Fractional powers of self-adjoint operators and Trotter-Kato product formula

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Abstract

We study error bounds in the operator norm topology for the Trotter-Kato product formula. We prove that they depend on fractional power conditions (domains and relative boundedness) for operators involved in this formula.

Mathematics Subject Classification

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1 Introduction

Let A and B two non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$, i.e.

$$A \geq 0 \qquad \text{and} \qquad B \geq 0. \tag{1.1}$$

For simplicity we consider the case

$$\overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})} = \mathcal{H}. \tag{1.2}$$

Denoting by H the form-sum of A and B, i.e. $H := A \dot{+} B$, which is well defined on $\mathcal{H}$ by (1.2). Kato has shown in [6] that under these conditions the Trotter product formula strongly converges to e^{-tH} uniformly in $t \in [0, T]$, $0 < T < +\infty$, i.e., one has

$$s - \lim_{n\to\infty} \left(e^{-tA/n} e^{-tB/n}\right)^n = e^{-tH}. \tag{1.3}$$

Furthermore, it was found by Kato in [5], [6] that the Trotter product formula [11] is true not only for the exponential function e^{-x}, $x \in [0, +\infty)$ but for a whole class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$ which are defined on $[0, +\infty)$ and characterized by

$$0 \leq f(x) \leq 1, \qquad f(0) = 1, \qquad f'(0) = -1 \tag{1.4}$$

$$0 \leq g(x) \leq 1, \qquad g(0) = 1, \qquad g'(0) = -1. \tag{1.5}$$

In [6] it was proved that besides (1.3) one always has

$$s - \lim_{n\to\infty} \left(f(tA/n) g(tB/n)\right)^n = e^{-tH} \tag{1.6}$$

uniformly in $t \in [0, T]$, $0 < T < +\infty$. Product formulae of this type we call Trotter-Kato product formulae in the following.

In general the strong convergence in (1.3) and (1.6) cannot be replaced by the the operator norm convergence. However, under special circumstances it is possible even in the trace norm topology [7], [3]. For instance, if A and B are arbitrary bounded operators on $\mathcal{H}$ a straightforward computation shows that there is a constant $C > 0$, which depends on $\|A\|$, $\|B\|$ and T, $0 < T < +\infty$, such that one has

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| \leq C \frac{1}{n}, \qquad n = 1, 2, \ldots, \tag{1.7}$$

uniformly in $t \in [0, T]$ (see e.g. [9]). It seems that Rogava was the first who has made an attempt to extend norm estimates to unbounded self-adjoint operators A and B. In [10] he claimed that there is a constant $C > 0$ such that

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| \leq C \frac{\log(n)}{n^{1/2}}, \qquad n = 2, 3, \ldots, \tag{1.8}$$

uniformly in $t \in [0,T]$, $0 < T < +\infty$, provided that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and the self-adjoint operator H is the algebraic sum of A and B, i.e. $H = A + B$. For a symmetrized version of the Trotter product formula a better estimate was proven by Ichinose and Tamura [4] under stronger assumption that

$$\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B), \qquad 0 \leq \alpha < 1. \tag{1.9}$$

They proved that in this case there is a constant $C > 0$ such that

$$\left\| \left(e^{-tA/2n} e^{-tB/n} e^{-tA/2n}\right)^n - e^{-tH} \right\| \leq C \frac{1}{n^{2/(3+\alpha)}}, \qquad n = 1, 2, \ldots, \tag{1.10}$$

uniformly in $t \in [0,T]$, $0 < T < +\infty$.

In [8] the both results were improved assuming

$$A \geq I \qquad \text{and} \qquad B \geq I, \tag{1.11}$$

$$\mathrm{dom}(A) \quad \subseteq \quad \mathrm{dom}(B) \tag{1.12}$$

and

$$\|Bf\| \leq a\|Af\|, \qquad f \in \mathrm{dom}(A), \qquad 0 < a < 1. \tag{1.13}$$

for the class of Borel functions (1.4) and (1.5) satisfying conditions

$$C_1 := \sup_{x>0} \frac{1 - f(x)}{x} < +\infty, \tag{1.14}$$

$$C_2 := \sup_{x>0} \left| \left(f(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < +\infty, \tag{1.15}$$

$$S_1 := \sup_{x>0} \frac{1 - g(x)}{x} < +\infty, \tag{1.16}$$

$$S_2 := \sup_{x>0} \left| \left(g(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < +\infty. \tag{1.17}$$

If in addition to (1.11) - (1.17) one has

$$C_0 := \sup_{x>0} \frac{x f(x)^{1/2}}{1 - f(x)} < +\infty \tag{1.18}$$

and

$$0 < aC_0 S_1 < 1, \tag{1.19}$$

then [8] there are constants $C > 0$ and $C' > 0$ such that

$$\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/2n)^{1/2} \right)^n - e^{-tH} \right\| \leq C \frac{\log(n)}{n}, \qquad n = 4, 5, \ldots, \tag{1.20}$$

and

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \leq C'\frac{\log(n)}{n}, \qquad n = 4, 5, \ldots, \tag{1.21}$$

uniformly in $t \in [0, +\infty)$, where in this case the form-sum H coincides with the algebraic sum $H = A + B$. We note that the Ichinose-Tamura condition (1.9) implies (1.12) and (1.13), see [8]. Moreover, the standard functions $f(x) = g(x) = e^{-x}$, $x \geq 0$, satisfy (1.14) - (1.19).

In the present paper we generalize these results assuming instead of (1.11) - (1.13) that

$$A \geq I \qquad \text{and} \qquad B \geq 0, \tag{1.22}$$

$$\mathrm{dom}(A^\alpha) \quad \subseteq \quad \mathrm{dom}(B^\alpha), \qquad \frac{1}{2} < \alpha < 1, \tag{1.23}$$

and

$$\|B^\alpha f\| \leq a\|A^\alpha f\|, \qquad f \in \mathrm{dom}(A^\alpha), \qquad 0 < a < 1. \tag{1.24}$$

Note that (1.9) implies (1.23) and (1.24). If in addition to (1.4), (1.5), (1.14) - (1.17) one modifies (1.18) and (1.19) to

$$C_0 := \sup_{x>0} \frac{xf(x)^{1/2\alpha}}{1 - f(x)} < +\infty, \tag{1.25}$$

and

$$0 < a^{1/\alpha}C_0S_1 < 1, \tag{1.26}$$

then we will prove that there are constants $C > 0$ and $C' > 0$ such that

$$\left\|\left(f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}\right)^n - e^{-tH}\right\| \leq C\frac{1}{n^{\alpha-1/2}}, \qquad n = 2, 3, \ldots, \tag{1.27}$$

and

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \leq C'\frac{1}{n^{\alpha-1/2}}, \qquad n = 2, 3, \ldots, \tag{1.28}$$

uniformly in $t \in [0, +\infty)$. Moreover, one has

$$\left\|\left(f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}\right)^n - e^{-tH}\right\| \leq C_\epsilon\frac{\log(n)}{n^{2\alpha-1}}, \qquad n = 4, 5, \ldots, \tag{1.29}$$

and

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \leq C'_\epsilon\frac{\log(n)}{n^{2\alpha-1}}, \qquad n = 4, 5, \ldots, \tag{1.30}$$

uniformly in $t \in [\epsilon, +\infty)$, $\epsilon > 0$, where H is, of course, the form sum of A and B.

If in addition to (1.23) we have

$$\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha), \qquad \frac{1}{2} < \alpha < 1, \tag{1.31}$$

then there are constants $D > 0$ and $D' > 0$ such that

$$\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tH} \right\| \leq D \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{1.32}$$

and

$$\left\| (f(tA/n) g(tB/n))^n - e^{-tH} \right\| \leq D' \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{1.33}$$

uniformly in $t \in [0, +\infty)$. Below we prove that condition (1.31) is ensured for $H = A \dot{+} B$ if

$$\mathrm{dom}(A^\beta) \subseteq \mathrm{dom}(B^\alpha), \qquad 0 < \beta < \alpha < 1, \tag{1.34}$$

in particular, if

$$\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^\alpha), \qquad \frac{1}{2} < \alpha < 1. \tag{1.35}$$

We note that the proof below does not cover the limits $\alpha = 1$ and $\alpha = \frac{1}{2}$. In the first case one has to follow the line of reasoning of our paper [8]. While it seems that for $\alpha \leq \frac{1}{2}$ and unbounded B one cannot ensure convergence of the Trotter-Kato product formula in the operator norm topology in general.

2 Domain Considerations

First we indicate some simple consequences of conditions (1.23) and (1.24). Applying Theorem X.4.3 of [1] we get

$$\mathrm{dom}(A^\theta) \subseteq \mathrm{dom}(B^\theta), \qquad 0 < \theta \leq \alpha, \tag{2.1}$$

and

$$\|B^\theta f\| \leq a^{\theta/\alpha} \|A^\theta f\|, \qquad f \in \mathrm{dom}(A^\theta). \tag{2.2}$$

In particular, one has

$$\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2}) \tag{2.3}$$

and

$$\|B^{1/2} f\| \leq a^{1/2\alpha} \|A^{1/2} f\|, \qquad f \in \mathrm{dom}(A^{1/2}), \tag{2.4}$$

as well as

$$\mathrm{dom}(A^{1-\alpha}) \subseteq \mathrm{dom}(B^{1-\alpha}) \tag{2.5}$$

and

$$\|B^{1-\alpha} f\| \leq a^{\frac{1-a}{a}} \|A^{1-\alpha} f\|, \qquad f \in \mathrm{dom}(A^{1-\alpha}). \tag{2.6}$$

Taking into account (2.3) we find that

$$\mathrm{dom}(H^{1/2}) = \mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2}) = \mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2}), \tag{2.7}$$

which guarantees that the form-sum H is densely defined in (1.27) - (1.30). In particular, property (2.7) yields that

$$\|A^{1/2}H^{-1/2}\| \leq 1 \qquad \text{and} \qquad \|B^{1/2}H^{-1/2}\| \leq 1. \tag{2.8}$$

Moreover, applying again Theorem X.4.3 of [1] to (2.7) and (2.8) we get

$$\mathrm{dom}(H^\theta) = \mathrm{dom}(A^\theta) \subseteq \mathrm{dom}(B^\theta), \qquad 0 < \theta \leq \frac{1}{2}, \tag{2.9}$$

as well as

$$\|A^\theta H^{-\theta}\| \leq 1 \qquad \text{and} \qquad \|B^\theta H^{-\theta}\| \leq 1, \qquad 0 < \theta \leq \frac{1}{2}. \tag{2.10}$$

In particular, one has

$$\mathrm{dom}(H^{1-\alpha}) = \mathrm{dom}(A^{1-\alpha}) \subseteq \mathrm{dom}(B^{1-\alpha}), \qquad \frac{1}{2} < \alpha < 1, \tag{2.11}$$

and

$$\|A^{1-\alpha}H^{-(1-\alpha)}\| \leq 1 \qquad \text{and} \qquad \|B^{1-\alpha}H^{-(1-\alpha)})\| \leq 1, \qquad \frac{1}{2} < \alpha < 1 \tag{2.12}$$

Lemma 2.1 *Let A and B be self-adjoint operators which obey (1.22) - (1.24). Then*

$$\mathrm{dom}(H) \subseteq \mathrm{dom}(A^\alpha). \tag{2.13}$$

Moreover, there is a bounded operator Y obeying $\|Y\| < 1$ such that

$$H^{-1} = A^{-(1-\alpha)}(I+Y)^{-1}A^{-\alpha} = A^{-\alpha}(I+Y^*)^{-1}A^{-(1-\alpha)}. \tag{2.14}$$

Proof: We set

$$Y := (B^\alpha A^{-\alpha})^*(B^{1-\alpha}A^{-(1-\alpha)}). \tag{2.15}$$

By (1.24) and (2.6) one has

$$\|Y\| \leq a^{1/\alpha} < 1. \tag{2.16}$$

¿From (2.7) we obtain $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(H^{1/2}) \subseteq \mathrm{dom}(A^{1-\alpha})$. Therefore, for $h \in \mathrm{dom}(A^\alpha)$ and $g \in \mathrm{dom}(H^{1/2})$ one has

$$\begin{aligned}(H^{1/2}h, H^{1/2}g) = (A^{1/2}h, A^{1/2}g) + (B^{1/2}h, B^{1/2}g) = \\ (A^\alpha h, A^{1-\alpha}g) + (B^\alpha A^{-\alpha}A^\alpha h, B^{1-\alpha}A^{-(1-\alpha)}A^{1-\alpha}g)\end{aligned} \tag{2.17}$$

Hence

$$(H^{1/2}h, H^{1/2}g) = (A^\alpha h, \{I+Y\}A^{1-\alpha}g) \tag{2.18}$$

If $g \in \mathrm{dom}(H)$, then

$$(h, Hg) = (A^\alpha h, \{I+Y\}A^{1-\alpha}g). \tag{2.19}$$

Hence

$$\{I + Y\}A^{1-\alpha}g \in \mathrm{dom}(A^\alpha), \qquad g \in \mathrm{dom}(H), \tag{2.20}$$

and

$$Hg = A^\alpha\{I + Y\}A^{1-\alpha}g, \qquad g \in \mathrm{dom}(H). \tag{2.21}$$

