CRITICAL SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS*†‡

R. Hardt[1]
M. Hoffmann-Ostenhof[2]
T. Hoffmann-Ostenhof[3,4]
N. Nadirashvili[5]

Mathematics Department, Rice University, Houston[1]
Institut für Mathematik, Universität Wien[2]
Institut für Theoretische Chemie, Universität Wien[3]
International Erwin Schrödinger Institute for Mathematical Physics[4]
Department of Mathematics, MIT[5]

Abstract

Let $u \not\equiv$ const satisfy an elliptic equation $L_0 u \equiv \sum a_{ij} D_{ij} u + \sum b_j D_j u = 0$ with smooth coefficients in a domain in $\mathbf{R}^n$. It is shown that the critical set $|\nabla u|^{-1}\{0\}$ has locally finite $n-2$ dimensional Hausdorff measure. This implies in particular that for a solution $u \not\equiv 0$ of $(L_0+c)u = 0$, with $c \in C^\infty$, the critical zero set $u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$ has locally finite $n-2$ dimensional Hausdorff measure.

1. Introduction and Main Results

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 3$ and let $u \not\equiv 0$ be a real-valued classical solution of the elliptic partial differential equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_j D_j u + cu = 0 \quad \text{in } \Omega \tag{1.1}$$

*Supported by Ministerium für Wissenschaft und Verkehr der Republik Österreich
†Work supported by the European Union TMR grant FMRX-CT 96-0001
‡Research partially supported by the NSF

where the real valued coefficients a_{ij}, b_j, c are C^∞ functions in Ω. We denote

$$\Sigma(u) = |\nabla u|^{-1}\{0\}, \text{ and } \Sigma_0(u) = \Sigma(u) \cap u^{-1}\{0\}.$$

In the following we shall show that locally the singular set Σ_0 of u has finite $(n-2)$ - dimensional Hausdorff measure, i.e. $\mathcal{H}^{n-2}(\Sigma_0(u) \cap K) < \infty$ for all compact $K \subset \Omega$. The first author and the remaining three authors independently wrote preprints proving this result and the present paper is a combination of these two works.

For $n = 2$, $\Sigma(u)$ is well-known to consist of isolated points. For $n \geq 3$ an elementary argument (see [HS], §1.9), first given by L. Caffarelli and A. Friedman [CF] for $\Delta u + f(x, u) = 0$, shows that $\Sigma(u)$ is contained in a countable union of smooth $n-2$ dimensional submanifolds. Q. Han [Hn] obtains similar structural results with much weaker assumptions on the smoothness of the coefficients, In particular, he proved that $\Sigma(u)$ is essentially contained in a countable union of $\mathcal{C}^{1,\alpha}$ graphs if the coefficients are Lipschitz. But, even for smooth coefficients, the question remained concerning the size of $\Sigma(u)$. Last year it was shown in [HOHON] that for $n = 3$, $\Sigma_0(u)$ has locally finite 1-dimensional Hausdorff measure.

Here we generalize the result to $n \geq 3$ dimensions. The result is obtained by showing that the critical set Σ of a solution of (1.1) with $c \equiv 0$ has locally finite $n-2$ dimensional Hausdorff measure.

Recently there is a rather rich literature describing the 'size' of the zero set and in particular the singular set Σ_0 of solutions to elliptic equations in terms of the appropriate Hausdorff measure, respectively, Hausdorff dimension. See the list of references in the introduction of [HOHON]. The size of the nodal set was considered in the conjecture of S.T. Yau [Y] that $\mathcal{H}^{n-1}(u_\lambda^{-1}\{0\}) \sim c\sqrt{\lambda}$ for the λ-eigenfunction u_λ on a compact Riemannian manifold. This was established for real analytic metrics by H. Donnelly and C. Fefferman [DF1]. Note that, for real analytic coefficients, the local finiteness, without estimates, of $\mathcal{H}^{n-1}(u^{-1}\{0\})$ (or of $\mathcal{H}^{n-2}(\Sigma(u))$) follows just from the real analyticity of u [F], 3.4.8. For the nonanalytic case, R. Hardt and L. Simon [HS] proved the local finiteness of $\mathcal{H}^{n-1}(u^{-1}\{0\})$ with the coefficients being only Lipschitz smooth. However, for the Riemannian manifold application, their upper estimate $C\lambda^{c\sqrt{\lambda}}$ is weaker than Yau's conjecture. F.H. Lin and Q. Han [L], [HnL1], [HnL2] proved a parabolic nodal set estimate (with time-independent coefficients), simplified several arguments in [DF1] and [HS], and made estimates involving the frequency (or order) $N_R \equiv \big[R\int_{\mathbf{B}_R} |\nabla u|^2\,dx\big]/\big[\int_{\partial\mathbf{B}_R} u^2\,d\mathcal{H}^{n-1}\big]$. Lin [L] also conjectured that

$$\mathcal{H}^{n-1}(u^{-1}\{0\}\cap\mathbf{B}_{R/2}) \le CN_R \text{ and } \mathcal{H}^{n-2}(\Sigma(u)\cap\mathbf{B}_{R/2}) \le CN_R^2\,.$$

While more precise results are known in 2 dimensions [AL, D, DF2, N], the general Yau and Lin conjectures remain open. Two very recent preprints give some nonexplicit bounds. [HHL1] treats coefficients with finite differentiability, and [HHL2] treats higher order equations.

