New representations of the Poincaré group describing two interacting bosons

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Abstract

New representations of the Poincaré group are given, which describe two bosons with interaction in four space-time dimensions. The quantum frame is the Schrödinger picture in momentum space. More precisely we start from the relativistic free model with Hilbert space $L^2(I\!R^3 \times I\!R^3, \sigma_2)$, where σ_2 is the Lorentz invariant measure. We add to the free Hamiltonian and the free Lorentz generators new interaction terms, without changing the Poincaré algebra commutation rules, and such that the algebra representation can be integrated to a unitary representation of the group on $L^2(I\!R^6, \sigma_2)$. The physics of these models can be investigated through the bound state equation (a *relativistic Schrödinger equation*) and through the scattering matrix. Asymptotic completeness is obtained in some cases. Finally we give an example for which a bound state exists and for which the scattering matrix can be written down explicitely. This example assures that an interaction between the particles can effectively occur in these models.

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1. Introduction

A new family of unitary representations of the Poincaré group in four space-time dimensions is given, which describe two boson systems with interaction. The quantum frame is the Schrödinger picture of Quantum Mechanics in momentum space. The physics of these models can be obtained from the scattering operator and the bound state equation, an eigenvalue equation of the kind of the Schrödinger equation. The existence of a bound state and the non triviality of the scattering matrix confirm that the interaction is effective. To our knowledge, this is the first example of relativistic quantum models for a finite number of interacting particles. Moreover they admit a rather simple mathematical construction.

The quantum frame of the Schrödinger picture means that the Hilbert state space is simply made of functions of the momenta. In particular the time variable does not appear explicitly. However, these models satisfy the relativistic principles, because they consist of unitary representations of the Poincaré group. An elementar example of such a model is given by the free model for two bosons ('free' means without interaction). Our models are obtained by perturbations of the Hamiltonian and the Lorentz generators of the free model, in such a way that the commutation rules of the Poincaré algebra remain satisfied. The exisence of solutions to this problem is rather surprising. Indeed, such a perturbation of the Hamiltonian and the Lorentz generators has already been proposed by Dirac in the late 40's [1], in the framework of classical mechanics. Dirac thought that the classical theory should be established first, and that the quantum version would be obtained afterwards by applying the canonical quantization. Therefore his paper concerns only classical physics, and it does not really obtain an interesting conclusion. In the early 60's, Currie continued this approach, still in the classical frame. He obtained in [2] that, under additional natural assumptions, such modifications cannot generate an interaction (this is known as the 'non-interaction theorem'). The present paper shows that by going directly to the quantum frame, this difficulty can be overcome.

The result given here is the generalization of similar results in 2-d (two space-time dimensions) [3] and in 3-d [4]. Although the family of models in [3] is very general (the kernel of the interaction part of the Hamiltonian can be chosen almost arbitrarily in the centre-of-mass frame), in 3-d and 4-d we have only managed to construct models with special interaction terms (unfortunately, local interactions by a potential are not included). Moreover, the generalization to more than two particles [5] and to bosons with different masses [6], which is easy in 2-d, seems to be very difficult in higher space-time dimensions.

Originally, the result in 2-d was suggested by an new approch of the bound state problem in 2-d Quantum Field Theory (QFT). Let us briefly describe this new approach.

The mathematical construction of bosonic 2-d QFT models (the so-called 'weakly coupled $P(\varphi)_2$ models') by Glimm, Jaffe, Spencer [7], has made possible a rigorous approach of the relativistic bound state problem. In this framework the Bethe-Salpeter method has been rigorously established and extensively studied [8]. This method provides a clear and precise theory for relativistic quantum particles with interaction at low energy, which is satisfactory both from the physical and the mathematical point of view. However, there are some drawbacks. First, the numerical calculation of the bound state mass from the Bethe-Salpeter equation is extremely difficult. Second, the connection to the corresponding problem in Quantum Mechanics (QM) is not completely made: only the non-relativistic limit is established, but not its relativistic corrections. In particular, it is not known how to get the relativistic corrections to the Schrödinger equation. This problem could be related to the famous 'inverse problem' in QM, because the Bethe-Salpeter method consists in the search of the bound states by analysing the scattering amplitudes, while the corrections to the Schrödinger equation refers to a 'potential approach', which has never been obtained clearly. Finally, the mathematical rigour of this method depends on the mathematical construction of the QFT model in question, which is missing at four dimensional space-time and for gauge interactions.

Before trying to study the last mentioned difficulties, it should be natural to improve first the bound state description of the $P(\varphi)_2$ models, in order to develop intuition and new ideas. To treat the bound state problem from an new point of vue, one can try to construct the eigenspace of a bound state. This leads indeed to an original approach, because the standard strategy of QFT is precisely to avoid working with states of interacting particles (because the notion of 'interacting particle' itself is not defined, and therefore only the vacuum state and the asymptotic states are used in QFT). Now, in a relativistic theory, the mass of a bound state is an eigenvalue of the mass operator M, and thus its states are eigenvectors of M, which is an unambiguous notion. A new method can be developed, adapted to QFT models in which the existence of bound states is already known (from the Bethe-Salpeter method). This has been done by combining the variational and the perturbation methods ([9], [10]). Let us explain in a few words how this works. The aim is to minimize the Rayleigh quotient $R(\psi) = (\psi, M\psi)\ \|\psi\|^{-2}$ over the largest set of vectors $\psi \in D(M)$, $\psi \neq 0$, orthogonal to the vacuum state and to the one-particle states. $R(\psi)$ is calculated by perturbation with respect to the coupling constant. More precisely, it is calculated explicitely at first orders, and the remainder is estimated, thanks to the mathematical construction of these models. In doing this, it appears that it is sufficient to restrict ourselves to a subspace made of vectors called 'zero-time vectors', which do not involve time variables. Then the minimization of the explicitly calculated terms of $R(\psi)$ is performed, in varying ψ in a subset of zero-time vectors for which the remainder is uniformly bounded. The vectors ψ_0 exhibited by the minimization can be considered as perturbation of eigenvectors. This gives a method to construct approximations of the eigenspace, once the existence of a bound state is known.