However, from (2.21) and (2.16) we immediately obtain the representation (2.14). Finally, from (2.14) we get (2.13). □

Note that in virtue of (1.22), (1.24), (2.14) and (2.16) we find

$$\|A^\alpha H^{-1}\| \le \frac{1}{1 - a^{1/\alpha}} \qquad \text{and} \qquad \|B^\alpha H^{-1}\| \le \frac{a}{1 - a^{1/\alpha}}. \tag{2.22}$$

Lemma 2.2 *Let A and B be self-adjoint operators which obey (1.22) - (1.24). If condition (1.34) is satisfied, then one has (1.31) for the operator H.*

Proof: We set

$$Y(x) := (B^\alpha(A + x)^{-\alpha})^* B^{1-\alpha}(A + x)^{-(1-\alpha)}, \qquad x \ge 0. \tag{2.23}$$

Since

$$B^\alpha(A + x)^{-\alpha} = B^\alpha A^{-\alpha}(I + xA^{-1})^{-\alpha}, \qquad x \ge 0, \tag{2.24}$$

and

$$B^{1-\alpha}(A + x)^{-(1-\alpha)} = B^{1-\alpha}A^{-(1-\alpha)}(I + xA^{-1})^{-(1-\alpha)}, \qquad x \ge 0, \tag{2.25}$$

one gets

$$Y(x) = (I + xA^{-1})^{-\alpha}Y(I + xA^{-1})^{-(1-\alpha)}, \qquad x \ge 0, \tag{2.26}$$

By (2.16) this implies the estimate

$$\|Y(x)\| \le a^{1/\alpha} < 1, \qquad x \ge 0. \tag{2.27}$$

Since $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(H^{1/2}) \subseteq \mathrm{dom}(A^{1-\alpha})$ we find

$$\begin{aligned} ((H + x)^{1/2}h, (H + x)^{1/2}g) = ((A + x)^{1/2}h, (A + x)^{1/2}g) + (B^{1/2}h, B^{1/2}g) = \\ ((A + x)^\alpha h, (A + x)^{(1-\alpha)}g) + \\ (B^\alpha(A + x)^{-\alpha}(A + x)^\alpha h, B^{1-\alpha}(A + x)^{-(1-\alpha)}(A + x)^{1-\alpha}g) \end{aligned} \tag{2.28}$$

for $h \in \mathrm{dom}(A^\alpha)$ and $g \in \mathrm{dom}(H^{1/2})$. Therefore we have

$$(H^{1/2}h, H^{1/2}g) = ((A + x)^\alpha h, \{I + Y(x)\}(A + x)^{1-\alpha}g), \qquad x \ge 0. \tag{2.29}$$

If $g \in \mathrm{dom}(H)$, then

$$\{I + Y(x)\}(A + x)^{1-\alpha}g \in \mathrm{dom}(A^\alpha), \qquad x \ge 0, \tag{2.30}$$

and

$$(H+x)g = (A+x)^{\alpha}\{I + Y(x)\}(A+x)^{1-\alpha}g, \qquad x \ge 0. \tag{2.31}$$

Hence

$$(H+x)^{-1} = (A+x)^{-(1-\alpha)}(I+Y(x))^{-1}(A+x)^{-\alpha}, \qquad x \ge 0. \tag{2.32}$$

By Theorem X.4.1 of [1] one has

$$H^{-\alpha}h = c_\alpha \int_0^{+\infty} dx\, x^{-\alpha}(H+x)^{-1}h, \qquad h \in \mathcal{H}, \tag{2.33}$$

where $c_\alpha = \pi^{-1}\sin(\pi\alpha)$. Inserting (2.32) into (2.33) we find

$$H^{-\alpha}h = c_\alpha \int_0^{+\infty} dx\, x^{-\alpha}(A+x)^{-(1-\alpha)}(I+Y(x))^{-1}(A+x)^{-\alpha}h. \tag{2.34}$$

Since

$$(I+Y(x))^{-1} = I - Y(x)(I+Y(x))^{-1}, \qquad x \ge 0, \tag{2.35}$$

one gets that

$$\begin{aligned} H^{-\alpha}h = c_\alpha &\int_0^{+\infty} dx\, x^{-\alpha}(A+x)^{-1}h - \\ &c_\alpha \int_0^{+\infty} dx\, x^{-\alpha}(A+x)^{-(1-\alpha)}Y(x)(I+Y(x))^{-1}(A+x))^{-\alpha}h \end{aligned} \tag{2.36}$$

which implies

$$H^{-\alpha}h = A^{-\alpha}h - c_\alpha \int_0^{+\infty} dx\, x^{-\alpha}(A+x)^{-(1-\alpha)}Y(x)(I+Y(x))^{-1}(A+x)^{-\alpha}h. \tag{2.37}$$

If condition (1.34) is satisfied, then

$$B^\alpha(A+x)^{-\alpha} = B^\alpha(A+x)^{-\beta}(A+x)^{-(\alpha-\beta)}, \qquad x \ge 0. \tag{2.38}$$

Since

$$B^\alpha(A+x)^{-\beta} = B^\alpha A^{-\beta}(I+xA^{-1})^{-\beta}, \qquad x \ge 0, \tag{2.39}$$

one gets

$$\sup_{x>0} \|B^\alpha(A+x)^{-\beta}\| \le \|B^\alpha A^{-\beta}\| < +\infty \tag{2.40}$$

which yields that

$$\hat{Y}(x) := (B^\alpha(A+x)^{-\beta})^* B^{1-\alpha}(A+x)^{-(1-\alpha)}, \qquad x \ge 0, \tag{2.41}$$

obeys (cf. (2.6))

$$\sup_{x>0} \|\hat{Y}(x)\| \le a^{\frac{1-\alpha}{\alpha}} \|B^\alpha A^{-\beta}\| < +\infty. \tag{2.42}$$

Since

$$Y(x) = (A+x)^{-(\alpha-\beta)}\hat{Y}(x), \qquad x \ge 0, \tag{2.43}$$

we have

$$H^{-\alpha}h = A^{-\alpha}h - c_\alpha \int_0^{+\infty} dx\; x^{-\alpha}(A+x)^{-(1-\beta)}\hat{Y}(x)(I+Y(x))^{-1}(I+xA^{-1})^{-\alpha}A^{-\alpha}h. \tag{2.44}$$

By (1.22), (2.27) and (2.42) one gets

$$\|(A+x)^{-(1-\beta)}\hat{Y}(x)(I+Y(x))^{-1}(I+xA^{-1})^{-\alpha}\| \le \frac{a^{\frac{1-\alpha}{\alpha}}}{1-a^{1/\alpha}}(1+x)^{-(1-\beta)}\|B^\alpha A^{-\beta}\| \tag{2.45}$$

Since $x \ge 0$ and $x^{-\alpha}(1+x)^{-(1-\beta)} \in L^1([0,+\infty))$, the operator

$$S := c_\alpha \int_0^{+\infty} dx\; x^{-\alpha}(A+x)^{-(1-\beta)}\hat{Y}(x)(I+Y(x))^{-1}(I+xA^{-1})^{-\alpha} \tag{2.46}$$

is well-defined and bounded. Hence we find the representation

$$H^{-\alpha} = \{I - S\}A^{-\alpha} = A^{-\alpha}\{I - S^*\} \tag{2.47}$$

which implies that $\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha)$. □

3 Auxiliary Scalar Estimates

By a simple calculation from (1.14) and (1.16) we obtain

$$\sup_{x>0} \frac{1-f(x)}{x^\gamma} \le C_1, \qquad 0 \le \gamma \le 1, \tag{3.1}$$

and

$$\sup_{x>0} \frac{1-g(x)}{x^\gamma} \le S_1, \qquad 0 \le \gamma \le 1. \tag{3.2}$$

Further we note that by the identity

$$\frac{1-f(x)^{1/2}}{x^\gamma} = \frac{1-f(x)}{x^\gamma}\frac{1}{1+f(x)^{1/2}}, \qquad x > 0, \quad 0 \le \gamma \le 1, \tag{3.3}$$

one gets that (1.4) and (3.1) imply

$$\sup_{x>0} \frac{1-f(x)^{1/2}}{x^\gamma} \le C_1, \qquad 0 \le \gamma \le 1. \tag{3.4}$$

Moreover, from (1.15) and (1.17) we get

$$\sup_{x>0} \left|\left(f(x) - \frac{1}{1+x}\right)\frac{1}{x^\gamma}\right| \le D_2 := \max\{1, C_2\}, \qquad 0 \le \gamma \le 2, \tag{3.5}$$

and

$$\sup_{x>0} \left|\left(g(x) - \frac{1}{1+x}\right)\frac{1}{x^\gamma}\right| \le E_2 := \max\{1, S_2\}, \qquad 0 \le \gamma \le 2. \tag{3.6}$$

In particular, we have

$$\sup_{x>0} \frac{1-e^{-x}}{x^\gamma} \leq 1, \qquad 0 \leq \gamma \leq 1, \tag{3.7}$$

and

$$\sup_{x>0} \left| \left(e^{-x} - \frac{1}{1+x} \right) \frac{1}{x^\gamma} \right| \leq 1, \qquad 0 \leq \gamma \leq 2. \tag{3.8}$$

In the following we need some simple auxiliary estimates of certain types.

Lemma 3.1 *If* $n = 2, 3, \ldots$, *then*

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m^\kappa} \leq 4 \left\{ \frac{1}{1-\alpha} + \frac{1}{1-\kappa} \right\} \frac{1}{n^{\alpha+\kappa-1}} \tag{3.9}$$

for $\frac{1}{2} \leq \alpha < 1$ *and* $\frac{1}{2} \leq \kappa < 1$.

Proof: We have

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m^\kappa} = \tag{3.10}$$

$$\frac{1}{n-1+\alpha} \sum_{m=1}^{n-1} \left\{ \frac{m^{1-\kappa}}{(n-1-m+\alpha)^\alpha} + \frac{(n-m-1+\alpha)^{1-\alpha}}{m^\kappa} \right\}.$$

Using estimates $m^{1-\kappa} \leq (n-1)^{1-\kappa}$ and $(n-1-m+\alpha)^{1-\alpha} \leq (n-2+\alpha)^{1-\alpha}$ for $m = 1, 2, \ldots, n-1$, we get

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m^\kappa} \leq \tag{3.11}$$

$$\frac{(n-1)^{1-\kappa}}{n-1+\alpha} \sum_{p=0}^{n-2} \frac{1}{(p+\alpha)^\alpha} + \frac{(n-2+\alpha)^{1-\alpha}}{n-1+\alpha} \sum_{m=1}^{n-1} \frac{1}{m^\kappa}.$$

Since $\frac{1}{n-1+\alpha} \leq \frac{2}{n}$, $n = 1, 2, \ldots$ we obtain

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m^\kappa} \leq 2 \left\{ \frac{1}{n^\kappa} \sum_{p=0}^{n-2} \frac{1}{(p+\alpha)^\alpha} + \frac{1}{n^\alpha} \sum_{m=1}^{n-1} \frac{1}{m^\kappa} \right\}, \quad n = 2, 3, \ldots. \tag{3.12}$$

Further, since

$$\sum_{p=0}^{n-2} \frac{1}{(p+\alpha)^\alpha} \leq 2 \int_{-\alpha}^{n-2} \frac{1}{(x+\alpha)^\alpha} dx \leq \frac{2}{1-\alpha} n^{1-\alpha}, \qquad n = 2, 3, \ldots, \tag{3.13}$$

and

$$\sum_{m=1}^{n-1} \frac{1}{m^\kappa} \leq \int_0^{n-1} \frac{1}{x^\kappa} dx \leq \frac{1}{1-\kappa} n^{1-\kappa}, \qquad n = 2, 3, \ldots, \tag{3.14}$$

from (3.12) we obtain (3.9). □

Corollary 3.2 *If* $n = 4, 5, \ldots$, *then*

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m} \le \frac{7}{1-\alpha} \frac{\log(n)}{n^\alpha}. \tag{3.15}$$

Proof: We use the representation

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m} = \tag{3.16}$$
$$\frac{1}{n-1+\alpha} \sum_{m=1}^{n-1} \left\{ \frac{1}{(n-1-m+\alpha)^\alpha} + \frac{(n-1-m+\alpha)^{1-\alpha}}{m} \right\}.$$

Since $(n-1-m+\alpha)^{1-\alpha} \le (n-2+\alpha)^{1-\alpha}$ for $m = 1, 2, \ldots$, we get

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m} \le \tag{3.17}$$
$$\frac{1}{n-1+\alpha} \sum_{p=0}^{n-2} \frac{1}{(p+\alpha)^\alpha} + \frac{1}{(n-1+\alpha)^\alpha} \sum_{m=1}^{n-1} \frac{1}{m}.$$

By

$$\sum_{m=1}^{n-1} \frac{1}{m} \le 1 + \int_1^{n-1} \frac{1}{x} dx = 1 + \log(n-1), \qquad n = 2, 3, \ldots, \tag{3.18}$$

the estimate (3.13) and $\frac{1}{n-1+\alpha} \le \frac{2}{n}$, $n = 1, 2, \ldots$, we find

$$\sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m} \le \frac{5}{1-\alpha} \frac{1}{n^\alpha} + 2\frac{\log(n)}{n^\alpha}, \qquad n = 2, 3, \ldots, \tag{3.19}$$

which immediately implies (3.15). □

4 Auxiliary Operator Estimates

In this section we prove a series of lemmata and corollaries which will be necessary to prove in Section 5 the error estimates announced in Section 1. They generalize corresponding results of [8].

