waveguide through small openings we construct a model in which these small apertures are replaced by point-like ones. The model is based on the theory of self-adjoint extensions of symmetric operators. It allows one to obtain the dispersion equation in an explicit form, and, hence, to find the eigenvalue in question. It has been proved earlier that the model solution is the main term of the asymptotics in the width of the aperture of the solution of corresponding realistic problem. The model has an advantage in comparison with the conventional approach. In the framework of conventional approach the existence of the eigenvalues only can be proved and some estimates can be obtained. The model allows one to compute the trapped mode and investigate its behaviour. More over, the model gives us a possibility to consider both Neumann and Dirichlet boundary conditions, i.e. both acoustic and quantum waveguides.

In the second part of the Letter we deal with periodic system of resonators connected with a waveguide. It is related with the problem of nanoelectronics. The discovery of coherent ballistic transport in narrow channels formed by imposing a potential on a two-dimensional electron gas has generated much interest in conduction in small ballistic structures [9]. Recently Lent et al [10], [11] studied the system which is analogous to our: the quantum channel with a periodically modulated width.

For our model periodic system the dispersion equation is obtained in an explicit form and the existence of a band imbedded in the continuous spectrum is shown. The model is based on the theory of self-adjoint extensions of symmetric operators and is analogous to the well-known zero-range potential method [12], [13]. The operator extension approach allows one to extend the range of applications of the method [14], [15], and, particularly, to apply it to the problem of quantum transport in nanostructures [16], [17].

Let us describe briefly the structure of the model for the case of one two-dimensional resonator connected through a small opening with a waveguide. Let us consider the Laplace operator in a half plane Ω_w with the Neumann boundary condition. Let r_0 be the point of the boundary common with the boundary of a bounded domain Ω_0. Restrict the Neumann Laplacian in $L_2\left(\Omega_0 \oplus \Omega_w\right)$ to the set of smooth functions vanishing at the point r_0. The closure of the operator so obtained is symmetric with the dificiency indices (2, 2). It has self-adjoint extensions. It is this operator that gives us the model in question. We shall consider only one (the most natural) of these operators. Its domain consists of the elements of the form

$$u(r) = \begin{cases} a_w G_w(r, r_0, \lambda_0) + u_w(r), r \in \Omega_w, \\ a_0 G_0(r, r_0, \lambda_0) + u_0(r), r \in \Omega_0, \end{cases}$$

where $G_0(r, r_0, \lambda_0)$ be the Green's function of the Neumann Laplacian $(-\Delta_0)$ for the domain Ω_0, $G_w(r, r_0, \lambda)$ be the Green's function of the Laplace operator with Neumann boundary condition for the waveguide, $k^2 = \lambda, \quad u_w \in W_2^2(\Omega_w), u_0 \in W_2^2(\Omega_0), \quad a_w = -a_0, \quad u_w(r_0) = u_0(r_0)$, λ_0 be a regular value of the operator. Here W_2^2 is Sobolev space.

Remark. From a physical point of view this choice of the extension means that the condition of "the conservation of the flux through the aperture" takes place (that is why we call this extension "the natural" one). It was shown [18] that one can choose the model parameter λ_0 in such a way that the model solution concises with the main term of the asymptotic expansion (in δ) of the solution of corresponding "realistic" problem for domains coupled through small (but non-zero) aperture of the diameter δ. Namely, for two-dimensional problem we must choose $\lambda_0 = \delta^{-2} \exp -2\gamma, \gamma = 0.5772...$ is Euler constant.