Basic for all these investigations is the asymptotic behaviour of $u(x)$ for $x \to x_0$, where $u(x_0) = 0$. Let $\mathcal{O} \in \Omega$, and let u be a solution of (1.1), then it is well known (see e.g. [B]) that

$$u(x) = p_M + O(|x|^{M+1}) \text{ as } |x| \to 0 \tag{1.2}$$

where $p_M \not\equiv 0$ is a homogeneous polynomial of degree M satisfying the osculating equation

$$\sum_{i,j} a_{ij}(\mathcal{O})D_{ij}p_M = 0.$$

Assume without loss of generality that $a_{ij}(\mathcal{O}) = \delta_{ij}$, then p_M is harmonic. Therefore the investigations of the zero set respectively singular set of a solution of (1.1) are motivated by the desire to understand to which extent these sets can be described locally by the zero sets respectively critical sets of harmonic homogeneous polynomials. For a harmonic polynomial P_M of degree M in n variables it is known (see e.g. [HS]) that for some $C(n) < \infty$

$$\mathcal{H}^{n-2}(\Sigma(P_M)\cap B_1) \le C(n)M^2, \tag{1.3}$$

B_1 denoting a ball with radius 1.

On the other hand there are examples showing that the singular set of a solution of an elliptic equation can be rather wild. See [HOHON], section 1. Conversations with L. Simon also led to the following simple example: For any closed subset K of $\mathbf{R}$, let f be a nonnegative smooth function vanishing exactly on K with $|ff''| + |f'^2| < 1/4$. Then $u(x,y,z) = xy + f^2(z)$ satisfies the elliptic equation $u_{xx} + u_{yy} + u_{zz} - (f^2)''(z)u_{xy} = 0$, and has singular set equaling $\{(0,0)\}\times K$.

To state now our main results we define the elliptic operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 = \mathcal{L} - c$$

with $\mathcal{L}$ and c given according to (1.1).

Theorem 1.1. *Let* $u \not\equiv \text{const}$ *satisfy*

$$\mathcal{L}_0 u = 0 \quad \text{in } \Omega, \quad \Omega \subset \mathbf{R}^n. \tag{1.1'}$$

Then for every compact subset K *of* Ω

$$\mathcal{H}^{n-2}(\Sigma(u) \cap K) < \infty.$$

Corollary 1.1. *Let* $u \not\equiv 0$ *satisfy equation (1.1). Then for every compact subset* K *of* Ω

$$\mathcal{H}^{n-2}(\Sigma_0(u) \cap K) < \infty.$$

The Corollary is a rather immediate consequence of Theorem 1.1:

Proof of the Corollary. Given $x_0 \in \Omega$ there is a neighbourhood $U(x_0)$ and a $u_0 \in C^\infty(U(x_0))$ with $u_0 > 0$ and $\mathcal{L}u_0 = 0$ in $U(x_0)$. See e.g. [BJS], p.228. It is easily seen that $\mu \equiv uu_0^{-1}$ satisfies in $U(x_0)$ an equation of type (1.1), so that by Theorem 1.1, $H^{n-2}(\Sigma(\mu) \cap U') < \infty$ for every compact subset U' of $U(x_0)$. Furthermore the singular set of u is a subset of the critical set of μ.■

Remark: That the assertion of Theorem 1.1 is false if $\mathcal{L}_0$ is replaced by $\mathcal{L}$ can be seen from the following example: Let $v \in C^\infty(B)$, $B \subset \mathbf{R}^n$, with $|v| < 1$, then with $u = v^2 + 1$ and $c = (\Delta v^2)(v^2+1)^{-1}$, $\Delta u + cu = 0$ and $\Sigma(u) = v^{-1}\{0\}$. But every closed subset of $\mathbf{R}^n$ can be the zero set of a C^∞-function (see e.g. [T])!

The structure of the proof of Theorem 1.1 is similar to the 3-dimensional case in [HOHON]. For this it was crucial to show (Theorem 3.1) that *in 3 dimensions, the complex dimension of the complex critical set of a homogeneous real harmonic polynomial is at most one.* Here it is shown that the complex critical set of a homogeneous harmonic polynomial P with real coefficients has at most complex dimension $n-2$ (Theorem 2.1). With this result it can be proven that for suitable complex 2-planes ϵ_{ij}, $1 \le i < j \le n$, $P|_{\epsilon_{ij}}$

has an isolated critical point in the origin of $\mathbf{C}^2$ for all i, j. Using results from singularity theory, [AGV], this implies that the algebraic multiplicity of the gradient map of $P|_{\epsilon_{ij}}$ at the origin is finite. Further looking at the restriction of the solution u to affine 2-planes it follows via a C^∞-perturbation argument that the number of critical points of u restricted to these affine 2-plane slices is uniformly bounded in a small enough neighborhood of the origin. This estimate together with the countable rectifiability of $\Sigma(u)$ (which follows immediately using the arguments from the proof of Lemma 1.9 in [HS], or see also [CF]) allows us to apply a geometric measure inequality of Federer [F] which yields the desired result.