Lemma 4.1 *If* X *is a self-adjoint operator such that* $0 \le X \le I$, *then*

$$\|X^k (I-X)^\beta\| \le (k+\beta)^{-\beta}, \qquad k = 0, 1, 2, \ldots, \quad 0 < \beta \le 1. \tag{4.1}$$

Proof: By

$$|x^k(1-x)^\beta| \le (k+\beta)^{-\beta}, \qquad x \ge 0, \quad k=0,1,2,\dots, \quad 0<\beta\le 1, \tag{4.2}$$

and the spectral theorem for self-adjoint operators we obtain (4.1). □

Lemma 4.2 *If X is a self-adjoint operator such that $X \ge 0$, then*

$$\|X^r e^{-qX}\| \le \frac{1}{q^r}, \qquad q>0, \qquad 0<r\le 1. \tag{4.3}$$

Proof: Since

$$x^r e^{-qx} \le r^r q^{-r} e^{-r} \le \frac{1}{q^r}, \qquad q>0, \quad 0<r\le 1, \tag{4.4}$$

we obtain (4.3) by the spectral theorem for self-adjoint operators. □

Lemma 4.3 *Let A be a self-adjoint operator such that $A \ge I$. If the Borel function $f(\cdot)$ satisfy the conditions (1.4) and (1.25), then the operator $I - f(\tau A)$ is boundedly invertible for each $\tau > 0$ and*

$$\|(I-f(\tau A))^{-\gamma}h\| \le \|h\| + C_0^\gamma \|A^{-\gamma}h\|\,\tau^{-\gamma}, \qquad 0<\gamma\le 1, \qquad h \in \mathcal{H}. \tag{4.5}$$

uniformly in $\tau > 0$.

Proof: By (1.4) one has $f(x)^{1/2\alpha} \ge f(x)$ for $x \ge 0$. Hence by (1.25) we find

$$\frac{xf(x)}{1-f(x)} \le C_0, \qquad x>0, \tag{4.6}$$

which yields

$$1-f(x) \ge \frac{x}{C_0+x}, \qquad x>0. \tag{4.7}$$

Since $x(C_0+x)^{-1}$ is a non-decreasing function and $\tau A \ge \tau I$, $\tau > 0$, one gets

$$I - f(\tau A) \ge \tau A(C_0+\tau A)^{-1} \ge \frac{\tau}{C_0+\tau} I, \qquad \tau>0. \tag{4.8}$$

Hence the operator $I - f(\tau A)$, $\tau > 0$, is boundedly invertible for each $\tau > 0$. Using the representation

$$(I-f(x)^{-1} = 1 + f(x)(1-f(x))^{-1}, \qquad x>0, \tag{4.9}$$

one gets that

$$(1-f(x))^{-\gamma} \le 1 + x^\gamma f(x)^\gamma (1-f(x))^{-\gamma} x^{-\gamma}, \qquad x>0, \quad 0<\gamma\le 1. \tag{4.10}$$

By (4.6) one estimates

$$(1 - f(x))^{-\gamma} \le 1 + C_0^\gamma x^{-\gamma}, \qquad x > 0, \quad 0 < \gamma \le 1. \tag{4.11}$$

Hence we find

$$\|(I - f(\tau A))^{-\gamma} h\| \le \|(I + C_0^\gamma \tau^{-\gamma} A^{-\gamma})h\| \le \|h\| + C_0^\gamma \|A^{-\gamma} h\| \, \tau^{-\gamma} \tag{4.12}$$

for $h \in \mathcal{H}$ and $0 < \gamma \le 1$ which proves (4.5). $\square$

We set

$$T(\tau) := f(\tau A)^{1/2} g(\tau B) f(\tau A)^{1/2}, \qquad \tau \ge 0. \tag{4.13}$$

Lemma 4.4 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy conditions (1.4), (1.5), (1.14) and (1.16), then there is a constant $L_1 > 0$ such that*

$$\|(I - T(\tau))^{1-\alpha} A^{-(1-\alpha)}\| \le L_1 \, \tau^{1-\alpha}. \tag{4.14}$$

uniformly in $\tau > 0$.

Proof: By the identity

$$I - T(\tau) = I - f(\tau A) + f(\tau A)^{1/2}(I - g(\tau B)) f(\tau A)^{1/2} \tag{4.15}$$

one gets

$$\begin{aligned} &(\tau A)^{-1/2}(I - T(\tau))(\tau A)^{-1/2} = \\ &\qquad (I - f(\tau A))(\tau A)^{-1} + f(\tau A)^{1/2} A^{-1/2} B^{1/2} (I - g(\tau B))(\tau B)^{-1} B^{1/2} A^{-1/2} f(\tau A)^{1/2}. \end{aligned} \tag{4.16}$$

By (2.3), (2.4), (1.14), (1.16) we find the estimate

$$\Big((\tau A)^{-1/2}(I - T(\tau))(\tau A)^{-1/2} h, h\Big) \le \{C_1 + a^{1/\alpha} S_1\} \|h\|^2, \qquad h \in \mathcal{H}. \tag{4.17}$$

Hence

$$\|(I - T(\tau))^{1/2} h\| \le \{C_1 + a^{1/\alpha} S_1\}^{1/2} \|A^{1/2} h\| \, \tau^{1/2}. \tag{4.18}$$

Applying Theorem X.4.3 of [1] one gets

$$\|(I - T(\tau))^{1-\alpha} h\| \le \{C_1 + a^{1/\alpha} S_1\}^{(1-\alpha)} \|A^{1-\alpha} h\| \, \tau^{(1-\alpha)} \tag{4.19}$$

where we have used that $2(1-\alpha) \le 1$. Setting $L_1 := \{C_1 + a^{1/\alpha} S_1\}^{(1-\alpha)}$ we find

$$\|(I - T(\tau))^{1-\alpha} h\| \le L_1 \|A^{1-\alpha} h\| \, \tau^{1-\alpha}, \qquad h \in \mathrm{dom}(A^{1-\alpha}), \tag{4.20}$$

which proves (4.14). $\square$

Corollary 4.5 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14), (1.16) and (1.25), then there is a constant $L_2 > 0$ such that*

$$\|(I - T(\tau))^{1-\alpha}(I - f(\tau A))^{-(1-\alpha)}\| \leq L_2 \tag{4.21}$$

uniformly in $\tau > 0$.

Proof: By Lemma 4.3 the operator $I - f(\tau A)$ is boundedly invertible for each $\tau > 0$. Taking into account (4.5) one gets

$$\begin{aligned} &\|(I - f(\tau A))^{-(1-\alpha)}(I - T(\tau))^{1-\alpha}h\| \leq \\ &\qquad \|(I - T(\tau))^{1-\alpha}h\| + C_0^{1-\alpha}\|A^{-(1-\alpha)}(I - T(\tau))^{1-\alpha}h\|\,\tau^{-(1-\alpha)}. \end{aligned} \tag{4.22}$$

Using $\|(I - T(\tau))^{1-\alpha}\| \leq 1$ and applying Lemma 4.4 we get

$$\|(I - f(\tau A))^{-(1-\alpha)}(I - T(\tau))^{1-\alpha}h\| \leq \{1 + C_0^{1-\alpha}L_1\}\|h\|, \qquad h \in \mathcal{H}. \tag{4.23}$$

Setting $L_2 := 1 + C_0^{1-\alpha}L_1$ we verify (4.21). □

The following lemma has no analogy with those of [8].

Lemma 4.6 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14), (1.16), (1.25) and (1.26), then for each $\tau > 0$ the operator $I - T(\tau)$ is boundedly invertible. Moreover, there is a family of bounded operators $\{Z(\tau)\}_{\tau>0}$ obeying*

$$\sup_{\tau>0}\|Z(\tau)\| < a^{1/\alpha}C_0S_1 < 1 \tag{4.24}$$

such that the representation

$$(I - T(\tau))^{-1} = (I - f(\tau A))^{-(1-\alpha)}(I + Z(\tau))^{-1}(I - f(\tau A))^{-\alpha} \tag{4.25}$$

holds for each $\tau > 0$.

Proof: Let us introduce the family

$$\begin{aligned} Z(\tau) := (\tau A)^{\alpha}f(\tau A)^{1/2}(I - f(\tau A))^{-\alpha}(B^{\alpha}A^{-\alpha})^{*}(I - g(\tau B))(\tau B)^{-1}\times \\ B^{1-\alpha}A^{-(1-\alpha)}(\tau A)^{1-\alpha}f(\tau A)^{1/2}(I - f(\tau A))^{-(1-\alpha)}. \end{aligned} \tag{4.26}$$

One estimates

$$\begin{aligned} &\|Z(\tau)\| \leq \|((\tau A)f(\tau A)^{1/2\alpha}(I - f(\tau A))^{-1})^{\alpha}\|\;\|B^{\alpha}A^{-\alpha}\|\times \\ &\quad \|(I - g(\tau B))(\tau B)^{-1}\|\;\|B^{1-\alpha}A^{-(1-\alpha)}\|\;\|((\tau A)f(\tau A)^{1/2(1-\alpha)}(I - f(\tau A))^{-1})^{-(1-\alpha)}\|. \end{aligned} \tag{4.27}$$

By (1.16), (1.24), (1.25), (2.6) and

$$f(\tau A)^{1/(1-\alpha)} \leq f(\tau A)^{1/\alpha}, \qquad \tau > 0, \tag{4.28}$$

one gets

$$\|Z(\tau)\| \leq a a^{(1-\alpha)/\alpha} C_0^\alpha S_1 C_0^{1-\alpha} = a^{1/\alpha} C_0 S_1. \tag{4.29}$$

Taking into account (1.26) we prove (4.24). Since

$$I - T(\tau) = (I - f(\tau A))^\alpha (I + Z(\tau))(I - f(\tau A))^{1-\alpha}, \qquad \tau > 0, \tag{4.30}$$

we obtain that the operator $I - T(\tau)$ is boundedly invertible for each $\tau > 0$ and, moreover, that the representation (4.25) is valid. □

In the following we set

$$U(\tau) := e^{-\tau H}, \qquad \tau \geq 0. \tag{4.31}$$

Lemma 4.7 *Let A and B be self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) and (1.16), then there is a constant $L_3 > 0$ such that*

$$\|(T(\tau) - U(\tau))h\| \leq L_3 \|H^{1/2}h\| \, \tau^{1/2}, \qquad h \in \mathrm{dom}(H^{1/2}), \tag{4.32}$$

uniformly in $\tau > 0$.

Proof: We use the representation

$$(T(\tau) - U(\tau))h =$$

$$-(I - f(\tau A))h + \tag{4.33}$$

$$(I - e^{-\tau H})h - \tag{4.34}$$

$$f(\tau A)^{1/2}(I - g(\tau B)) f(\tau A)^{1/2} h, \qquad h \in \mathcal{H}. \tag{4.35}$$

To estimate (4.33) we represent it as follows:

$$(I - f(\tau A))h = \tau^{1/2}(I - f(\tau A))(\tau A)^{-1/2} A^{1/2} H^{-1/2} H^{1/2} h. \tag{4.36}$$

By (2.8) and (3.1) we get

$$\|(I - f(\tau A))h\| \leq C_1 \|H^{1/2}h\| \, \tau^{1/2}, \qquad H \in \mathrm{dom}(H^{1/2}). \tag{4.37}$$

To handle (4.34) we represent this term as

$$(I - e^{-\tau H})h = \tau^{1/2}(I - e^{-\tau H})(\tau H)^{-1/2} H^{1/2} h. \tag{4.38}$$

By (3.7) we get the estimate

$$\|(I - e^{-\tau H})h\| \leq \|H^{1/2}h\|\, \tau^{1/2}, \qquad h \in \mathrm{dom}(H^{1/2}). \tag{4.39}$$

Finally, since

$$\begin{aligned} & f(\tau A)^{1/2}(I - g(\tau B))f(\tau A)^{1/2}h = \\ & \qquad \tau^{1/2} f(\tau A)^{1/2}(I - g(\tau B))(\tau B)^{-1/2} B^{1/2} A^{-1/2} f(\tau A)^{1/2} A^{1/2} H^{-1/2} H^{1/2} h \end{aligned} \tag{4.40}$$

by (1.4), (2.4), (2.8) and (3.2) we find

$$\|f(\tau A)^{1/2}(I - g(\tau B))f(\tau A)^{1/2}h\| \leq a^{1/2\alpha} S_1 \|H^{1/2}h\|\, \tau^{1/2}, \qquad h \in \mathrm{dom}(H^{1/2}). \tag{4.41}$$

By

$$\begin{aligned} & \|(T(\tau) - U(\tau))h\| \leq \|(I - f(\tau A))h\| + \\ & \qquad \|(I - e^{-\tau H})h\| + \|f(\tau A)^{1/2}(I - g(\tau A))f(\tau A)^{1/2}h\| \end{aligned} \tag{4.42}$$

and (4.37), (4.39) and (4.41) one immediately gets the estimate (4.32) where $L_3 := 1 + C_1 + a^{1/2\alpha} S_1$. □

Corollary 4.8 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) and (1.16), then there is a constant $L_4 > 0$ such that*

$$\|A^{-\alpha}(T(\tau) - U(\tau))h\| \leq L_4 \|h\|\, \tau^{\alpha}, \qquad h \in \mathcal{H}, \tag{4.43}$$

uniformly in $\tau > 0$.