Let there are two identical resonators $\Omega_{0\pm}$ connected with the waveguide $\Omega_w, \Omega_w = \{(x, y) : |d| < d\}$, the boundary of which is represented in Cartesian coordinates (x, y) by two parallel lines $\Gamma_\pm$. The points of connections are $r_{0\pm}$. Let the system is symmetric about the centreline $y = 0$ of the waveguide. We consider the Laplace operator with Neumann boundary condition in the space $\mathcal{H} = L_2(\Omega_w \oplus \Omega_{0+} \oplus \Omega_{0-}$. We construct model operator $-\Delta$ using "restriction - extension" procedure described above. The space can be represented in a form of orthogonal sum $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_a$, where $\mathcal{H}_s(\mathcal{H}_a)$ is the corresponding subspace of symmetric (antisymmetric) functions in respect to variable y. These subspaces are invariant in respect to operator $-\Delta$. Hence,

$$-\Delta = -\Delta|_{\mathcal{H}_a} \oplus (-\Delta|_{\mathcal{H}_s}).$$

Note that functions from the domain of the operator $-\Delta|_{\mathcal{H}_a}$ satisfy Dirichlet condition on the centreline $y = 0$. Consequently, the continuous spectrum are the following

$$\sigma_c(-\Delta|_{\mathcal{H}_a}) = [\pi^2(2d)^{-2}, \infty), \sigma_c(-\Delta|_{\mathcal{H}_s}) = [0, \infty), \sigma_c(-\Delta) = [0, \infty).$$

Using well-known representation for the Green function for Neumann Laplacian in the waveguide,

$$G_w(x, y, x*, y*, k) = \sum_{n=0}^{\infty} \frac{\cos(\pi n(y + d)(2d)^{-1}) \cos(\pi n(y * + d)(2d)^{-1})}{2p_n} e^{-p_n|x-x*|},$$

$$p_n(\lambda) = \left\{ \begin{array}{ll} \sqrt{(\pi n/(2d))^2 - k^2} & for \quad (\pi n/(2d))^2 > k^2, \\ i\sqrt{k^2 - (\pi n/(2d))^2} & \quad for \quad (\pi n/(2d))^2 > k^2. \end{array} \right. ,$$

one obtains the dispersion equation for the model operator $-\Delta|_{\mathcal{H}_a}$ in an explicit form:

$$\sum_{n,m=1}^{\infty} \frac{1}{(\lambda_{nm} - \lambda)(\lambda_{nm} - \lambda_0)} = \sum_{n=1}^{\infty} (\frac{1}{2p_n(\lambda)} - \frac{1}{p_n(\lambda_0)}), \tag{1}$$

where λ_{nm} are eigenvalues for the resonator Ω_{0+}.

Let

$$min_{m,n}\lambda_{nm} < \pi^2/(4d^2). \tag{2}$$

Then for sufficiently great λ_0 (i.e. for sufficiently small diameter of connection aperture) the equation (1) has real root less then the lower bound of the continuous spectrum $\pi^2/(4d^2)$. Hence, we have an eigenvalue for the operator $-\Delta|_{\mathcal{H}_a}$ and an eigenvalue imbedded in the continuous spectrum of the operator $-\Delta$.

Note that in the framework of conventional approach authors can prove only that there exists an eigenvalue of the operator $-\Delta|_{\mathcal{H}_a}$ less then $\pi^2/(4d^2)$. Namely, they show that there exists test function ψ such that

$$\frac{\int_{\Omega_w \cup \Omega_{0+} \cup \Omega_{0-}} |\nabla \psi|^2 dS}{\int_{\Omega_w \cup \Omega_{0+} \cup \Omega_{0-}} |\psi|^2 dS} < \pi^2/(4d^2).$$

Hence, in accordance with variational principle there exists an eigenvalue in question. Thus, they have a proof of existence of an eigenvalue imbedded in the continuous spectrum but have no effective algorithm for its determination. Our approach allows one not only to prove the existence of such eigenvalues, but also to determine it approximately using simple procedure. Moreover, the conventional approach allows one to consider the Neumann boundary condition only. The reason is the following. If one proves that there exist an eigenvalue of the operator $-\Delta|_{\mathcal{H}_a}$ less then the lower bound of the continuous spectrum of this operator he does not conclude that there is an eigenvalue imbedded in the continuous spectrum because the continuous spectrum of the full operator in the case of Dirichlet condition is not the half axis $[0,\infty)$, but only $[\pi^2/(4d^2),\infty)$.