2. The critical set of a homogeneous polynomial

A *homogeneous polynomial* of degree $k \geq 1$ on $\mathbf{R}^n$ is a nonzero function in the form

$$u(x) \ = \ \sum_{\|\alpha\|=k} a_\alpha x^\alpha$$

where $a_\alpha \in \mathbf{R}$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots\}^n$, and $\|\alpha\| \equiv \alpha_1 + \cdots + \alpha_n$.

The critical set $\Sigma(v)$ of a polynomial $v(x) = \sum_{\|\alpha\| \leq k} b_\alpha x^\alpha$ is a real algebraic variety that is a cone in case v is homogeneous. Extending v to a complex-valued polynomial, also denoted v, on $\mathbf{C}^n$ by replacing each x_i by z_i, we also have the *complex critical (zero) set*

$$\Sigma_{\mathbf{C}}(v) \ \equiv \ \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n \ : \ v(0) = \frac{\partial v}{\partial z_1}(z) = \cdots = \frac{\partial v}{\partial z_n}(z) = 0\} \ ,$$

which is a complex algebraic variety that satisfies

$$\Sigma_{\mathbf{C}}(v) \cap \mathbf{R}^n = \ \Sigma(v) \ .$$

Analogously we denote $\Sigma_{0\mathbf{C}}(v) = \Sigma_{\mathbf{C}}(v) \cap v^{-1}\{0\}$.

For a nonconstant polynomial v on $\mathbf{R}^n$, one thus always has the rough estimates

$$\dim_{\mathbf{R}} \Sigma(v) \ \leq \ n-1 \quad \text{and} \quad \dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(v) \ \leq \ n-1 \ .$$

Suppose now that v is a nonconstant *harmonic* polynomial on $\mathbf{R}^n$. From [CF], [HS], we know that

$$\dim_{\mathbf{R}} \Sigma(v) \;\leq\; n-2 \;.$$

[HOHON], Th.3.1, also shows that

$$\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(v) \;\leq\; 1 \;\text{ in case }\; n=3 \;,$$

and v is homogeneous.

For the proof of Theorem 1.1 we need the generalisation of this result to n dimensions, which is given below. Thereby we thank H. Knörrer for crucial remarks.

Theorem 2.1. *Let P be a harmonic homogeneous polynomial in $\mathbf{C}^n$ with real coefficients, $P \not\equiv \mathrm{const}$, then $\dim \Sigma_{\mathbf{C}}(P) \leq n-2$.*

Proof of Theorem 2.1. For an irreducible polynomial in $\mathbf{C}^n$ the assertion is true. See e.g. [S, Chap. II, 1.4].

Let P be reducible. We assume for contradiction that $\dim\Sigma(P) = n-1$. Then P can be represented as

$$P = p^2 q, \text{where p and q are homogenous and p is irreducible.} \qquad (2.1)$$

This can be seen as follows: Let $P = \prod_{j=1}^{k} q_j$, q_j irreducible $\forall j$, with $k \geq 2$, and denote $N_j = N(q_j)$. If for $i \neq j$, $\dim N_i \cap N_j < n-1$, then clearly $\dim \Sigma(P) < n-1$. Without loss we assume $\dim N_1 \cap N_2 = n-1$. Since q_1, q_2 are irreducible this implies (see e.g. [M] Lemma 2.5) $q_1 = \mathrm{const}\, q_2$, proving (2.1).

(2.1) implies

$$P = \tilde{p}^2 \tilde{q}, \text{where } \tilde{p}, \tilde{q} \text{ are homogeneous polynomials with real coefficients and } \tilde{p} \not\equiv \mathrm{const}\,. \qquad (2.2)$$

This can be seen as follows: Let $\underline{f}$ denote the polynomial which is obtained from the polynomial f by complex conjugations of its coefficients. Since P has real coefficients we conclude from (2.1) that $P = \underline{p}^2 \underline{q} = p^2 q$. Since p is irreducible $q = \underline{p}^2 \tilde{q}$ follows for some homogeneous polynomial $\tilde{q}$. Hence $P = (p\underline{p})^2 \tilde{q}$ and $p\underline{p}$ has real coefficients.