Proof: From (4.33) - (4.35) one gets the representation

$$A^{-\alpha}(T(\tau) - U(\tau))h =$$

$$-A^{-\alpha}(I - f(\tau A))h + \tag{4.44}$$

$$A^{-\alpha}(I - e^{-\tau H})h - \tag{4.45}$$

$$f(\tau A)^{1/2} A^{-\alpha}(I - g(\tau B))f(\tau A)^{1/2}h, \qquad h \in \mathcal{H}. \tag{4.46}$$

By (3.1) we find

$$\|A^{-\alpha}(I - f(\tau A))h\| \leq C_1 \|h\|\, \tau^{\alpha}. \tag{4.47}$$

To estimate (4.46) we use the representation

$$A^{-\alpha} f(\tau A)^{1/2}(I - g(\tau B))f(\tau A)^{1/2}h = \tau^{\alpha} f(\tau A)^{1/2} A^{-\alpha} B^{\alpha}(I - g(\tau B))(\tau B)^{-\alpha} f(\tau A)^{1/2}h. \tag{4.48}$$

By (1.4), (1.24), and (3.2) we get

$$\|A^{-\alpha}f(\tau A)^{1/2}(I-g(\tau B))f(\tau A)^{1/2}h\| \le aS_1\|h\|\,\tau^{\alpha}. \tag{4.49}$$

It remains to estimate (4.45). For $h \in \mathrm{dom}(H)$ and $g \in \mathcal{H}$ one has

$$(A^{-\alpha}(I-e^{-\tau H})h,g) = \int_0^\tau ds(e^{-sH}Hh, A^{-\alpha}g). \tag{4.50}$$

By (2.7) and $\frac{1}{2} < \alpha < 1$ we have $A^{-\alpha}g \in \mathrm{dom}(A^{1/2}) = \mathrm{dom}(H^{1/2})$ which yields the representation

$$\begin{aligned}(e^{-sH}Hh, A^{-\alpha}g) = (H^{1/2}e^{-sH}h, H^{1/2}A^{-\alpha}g) = \\ (A^{1/2}e^{-sH}h, A^{1/2}A^{-\alpha}g) + (B^{1/2}e^{-sH}h, B^{1/2}A^{-\alpha}g).\end{aligned} \tag{4.51}$$

Taking into account (2.9) we find

$$\begin{aligned}&(e^{-sH}Hh, A^{-\alpha}g) = \\ &\quad (A^{1-\alpha}H^{-(1-\alpha)}H^{1-\alpha}e^{-sH}h, g) + (B^{1-\alpha}H^{-(1-\alpha)}H^{1-\alpha}e^{-sH}h, B^{\alpha}A^{-\alpha}g).\end{aligned} \tag{4.52}$$

Then by (1.24) and (2.12) we get

$$|(e^{-sH}Hh, A^{-\alpha}g)| \le \{1+a\}\|H^{1-\alpha}e^{-sH}h\|\,\|g\| \tag{4.53}$$

which implies

$$|(A^{-\alpha}(I-e^{-\tau H})h,g)| \le (1+a)\int_0^\tau ds\;\|H^{1-\alpha}e^{-sH}h\|\,\|g\|. \tag{4.54}$$

By Lemma 4.2 we get the estimate

$$\|H^{1-\alpha}e^{-sH}\| \le s^{-(1-\alpha)}\|h\|. \tag{4.55}$$

Hence

$$|(A^{-\alpha}(I-e^{-\tau H})h,g)| \le \frac{1+a}{\alpha}\int_0^\tau ds\; s^{-(1-\alpha)}\|h\|\,\|g\|, \tag{4.56}$$

which yields

$$|(A^{-\alpha}(I-e^{-\tau H})h,g)| \le \frac{1+a}{\alpha}\|h\|\,\|g\|\,\tau^{\alpha}. \tag{4.57}$$

Since $g \in \mathcal{H}$ arbitrary we finally get the estimate

$$\|A^{-\alpha}(I-e^{-\tau H})h\| \le \frac{1+a}{\alpha}\|h\|, \qquad h \in \mathcal{H}. \tag{4.58}$$

By (4.47), (4.49) and (4.58) we find (4.43) where $L_4 := \frac{1+a}{\alpha} + C_1 + aS_1$. $\square$

Lemma 4.9 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) and (1.16), then there is a constant $L_5 > 0$ such that*

$$\|(T(\tau)-U(\tau))h\| \le L_5\|Hh\|\,\tau^{\alpha}, \qquad h \in \mathrm{dom}(H), \tag{4.59}$$

uniformly in $\tau > 0$.

Proof: By Lemma 2.1 we have $\mathrm{dom}(H) \subseteq \mathrm{dom}(A^\alpha)$. Therefore the representation

$$(I - f(\tau A))h = \tau^\alpha (I - f(\tau A))(\tau A)^{-\alpha} A^\alpha H^{-1} Hh \tag{4.60}$$

makes sense. By (1.14), (2.22) and (3.1) one estimates that

$$\|(I - f(\tau A))h\| \leq C_1 (1 - a^{1/\alpha})^{-1} \|Hh\| \, \tau^\alpha. \tag{4.61}$$

Obviously, we have

$$(I - e^{-\tau H})h = \tau^\alpha (I - e^{-\tau H})(\tau H)^{-\alpha} H^\alpha h. \tag{4.62}$$

Taking into account (3.7) we find

$$\|(I - e^{-\tau H})h\| \leq \|H^\alpha h\| \, \tau^\alpha \leq \|Hh\| \, \tau^\alpha. \tag{4.63}$$

Further, by

$$\begin{aligned} &f(\tau A)^{1/2}(I - g(\tau B)) f(\tau A)^{1/2} h = \\ &\qquad \tau^\alpha f(\tau A)^{1/2}(I - g(\tau B))(\tau B)^{-\alpha} B^\alpha A^{-\alpha} f(\tau A)^{1/2} A^\alpha H^{-1} Hh \end{aligned} \tag{4.64}$$

and (1.4), (1.24), (2.22) and (3.2) we find

$$\|f(\tau A)^{1/2}(I - g(\tau B)) f(\tau A)^{1/2} h\| \leq a S_1 (1 - a^{1/\alpha})^{-1} \|H^\alpha h\| \, \tau^\alpha. \tag{4.65}$$

Taking into account (4.42), (4.61), (4.63) and (4.65) we obtain (4.60) where $L_5 := \frac{1}{1-a^{1/\alpha}}\{1 + C_1 + aS_1 - a^{1/\alpha}\}$. □

Corollary 4.10 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) and (1.16) and in addition for the operator H the condition (1.31) is valid, then there is a constant $L_6 > 0$ such that*

$$\|(T(\tau) - U(\tau))h\| \leq L_6 \|H^\alpha h\| \, \tau^\alpha, \qquad h \in \mathrm{dom}(H^\alpha), \tag{4.66}$$

uniformly in $\tau > 0$.

Proof: By (1.31) the representation

$$(I - f(\tau A))h = \tau^\alpha (I - f(\tau A))(\tau A)^{-\alpha} A^\alpha H^{-\alpha} H^\alpha h \tag{4.67}$$

is valid. By (3.1) this leads to the estimate

$$\|(I - f(\tau A))h\| \leq C_1 \|A^\alpha H^{-\alpha}\| \, \|H^\alpha h\| \, \tau^\alpha. \tag{4.68}$$

Next, by

$$f(\tau A)^{1/2}(I-g(\tau B))f(\tau A)^{1/2}h = \tau^{\alpha} f(\tau A)^{1/2}(I-g(\tau B))(\tau B)^{-\alpha}B^{\alpha}A^{-\alpha}f(\tau A)^{1/2}A^{\alpha}H^{-\alpha}H^{\alpha}h \tag{4.69}$$

and (1.4), (1.24) and (3.2) we find

$$\|f(\tau A)^{1/2}(I-g(\tau B))f(\tau A)^{1/2}h\| \le aS_1\|A^{\alpha}H^{-\alpha}\|\,\|H^{\alpha}h\|\,\tau^{\alpha}. \tag{4.70}$$

Again taking into account (4.42), (4.63), (4.68) and (4.70) we obtain (4.66) where $L_6 := 1 + \|A^{\alpha}H^{-\alpha}\|\{C_1 + aS_1\}$. □

Lemma 4.11 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), then there is a constant $L_7 > 0$ such that*

$$\|A^{-\alpha}(T(\tau)-U(\tau))h\| \le L_7\|H^{1/2}h\|\,\tau^{\alpha+1/2}, \qquad h\in \mathrm{dom}(H^{1/2}). \tag{4.71}$$

uniformly in $\tau > 0$.

Proof: We use the representation

$$A^{-\alpha}(T(\tau)-U(\tau))h =$$

$$A^{-\alpha}(f(\tau A)-(I+\tau A)^{-1}) - \tag{4.72}$$

$$A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h + \tag{4.73}$$

$$A^{-\alpha}(I-f(\tau A)^{1/2})(I-g(\tau B))f(\tau A)^{1/2}h + \tag{4.74}$$

$$A^{-\alpha}(I-g(\tau B))(I-f(\tau A)^{1/2})h + \tag{4.75}$$

$$A^{-\alpha}(g(\tau B)-(I+\tau B)^{-1})h + \tag{4.76}$$

$$A^{-\alpha}\left(\tau H(I+\tau H)^{-1}h - \tau A(I+\tau A)^{-1}h - \tau B(I+\tau B)^{-1}\right). \tag{4.77}$$

To estimate (4.72) we represent this term as

$$A^{-\alpha}(f(\tau A)-(I+\tau A)^{-1})h = \tau^{\alpha+1/2}(f(\tau A)-(I+\tau A)^{-1})(\tau A)^{-(\alpha+1/2)}A^{1/2}H^{-1/2}H^{1/2}H^{1/2}. \tag{4.78}$$

By (2.8) and (3.5) we find

$$\|A^{-\alpha}(f(\tau A)-(I+\tau A)^{-1})h\| \le D_2\|H^{1/2}h\|\,\tau^{\alpha+1/2}. \tag{4.79}$$

To estimate (4.73) we take into account Lemma 2.1. Thus we get the representation

$$A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h = (I+Y)A^{1-\alpha}H^{-1}(e^{-\tau H}-(I+\tau H)^{-1})h. \tag{4.80}$$

By (2.11) we have

$$A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h=(I+Y)A^{1-\alpha}H^{-(1-\alpha)}(e^{-\tau H}-(I+\tau H)^{-1})H^{-\alpha}h \tag{4.81}$$

which yields

$$\begin{aligned}&A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h=\\&\qquad\tau^{\alpha+1/2}(I+Y)A^{1-\alpha}H^{-(1-\alpha)}(e^{-\tau H}-(I+\tau H)^{-1})(\tau H)^{-(\alpha+1/2)}H^{1/2}h.\end{aligned} \tag{4.82}$$

By (2.12), (2.16) and (3.8) we get

$$\|A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h\|\le\{1+a^{1/\alpha}\}\|H^{1/2}h\|\,\tau^{\alpha+1/2}. \tag{4.83}$$

Next, the term (4.74) can be estimated as follows. By

$$\begin{aligned}&A^{-\alpha}(I-f(\tau A)^{1/2})(I-g(\tau B))f(\tau A)^{1/2}h=\\&\quad\tau^{\alpha+1/2}(I-f(\tau A)^{1/2})(\tau A)^{-\alpha}(I-g(\tau B))(\tau B)^{-1/2}B^{1/2}A^{-1/2}f(\tau A)^{1/2}A^{1/2}H^{-1/2}H^{1/2}h\end{aligned} \tag{4.84}$$

and (2.4), (2.8), (3.2) and (3.4) we get

$$\|A^{-\alpha}(I-f(\tau A)^{1/2})(I-g(\tau B))f(\tau A)^{1/2}h\|\le a^{1/2\alpha}C_1S_1\|H^{1/2}h\|\,\tau^{\alpha+1/2}. \tag{4.85}$$

Similarly, we estimate the term (4.75). We have

$$\begin{aligned}&A^{-\alpha}(I-g(\tau B))(I-f(\tau A)^{1/2})h=\\&\quad=\tau^{\alpha+1/2}A^{-\alpha}B^{\alpha}(I-g(\tau B))(\tau B)^{-\alpha}(I-f(\tau A)^{1/2})(\tau A)^{-1/2}A^{1/2}H^{-1/2}H^{1/2}h.\end{aligned} \tag{4.86}$$