The interest of this methods is that it brings several surprises. First, it leads to an eigenvalue equation for the bound state mass, which is similar to the Schrödinger equation (made of a 'kinetic' plus an 'interaction' terms) and which can be considered as an approximation of a 'relativistic Schrödinger equation' ([9], [11]). Second, a subspace is found which is stable under the Poincaré transformation (at first perturbation orders) and nevertheless carries interaction (sect. 3 of [7], [11]). Such representations of the Poincaré group can be studied for themselves, without reference to QFT. The third surprise is that such representations can be constructed non-perturbatively [3]. They provide a set of models for two particles with effective interaction in the Schrödinger picture (the time variable is not introduced, due to the restriction to zero-time vectors). Moreover, the complete set of such representations can be given, in the weak coupling regime. They are all constructed from an interaction kernel which, in the centre-of-mass frame, can be chosen arbitrarily (apart from some analytic conditions) [3].

The last surprise is that generalizations are possible, to $N > 2$ particle systems [5] and to higher dimensional space-time. The case of two particles in three space-time dimensions has been obtained for special interactions (which involve the projection on the s-waves) [4]. With such an interaction, the direct connection with QFT fails. The present paper concerns the generalization to four space-time dimensions, for two particle systems. The difficulty in passing from 3-d to 4-d comes from the highter dimension of the Poincaré group (passing from 7 to 10 dimensions) and from the non-integrability of some kernels. This has the consequence that, in order to follow the same general exposition as for the 3-d case [4], all the proofs have had to be completely rewritten. However, provided the interaction satisfies again the special property given above, unitary representations of the Poincaré group can be constructed. Moreover, the study of the physical content of these models (bound state equation, scattering matrix) confirms that they describe two particles systems with interaction in four space-time dimensions. Note that these are probably the first relativistic quantum models for two interacting particles in four space-time dimensions, and that they admit a rigorous mathematical construction which is remarkably simple.

The paper is organized as follows. Section 2 presents the models, given by a family of operators which (formally) satisfy the commutation rules of the Poincaré algebra, provided some conditions on the 'interaction kernel' h (the kernel of the interaction term of the Hamiltonian) hold. In particular, if h satisfies the special property given above, then these conditions reduce to a single 'fundamental equation' for h. In section 3 the complete set of solutions to this equation is given, in the weak coupling regime. Then we focus on a particular subset of solutions which satisfy all the analytic properties we need for the following proofs. Section 4 shows that the formal representations of the Poincaré algebra, we have just obtained, are mathematically well defined and can be integrated to give unitary representations of the Poincaré group. Section 5 gives the eigenvalue equation for the bound states (the 'relativistic Schrödinger equation') and section 6 establishes the existence of two-particle asymptotic states and of the scattering operator. Asymptotic completness is obtained for special h. Finally, section 7 gives a simple example (one-dimensional perturbation) for which the bound state equation admits a solution and for which the scattering matrix can be written explicitly. Its non triviality can easily be seen, assuring the non triviality of this set of models.

2. The models

The Lie algebra $\mathcal{G}$ of the Poincaré group $\mathcal{P}_+^\uparrow$ in four space-time dimensions is generated by the ten operators H (Hamiltonian), $\vec{P} = (P_1, P_2, P_3)$ (momentum), $\vec{J} = (J_1, J_2, J_3)$ (angular momentum) and $\vec{L} = (L_1, L_2, L_3)$ (generators of the Lorentz transformations), satisfying the 45 commutation rules

$$[P_j, P_k] = 0 \qquad [P_j, J_k] = i\varepsilon_{j,k,\ell} P_\ell \qquad [J_j, J_k] = i\varepsilon_{j,k,\ell} J_\ell \tag{1}$$

$$[P_j, H] = 0 \qquad [H, J_j] = 0 \tag{2}$$

$$[P_j, L_k] = i\delta_{j,k} H \qquad [J_j, L_k] = i\varepsilon_{j,k,\ell} L_\ell \tag{3}$$

$$[H, L_j] = iP_j \qquad\qquad [L_j, L_k] = -i\varepsilon_{j,k,\ell} J_\ell \tag{4}$$

for all $1 \leq j, k \leq 3$ where $\delta_{i,j}$ is the Kronecker tensor and $\varepsilon_{j,k,\ell} \in \{0, 1\}$ is the totally antisymetric tensor with $\varepsilon_{1,2,3} = 1$ (we have used the Einstein summation convention). This algebra admits two Casimir operators, the mass operator M, the square of which is given by

$$M^2 \ := \ H^2 \ - \vec{P}^2 \tag{5}$$

and the Pauli-Lubanski operator

$$W^2 \ := \ \left(\vec{P} \cdot \vec{J}\,\right)^2 \ - \ \left(-H\vec{J} + \vec{P} \wedge \vec{L}\,\right)^2 \tag{6}$$

where the dot "$\cdot$" and the wedge "$\wedge$" denote the ordinary scalar and vector products respectively.

We start from the representation describing one spinless particle of mass $m > 0$, in the Schrödinger picture, given by the following choice of operators

$$\begin{aligned} \vec{P}\phi(\vec{p}) &= \vec{p}\,\phi(\vec{p}) \\ H\phi(\vec{p}) &= \omega(\vec{p})\phi(\vec{p}) \\ \vec{J}\phi(\vec{p}) &= -i\,\vec{p} \wedge \vec{\nabla}\phi(\vec{p}) \\ \vec{L}\phi(\vec{p}) &= -i\omega(\vec{p})\,\vec{\nabla}\phi(\vec{p}) \end{aligned} \tag{7}$$

for almost all $\vec{p} = (p_1, p_2, p_3) \in I\!R^3$ and suitable three-variable functions ϕ, where we have put $\omega(\vec{p}) := \sqrt{\vec{p}^{\,2} + m^2}$. We have used the notation $\vec{\nabla} := (\partial_{p_1}, \partial_{p_2}, \partial_{p_3})$. These operators define a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary, continuous and irreducible representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space $L^2(I\!R^3, \sigma)$, where σ is the measure $d\sigma(\vec{p}) := d\vec{p}\,[2\omega(\vec{p})]^{-1}$ (see appendix A). The Casimir operators of the representation (7) are simply $M = m$ (times the identity operator) and $W = 0$.