Finally we use:

Proposition 2.2. *Let $P \not\equiv$ const be a harmonic polynomial in $\mathbf{R}^n$ given by $P = p^2 q, p, q$ polynomials with real coefficients and $p \not\equiv$ const, then $P \equiv 0$.*

Proof of Proposition 2.2: Let M denote the degree of P, then for some spherical harmonic $Y_M(x|x|^{-1})$, $P(x) = |x|^M Y_M$ in polar coordinates. Further we have

$$q|_{S^{n-1}} = \sum_{j=1}^{M-1} a_j Y_j,$$

where the Y_j's are spherical harmonics of degree $\leq M-1$, which can be taken orthonormal on S^{n-1} and $a_j \in \mathbf{R}$, $\forall j$. Hence $\int_{S^{n-1}} Y_j Y_M d\omega = 0$ for $j \neq M$, and

$$\int_{S^{n-1}} p^2 q^2 d\omega = \int_{S^{n-1}} Pq d\omega = 0$$

implying $p = q = 0$. ∎

Together with (2.2) this gives $P \equiv 0$ which is a contradiction to $P \not\equiv$ const and finishes the proof of Theorem 2.1. ∎

3. Restriction to 2-plane slices

Suppose now $p : \mathbf{C}^n \to \mathbf{C}$ is a complex *homogeneous* polynomial; hence,

$$\Sigma_{0\mathbf{C}}(p) = \Sigma_{\mathbf{C}}(p).$$

For any complex 2-dimensional subspace $\epsilon \subset \mathbf{C}^n$, the restriction $p|\epsilon$ is essentially a complex homogeneous polynomial of two variables. Moreover, *for $z \in \epsilon \setminus \{0\}$,*

$$\nabla(p|_\epsilon)(z) = 0$$

if and only if either

$$z \in \Sigma_{\mathbf{C}}(p)$$

or

$$z \in p^{-1}\{0\} \setminus \Sigma_{\mathbf{C}}(p) \;\text{ and }\; \epsilon \text{ is tangent to } p^{-1}\{0\} \text{ at } z\,. \tag{3.1}$$

For each pair i, j of integers with $1 \leq i < j \leq n$ and point $(z_1, \ldots, z_n) \in \mathbf{C}^n$, let

$$\pi_{ij}(z_1, \ldots, z_n) \;=\; (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \;\in\; \mathbf{C}^{n-2} \;.$$

For each real rotation $\gamma \in O(n)$, let $\gamma : \mathbf{C}^n \to \mathbf{C}^n$ also denote the complex linear extension of γ. Thus each set $(\pi_{ij} \circ \gamma)^{-1}\{y\}$, for $y \in \mathbf{C}^{n-2}$, is a complex affine 2 plane in $\mathbf{C}^n$.

3.1 Lemma. *For any nonconstant homogeneous polynomial $p : \mathbf{C}^n \to \mathbf{C}$ having $\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(p) \;\leq\; n-2$, there exists a rotation $\gamma \in O(n)$ so that, for all integers $1 \leq i < j \leq n$,*

$$\Sigma_{\mathbf{C}}(p) \cap (\pi_{ij} \circ \gamma)^{-1}\{0\} \;=\; \{0\}$$

and each complex 2-plane $(\pi_{ij} \circ \gamma)^{-1}\{0\}$ is transverse to $p^{-1}\{0\} \setminus \Sigma_{\mathbf{C}}(p)$, hence,

$$\left[\nabla p|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}\right]^{-1}\{0\} \;=\; \{0\} \;.$$

Proof : For $a \in \mathbf{S}^{n-1}$, let π_a be the complex rank $n-1$ projection of $\mathbf{C}^n$ corresponding to the orthogonal projection of $\mathbf{R}^n$ onto $a^\perp$. Thus, $\pi_a(z) = z - (z \cdot a)a$, and π_a has kernel $\pi_a^{-1}\{0\}$ equaling the complex span of a and image $\pi_a(\mathbf{C}^n)$ equaling the complex span of $a^\perp$ in $\mathbf{C}^n$.

We also define, for $y \in \pi_a(\mathbf{C}^n)$,

$$D_a(y) \;=\; \prod_{z \in \pi_a^{-1}(y) \cap p^{-1}\{0\}} \left[a_1 \frac{\partial p}{\partial z_1}(z) + \cdots + a_n \frac{\partial p}{\partial z_n}(z)\right] \;,$$

which is the the discriminant of p, with respect to the direction a. Then D_a is a polynomial that is homogeneous because, for $0 \neq \lambda \in \mathbf{C}$,

$$z \in \pi_a^{-1}(\lambda y) \cap p^{-1}\{0\} \text{ if and only if } \lambda^{-1} z \in \pi_a^{-1}(y) \cap p^{-1}\{0\}$$

and $\frac{\partial p}{\partial z_i}(z) \;=\; \lambda^{k-1} \frac{\partial p}{\partial z_i}(\lambda^{-1} z)$. Moreover, $D_a \not\equiv 0$, because, otherwise, ∇p would vanish on a complex $n-1$ dimensional stratum of $p^{-1}\{0\}$, contradicting that $\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(p) \leq n-2$.