By (1.24), (2.8) (3.2) and (3.4) the term (4.86) can be estimated by

$$\|A^{-\alpha}(I-g(\tau B))(I-f(\tau A)^{1/2})h\|\le aS_1C_1\|H^{1/2}h\|\,\tau^{\alpha+1/2}. \tag{4.87}$$

The term (4.76) admits the representation

$$\begin{aligned}&A^{-\alpha}(g(\tau B)-(I+\tau B)^{-1})h=\\&\qquad A^{-\alpha}B^{\alpha}(g(\tau B)-(I+\tau B)^{-1})(\tau B)^{\alpha+1/2}B^{1/2}H^{-1/2}H^{1/2}h.\end{aligned} \tag{4.88}$$

Thus by (1.24), (2.8) and (3.6) we get

$$\|A^{-\alpha}(g(\tau B)-(I+\tau B)^{-1})h\|\le aE_2\|H^{1/2}h\|. \tag{4.89}$$

To estimate (4.77) we note that by (2.7) one has

$$\begin{aligned}&\tau\left(A^{-\alpha}\{H(I+\tau H)^{-1}-A(I+\tau A)^{-1}-B(I+\tau B)^{-1}\}h,g\right)=\\&\qquad\tau\left((I+\tau H)^{-1}H^{1/2}h,H^{1/2}A^{-\alpha}g\right)-\tau\left((I+\tau A)^{-1}A^{1/2}h,A^{1/2}A^{-\alpha}g\right)-\\&\qquad\tau\left((I+\tau B)^{-1}B^{1/2}h,B^{1/2}A^{-\alpha}g\right)\end{aligned} \tag{4.90}$$

for $h \in \mathrm{dom}(H^{1/2})$ and $g \in \mathcal{H}$. Therefore, we find

$$\tau\left(A^{-\alpha}\{H(I+\tau H)^{-1} - A(I+\tau A)^{-1} - B(I+\tau B)^{-1}\}h, g\right) = \tag{4.91}$$
$$\tau\left(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}A^{-\alpha}g\right) - \tau\left(\{(I+\tau A)^{-1} - I\}A^{1/2}h, A^{1/2}A^{-\alpha}g\right) -$$
$$\tau\left(\{(I+\tau B)^{-1} - I\}B^{1/2}h, B^{1/2}A^{-\alpha}g\right).$$

We have

$$\tau\left(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}A^{-\alpha}g\right) = \tag{4.92}$$
$$-\tau^2\left(H(I+\tau H)^{-1}H^{1/2}h, H^{1/2}A^{-\alpha}(I+Y^*)^{-1}A^{-(1-\alpha)}A^{1-\alpha}(I+Y^*)g\right)$$

for $g \in (I+Y^*)^{-1}\mathrm{dom}(A^{1-\alpha})$. By Lemma 2.1 one gets

$$\tau\left(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}A^{-\alpha}g\right) = \tag{4.93}$$
$$-\tau^2\left(H(I+\tau H)^{-1}H^{1/2}h, H^{-1/2}A^{1-\alpha}(I+Y^*)g\right) =$$
$$-\tau^2\left(H(I+\tau H)^{-1}H^{1/2}h, H^{-(\alpha-1/2)}H^{-(1-\alpha)}A^{(1-\alpha)}(I+Y^*)g\right)$$

which yields

$$\tau\left(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}A^{-\alpha}g\right) = \tag{4.94}$$
$$-\tau^{\alpha+1/2}\left((\tau H)^{3/2-\alpha}(I+\tau H)^{-1}H^{1/2}h, H^{-(1-\alpha)}A^{1-\alpha}(I+Y^*)g\right)$$

for $h \in \mathrm{dom}(H^{1/2})$ and $g \in (I+Y^*)\mathrm{dom}(A^{1-\alpha})$. Taking into account (2.12) and (2.16) one gets the estimate

$$\left|\tau\left(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}A^{-\alpha}g\right)\right| \le (1+a^{1/\alpha})\|H^{1/2}h\|\,\|g\|\,\tau^{\alpha+1/2} \tag{4.95}$$

for $h \in \mathrm{dom}(H^{1/2})$ and $g \in \mathcal{H}$ where we have used that

$$\sup_{x>0} \frac{x^\gamma}{1+x} \le 1, \qquad 0 < \gamma \le 1. \tag{4.96}$$

Further, one has

$$\tau\left(\{(I+\tau A)^{-1} - I\}A^{1/2}h, A^{1/2}A^{-\alpha}g\right) = \tag{4.97}$$
$$-\tau^{\alpha+1/2}\left((\tau A)^{3/2-\alpha}(I+\tau A)^{-1}A^{1/2}H^{-1/2}H^{1/2}h, g\right).$$

By (2.8) and (4.96) we find

$$\left|\tau\left(\{(I+\tau A)^{-1} - I\}A^{1/2}h, A^{1/2}A^{-\alpha}g\right)\right| \le \|H^{1/2}h\|\,\|g\|\,\tau^{\alpha+1/2}. \tag{4.98}$$

Next, from

$$\tau\left(\{(I+\tau B)^{-1} - I\}B^{1/2}h, B^{1/2}A^{-\alpha}g\right) = \tag{4.99}$$
$$-\tau^{\alpha+1/2}\left(\{(\tau B)^{3/2-\alpha}(I+\tau B)^{-1}B^{1/2}H^{-1/2}H^{1/2}h, B^{\alpha}A^{-\alpha}g\right),$$

(1.24), (2.8) and (4.96) we obtain

$$\left|\tau\left(\{(I+\tau B)^{-1}-I\}B^{1/2}h, B^{1/2}A^{-\alpha}g\right)\right| \leq a\|H^{1/2}h\|\,\|g\|\,\tau^{\alpha+1/2}. \tag{4.100}$$

Summing up (4.91), (4.95), (4.98) and (4.100) we finally get

$$\begin{aligned}\left|\tau\left(A^{-\alpha}\{H(I+\tau H)^{-1}-A(I+\tau A)^{-1}-B(I+\tau B)^{-1}\}h, g\right)\right| \leq \\ \{2+a+a^{1/\alpha}\}\|H^{1/2}h\|\,\|g\|\,\tau^{\alpha+1/2}\end{aligned} \tag{4.101}$$

for $h \in \mathrm{dom}(H^{1/2})$ and $g \in \mathcal{H}$. Hence

$$\begin{aligned}\|\tau A^{-\alpha}\{H(I\tau H)^{-1}-A(I+\tau A)^{-1}-B(I+\tau B)^{-1}\}\| \leq \\ \{2+a+a^{1/\alpha}\}\|H^{1/2}h\|\,\tau^{\alpha+1/2}\end{aligned} \tag{4.102}$$

Therefore from (4.72) - (4.77), (4.79), (4.83), (4.85), (4.87), (4.89) and (4.102) we get

$$\|A^{-\alpha}(T(\tau)-U(\tau))h\| \leq L_7\|H^{1/2}h\|\,\tau^{\alpha+1/2}, \qquad h \in \mathrm{dom}(H^{1/2}), \tag{4.103}$$

where

$$L_7 := 3 + D_2 + a\{1 + +E_2 + C_1S_1\} + 2a^{1/\alpha} + a^{1/2\alpha}C_1S_1 \tag{4.104}$$

which proves (4.71). □

Lemma 4.12 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), then there is a constant $L_8 > 0$ such that*

$$\|A^{-\alpha}(T(\tau)-U(\tau))h\| \leq L_8\|Hh\|\,\tau^{2\alpha}, \qquad h \in dom(H), \tag{4.105}$$

uniformly in $\tau > 0$.

Proof: To estimate (4.72) we use Lemma 2.1 and the representation

$$A^{-\alpha}(f(\tau A)-(I+\tau A)^{-1})h = \tau^{2\alpha}(f(\tau A)-(I+\tau A)^{-1})(\tau A)^{-2\alpha}A^{\alpha}H^{-1}Hh. \tag{4.106}$$

By (2.22) and (3.5) this leads to the estimate

$$\|A^{-\alpha}(f(\tau A)-(I+\tau A)^{-1})h\| \leq D_2(1-a^{1/\alpha})^{-1}\|Hh\|\,\tau^{2\alpha}, \qquad h \in \mathrm{dom}(H). \tag{4.107}$$

¿From (4.82) we obtain the representation

$$\begin{aligned}A^{-\alpha}(e^{-\tau H}-(I+\tau H)^{-1})h = \\ \tau^{2\alpha}(I+Y)A^{1-\alpha}H^{-(1-\alpha)}(e^{-\tau H}-(I+\tau H)^{-1})(\tau H)^{-2\alpha}H^{\alpha}.\end{aligned} \tag{4.108}$$

Using (2.12), (2.16) and (3.8) the term (4.73) can be estimated by

$$\|A^{-\alpha}(e^{-\tau H} - (I+\tau H)^{-1})h\| \le (1+a^{1/\alpha})\|H^\alpha h\|\,\tau^{2\alpha} \le (1+a^{1/\alpha})\|Hh\|\,\tau^{2\alpha}. \tag{4.109}$$

To estimate (4.74) we represent this term as

$$\begin{aligned} &A^{-\alpha}(I - f(\tau A)^{1/2})(I - g(\tau B))f(\tau A)^{1/2}h = \\ &\quad \tau^{2\alpha}(I - f(\tau A)^{1/2})(\tau A)^{-\alpha}(I - g(\tau B))(\tau B)^{-\alpha}B^\alpha A^{-\alpha}f(\tau A)^{1/2}A^\alpha H^{-1}Hh. \end{aligned} \tag{4.110}$$

By (1.24), (2.22), (3.2) and (3.4) we find

$$\|A^{-\alpha}(I - f(\tau A)^{1/2})(I - g(\tau B))f(\tau A)^{1/2}h\| \le a(1-a^{1/\alpha})^{-1}C_1S_1\|Hh\|\tau^{2\alpha}. \tag{4.111}$$

Simialarly, we estimate (4.75) and find

$$\|A^{-\alpha}(I - g(\tau B))(I - f(\tau A)^{1/2})h\| \le a(1-a^{1/\alpha})^{-1}C_1S_1\|Hh\|\,\tau^{2\alpha}. \tag{4.112}$$

For the term (4.76) we have the representation

$$A^{-\alpha}(g(\tau B) - (I+\tau B)^{-1})h = \tau^{2\alpha}A^{-\alpha}B^\alpha(g(\tau B) - (I+\tau B)^{-1})(\tau B)^{-2\alpha}B^\alpha H^{-1}Hh. \tag{4.113}$$

By (1.24), (2.22) and (3.6) we get

$$\|A^{-\alpha}(g(\tau B) - (I+\tau B)^{-1})h\| \le a^2(1-a^{1/\alpha})^{-1}E_2\|Hh\|\,\tau^{2\alpha}. \tag{4.114}$$

Let us estimate (4.77). From (4.94) we find

$$\begin{aligned} &\tau(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}h) = \\ &\quad -\tau^{2\alpha}((\tau H)^{2(1-\alpha)}(I+\tau H)^{-1}H^\alpha h, H^{-(1-\alpha)}A^{1-\alpha}(I+Y^*)g) \end{aligned} \tag{4.115}$$

for $h \in \mathrm{dom}(H^\alpha)$ and $g \in (I+Y^*)^{-1}\mathrm{dom}(A^{1-\alpha})$. Taking into account (2.12), (2.16) and (4.96) we have

$$\begin{aligned} &\left|\tau(\{(I+\tau H)^{-1} - I\}H^{1/2}h, H^{1/2}g)\right| \le \\ &\quad (1+a^{1/\alpha})\|H^\alpha h\|\,\|g\|\,\tau^{2\alpha} \le (1+a^{1/\alpha})\|Hh\|\,\|g\|\,\tau^{2\alpha} \end{aligned} \tag{4.116}$$

for $h \in \mathrm{dom}(H^\alpha)$ and $g \in \mathcal{H}$. From (4.97) one obtains the representation

$$\tau(\{(I+\tau A)^{-1} - I\}A^{1/2}h, A^{1/2}A^{-\alpha}g) = -\tau^{2\alpha}((\tau A)^{2(1-\alpha)}(I+\tau A)^{-1}A^\alpha H^{-1}Hh, g). \tag{4.117}$$

By (2.22) and (4.96) this implies the estimate

$$\left|\tau(\{(I+\tau A)^{-1} - I\}A^{1/2}h, A^{1/2}A^{-\alpha}g)\right| \le (1-a^{1/\alpha})^{-1}\|Hh\|\,\|g\|\,\tau^{2\alpha}. \tag{4.118}$$

Next, from (4.99) we obtain

$$\tau(\{(I+\tau B)^{-1}-I\}B^{1/2}h, B^{1/2}A^{-\alpha}g) = \\ -\tau^{2\alpha}((\tau B)^{2(1-\alpha)}(I+\tau B)^{-1}B^{\alpha}H^{-1}Hh, B^{\alpha}A^{-\alpha}g). \tag{4.119}$$

By (1.24), (2.22) and (4.96) which yields

$$\left|\tau(\{(I+\tau B)^{-1}-I\}B^{1/2}h, B^{1/2}A^{-\alpha}g)\right| \le a^2(1-a^{1/\alpha})^{-1}\|Hh\|\,\|g\|\,\tau^{2\alpha}. \tag{4.120}$$

Summing up (4.91), (4.116), (4.118) and (4.120) we get the estimate

$$\|\tau A^{-\alpha}\{H(I+\tau H)^{-1} - A(I+\tau A)^{-1} - B(I+\tau B)^{-1}\}h\| \le \\ \{2-a^{2/\alpha}+a^2\}(1-a^{1/\alpha})^{-1}\|Hh\|\,\tau^{2\alpha}. \tag{4.121}$$

Therefore from (4.72) - (4.77), (4.107), (4.109), (4.111), (4.112), (4.114) and (4.121) we find

$$\|A^{-\alpha}(T(\tau)-U(\tau))h\| \le L_8\|Hh\|\,\tau^{2\alpha} \tag{4.122}$$

where

$$L_8 := (1-a^{1/\alpha})^{-1}\{3+D_2+2aC_1S_1+a^2E_2-2a^{2/\alpha}+a^2\} \tag{4.123}$$

which proves (4.105). □

Corollary 4.13 *Let A and B be two self-adjoint operators which obey conditions (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17) and, in addition the condition (1.31) is valid, then there is a constant $L_9 > 0$ such that*

$$\|A^{-\alpha}(T(\tau)-U(\tau))h\| \le L_9\|H^{\alpha}h\|\,\tau^{2\alpha}, \qquad h \in dom(H), \tag{4.124}$$

uniformly in $\tau > 0$.