The representation describing two identical bosons of mass $m > 0$ without interaction, in the Schrödinger picture, is given by the symmetrical tensor product of two copies of the representation (7) (the symmetry refers to the case where the particles are identical, but this restriction is not essential and could be removed). The action of the operators becomes

$$\begin{aligned} \vec{P}\,\phi(\vec{p}_1, \vec{p}_2) &= (\vec{p}_1 + \vec{p}_2)\ \phi(\vec{p}_1, \vec{p}_2) \\ H_0\,\phi(\vec{p}_1, \vec{p}_2) &= (\omega(\vec{p}_1) + \omega(\vec{p}_2))\ \phi(\vec{p}_1, \vec{p}_2) \\ \vec{J}\,\phi(\vec{p}_1, \vec{p}_2) &= -i\left(\vec{p}_1 \wedge \vec{\nabla}_1 + \vec{p}_2 \wedge \vec{\nabla}_2\right)\ \phi(\vec{p}_1, \vec{p}_2) \\ \vec{L}_0\,\phi(\vec{p}_1, \vec{p}_2) &= -i\left(\omega(\vec{p}_1)\vec{\nabla}_1 + \omega(\vec{p}_2)\vec{\nabla}_2\right)\ \phi(\vec{p}_1, \vec{p}_2) \end{aligned} \tag{8}$$

for almost all $(\vec{p}_1, \vec{p}_2) \in I\!R^6$ and suitable six variable functions ϕ, where $\vec{\nabla}_i := (\partial_{(\vec{p}_i)_1}, \partial_{(\vec{p}_i)_2}, \partial_{(\vec{p}_i)_3})$, $i \in \{1, 2\}$. Because of the properties of the tensor product, (8) defines a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary and continuous representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space

$$\mathcal{H} \ := \ L^2_{Sym}(I\!R^6, \sigma_2), \qquad d\sigma_2(\vec{p}_1, \vec{p}_2) := \frac{d\vec{p}_1}{2\omega(\vec{p}_1)} \, \frac{d\vec{p}_2}{2\omega(\vec{p}_2)} \tag{9}$$

made of symmetrical functions (i.e. $\phi(\vec{p_1}, \vec{p_2}) = \phi(\vec{p_2}, \vec{p_1})$). Note that $\sigma_2 = \sigma \otimes \sigma$ and recall that $\omega(\vec{p_i}) = \sqrt{\vec{p_i}^{\,2} + m^2}$ for each $i \in \{1, 2\}$. $\mathcal{H}$ describes two particles without interaction (therefore we have put an index 0 to H_0 and $\vec{L}_0$).

The Casimir operator of the representation (8) are given by

$$\begin{aligned} M_0^2 \ \phi(\vec{p_1}, \vec{p_2}) \ &= \ \left[(\omega(\vec{p_1}) + \omega(\vec{p_2}))^2 \ - \ (\vec{p}_1 + \vec{p}_2)^2 \right] \ \phi(\vec{p_1}, \vec{p_2}) & (10) \\ W_0^2 \ \phi(\vec{p_1}, \vec{p_2}) \ &= \ - \Big[\Big((\vec{p}_1 \wedge \vec{p}_2) \cdot (\vec{\nabla}_1 - \vec{\nabla}_2) \Big)^2 \\ &\qquad - \Big(\omega(\vec{p_1}) \vec{p}_2 \wedge \vec{\nabla}_1 + \omega(\vec{p_2}) \vec{p}_1 \wedge \vec{\nabla}_2 \Big)^2 \Big] \ \phi(\vec{p_1}, \vec{p_2}). & (11) \end{aligned}$$

Now we want to modify these operators in order to introduce the interaction. To remain in the Schrödinger picture we keep $\vec{P}$ and $\vec{J}$ unchanged and modify only H_0 and $\vec{L}_0$. We introduce a symmetric operator $\mathcal{O}$, the *interaction operator*, on which we impose only, for the moment, the formal commutation properties

$$[\mathcal{O}, \vec{P}] = 0 \qquad\qquad [\mathcal{O}, \vec{J}] = 0. \tag{12}$$

The interaction representation is defined as follows

$$\begin{array}{ccl} & \vec{P}, \ \vec{J} & \quad as \ \ in \ \ (8) \\ H & := & H_0 \ + \ \{\mathcal{O}, H_0\} \\ \vec{L} & := & \vec{L}_0 \ + \ \{\mathcal{O}, \vec{L}_0\} \end{array} \tag{13}$$

where we have used the notation $\{A, B\} = AB + BA$. The particular form of the 'interaction parts' of (13) together with (12) have the following nice consequence.

Lemma 1 *The operators* (13) *satisfy formally the commutation rules* (1), (2) *and* (3) *of the algebra* $\mathcal{G}$.

Proof. (1) has not been modified; (2) is an immediate consequence of (12); (3) follows from (12), from the fact that the free algebra $(\vec{P}, \vec{J}, H_0, \vec{L}_0)$ satisfies also (3), and from some algebraic calculations (see [3] or [5] for more details). []

Note the particular role of (3) : it gives the new operators H, L_1, L_2 after the change of L_3 only. Until now no condition on $\mathcal{O}$, except (12), were needed. But the last commutation rules (4) impose the following equations

$$0 \ = \ [\{\mathcal{O}, H_0\}, L_{0,j}] \ + \ [H_0, \{\mathcal{O}, L_{0,j}\}] \ + \ [\{\mathcal{O}, H_0\}, \{\mathcal{O}, L_{0,j}\}] \tag{14}$$

$$0 \ = \ [\{\mathcal{O}, L_{0,j}\}, L_{0,k}] \ + \ [L_{0,j}, \{\mathcal{O}, L_{0,k}\}] \ + \ [\{\mathcal{O}, L_{0,j}\}, \{\mathcal{O}, L_{0,k}\}] \tag{15}$$

for all $j \neq k \in \{1,2,3\}$. We get six equations for $\mathcal{O}$ of the same form (made of a linear term plus a bilinear term), which must hold simultaneously. We have now to find solutions to this system.