For $b \in \mathbf{S}^{n-1}$ with $a \cdot b = 0$, $\pi_b^{-1}\{0\} \subset \pi_a(\mathbf{C}^n)$ and

$$\epsilon_{a,b} \;=\; \pi_a^{-1}\left[\pi_b^{-1}\{0\}\right]$$

is the complex span of $\{a,b\}$ in $\mathbf{C}^n$. Note that this complex 2-plane is transverse to $p^{-1}\{0\}\setminus\{0\}$ if $D_a(b)\neq 0$. In fact, if $z\in\epsilon_{a,b}\cap p^{-1}\{0\}\setminus\{0\}$, then $\pi_a(z)=\lambda b$ for some $0\neq\lambda\in\mathbf{C}$. Since $D_a(\lambda b)\neq 0$, $a\cdot(\nabla p)(z)\neq 0$, and, by the complex implicit function theorem, $p^{-1}\{0\}$, is, locally near z, a holomorphic graph over a domain in $\pi_a(\mathbf{C}^n)$. In particular, $p^{-1}\{0\}$ is transverse at z to $\pi_a^{-1}\big[\pi_b^{-1}\{0\}\big]=\epsilon_{a,b}$.

In the set of all pairs

$$\mathcal{A}=\{(a,b)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}\ :a\cdot b=0\}\ ,$$

we are now interested in the "bad" set

$$\mathcal{B}\ =\{(a,b)\in\mathcal{A}\ :\ \text{either }\ \Sigma_{\mathbf{C}}(p)\cap\epsilon_{a,b}\setminus\{0\}\neq\emptyset\ \text{or}\ D_a(b)=0\}\ .$$

Since $\mathcal{B}$ is a semi-algebraic set [BR], we may show that $\dim_{\mathbf{R}}\mathcal{B}<\dim_{\mathbf{R}}\mathcal{A}=(n-1)(n-2)$ by simply verifying that $\mathcal{B}$ contains no nonempty open subset U of $\mathcal{A}$.

Suppose, for contradiction, that there were such a U. Since $p\not\equiv 0$, $p|_{\mathbf{S}^{n-1}}\not\equiv 0$, because the coefficients of p are determined by $p|_{\mathbf{R}^n}$. Thus,

$$\dim_{\mathbf{R}}\big(\mathbf{S}^{n-1}\cap p^{-1}\{0\}\big)\ \leq\ n-2\ ,$$

and we may chose a pair $(a,b)\in U$ with $p(a)\neq 0$. By homogeneity, $p(\lambda a)\neq 0$ for all $0\neq\lambda\in\mathbf{C}$, hence,

$$p^{-1}\{0\}\cap\pi_a^{-1}\{0\}\ =\ \{0\}\ .$$

The projection $\pi_a(\Sigma_{\mathbf{C}}(p))$ is, by the Proper Mapping Theorem and Chow's Theorem [GR], pp.162,170, a complex homogeneous algebraic subvariety of $\pi_a(\mathbf{C}^n)$ of complex dimension $\leq n-2$. Moreover, the discriminant locus $D_a^{-1}\{0\}$ is also a complex homogeneous algebraic subvariety of $\pi_a(\mathbf{C}^n)$ of complex dimension $\leq n-2$. Thus,

$$\pi_a(\Sigma_{\mathbf{C}}(p))\cup D_a^{-1}\{0\}\ \subset\ q^{-1}\{0\}$$

for some not-identically-zero complex homogeneous polynomial q on $\pi_a(\mathbf{C}^n)$, and we may similarly find a point $c\in\mathbf{S}^{n-1}\cap a^{\perp}$ near b so that $(a,c)\in U\subset\mathcal{B}$ and $q(c)\neq 0$, hence, $q^{-1}\{0\}\cap\pi_c^{-1}\{0\}=\{0\}$. But then

$$\begin{aligned}\Sigma_{\mathbf{C}}(p)\cap\epsilon_{a,c}\ &\subset\ p^{-1}\{0\}\cap\pi_a^{-1}\big[\pi_a\big(\Sigma_{\mathbf{C}}(p)\big)\big]\cap\pi_a^{-1}\big[\pi_c^{-1}\{0\}\big]\\ &\subset\ p^{-1}\{0\}\cap\pi_a^{-1}\big[q^{-1}\{0\}\cap\pi_c^{-1}\{0\}\big]\ \subset\ p^{-1}\{0\}\cap\pi_a^{-1}\{0\}\ =\{0\}\ ,\end{aligned}$$

and $D_a(c) \neq 0$, contradicting that $(a,c) \in \mathcal{B}$.