Proof: One has only to modify a bit the proof of Corollary 4.12. □

5 Error Estimates

Theorem 5.1 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26), then there is a constant $C > 0$ such that*

$$\left\|\left(f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}\right)^n - e^{-tH}\right\| \le C\frac{1}{n^{\alpha-1/2}}, \qquad n = 2,3,\ldots, \tag{5.1}$$

uniformly in $t > 0$.

Proof: We use the representation

$$(T(\tau)^n - U(\tau)^n)h = \sum_{m=0}^{n-1} T(\tau)^{n-m-1}(T(\tau) - U(\tau))U(\tau)^m h, \qquad n = 1, 2, \ldots. \tag{5.2}$$

By Lemma 4.6 the operator $I - T(\tau)$ is boundedly invertible. Therefore, we find

$$\begin{aligned}(T(\tau)^n - U(\tau)^n)h = \\ \sum_{m=0}^{n-1} T(\tau)^{n-m-1}(I - T(\tau))(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m h.\end{aligned} \tag{5.3}$$

Using (4.25) we get

$$\begin{aligned}(T(\tau)^n - U(\tau)^n)h = \\ \sum_{m=0}^{n-1} T(\tau)^{n-m-1}(I - T(\tau))^\alpha (I - T(\tau))^{1-\alpha}(I - f(\tau A))^{-(1-\alpha)} \times \\ (I + Z(\tau))^{-1}(I - f(\tau A))^{-\alpha}(T(\tau) - U(\tau))U(\tau)^m h\end{aligned} \tag{5.4}$$

which leads to the estimate

$$\begin{aligned}\|(T(\tau)^n - U(\tau)^n)h\| \leq \\ \sum_{m=0}^{n-1} \|T(\tau)^{n-m-1}(I - T(\tau))^\alpha\| \, \|(I - T(\tau))^{1-\alpha}(I - f(\tau A))^{-(1-\alpha)}\| \times \\ \|(I + Z(\tau))^{-1}\| \, \|(I - f(\tau A))^{-\alpha}(T(\tau) - U(\tau))U(\tau)^m h\|\end{aligned} \tag{5.5}$$

By Lemma 4.1, Corollary 4.5 and (4.24) one gets

$$\begin{aligned}\|(T(\tau)^n - U(\tau)^n)h\| \leq \\ M_1 \sum_{m=0}^{n-1} \frac{1}{(n - 1 - m + \alpha)^\alpha} \|(I - f(\tau A))^{-\alpha}(T(\tau) - U(\tau))U(\tau)^m h\|.\end{aligned} \tag{5.6}$$

where

$$M_1 := \frac{L_2}{1 - a^{1/\alpha} C_0 S_1} \tag{5.7}$$

¿From Lemma 4.3 we obtain

$$\begin{aligned}\|(T(\tau)^n - U(\tau)^n)h\| \leq \\ M_1 \sum_{m=0}^{n-1} \frac{1}{(n - 1 - m + \alpha)^\alpha} \times \\ \left\{\|(T(\tau) - U(\tau))U(\tau)^m h\| + C_0^\alpha \|A^{-\alpha}(T(\tau) - U(\tau))U(\tau)^m h\| \, \tau^{-\alpha}\right\}.\end{aligned} \tag{5.8}$$

which can be rewritten as

$$\begin{aligned}
&\|(T(\tau)^n - U(\tau)^n)h\| \leq \\
&\qquad M_1 \frac{1}{(n-1+\alpha)^\alpha}\left\{\|(T(\tau)-U(\tau))h\| + C_0^\alpha\|A^{-\alpha}(T(\tau)-U(\tau))h\|\,\tau^{-\alpha}\right\} + \\
&\qquad M_1 \sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\times \\
&\qquad \left\{\|(T(\tau)-U(\tau))U(\tau)^m h\| + C_0^\alpha\|A^{-\alpha}(T(\tau)-U(\tau))U(\tau)^m h\|\,\tau^{-\alpha}\right\}.
\end{aligned} \tag{5.9}$$

for $n = 2, 3; \ldots$. Applying Corollary 4.8 and taking into account that $\|(T(\tau)-U(\tau))h\| \leq 2\|h\|$ we get

$$\begin{aligned}
&\|(T(\tau)^n - U(\tau)^n)h\| \leq M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\|h\| + \\
&\qquad M_1 \sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\times \\
&\qquad \left\{\|(T(\tau)-U(\tau))U(\tau)^m h\| + C_0^\alpha\|A^{-\alpha}(T(\tau)-U(\tau))U(\tau)^m h\|\,\tau^{-\alpha}\right\}.
\end{aligned} \tag{5.10}$$

Furthermore, by Lemma 4.7 and Lemma 4.11 we find

$$\begin{aligned}
&\|(T(\tau)^n - U(\tau)^n)h\| \leq M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\|h\| + \\
&\qquad M_1(L_3 + C_0^\alpha L_7)\sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\|H^{1/2}U(\tau)^m h\|\,\tau^{1/2}
\end{aligned} \tag{5.11}$$

for $n = 2, 3, \ldots$. By Lemma 4.2 we finally get

$$\begin{aligned}
&\|(T(\tau)^n - U(\tau)^n)h\| \leq \\
&\qquad M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\,\|h\| + M_1(L_3 + C_0^\alpha L_7)\sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\frac{1}{m^{1/2}}\,\|h\|.
\end{aligned} \tag{5.12}$$

Using Lemma 3.1 for $\kappa = 1/2$ we find

$$\begin{aligned}
&\|(T(\tau)^n - U(\tau)^n)h\| \leq \\
&\qquad M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\,\|h\| + 4M_1(L_3 + C_0^\alpha L_7)\left\{2 + \frac{1}{1-\alpha}\right\}\frac{1}{n^{\alpha-1/2}}\,\|h\|.
\end{aligned} \tag{5.13}$$

for $n = 2, 3, \ldots$. Since $(n-1+\alpha)^{-\alpha} \leq 2^\alpha n^{-\alpha} \leq 2^\alpha n^{-(\alpha-1/2)}$, $n = 1, 2, \ldots$, we obtain

$$\|(T(\tau)^n - U(\tau)^n)h\| \leq C\frac{1}{n^{\alpha-1/2}}\,\|h\|, \qquad n = 2, 3, \ldots, \tag{5.14}$$

where

$$C := 2^\alpha M_1(2 + C_0^\alpha L_4) + 4M_1(L_3 + C_0^\alpha L_7)\left\{2 + \frac{1}{1-\alpha}\right\}. \tag{5.15}$$

which proves (5.1). □

Corollary 5.2 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26), then there is a constant $C' > 0$ such that*

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \le C' \frac{1}{n^{\alpha-1/2}}, \qquad n = 2, 3, \ldots, \tag{5.16}$$

uniformly in $t > 0$.

Proof: We note that

$$(f(\tau A)g(\tau B))^n = f(\tau A)^{1/2} T(\tau)^{n-1} f(\tau A)^{1/2} g(\tau B), \qquad n = 1, 2, \ldots . \tag{5.17}$$

Therefore, one gets that

$$\begin{aligned}
&(f(\tau A)g(\tau B))^n - U(\tau)^n)\, h = \\
&\qquad f(\tau A)^{1/2}\{T(\tau)^{n-1} - U(\tau)^{n-1}\} f(\tau A)^{1/2} g(\tau B) h - &(5.18)\\
&\qquad (I - f(\tau A)^{1/2}) U(\tau)^{n-1} f(\tau A)^{1/2} g(\tau B) h - &(5.19)\\
&\qquad U(\tau)^{n-1}(I - f(\tau A)^{1/2}) g(\tau B) h - &(5.20)\\
&\qquad U(\tau)^{n-1}(I - g(\tau B)) h. &(5.21)
\end{aligned}$$

The term (5.18) can be estimated by

$$\begin{aligned}
&\|f(\tau A)^{1/2}(T(\tau)^{n-1} - U(\tau)^{n-1}) f(\tau A)^{1/2} g(\tau B) h\| \le \\
&\qquad\qquad \|(T(\tau)^{n-1} - U(\tau)^{n-1}) f(\tau A)^{1/2} g(\tau B) h\|.
\end{aligned} \tag{5.22}$$

Applying Theorem 5.1 we obtain

$$\begin{aligned}
&\|f(\tau A)^{1/2}(T(\tau)^{n-1} - U(\tau)^{n-1}) f(\tau A)^{1/2} g(\tau B) h\| \le \\
&\qquad\qquad C \frac{1}{(n-1)^{\alpha-1/2}} \|f(\tau A)^{1/2} g(\tau B) h\|
\end{aligned} \tag{5.23}$$

Since $\frac{1}{n-1} \le \frac{2}{n}$, $n = 1, 2, \ldots$, and $\|f(\tau A)^{1/2} g(\tau B) h\| \le \|h\|$ we get

$$\|f(\tau A)^{1/2}(T(\tau)^{n-1} - U(\tau)^{n-1}) f(\tau A)^{1/2} g(\tau B) h\| \le 2^{\alpha-1/2} C \frac{1}{n^{\alpha-1/2}} \|h\| \tag{5.24}$$

for $n = 2, 3, \ldots$. To estimate (5.19) and (5.20) we start with the term $(I - f(\tau A)^{1/2}) U(\tau)^{n-1} h$. By (2.9) and (2.10) we get the representation

$$\begin{aligned}
&(I - f(\tau A)^{1/2}) U(\tau)^{n-1} h = \\
&\qquad (I - f(\tau A)^{1/2})(\tau A)^{-(\alpha-1/2)} A^{\alpha-1/2} H^{-(\alpha-1/2)} (\tau H)^{\alpha-1/2} U(\tau)^{n-1} h
\end{aligned} \tag{5.25}$$

which leads to the estimate

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}h\| \leq \tag{5.26}$$
$$\|(I - f(\tau A)^{1/2})(\tau A)^{-(\alpha-1/2)}\| \; \|A^{\alpha-1/2}H^{-(\alpha-1/2)}\| \; \|(\tau H)^{\alpha-1/2}U(\tau)^{n-1}h\|$$

Taking into account (2.10), (3.4) and Lemma 4.2 we find

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}h\| \leq C_1 \frac{1}{(n-1)^{\alpha-1/2}}\|h\|, \qquad n = 2, 3, \ldots, \tag{5.27}$$

Since $\frac{1}{n-1} \leq \frac{2}{n}$, $n = 1, 2, \ldots$, we finally obtain

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}\| \leq 2^{\alpha-1/2} C_1 \frac{1}{n^{\alpha-1/2}}, \qquad n = 2, 3, \ldots . \tag{5.28}$$

Using (5.28) we estimate (5.19) and (5.20) by

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1} f(\tau A)^{1/2} g(\tau B)h\| \leq 2^{\alpha-1/2} C_1 \frac{1}{n^{\alpha-1/2}}\|h\|, \qquad n = 2, 3, \ldots \tag{5.29}$$

and

$$\|U(\tau)^{n-1}(I - f(\tau A)^{1/2})g(\tau B)h\| \leq 2^{\alpha-1/2} C_1 \frac{1}{n^{\alpha-1/2}}\|h\| \qquad n = 2, 3, \ldots . \tag{5.30}$$

Taking into account (2.9), (2.10), (3.2) and Lemma 4.2 and following the line of reasoning in (5.25) - (5.28) one gets

$$\|(I - g(\tau B))U(\tau)^{n-1}\| \leq 2^{\alpha-1/2} S_1 \frac{1}{n^{\alpha-1/2}} \qquad n = 2, 3, \ldots . \tag{5.31}$$

Hence (5.21) can be estimated by

$$\|U(\tau)^{n-1}(I - g(\tau B))h\| \leq 2^{\alpha-1/2} S_1 \frac{1}{n^{\alpha-1/2}}\|h\|, \qquad n = 2, 3, \ldots, . \tag{5.32}$$

By (5.18) - (5.21), (5.24), (5.29), (5.30) and (5.32) we find

$$\|(f(\tau A)g(\tau B))^n - U(\tau)^n)h\| \leq C' \frac{1}{n^{\alpha-1/2}}\|h\|, \qquad n = 2, 3, \ldots, \tag{5.33}$$

where

$$C' := 2^{\alpha-1/2}(C + 2C_1 + S_1), \tag{5.34}$$

which proves (5.16). □

Theorem 5.3 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26), then for each $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tH} \right\| \leq C_\epsilon \frac{1}{n^{2\alpha-1}}, \qquad n = 4, 5, \ldots, \tag{5.35}$$

uniformly in $t \geq \epsilon$.