In order to have a better physical description we make the change of variables $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ given by

$$
\begin{aligned}
\vec{P} &= \vec{p}_1 + \vec{p}_2 \\
\vec{Q} &:= \frac{1}{2}(\vec{p}_1 - \vec{p}_2) - \frac{1}{2}\,\vec{P}\,\frac{\omega(\vec{p}_1) - \omega(\vec{p}_2)}{M_0(\vec{p}_1,\vec{p}_2) + \omega(\vec{p}_1) + \omega(\vec{p}_2)}
\end{aligned}
\tag{16}
$$

where $M_0(\vec{p}_1,\vec{p}_2) = \left[(\omega(\vec{p}_1)+\omega(\vec{p}_2))^2 - \vec{P}^2\right]^{1/2}$ is the function appearing in (10). The variable $\vec{P}$ is the total momentum while $\vec{Q}$ is the relative momentum, defined as the momentum in the centre-of-mass frame (more precisely, $\vec{Q}$ is a Lorentz transformation of $\frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ leading to the centre-of-mass frame, see appendix B for more explanations).

Under this change of variables the free representation (8) becomes

$$
\begin{aligned}
\vec{P}\,\phi(\vec{P},\vec{Q}) &= \vec{P}\,\phi(\vec{P},\vec{Q}) \qquad \text{(multiplication by the variable } \vec{P}\text{)} \\
H_0\,\phi(\vec{P},\vec{Q}) &= \Omega(\vec{P},\vec{Q})\,\phi(\vec{P},\vec{Q}) \\
\vec{J}\,\phi(\vec{P},\vec{Q}) &= \left(\vec{J}^{\vec{P}} + \vec{J}^{\vec{Q}}\right)\,\phi(\vec{P},\vec{Q}) \\
\vec{L}_0\,\phi(\vec{P},\vec{Q}) &= \left(-i\Omega(\vec{P},\vec{Q})\vec{\nabla}_{\vec{P}} - \frac{\vec{P}\wedge\vec{J}^{\vec{Q}}}{2\omega(\vec{Q}) + \Omega(\vec{P},\vec{Q})}\right)\,\phi(\vec{P},\vec{Q})
\end{aligned}
\tag{17}
$$

for almost all $(\vec{P},\vec{Q}) \in I\!\!R^6$ and suitable functions ϕ, where we have put

$$
\begin{aligned}
\Omega(\vec{P},\vec{Q}) &:= \sqrt{\vec{P}^2 + 4\omega(\vec{Q})^2} = \sqrt{\vec{P}^2 + 4\vec{Q}^2 + 4m^2} \\
\vec{J}^{\vec{P}} &:= -i\vec{P}\wedge\vec{\nabla}_{\vec{P}} \\
\vec{J}^{\vec{Q}} &:= -i\vec{Q}\wedge\vec{\nabla}_{\vec{Q}}
\end{aligned}
\tag{18}
$$

(see the calculations in appendix B). As for (8), these operators define a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary and continuous representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space (9), which in these variables becomes

$$
\mathcal{H} = L^2_{ev.\vec{Q}}(I\!\!R^3 \times I\!\!R^3, \mu), \qquad d\mu(\vec{P},\vec{Q}) := \frac{d\vec{Q}\,d\vec{P}}{2\omega(\vec{Q})\Omega(\vec{P},\vec{Q})} = \frac{d\sigma(\vec{Q})\,d\vec{P}}{\Omega(\vec{P},\vec{Q})}
\tag{19}
$$

made of even functions w.r.t. $\vec{Q}$ (i.e. $\phi(\vec{P},\vec{Q}) = \phi(\vec{P},-\vec{Q})$), see appendix B.

Let us discuss the representation (17). In the non-relativistic case, it is possible to find relative variables which are invariant under the proper Galilean transformations. In the relativistic case we have not managed to find such relative variables, and the Lorentz generators contain a term involving $\vec{J}^{\vec{Q}}$, i.e. acting on $\vec{Q}$. Note that only the angular variables of $\vec{Q}$ are concerned, because $\vec{Q}^2 =$

$\frac{1}{4}\left[(\vec{p}_1-\vec{p}_2)^2-(\omega(\vec{p}_1)\ -\ \omega(\vec{p}_2))^2\right]$ is an invariant quantity. This can also be seen from the Casimir operators, which are now only functions of $\vec{Q}^2$ and $(\vec{J}^{\vec{Q}})^2$:

$$
\begin{aligned}
M_0^2\ \phi(\vec{P},\vec{Q}) &= 4\omega(\vec{Q})^2\ \phi(\vec{P},\vec{Q}) && (20)\\
W_0^2\ \phi(\vec{P},\vec{Q}) &= 4\omega(\vec{Q})^2\ (\vec{J}^{\vec{Q}})^2\ \phi(\vec{P},\vec{Q}). && (21)
\end{aligned}
$$

To find solutions of (14), (15) we must write $\mathcal{O}$ more explicitly. The most general form compatible with (12) is

$$
\begin{aligned}
\mathcal{O}\ \phi(\vec{P},\vec{Q}') &:= \int_{I\!R^6} d\mu(\vec{P}',\vec{Q}')\ \ \phi(\vec{P}',\vec{Q}')\ \frac{\eta(\vec{P},\vec{Q},\vec{P}',\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P}',\vec{Q}')}\ \delta(\vec{P}-\vec{P}')\\
&= \int_{I\!R^3} \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\ \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')} && (22)
\end{aligned}
$$

where $h(\vec{P},\vec{Q},\vec{Q}'):=\eta(\vec{P},\vec{Q},\vec{P},\vec{Q}')$ is an arbitrary kernel (for the moment) which is even w.r.t $\vec{Q}$ and $\vec{Q}'$, that is $h(\vec{P},\vec{Q},\vec{Q}')=h(\vec{P},-\vec{Q},\vec{Q}')=h(\vec{P},\vec{Q},-\vec{Q}')$, and which satisfies the symmetry condition

$$
h(\vec{P},\vec{Q},\vec{Q}')^* \ =\ h(\vec{P},\vec{Q}',\vec{Q}) \qquad (23)
$$

where the star * means complex conjugation, all these conditions being necessary for $\mathcal{O}$ to be a symmetric operator on $\mathcal{H}$. The Dirac δ pseudo-function assures that $\mathcal{O}$ (formally) commutes with $\vec{P}$ and the last condition of (12) leads (formally) to

$$
\left(\vec{J}^{\vec{P}}+\vec{J}^{\vec{Q}}+\vec{J}^{\vec{Q}'}\right)\ h(\vec{P},\vec{Q},\vec{Q}')\ =\ 0. \qquad (24)
$$