Thus, $\dim_{\mathbf{R}} \mathcal{B} < (n-1)(n-2)$. For each pair of integers $1 \leq i < j \leq n$, we deduce that, in the space $\mathcal{C}$ of ordered orthonormal bases of $\mathbf{R}^n$, the set of ordered bases $(a_1, \ldots, a_n)$ with $(a_i, a_j) \in \mathcal{B}$ has dimension $< \dim_{\mathbf{R}} \mathcal{C} = (n-1)!$. In particular we are able to choose a "good" basis $(a_1, \ldots, a_n)$ so that $(a_i, a_j) \notin \mathcal{B}$ for every $1 \leq i < j \leq n$. Such a basis readily determines the desired rotation $\gamma \in O(n)$. ∎

Corollary 3.2.*Let $P : \mathbf{C}^n \to \mathbf{C}$ be a nonconstant homogeneous harmonic polynomial with real coefficients.*

Then for some real rotation $\gamma \in O(n)$, $P|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}$ has an isolated critical zero in the origin of $\mathbf{C}^2$, $\forall i,j$ with $1 \leq i < j \leq n$.

Proof of Corollary 3.2: Because of Theorem 2.1

$$\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(P) \leq n-2.$$

Therefore Lemma 3.1 is applicable and yields the desired result. ∎

4. Stability under smooth perturbations

Let $u \neq$ const satisfy (1.1'), i.e

$$\mathcal{L}_0 = 0 \quad \text{in } \Omega, \Omega \subset \mathbf{R}^n.$$

Without loss we assume that $\mathcal{O} \in \Omega$,

$$a_{ij}(\mathcal{O}) = \delta_{ij}, \quad 1 \leq i < j \leq n$$

and that u has a critical zero in $\mathcal{O}$. Then due to (1.2)

$$u(x) = P_M(x) + O(|x|^{M+1}) \quad \text{for } |x| \to 0 \tag{4.1}$$

for some harmonic homogeneous polynomial $P_M \not\equiv 0$ of degree $M \geq 2$.

Denoting the complexification of P_M for simplicity again by P_M it follows from Corollary 3.2 that

$$P_M|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}} \text{ has an isolated critical zero in the origin of } \mathbf{C}^2, \forall i,j. \tag{4.2}$$

This will be essential to show

Lemma 4.1. *There exists $R > 0$ such that*

$$\operatorname{card} \Sigma(u) \cap (\pi_{ij} \circ \gamma)^{-1}\{y\} \cap B_R \leq (M-1)^2 \tag{4.3}$$

for all $y \in B_R^{(n-2)}$ and for all i, j such that $1 \leq i < j \leq n$.

Proof of Lemma 4.1: The proof is similar to that of Lemma 2.3 in [HO-HON]. From there we use Proposition 2.2, namely:

Proposition 4.2. *Let $p(z_1, z_2)$ be a homogeneous polynomial in $\mathbf{C}^2$ of degree k with real coefficients, and assume that p has an isolated critical point in the origin in $\mathbf{C}^2$. Let further $\phi \in C^\infty(D_r(\mathcal{O}))$, $D_r(\mathcal{O}) = \{y \in \mathbf{R}^2 : |y| < r\}$, and $r > 0$, with*

$$\phi(y) = p(y) + o(|y|^k) \quad \textit{for } |y| \to 0$$

and let $\phi_t(y) \in C^\infty(D(\mathcal{O}) \times I)$ for $t \in I$ where $I = [-t_0, t_0]$, with $\phi_0 = \phi$. Then there exists $\tilde{r}$, $0 < \tilde{r} < r$ such that for $|t| \leq t_0$, $\quad t_0$ small enough, the number of critical points of $\phi_t(.)$ in $D_{\tilde{r}}(\mathcal{O})$ is uniformly bounded by $(k-1)^2$.

For the proof we used results in [AGV], namely: for a homogeneous polynomial $p(z)$, $z \in \mathbf{C}^2$ of degree k, with an isolated critical point in the origin $\mathcal{O}$ the algebraic multiplicity of the gradient map of p in $\mathcal{O}$ is $(k-1)^2$. This together with the subadditivity of the algebraic multiplicity yields the result, which can be stated as

$$\operatorname{card}(\Sigma(\phi_t) \cap D_r(\mathcal{O})) \leq (k-1)^2 \quad \forall t, \quad |t| \leq t_0.$$

We apply this to our case and identify $\forall i, j \quad 1 \leq i < j \leq n$

$$p = P_M|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}$$

and

$$\phi_0 = u|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}$$

Let $\sigma \in C^\infty$ denote a curve in $\mathbf{R}^{n-2}$ passing through the origin, parametrized such that $\sigma(0) = \mathcal{O}$. We define for $t \in I$, I an interval with $0 \in I$,

$$\phi_t = u|_{(\pi_{ij} \circ \gamma)^{-1}\{\sigma(t)\}}.$$

Due to Lemma 3.2 and (4.1) we can apply Proposition 4.2 and obtain for some $\tilde{r} > 0$

$$\operatorname{card}(\Sigma(u|_{(\pi_{ij}\circ\gamma)^{-1}\{\sigma(t)\}}) \cap D_{\tilde{r}}(\mathcal{O})) \leq (M-1)^2$$

for t, $|t| \leq t_0$, t_0 small enough. This implies further that for some $\overline{R} > 0$ and $\overline{t} > 0$

$$\operatorname{card}(\Sigma(u) \cap (\pi_{ij}\circ\gamma)^{-1}\{\sigma(t)\} \cap B_{\overline{R}}) \leq (M-1)^2 \quad \forall t, |t| \leq \overline{t}. \tag{4.4}$$