Proof: From (5.10), Lemma 4.9 and Lemma 4.12 we get the estimate

$$\|(T(\tau)^n - U(\tau)^n)h\| \le M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\|h\| + \\ M_1(L_5 + C_0^\alpha L_8)\sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\|(\tau H)U(\tau)^m h\|\,\tau^{-(1-\alpha)} \tag{5.36}$$

for $n = 2, 3, \ldots$. By Lemma 4.2 we get

$$\|(T(\tau)^n - U(\tau)^n)h\| \le \\ M_1(2 + C\alpha_0 L_4)\frac{1}{(n-1+\alpha)^\alpha}\|h\| + M_1(L_5 + C_0^\alpha L_8)\sum_{m=1}^{n-1}\frac{1}{(n-1-m+\alpha)^\alpha}\frac{1}{m}\|h\|\,\tau^{-(1-\alpha)}. \tag{5.37}$$

Using Corollary 3.2 we obtain

$$\|(T(\tau)^n - U(\tau)^n)h\| \le \\ M_1(2 + C_0^\alpha L_4)\frac{1}{(n-1+\alpha)^\alpha}\|h\| + 7M_1(L_5 + C_0^\alpha L_8)\frac{1}{1-\alpha}\frac{\log(n)}{n^\alpha}\|h\|\,\tau^{-(1-\alpha)} \tag{5.38}$$

for $n = 4, 5 \ldots$. Since for $\tau = t/n$, $t \ge \epsilon$, we have

$$\frac{\log(n)}{n^\alpha}\tau^{-(1-\alpha)} = \frac{1}{t^{1-\alpha}}\frac{\log(n)}{n^{2\alpha-1}} \le \frac{1}{\epsilon^{1-\alpha}}\frac{\log(n)}{n^{2\alpha-1}} \tag{5.39}$$

for $n = 1, 2, \ldots$ and $(n-1+\alpha)^{-\alpha} \le 2^\alpha n^{-\alpha} \le 2^\alpha n^{-(2\alpha-1)}$, $n = 1, 2, \ldots$, one gets

$$\|(T(\tau)^n - U(\tau)^n)h\| \le \\ 2^\alpha M_1(2 + C_0^\alpha L_4)\frac{1}{n^{2\alpha-1}}\|h\| + 7M_1(L_5 + C_0^\alpha L_8)\frac{1}{1-\alpha}\frac{1}{\epsilon^{1-\alpha}}\frac{\log(n)}{n^{2\alpha-1}}\|h\| \tag{5.40}$$

for $n = 4, 5, \ldots$ and $t \ge \epsilon$. Therefore, we obtain

$$\|(T(\tau)^n - U(\tau)^n)h\| \le C_\epsilon\frac{\log(n)}{n^{2\alpha-1}}\|h\| \tag{5.41}$$

for $n = 4, 5, \ldots$ and $t \ge \epsilon$ where

$$C_\epsilon := 2^\alpha M_1(2 + C\alpha_0 L_4) + 7M_1(L_5 + C_0^\alpha L_8)\frac{1}{1-\alpha}\frac{1}{\epsilon^{1-\alpha}}. \tag{5.42}$$

which proves (5.35). □

Corollary 5.4 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26), then for each $\epsilon > 0$ there is a constant $C'_\epsilon > 0$ such that*

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \le C'_\epsilon\frac{1}{n^{2\alpha-1}}, \qquad n = 4, 5, \ldots, \tag{5.43}$$

uniformly in $t \ge \epsilon$.

Proof: Following the line of reasoning (5.22) - (5.24) we obtain

$$\|f(\tau A)^{1/2}(T(\tau)^{n-1} - U(\tau)^{n-1})f(\tau A)^{1/2}g(\tau B)h\| \leq 2^{2\alpha-1}C_\epsilon \frac{1}{n^{2\alpha-1}}\|h\| \tag{5.44}$$

for $n = 4, 5, \ldots$ and $t \geq \epsilon$. Using the representation

$$(I - f(\tau A)^{1/2})U(\tau)^{n-1}h = \tau^{-(1-\alpha)}(I - f(\tau A)^{1/2})(\tau A)^{-\alpha}A^\alpha H^{-1}(\tau H)U(\tau)^{n-1}h \tag{5.45}$$

one gets the estimate

$$\begin{aligned}&\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}h\| \leq \\ &\qquad \|(I - f(\tau A)^{1/2})(\tau A)^{-\alpha}\| \, \|A^\alpha H^{-1}\| \, \|(\tau H)U(\tau)^{n-1}h\| \, \tau^{-(1-\alpha)}.\end{aligned} \tag{5.46}$$

Taking into account (2.22), (3.4) and Lemma 4.2 we find

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}h\| \leq C_1 \frac{1}{1 - a^{1/\alpha}} \frac{1}{n-1}\|h\| \, \tau^{-(1-\alpha)}, \qquad n = 2, 3, \ldots . \tag{5.47}$$

Since $\frac{1}{n-1} \leq \frac{2}{n}$ and $\frac{1}{n^\alpha} \leq \frac{1}{n^{2\alpha-1}}$ for $n = 1, 2, \ldots$, we get

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}\| \leq 2C_1 \frac{1}{1 - a^{1/\alpha}} \frac{1}{\epsilon^{1-\alpha}} \frac{1}{n^{2\alpha-1}} \tag{5.48}$$

for $n = 2, 3, \ldots$ and $t \geq \epsilon$. Hence, the estimates (5.29) and (5.30) can be replaced by

$$\|(I - f(\tau A)^{1/2})U(\tau)^{n-1}f(\tau A)^{1/2}g(\tau B)h\| \leq 2C_1 \frac{1}{1 - a^{1/\alpha}} \frac{1}{\epsilon^{1-\alpha}} \frac{1}{n^{2\alpha-1}}\|h\| \tag{5.49}$$

and

$$\|U(\tau)^{n-1}(I - f(\tau A)^{1/2})g(\tau B)h\| \leq 2C_1 \frac{1}{1 - a^{1/\alpha}} \frac{1}{\epsilon^{1-\alpha}} \frac{1}{n^{2\alpha-1}}\|h\| \tag{5.50}$$

for $n = 2, 3, \ldots$ and $t \geq \epsilon$. Using (2.22), (3.2) and Lemma 4.2 by analogy with the proof of (5.32) we show that

$$\|U(\tau)^{n-1}(I - g(\tau B))h\| \leq 2S_1 \frac{1}{1 - a^{1/\alpha}} \frac{1}{\epsilon^{1-\alpha}} \frac{1}{n^{2\alpha-1}}\|h\| \tag{5.51}$$

for $n = 2, 3, \ldots$ and $t \geq \epsilon$. Then by (5.18) - (5.21), (5.44) and (5.49) - (5.51) we obtain

$$\|(T(\tau)^n - U(\tau)^n)h\| \leq C'_\epsilon \frac{1}{n^{2\alpha-1}}\|h\| \tag{5.52}$$

for $n = 4, 5, \ldots$ and $t \geq \epsilon$ where

$$C'_\epsilon := 2^{2\alpha-1}C_\epsilon + 2\frac{1}{1 - a^{1/\alpha}} \frac{1}{\epsilon^{1-\alpha}}\{2C_1 + S_1\} \tag{5.53}$$

which proves (5.43). □

Theorem 5.5 *Let A and B be self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26) and if in addition the condition (1.31) takes place, then there is a constant $D > 0$ such that*

$$\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tH} \right\| \leq D \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.54}$$

uniformly in $t \geq 0$.

Proof: From (5.10), Corollary 4.10 and Corollary 4.13 one gets the estimate

$$\begin{aligned} \|(T(\tau)^n - U(\tau)^n)h\| \leq M_1(2 + C_0^\alpha L_4) \frac{1}{(n-1+\alpha)^\alpha} \|h\| + \\ M_1(L_6 + C_0^\alpha L_9) \sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \|(\tau H)^\alpha U(\tau)^m h\| \end{aligned} \tag{5.55}$$

for $n = 2, 3, \ldots$. By Lemma 4.2 we have

$$\begin{aligned} &\|(T(\tau)^n - U(\tau)^n)h\| \leq \\ &M_1(2 + C_0^\alpha L_4) \frac{1}{(n-1+\alpha)^\alpha} \|h\| + M_1(L_6 + C_0^\alpha L_9) \sum_{m=1}^{n-1} \frac{1}{(n-1-m+\alpha)^\alpha} \frac{1}{m^\alpha} \|h\|. \end{aligned} \tag{5.56}$$

Using Lemma 3.1 for $\kappa = \alpha$ we obtain

$$\begin{aligned} &\|(T(\tau)^n - U(\tau)^n)h\| \leq \\ &M_1(2 + C_0^\alpha L_4) \frac{1}{(n-1+\alpha)^\alpha} \|h\| + 8M_1(L_6 + C_0^\alpha L_9) \frac{1}{1-\alpha} \frac{1}{n^{2\alpha-1}} \|h\|. \end{aligned} \tag{5.57}$$

Since $(n - 1 + \alpha)^{-\alpha} \leq 2^\alpha n^{-(2\alpha-1)}$, $n = 1, 2, \ldots$, one gets

$$\|(T(\tau)^n - U(\tau)^n)h\| \leq D \frac{1}{n^{2\alpha-1}} \|h\| \tag{5.58}$$

for $n = 2, 3, \ldots$ where

$$D := 2^\alpha M_1(2 + C_0^\alpha L_4) + 8M_1(L_6 + C_0^\alpha L_9) \frac{1}{1-\alpha} \tag{5.59}$$

which proves (5.54) $\square$

Corollary 5.6 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26) and if in addition the condition (1.31) takes place, then there is a constant $D' > 0$ such that*

$$\left\| (f(tA/n) g(tB/n))^n - e^{-tH} \right\| \leq D' \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.60}$$

uniformly in $t \geq 0$.

Since the proof in essential coincides with considerations above we omit it.

Corollary 5.7 *Let A and B be two self-adjoint operators which obey (1.22) - (1.24). If the Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.4), (1.5), (1.14) - (1.17), (1.25) and (1.26) and if in addition the condition (1.34) is valid, then there are constant $D > 0$ and $D' > 0$ such that estimates (5.54) and (5.60) holds uniformly in $t \geq 0$.*

Proof: By Lemma 2.2 condition (1.31) is satisfied. Applying Theorem 5.5 and Corollary 5.6 one gets the proof. □

Remark 5.8 Suppose that A and B satisfy only (1.4), (1.5), (1.14) - (1.17), (1.22), (1.25) and (1.34), i.e. conditions (1.23), (1.24) and (1.26) are not a priori assumed. Then we introduce operator

$$A_b := A + bI, \qquad b > 0. \tag{5.61}$$

Since $\mathrm{dom}(A_b^\beta) = \mathrm{dom}(A^\beta)$ and

$$\mathrm{dom}(A_b^\alpha) \subseteq \mathrm{dom}(A_b^\beta) \subset \mathrm{dom}(B^\alpha) \tag{5.62}$$

for each $b > 0$ we find that the pair $\{A_b, B\}$ satisfies (1.22), (1.23) and (1.34) for each $b > 0$. Moreover, by

$$\|B^\alpha A_b^{-\alpha} h\| \leq \|B^\alpha A^{-\beta} A^\beta A_b^{-\alpha} h\| \leq b^{-(\alpha-\beta)} \|B^\alpha A^{-\beta}\| \, \|h\| \tag{5.63}$$

one gets that

$$\|B^\alpha f\| \leq b^{-(\alpha-\beta)} \|B^\alpha A^{-\beta}\| \, \|A^\alpha f\|, \qquad f \in \mathrm{dom}(A_b^\beta) = \mathrm{dom}(A^\alpha). \tag{5.64}$$

Setting $a := b^{-(\alpha-\beta)} \|B^\alpha A^{-\beta}\|$ and choosing b sufficiently large we see that conditions (1.24) and (1.26) are satisfied. Applying Corollary 5.7 we find constants $D > 0$ and $D' > 0$ such that

$$\left\| \left(f(tA_b/n)^{1/2} g(tB/n) f(tA_b/n)^{1/2} \right)^n - e^{-tH_b} \right\| \leq D \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.65}$$

or

$$\left\| (f(tA_b/n) g(tB/n))^n - e^{-tH_b} \right\| \leq D' \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.66}$$

uniformly in $t \geq 0$ where $H_b := H + bI$. In particular, if $f(x) = e^{-x}$, then one has

$$e^{-bt} \left\| \left(e^{-tA/2n} g(tB/n) e^{-tA/2n} \right)^n - e^{-tH} \right\| \leq D \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.67}$$

and

$$e^{-bt} \left\| \left(e^{-tA/n} g(tB/n) \right)^n - e^{-tH} \right\| \leq D' \frac{1}{n^{2\alpha-1}}, \qquad n = 2, 3, \ldots, \tag{5.68}$$

uniformly in $t \geq 0$ which yields that the Trotter-Kato product formula for $f(x) = e^{-x}$ converges uniformly on each compact interval of $[0, \infty)$.