The denominateur in (22) is necessary for h to be the kernel of the interaction part of H. In fact, the Hamiltonian (13) can be written

$$
H\ \phi(\vec{P},\vec{Q}')\ =\ \Omega(\vec{P},\vec{Q})\ \phi(\vec{P},\vec{Q})\ +\ \int_{I\!R^3}\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\ h(\vec{P},\vec{Q},\vec{Q}') \qquad (25)
$$

(note that h has no physical dimension) while $\vec{L}$ is given by

$$
\begin{aligned}
\vec{L}\ \phi(\vec{P},\vec{Q}') &= \vec{L}_0\phi(\vec{P},\vec{Q})\ +\ \int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \left[\vec{L}_0\phi(\vec{P},\vec{Q}')\right]\ \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} && (26)\\
&\quad -\ i\vec{P}\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\left[\frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|\Omega(\vec{P},\vec{Q}')}\ -\ \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')^2}\right]\\
&\quad +\ \int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\\
&\qquad \times\ \frac{\vec{P}\wedge\left[\dfrac{\vec{J}^{\vec{Q}}}{2\omega(\vec{Q})+\Omega(\vec{P},\vec{Q})}+\dfrac{\vec{J}^{\vec{Q}'}}{2\omega(\vec{Q}')+\Omega(\vec{P},\vec{Q}')}\right]}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}\ h(\vec{P},\vec{Q},\vec{Q}').
\end{aligned}
$$

The substitution of (22) in (14) and (15) leads to six non-linear, integro-differential equations for h. It seems difficult to find a general solution $h \neq 0$ unless we restrict ourselves to solutions satisfying the folowing condition

$$\vec{J}^{\vec{Q}}\, h(\vec{P},\vec{Q},\vec{Q}') \;=\; 0 \tag{27}$$

which leads to $\vec{J}^{\vec{P}} h = \vec{J}^{\vec{Q}'} h = 0$ because of (21) and (22). Thus (27) is equivalent to the requirement that $h(\vec{P},\vec{Q},\vec{Q}')$ is function of the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only. With this condition, the six equations (14) and (15) coincide. Let us introduce the differential operator D given by

$$Dh(\vec{P},\vec{Q},\vec{Q}') \;:=\; \frac{\Omega(\vec{P},\vec{Q})\;\Omega(\vec{P},\vec{Q}')}{\Omega(\vec{P},\vec{Q})\;+\;\Omega(\vec{P},\vec{Q}')}\,\partial_{\|\vec{P}\|}\, h(\vec{P},\vec{Q},\vec{Q}'). \tag{28}$$

In appendix C we show that (14), (15) and (27) lead to the single equation

$$\begin{aligned} 0 \;=\;& 2Dh(\vec{P},\vec{Q},\vec{Q}') \;+\; \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\left\{ -\frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}'')}\, h(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}') \right. \\ +\;& \left. Dh(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}') \;+\; h(\vec{P},\vec{Q},\vec{Q}'')Dh(\vec{P},\vec{Q}'',\vec{Q}') \right\}. \end{aligned} \tag{29}$$

This is the fundamental equation which guarantees the relativistic structure of the theory. Let us sum up what we have found in a proposition.

Proposition 2 *Let $h(\vec{P},\vec{Q},\vec{Q}')$ be a function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only which satisfies* (23) *and* (29). *Then the operators* (13), *with $\mathcal{O}$ given by* (22), *are symmetric and satisfy formally the commutation rules* (1) *to* (4) *of the algebra $\mathcal{G}$.*

3. Existence of solutions of the fundamental equation

In a first part we give the complete set of solutions of (29) lying in a ball of a Banach space, neglecting the conditions (23) and (27). In a second part we introduce a particular subset of solutions, which satisfy also (23) and (27), and we establish some of their analytic properties.

Definition 1 Let $\mathcal{K}$ be a compact set of $\mathbb{R}^3$ with Lebesgue-measure $|\mathcal{K}| \neq 0$. We denote by $\mathcal{B}$ the Banach space of continuous and bounded functions $h(\vec{P},\vec{Q},\vec{Q}')$ with support in $(\vec{P},\vec{Q},\vec{Q}') \in \mathbb{R}^3 \times \mathcal{K} \times \mathcal{K}$ for which $Dh(\vec{P},\vec{Q},\vec{Q}')$ exists and is also bounded and continuous, given the norm

$$|h|_{\mathcal{B}} \;:=\; \|h\|_\infty + \|Dh\|_\infty$$

where the differential operator D is given by (28).

The following result assures the existence of a large class of solutions of (29).

Proposition 3 *There exists* $K_1 \in (0,\infty)$ *such that, for all* $c \in C_0^0(\mathcal{K}\times\mathcal{K})$ *satisfying* $\|c\|_\infty < K_1$*, there exists one and only one* $h \in \mathcal{B}$ *satisfying*

1) $|h|_{\mathcal{B}} < 2K_1$
2) $h(0,\vec{Q},\vec{Q}') = c(\vec{Q},\vec{Q}')$ *for all* $\vec{Q},\vec{Q}' \in I\!\!R^3$
3) $h(\vec{P},\vec{Q},\vec{Q}')$ *satisfies* (29) *for all* $\vec{P},\vec{Q},\vec{Q}' \in I\!\!R^3$.

The proposition gives the complete set of solutions h of (29) lying in a certain ball of $\mathcal{B}$; they are uniquely defined by their restriction to $\vec{P}=0$, which can be chosen arbitrarily in a certain ball of $C_0^0(\mathcal{K}\times\mathcal{K})$. Thus, apart from this analytic condition, the restriction of h for $\vec{P}=0$ (in the centre-of-mass frame), is arbitrary.