Suppose now for contradiction that Lemma 4.1 is false, then for some i, j there are sequences $\{R_k\}$ and $\{y^{(k)}\}$ with $y^{(k)} \in \mathbf{R}^{n-2}$, $R_k \to 0$, $|y^{(k)}| \to 0$ for $k \to \infty$ such that

$$\operatorname{card}(\Sigma(u) \cap (\pi_{ij}\circ\gamma)^{-1}\{y^{(k)}\} \cap B_{R_k}) > (M-1)^2.$$

Proposition 4.3. *Let $\{y^{(k)}\}$ denote a sequence in $\mathbf{R}^n$ convergent to some $\overline{y}$. Then there is a subsequence which is a subset of a C^∞-curve in $\mathbf{R}^n$.*

Proof of Proposition 4.3: We use a result of Kriegl [Kr] (see also Lemma 4.2.15 in [FK]):

Let $x_m \in \mathbf{R}^n$, $x_m \to \overline{x}$ for $m \to \infty$ and let $t_m \in \mathbf{R}$, $t_m \downarrow 0$ for $m \to \infty$. If $\forall k$, $k \in \mathbf{N}$, $\{(x_m - x_{m+1})(t_m - t_{m+1})^{-k}\}$ is bounded, then for some C^∞-curve γ, $\gamma(t_m) = x_m$, $\forall m$ and $\gamma^{(j)}(t_m) = 0$, $\forall j \in \mathbf{N}$.

¿From any convergent sequence $\{y^{(k)}\}$ it is easily seen that we can pick a subsequence converging fast enough so that the assumptions above are satisfied. ■

Therefore we conclude: We can pick a subsequence of $\{y^{(k)}\}$ (again denoted by $\{y^{(k)}\}$) such that for some $\sigma \in C^\infty$, $\sigma(t_k) = y^{(k)}$, $\forall k$ and

$$\operatorname{card}(\Sigma(u) \cap (\pi_{ij}\circ\gamma)^{-1}\{\sigma(t_k)\} \cap B_{R_k}) > (M-1)^2 \quad \forall k. \tag{4.5}$$

On the other hand given σ, there are $\overline{R}, \overline{t} \geq 0$ such that (4.4) holds which is a contradiction to (4.5). This verifies Lemma 4.1. ■

5. Finiteness of the measure of the critical set.

We first need

Lemma 5.1. *Let $u \not\equiv$ const satisfy (1.1') and B be a ball with $\overline{B} \subset \Omega$, then $\Sigma(u) \cap B$ decomposes into to the countable union of subsets of a pairwise disjoint collection of smooth $n-2$ dimensional submanifolds, i.e $\Sigma(u) \cap B$ is a countably $(n-2)$-rectifiable subset in the sense of Federer [F].*

Proof of Lemma 5.1: The proof is in principle the same as the one of Lemma 1.9 in [HS]: Thereby the argument is essentially that used by Cafarelli and Friedman [CF]:

Let for $q = 1, 2, 3, \ldots$

$$S_q = \{x | D^\alpha u(x) = 0, \forall \alpha \text{ with } 0 < |\alpha| \leq q, \quad D^{q+1}u(x) \neq 0\}.$$

Let B_{2R} be a ball with $\overline{B}_{2R} \subset \Omega$, then $\forall x_0 \in \overline{B}_R$, $\mathcal{L}_0(u - u(x_0)) = 0$. Since the coefficients of the equation are smooth it follows via unique continuation that $u - u(x_0)$ vanishes of finite order $M(x_0)$ in x_0 and

$$\sup_{x_0 \in \overline{B}_{2R}} M(x_0) \equiv \overline{M} < \infty$$

Therefrom we obtain that $\forall a \in \Sigma(u) \cap B_R$

$$B_R(a) \cap \{x | \nabla u(x) = 0\} = B_R(a) \cap \bigcup_{q=1}^{\overline{M}} S_q.$$

The remaining part of the proof is the same as in e.g. [HS] or [CF]. ∎

Due to Lemma 5.1 we have in particular

$$\Sigma(u) \cap B_R = \cup_{m=1}^{\infty} E_m$$

where $E_1 \subset E_2 \subset \ldots$ are Borel subsets of $\Sigma(u)$ of finite $\mathcal{H}^{n-2}$-measure.