6 Examples

In this section let us demonstrate and compare the proven results in comparison with [4], [8] and [10] by means of one dimensional Schrödinger operator. We set $\mathcal{H} = L^2(\mathbf{R}^1)$ and

$$(Af)(x) = -\frac{d^2}{dx^2}f(x) + f(x), \qquad f \in \mathrm{dom}(A) = W^{2,2}(\mathbf{R}^1). \tag{6.1}$$

We add the identity to the Laplace operator in order to guarantee that $A \geq I$. By B_γ we denote the multiplication operator

$$\mathrm{dom}(B_\gamma) = \{f \in L^2(\mathbf{R}^1) : \frac{C}{|x-y|^\gamma}f(x) \in L^2(\mathbf{R}^1)\} \tag{6.2}$$

$$(B_\gamma)f(x) = \frac{C}{|x-y|^\gamma}f(x), \qquad f \in \mathrm{dom}(B), \tag{6.3}$$

where $0 < \gamma < 1$ and $y \in \mathbf{R}^1$. By means of Fourier transformation and Cauchy-Schwartz inequality one gets that

$$|f(x)|^2 \leq \frac{1}{2}\rho(\beta)\|A^\beta f\|^2, \qquad f \in \mathrm{dom}(A^\beta), \quad \beta > \frac{1}{4}, \quad x \in \mathbf{R}^1, \tag{6.4}$$

where

$$\rho(\beta) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{(1+t^2)^{2\beta}}dt. \tag{6.5}$$

This yields the estimate

$$\int_{|x-y|\leq 1} \frac{1}{|x-y|^{2\kappa}}|f(x)|^2 dx \leq \frac{1}{1-2\kappa}\rho(\beta)\|A^\beta f\|^2, \qquad f \in \mathrm{dom}(A^\beta), \tag{6.6}$$

for $\beta > \frac{1}{4}$, $y \in \mathbf{R}^1$ and $0 < \kappa < \frac{1}{2}$. Since $A \geq I$ we obtain

$$\int_{|x-y|\geq 1} \frac{1}{|x-y|^{2\kappa}}|f(x)|^2 dx \leq \int_{|x-y|\geq 1} |f(x)|^2 dx \leq \|f\|^2 \leq \|A^\beta f\|^2, \qquad f \in \mathrm{dom}(A^\beta). \tag{6.7}$$

Let $0 < \gamma < \frac{1}{2}$. Setting $\kappa = \gamma$ and $\beta = 1$ from (6.4) - (6.7) we get

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B_\gamma) \tag{6.8}$$

and

$$\|B_\gamma f\| \leq a\|Af\|, \qquad f \in \mathrm{dom}(A), \tag{6.9}$$

where

$$a := C\sqrt{1 + \frac{\rho(1)}{1-2\gamma}}. \tag{6.10}$$

Therefore, if C is sufficiently small, then conditions (1.11) - (1.13) are satisfied. Hence, from [8] it follows that the Trotter-Kato formula converges in the operator norm as $O(\log(n)/n)$ for $n \longrightarrow \infty$.

However, if $\frac{1}{2} \leq \gamma < 1$ the condition (6.8) is not valid. To see this let $f \in \mathrm{dom}(A) = W^{2,2}(\mathbf{R}^1)$ such that

$$\inf_{|x-y|\leq 1} |f(x)| \geq \epsilon > 0. \tag{6.11}$$

Of course, such elements always exist. Then obviously

$$\int_{-\infty}^{+\infty} \frac{1}{|x-y|^{2\gamma}} |f(x)|^2 dx = +\infty. \tag{6.12}$$

Therefore,

$$\mathrm{dom}(A) \not\subseteq \mathrm{dom}(B_\gamma). \tag{6.13}$$

Moreover, if $\alpha\gamma \geq \frac{1}{2}$ and $\beta > \frac{1}{4}$ one always has that

$$\mathrm{dom}(A^\beta) \not\subseteq \mathrm{dom}(B_\gamma^\alpha). \tag{6.14}$$

However, if $0 < \alpha\gamma < \frac{1}{2}$ and $\beta > \frac{1}{4}$ from (6.4) - (6.7) we obtain that

$$\mathrm{dom}(A^\beta) \subseteq \mathrm{dom}(B_\gamma^\alpha) \tag{6.15}$$

and

$$\|B_\gamma^\alpha f\| \leq a \|A^\beta f\|, \qquad f \in \mathrm{dom}(A^\beta), \tag{6.16}$$

where

$$a := C^\alpha \sqrt{1 + \frac{\rho(\beta)}{1 - 2\alpha\gamma}} \tag{6.17}$$

Therefore, if $\frac{1}{2} \leq \gamma < 1$ we can find α, $\frac{1}{2} < \alpha < 1$, such that $0 < \alpha\gamma < \frac{1}{2}$. If C is sufficiently small and $\alpha = \beta > \frac{1}{4}$ we satisfy conditions (1.22) - (1.24). Moreover, choosing β from $(\frac{1}{4}, \alpha)$ we see that condition (1.34) is valid. Hence, applying Corollary 5.7 we get that the Trotter-Kato product formula converges in the operator norm as $O(1/n^{2\alpha-1})$ where α has to satisfy the condition $2\alpha\gamma < 1$ and H is given by $H = A \dot{+} B_\gamma$.

In fact, the example can be improved. Let $\{x_n\}_{n=1}^\infty$ be set of all rational numbers on the real axis. We set

$$q_\gamma(x) = \sum_{n=1}^\infty \frac{C_n}{|x - x_n|^\gamma}, \qquad 0 < \gamma < 1, \quad x \in \mathbf{R}^1, \tag{6.18}$$

where $\{C_n\}_{n=1}^\infty$ is a sequence of positive numbers such that

$$S := \sum_{n=1}^\infty C_n < +\infty. \tag{6.19}$$

Since

$$\int_{x_0-1}^{x_0+1} \sum_{n=1}^N \frac{C_n}{|x-x_n|^\gamma} dx \leq \frac{2}{1-\gamma} \sum_{n=1}^N C_n, \qquad N = 1, 2, \ldots, \quad x_0 \in \mathbf{R}^1 \tag{6.20}$$

one immediately gets that potential $q_\gamma \in L^1_{loc}(\mathbf{R}^1)$. Furthermore, we note that by Jensen's inequality one has

$$\left(\sum_{n=1}^{N} \frac{C_n}{|x-x_n|^\gamma}\right)^\nu \leq S^{\nu-1} \sum_{n=1}^{N} \frac{C_n}{|x-x_n|^{\gamma\nu}}, \qquad N = 1,2,\ldots, \quad x \in \mathbf{R}^1, \tag{6.21}$$

for $\nu \geq 1$. We set

$$\begin{aligned} \mathrm{dom}(\hat{B}_\gamma) &= \{f \in L^2(\mathbf{R}^1) : q_\gamma(x)f(x) \in L^2(\mathbf{R}^1)\} & (6.22)\\ (\hat{B}_\gamma f)(x) &= q_\gamma(x)f(x), \qquad f \in \mathrm{dom}(B_\gamma). & (6.23) \end{aligned}$$

Let $0 < \gamma < \frac{1}{2}$. Then by (6.4) - (6.7) and (6.21) for $\nu = 2$ one gets

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(\hat{B}_\gamma). \tag{6.24}$$

and

$$\|\hat{B}_\gamma f\| \leq a\|Af\|, \qquad f \in \mathrm{dom}(A). \tag{6.25}$$

where

$$a := S\sqrt{1 + \frac{\rho(1)}{1-2\gamma}}. \tag{6.26}$$

If S is small enough, then we satisfy conditions (1.11) - (1.13). Hence, the Trotter-Kato product formula converges as $O(\log(n)/n)$ as $n \longrightarrow \infty$ [8]. If $\frac{1}{2} \leq \gamma < 1$, then

$$\frac{C_n}{|x-x_n|^\gamma} f(x) \in L^2(\mathbf{R}^1), \qquad f \in \mathrm{dom}(A) = W^{2,2}(\mathbf{R}^1), \tag{6.27}$$

implies $f(x_n) = 0$. Hence $f \in \mathrm{dom}(\hat{B}_\gamma) \cap \mathrm{dom}(A)$ implies that $f(x) \equiv 0$. Therefore,

$$\mathrm{dom}(\hat{B}_\gamma) \cap \mathrm{dom}(A) = \{0\}, \qquad \frac{1}{2} \leq \gamma < 1. \tag{6.28}$$

Furthermore, since

$$\frac{C_n^\alpha}{|x-x_n|^{\alpha\gamma}} \leq q_\gamma(x)^\alpha, \qquad n = 1,2,\ldots, \quad x \in \mathbf{R}^1, \tag{6.29}$$

one gets that

$$\mathrm{dom}(\hat{B}_\gamma^\alpha) \cap \mathrm{dom}(A^\beta) = \{0\}, \qquad \frac{1}{2} \leq \alpha\gamma, \quad \beta > \frac{1}{4}. \tag{6.30}$$

However, if $0 < \alpha\gamma < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$, $\beta > \frac{1}{4}$, then by (6.4) - (6.7) and (6.21) for $\nu = 2\alpha$ we obtain that

$$\mathrm{dom}(A^\beta) \subseteq \mathrm{dom}(\hat{B}_\gamma^\alpha) \tag{6.31}$$

and

$$\|\hat{B}_\gamma^\alpha f\| \leq a\|A^\beta f\|, \qquad f \in \mathrm{dom}(A^\beta), \tag{6.32}$$

where

$$a := S^{\alpha}\sqrt{1 + \frac{\rho(\beta)}{1 - 2\alpha\gamma}}. \tag{6.33}$$

As above, if $\frac{1}{2} \le \gamma < 1$ we find α, $\frac{1}{2} < \alpha < 1$, such that $0 < \alpha\gamma < \frac{1}{2}$. If S is sufficiently small and $\alpha = \beta > \frac{1}{4}$, then conditions (1.22) - (1.24) can be satisfied. Since (6.31) holds for $\beta \in (\frac{1}{4}, \alpha)$, condition (1.34) is valid and by Corollary 5.7 the Trotter-Kato product formula converges in operator norm as $O(1/n^{2\alpha-1})$ for $n \longrightarrow \infty$ and H given by $H = A \dot{+} \hat{B}_{\gamma}$.

Summing up above considerations let us introduce the critical value

$$\alpha_0 = \frac{1}{2\gamma}, \qquad 0 < \gamma < 1. \tag{6.34}$$

We note that $\alpha_0 > \frac{1}{2}$. If $\alpha_0 > 1$, then $\mathrm{dom}(A) \subseteq \mathrm{dom}(B_\gamma)$ and $\mathrm{dom}(A) \subseteq \mathrm{dom}(\hat{B}_\gamma)$. In this case the Trotter-Kato product formula converges as $O(\log(n)/n)$ [8]. If $\frac{1}{2} < \alpha_0 \le 1$, then for each $\alpha \in [\alpha_0, 1]$ we have $\mathrm{dom}(A^\beta) \not\subseteq \mathrm{dom}(B^\alpha_\gamma)$ in the first example and even $\mathrm{dom}(A^\beta) \cap \mathrm{dom}(\hat{B}^\alpha_\gamma) = \{0\}$ in the second example for $\beta > \frac{1}{4}$. Nevertheless, the Trotter-Kato product formula converges for both examples and its convergence rate is $O(1/n^{2\alpha-1})$ where $\frac{1}{2} < \alpha < \alpha_0$.

7 Conclusions

The aim of this paper was to describe the dependence of the operator-norm error bound in the Trotter-Kato product formula on relations between fractional powers of self-adjoint operators A and B involved in this formula in an abstract setting.

1. We show that the error estimate depends on $\frac{1}{2} < \alpha < 1$ which orders $\mathrm{dom}(A^\alpha)$ and $\mathrm{dom}(B^\alpha)$ or $\mathrm{dom}(H^\alpha)$ and $\mathrm{dom}(A^\alpha)$, see Section 1.

2. Examples of Section 6 illustrate this dependence and, in particular, the limit case $\alpha = 1$ considered in [8].

3. We conjecture that for $\alpha = \frac{1}{2}$ the Trotter formula does not converge in the operator norm topology in this abstract setting, while one has the strong convergence [6].

4. Examples coming from the exponential Trotter-Kato formula for Schrödinger operators show [2] that one can get error bounds of order $O(n^{-1})$ even in the case when $\mathrm{dom}(A^\alpha) \not\subseteq \mathrm{dom}(B^\alpha)$ for $\frac{1}{2} < \alpha \le 1$ (!) which means that the present abstract setting is not exhaustive.

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