Proof. Let us introduce a bilinear operator b on suitable functions f, g

$$\begin{aligned} b(g,h)(\vec{P},\vec{Q},\vec{Q}') \;&:= \\ -\;\frac{1}{2}\int_0^{\|\vec{P}\|} d\|\vec{\zeta}\| &\left(\frac{1}{\Omega(\vec{\zeta},\vec{Q})}+\frac{1}{\Omega(\vec{\zeta},\vec{Q}')}\right)\int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{\zeta},\vec{Q}'')^2}\Big\{Dg(\vec{\zeta},\vec{Q},\vec{Q}'')h(\vec{\zeta},\vec{Q}'',\vec{Q}') \\ &+ g(\vec{\zeta},\vec{Q},\vec{Q}'')Dh(\vec{\zeta},\vec{Q}'',\vec{Q}') \;-\; \frac{\|\vec{\zeta}\|}{\Omega(\vec{\zeta},\vec{Q}'')}\, g(\vec{\zeta},\vec{Q},\vec{Q}'')h(\vec{\zeta},\vec{Q}'',\vec{Q}')\Big\}. \end{aligned} \tag{30}$$

for all $(\vec{P},\vec{Q},\vec{Q}') \in I\!\!R^9$. By applying the derivation operator D (given by (28)), with right derivation at 0, we get simply

$$\begin{aligned} Db(g,h)(\vec{P},\vec{Q},\vec{Q}') \;&=\; -\frac{1}{2}\int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\Big\{-\frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}'')}\, f(\vec{P},\vec{Q},\vec{Q}'')g(\vec{P},\vec{Q}'',\vec{Q}') \\ +\quad Dg(\vec{P},\vec{Q},\vec{Q}'')&h(\vec{P},\vec{Q}'',\vec{Q}') \;+\; g(\vec{P},\vec{Q},\vec{Q}'')Dh(\vec{P},\vec{Q}'',\vec{Q}')\Big\}. \end{aligned} \tag{31}$$

The fundamental equation (29) can now be written $Dh - Db(h,h) = 0$. By integration this equation becomes

$$h(\vec{P},\vec{Q},\vec{Q}') \;=\; c(\vec{Q},\vec{Q}') \;+\; b(h,h)(\vec{P},\vec{Q},\vec{Q}') \;=:\; A(h)(\vec{P},\vec{Q},\vec{Q}') \tag{32}$$

where $c(\vec{Q},\vec{Q}')$ is an arbitrary function (the integration 'constant'). This is a fixed point equation which can be solved by applying the Banach fixed point theorem (see sect. 1.1 of [12]) in the Banach space $\mathcal{B}$. Obviously $b:\mathcal{B}\to\mathcal{B}$ and satisfies

$$|b(g,h)|_{\mathcal{B}} \;<\; K_1' \;|g|_{\mathcal{B}}\;|h|_{\mathcal{B}}. \tag{33}$$

where

$$\begin{aligned} K_1' \;&=\; \frac{1}{2}\int_0^\infty \frac{2\,d\zeta}{\sqrt{\zeta^2+4m^2}}\int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{\zeta^2+4\vec{Q}''^2+4m^2} \;+\; \frac{1}{2}\int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{4(\vec{Q}''^2+m^2)} \\ &<\; \frac{|\mathcal{K}|}{4m^3}. \end{aligned}$$

Let us take $K_1 = (4K_1')^{-1}$ and $c(\vec{Q},\vec{Q}') \in \mathcal{B}$ with $|c|_{\mathcal{B}} = \|c\|_\infty < K_1$. Then the operator $A(h)$ sends the ball $B(2K_1) := \{g \in \mathcal{B} \mid |g|_{\mathcal{B}} < 2K_1\}$ into itself and is a strict contraction on $B(2K_1)$ (see [3] or [5] for more details). In such a case, it follows from the Banach fixed point theorem that the equation $h = A(h)$ has one and only one solution in $B(2K_1)$ given by the convergent limit

$$h \;=\; \lim_{n\to\infty} A^n(0). \qquad [] \tag{34}$$

The previous proof constructs the solutions h in terms of their restriction to $\vec{P} = 0$, denoted by c, as the limit of a convergent sequence in $\mathcal{B}$, provided $\|c\|_\infty < m^3/|\mathcal{K}|$ (because the proof shows that we can take $K_1 = m^3/|\mathcal{K}|$). Let us compute the first terms of the sequence (34):

$$\begin{array}{rcl} A(0) & = & c \\ A^2(0) & = & c \,+\, b(A(0),A(0)) \;=\; c \,+\, b(c,c) \\ A^3(0) & = & c \,+\, b(A^2(0),A^2(0)) \\ & = & c \,+\, b(c,c) \,+ b(b(c,c),c) \,+\, b(c,b(c,c)) \,+\, b(b(c,c),b(c,c)) \\ & etc... & \end{array} \tag{35}$$

The limit can be seen as a convergent series in powers of c. More precisely, by replacing c by λc in the definition of A (32), the limit (34) becomes a power series of λ

$$h \;=\; \sum_{n=0}^{\infty} \lambda^n \, h_n \tag{36}$$

converging for $-1 < \lambda \leq 1$, where h_n is the sum of all terms in (34) of degree n in c. In fact, h_n is given by the sum of all different ways of writing $n-1$ bilinear functions b applied on n functions c. For example,

$$\begin{array}{rcl} h_3 & = & b(b(c,c),c) \,+\, b(c,b(c,c)) \\ h_4 & = & b(c,b(c,b(c,c))) \,+\, b(c,b(b(c,c),c)) \,+\, b(b(c,c),b(c,c)) \\ & & \quad + \, b(b(c,b(c,c)),c) \,+\, b(b(b(c,c),c),c) \end{array}$$

and so on.

We have now to extract from this large set of solutions a subset which realizes what we want to construct: unitary representations of the Poincaré group describing two particles with effective interaction. For that purpose we restrict ourselves to the following class of functions.