Without loss of generality we change coordinates to make $\gamma = \text{Id}$ in Lemma 4.1. Then we use the integral geometric inequality 3.2.27 in [F] and Lemma 4.1 to obtain the following estimate:

With $R > 0$ given in Lemma 4.1

$$\begin{aligned}\mathcal{H}^{n-1}(\Sigma(u)\cap\overline{B}_R) &= \lim_{m\to\infty}\mathcal{H}^{n-2}(E_m\cap\overline{B}_R)\\ &\le \limsup_{m\to\infty}\sum_{1\le i<j\le n}\int_{B_R^{n-2}}\mathrm{card}[\pi_{ij}^{-1}\{y\}\cap E_m\cap\overline{B}_R]d\mathcal{H}^{n-2}y\\ &\le \binom{n}{2}\mathcal{H}^{n-2}(B_R^{n-2})(M-1)^2.\end{aligned}$$

This finishes the proof of Theorem 1.1. ■

References

[AGV] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of differentiable maps*, Vol.I, Monographs in Math. Vol. 82, Birkhäuser, Berlin, 1985.

[AL] G. Allessandrini, *Singular solutions and the determination of conductivity by boundary measurements*, J. Diff. Equ. **84** (1990), 252-272.

[B] L. Bers, *Local behaviour of solutions of general linear elliptic equations*, Commun. Pure Appl. Math. **8** (1955), 473-496.

[BJS] L. Bers, F. John, Schechter, *Partial differential equations*, Lectures in Applied Mathematics, American Math. Soc., Providence, 1964

[BR] R. Benedetti, R. Risler, *Real algebraic sets. Actualites Mathématiques*, Current Mathematical Topics, Hermann, Paris, 1990.

[CF] L. Caffarelli, A. Friedman, *Partial regularity of the zero set of linear and superlinear elliptic equations*, J. Diff. Eqns **60** (1985), 420–439.

[D] R.-T. Dong, *Nodal sets of eigenfunctions on Riemann surfaces*, J. Diff. Geometry **36** (1992), 493–506.

[DF1] H. Donnelly, Ch. Fefferman *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.

[DF2] ______, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Am. Math. Soc.**3** (1990), 333–353.

[F] H. Federer, *Geometric measure theory*, Springer-Verlag, Berlin and New York, 1969.

[FK] A. Fröhlicher, A. Kriegl, *Linear spaces and differentiation theory*, Wiley, 1988.

[GR] R. Gunning, H. Rossi, *Analytic functions of several complex variables.* Prentice Hall, Englewood Cliffs, N.J. 1965

[HHL1] R. Hardt, Q. Han, F. H. Lin, *Geometric measure of singular sets of elliptic equations*, Preprint.

[HHL2] R. Hardt, Q. Han, F. H. Lin, *Singular sets of higher order equations*, Preprint.

[Hn] Q. Han, *Singular sets of solutions to elliptic equations.* Indiana Univ. Math. J. **43** (1994), no. 3, 983–1002.

[HnL1] Q. Han, F.H. Lin, *Nodal sets of solutions of parabolic equations II.* Comm. Pure Appl. Math. **47** (1994), no. 9, 1219–1238.

[HnL2] ______, *On the geometric measure of nodal sets of solutions.* J. Partial Differential Equations **7** (1994), no. 2, 111–131.

[HOHON] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili, *Critical sets of smooth solutions of elliptic equations in dimension 3*, Indiana Univ. Math. J. **45** (1996), no. 1, 15–37.

[HS] R. Hardt and L. Simon, *Nodal sets for solutions for elliptic equations*, J. Diff. Geom. **30** (1989), 505–522.

[KR] A. Kriegl, *Die richtigen Räume für Analysis im Unendlich- Dimensionalen*, Monatshefte f. Math. **94** (1982), 109-124.

[L] F.H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure and Appl. Math. **XLIV** (1991), 287–308.

[M] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Math. Studies 61, Princeton University Press, Princeton, 1968.

[N] N. Nadirashvili, *Uniqueness and stability of the solution of an elliptic equation from a set to a domain*, Math. Notes of the Acad. Sci. USSR

40(1986), 623-627.

[S] I-R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, Berlin and New York, 1977.

[T] J-C. Tougeron, *Idéaux de fonctions différentiables*, Ergebnisse der Mathematik und ihrer Grenzgebiete 71, Springer-Verlag, Berlin- Heidelberg-New York 1972.

[Y] S.T. Yau, *Open problems in geometry*, Proc. Symp. Pure Math. **54,1** (1993), 1–28.

R. Hardt: Mathematics Department, Rice University, Houston, TX 777251-1892 USA

E-mail address: hardt@math.rice.edu

M. Hoffmann-Ostenhof: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: mho@nelly.mat.univie.ac.at

T. Hoffmann-Ostenhof: Institut für Theoretische Chemie, Universität Wien, Währinger Strasse 17, A-1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria

E-mail address: thoffman@esi.ac.at

N. Nadirashvili: Institute for Problems of Information Transmission, Bol'shoi Karetnyi 19, 101447 Moscow, Russia

Current address: Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, 02139-4307 MA, USA

E-mail address: nicolai@math.mit.edu