In our model the Dirichlet case may be considered absolutely parallel to the Neumann one. The model for the Dirichlet boundary condition is slightly more complicated then the variant described above. In this case we had to extend the initial space L_2 to construct the model operator. One can find the description of this variant of the model in [15]. The model dispersion equation is analogous to (1). The alteration is the following: we must replace the term $p_n(\lambda)$ by $p_n^D(\lambda), p_n^D(\lambda) = \sqrt{\lambda - (\pi n/d)^2}$. The sufficient condition for existence of eigenvalues imbedded in the continuous spectrum in the Dirichlet case is

$$\pi^2(2d)^{-2} < min_{m,n}\lambda_{nm} < \pi^2 d^{-2}. \tag{3}$$

The dispersion equation allows one to determine this eigenvalue.

One can consider the corresponding periodic system, i.e. when there is a periodic system (with the period a) of identical resonators $\Omega_{j\pm}, j = 0,\pm1,\pm2,...$, connected with the waveguide Ω_w (note, that analogous periodic system of resonators connected with a half plain is studied in [19]). In this situation we have for the Neumann case the following representation for an eigenfunction:

$$\psi(x,k) = \begin{cases} \alpha_j G_j(r,r_j,k), & r\in\Omega_j, \\ \sum_{j=-\infty}^{\infty}\alpha_j^w G_w(r,r_j,k), & r\in\Omega_w, \end{cases}$$

The periodicity of the system leads to Bloch's condition for the function ψ:

$$\psi(x+a,k) = \exp{(ipa)}\psi(x,k),$$

where p is quasimomentum. Hence, one obtains the additional condition for the coefficients α_j^w:

$$\alpha_j^w = \exp{(ipaj)}\alpha_0^w. \tag{4}$$

Using the choice of the extension operator described above and (4) one gets the dispersion equation in the form:

$$-\left(G_{0+}(r,r_0,\lambda) - G_{0+}(r,r_0,\lambda_0)\right)\Big|_{r=r_0} =$$
$$\left(G_w(r,r_0,\lambda) - G_w(r,r_0,\lambda_0)\right)\Big|_{r=r_0} + \sum_{j\neq0,j=-\infty}^{\infty}\exp{(ipaj)}G_w(r_0,r_j,k).$$

Using the expression for G_w, one gets

$$\left(G_{0+}(r,r_0,\lambda) - G_{0+}(r,r_0,\lambda_0)\right)\Big|_{r=r_0} =$$
$$\sum_{n=0}^{\infty}(p_n^0 - p_n)(2p_np_n^0)^{-1} + \sum_{n=0}^{\infty}2exp(-p_na)\frac{\cos pa - \exp{(-p_na)}}{1-2\exp{(-p_na)}\cos pa + \exp{(-2p_na)}},$$

where p_n^0 is the value of p_n for $\lambda = \lambda_0$. Using the representation for Green function of the resonator, one obtains

$$\sum_{n,m=1}^{\infty} \frac{1}{(\lambda_{nm}-\lambda)(\lambda_{nm}-\lambda_0)} =$$
$$\sum_{n=0}^{\infty}(p_n^0 - p_n)(2p_n p_n^0)^{-1} + \sum_{n=0}^{\infty} 2exp(-p_n a)\frac{\cos pa - \exp(-p_n a)}{1-2\exp(-p_n a)\cos pa + \exp(-2p_n a)}, \tag{5}$$

One can see that if the condition (2) takes place, there exist a band for the operator $-\Delta|_{\mathcal{H}_a}$, i.e. the band imbedded in the continuous spectrum of the operator $-\Delta$. One can determine its parameters by solving equation (5). The analogous consideration is for the Dirichlet boundary condition. In this case the sufficient condition for the existence of the corresponding band is (3).

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