Definition 2. Let $R > 0$ and let $\mathcal{K}$ be the ball of $I\!R^3$ of centre 0 and radius R. Let $\mathcal{V}$ be the set of functions $c(\vec{Q},\vec{Q}') \in C_0^2(\mathcal{K}\times\mathcal{K})$ which satisfy $c(\vec{Q},\vec{Q}')^* = c(\vec{Q}',\vec{Q})$, which depend only on the norms $\|\vec{Q}\|$ and $\|\vec{Q}'\|$, and such that

$$\|c\|_\infty \;\leq\; \frac{m^3}{2\,|\mathcal{K}|}. \tag{37}$$

Note that $\mathcal{V}$ depends on the two positive parameters m and R.

Let us recall that c is the kernel of the interaction hamiltonian in the centre-of-mass frame, as function of the momenta. The set $\mathcal{V}$ is made of kernels c with compact support and bounded by a constant depending on the support. Moreover the functions $c(\vec{Q},\vec{Q}')$ must depend on $\|\vec{Q}\|$ and $\|\vec{Q}'\|$ only. All these requirements take us away from physical applications.

Each $c \in \mathcal{V}$ satisfies the hypothesis of proposition 3 and moreover satisfies

$$\|c\|_{L^2(I\!\!R^6)} \;\leq\; \frac{m^3}{2} \tag{38}$$

where $L^2(I\!\!R^6)$ refers to the ordinary Lebesgue measure (not the σ_2 measure). The following set of properties hold on $\mathcal{V}$.

Proposition 4 *Let $c \in \mathcal{V}$. The function h deduced from c by* (34) *(or* (35)*) has the following properties:*

1) *h satisfies* (23),

2) *$h(\vec{P},\vec{Q},\vec{Q}')$ depends only on the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$,*

3) *for all $\vec{Q},\vec{Q}' \in I\!\!R^3$, the function $\vec{P} \to h(\vec{P},\vec{Q},\vec{Q}')$ is a function of $\vec{P}^2$ analytic in a $\mathbb{C}$-neighbourhood of $I\!\!R_+$,*

4) *for all $\vec{P}^2$ in some $\mathbb{C}$-neighbourhood of $I\!\!R_+$, the function $\vec{Q},\vec{Q}' \to h(\vec{P},\vec{Q},\vec{Q}')$ has compact support and belongs to $C^2(I\!\!R^6)$,*

5) *the partial derivatives of $h(\vec{P},\vec{Q},\vec{Q}')$, of any order w.r.t. $\vec{P}^2$ and of order ≤ 2 w.r.t. $\vec{Q},\vec{Q}'$, belong to $\mathcal{B}$,*

6) *h and Dh satisfy the estimates*

$$\begin{aligned}
\sup_{\vec{P}\in I\!\!R^3} \int_{I\!\!R^6} d\vec{Q}d\vec{Q}'|h(\vec{P},\vec{Q},\vec{Q}')|^2 \;&\leq\; \left(\frac{2}{3}\right)^2 m^6 \\
\sup_{\vec{P}\in I\!\!R^3} \int_{I\!\!R^6} d\vec{Q}d\vec{Q}'|Dh(\vec{P},\vec{Q},\vec{Q}')|^2 \;&\leq\; \left(\frac{1}{20}\right)^2 m^6.
\end{aligned} \tag{39}$$

Proof. First we establish the convergence of $h = \sum_{n=0}^{\infty} h_n$, where the h_n are introduced by (36), by another way. From the estimate on b given by (33) it follows that $|h_n|_{\mathcal{B}} < \mathcal{N}_n {K'}_1^{n-1} |c|_{\mathcal{B}}^n$, where $\mathcal{N}_n$ is the number of terms of h_n. To get an estimate on $\mathcal{N}_n$ we consider the trivial equation : $x = u + x^2$ with $u \in \mathbb{C}$ and small $|u|$, which is a caricature of the equation (32). The fixed-point theorem leads to a solution $x = \sum_{n=1}^{\infty} \mathcal{N}_n u^n$, convergent for $|u| < 1/4$ and unique in the disc $|x| < 1/2$, where the $\mathcal{N}_n$ are given by $\mathcal{N}_1 = 1$ and $\mathcal{N}_n = \sum_{i=1}^{n-1} \mathcal{N}_i \mathcal{N}_{n-i}$ for all $n > 1$. On the other hand, an elementary calculation gives $x = \frac{1}{2}(1 - \sqrt{1-4u})$ for $|u| \leq 1/4$ as the unique solution near the origin. By expansion of this solution in power series we obtain explicit expressions for $\mathcal{N}_n$, from which follows easily the estimate $\mathcal{N}_n < 4^n$ for all $n \geq 1$. By summing up these estimates we get

$$|h_n|_{\mathcal{B}} \;<\; {K'}_1^{n-1}\, \mathcal{N}_n\, |c|_{\mathcal{B}}^n \;<\; \frac{1}{K'_1}(4K'_1|c|_{\mathcal{B}})^n \tag{40}$$

which assures the convergence of $h = \sum_{n=0}^{\infty} h_n$ provided $4K_1'|c|_{\mathcal{B}} < 1$, in agreement with the proof of proposition 3.

Proof of 1). Because $A(0) = c$ we have $A(0)(\vec{Q}, \vec{Q}') = A(0)(\vec{Q}', \vec{Q})^*$. If $g \in \mathcal{B}$ satisfies $g(\vec{P}, \vec{Q}, \vec{Q}') = g(\vec{P}, \vec{Q}', \vec{Q})^*$, so does $b(g, g)$ and $A(g)$. By an induction argument it follows that $A^n(0)$ satisfies this symmetry for all n, and so does the uniform limit $lim_{n\to\infty} A^n(0)$.

Proof of 2). If $f(\vec{P}, \vec{Q}, \vec{Q}')$, $g(\vec{P}, \vec{Q}, \vec{Q}')$ are only function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$, so are $b(f, g)(\vec{P}, \vec{Q}, \vec{Q}')$ and $Db(f, g)(\vec{P}, \vec{Q}, \vec{Q}')$. Thus, if $c(\vec{Q}, \vec{Q}')$ is only function of $\|\vec{Q}\|$, $\|\vec{Q}'\|$, then $h_n(\vec{P}, \vec{Q}, \vec{Q}')$ is only function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ for all n. This property passes obviously to the uniform limit $h = lim_{N\to\infty} \sum_{n=0}^{N} h_n$.

Proof of 3). Let us fix $(\vec{Q}, \vec{Q}') \in I\!R^6$ and put $x = \|\vec{P}\|$. In h_n, x appears only in kernels $\sqrt{x^2 + 4\omega(\vec{\xi})^2}$ or as limit of integrals $\int_0^x dy$ over a variable y appearing only in kernels $\sqrt{y^2 + 4\omega(\vec{\chi})^2}$ or as limit of integrals $\int_0^y dz$, and so on, for various variables $\vec{\xi}, \vec{\chi}$, etc... All these kernels are analytic in a strip $\{x \in \mathbb{C} |\; |Imx| < 2m\}$. Moreover the derivation of $(x^2 + 4\omega(\vec{\xi})^2)^{k/2}$, for any $k \in \mathbb{Z}$, gives a function bounded by a constant times $(x^2 + 4\omega(\vec{\xi})^2)^{k/2}$. Thus the integrability is preserved. From all these properties follows that $h_n(\vec{P}, \vec{Q}, \vec{Q}')$ is an analytic function of $\|\vec{P}\|$ in the strip given above, for all $(\vec{Q}, \vec{Q}') \in I\!R^6$.

To study the convergence for $n \to \infty$ we note that $\partial^j_{\|\vec{P}\|} h_n$ satisfies the bound (40) multiplied by a constant due to the derivations of the kernels, and by the factor $(n-1)^j$ for the number of terms caused by j derivations, according to the Leibniz formula. These new factors have no influence on the convergence, and thus $\partial^j_{\|\vec{P}\|} h_n$ converges to an element of $\mathcal{B}$, which is nothing else than $\partial^j_{\|\vec{P}\|} h$ because of the uniform convergence.

The analyticity is obtained by giving a small fixed imaginary part to $\|\vec{P}\|$, taking $\|\vec{P}\| + i\eta$ in the definition of b (30), with $|\eta|$ small enough to have again the estimate (33). The same argument as before, based on (40) which is still valid, shows the convergence of h_n and $\partial^j_{\|\vec{P}\|} h_n$, for all n, uniformly in η.

To see that h is an analytic function of $\vec{P}^2$ (and not only of $\|\vec{P}\|$) we note that, if $f(\vec{P}, \vec{Q}, \vec{Q}')$, $g(\vec{P}, \vec{Q}, \vec{Q}') \in \mathcal{B}$ are functions of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ that are even in $\|\vec{P}\|$, the same is true for $b(f, g)(\vec{P}, \vec{Q}, \vec{Q}')$. A glance to (36) suffices to convince ourselves that all the terms of h_n, and thus the limit h itself, are even functions of $\|\vec{P}\|$.

Proof of 4). The functions h_n can be written as

$$\begin{aligned} h_n(\vec{P}, \vec{Q}, \vec{Q}') \;&=\; \int d\sigma(\vec{Q}''_1) \cdots d\sigma(\vec{Q}''_{n-1})\; c(\vec{Q}, \vec{Q}''_1)\; c(\vec{Q}''_1, \vec{Q}''_2) \cdots \\ &\qquad \cdots c(\vec{Q}''_{n-1}, \vec{Q}')\; \Theta_n(\vec{P}, \vec{Q}, \vec{Q}'; \vec{Q}''_1, \ldots, \vec{Q}''_{n-1}) \end{aligned}$$

where the kernels Θ_n can be deduced from (34), (30), (31) by an easy (and tedius) calculation. This expression shows clearly that the support property of c leads to a support property for all h_n, and thus for h, w.r.t. $\vec{Q}, \vec{Q}'$.

For $j = 1$ or 2, the jth derivation of h_n w.r.t. $\vec{Q}$ or $\vec{Q}'$ is bounded by the estimate (40) multiplied by $(\|\partial c\|_\infty/\|c\|_\infty)^j$ or $\|\partial^j c\|_\infty/\|c\|_\infty$, by a constant due to the derivations of the kernels $\sqrt{z+4\vec{Q}^2}$, $\sqrt{z'+4\vec{Q}'^2}$ (for various $z, z' \in \mathbb{C}$ with $Re z, Re z' > 0$) w.r.t. $\vec{Q}$ or $\vec{Q}'$, and the factor $(n-1)^j$ for the number of terms caused by j derivations, according to the Leibniz formula. As before, these new factors have no influence on the convergence, and then these functions converges to an element of $\mathcal{B}$, which is nothing else than the corresponding derivation of h.

Proof of 5). We have already seen in the proofs of 2) and 3) that the derivation of h at any order w.r.t. $\|\vec{P}\|$ and at order ≤ 2 w.r.t. $\vec{Q}, \vec{Q}'$, belong to $\mathcal{B}$. The derivations w.r.t. $\vec{P}^2$ are bounded by the derivations w.r.t. $\|\vec{P}\|$ for large $\|\vec{P}\|$; for small $\|\vec{P}\|$ the boundedness is assured by the analyticity of h w.r.t. $\vec{P}^2$.

Proof of 6). Let us introduce the norm

$$\|f\|_* \;:=\; m^{-3} \sup_{\vec{P}\in\mathbb{R}^3} \sqrt{\int_{\mathbb{R}^6} d\vec{Q}\; d\vec{Q}'\; |f(\vec{P},\vec{Q},\vec{Q}')|^2} \tag{41}$$

for all $f \in \mathcal{B}$. From an easy estimate involving the Cauchy-Schwartz inequality we get

$$\begin{aligned} \|b(f,f)\|_* \;&<\; \frac{\sqrt{3}}{8}\;\left(\|f\|_*^2 + \|Df\|_*^2\right) \\ \|Db(f,f)\|_* \;&<\; \frac{\sqrt{3}}{16}\;\left(\|f\|_*^2 + \|Df\|_*^2\right) \end{aligned} \tag{42}$$

for all $f \in \mathcal{B}$. Now recall that $h = \lim_{n\to\infty} A^n$. Let us suppose that for some $n \in \mathbb{N}$ we know that $\|A^n\|_* \leq 2/3$ and $\|DA^n\|_* \leq 1/20$ (because of (38) this is true for $n=0$). Then from (38) and (42) we obtain

$$\begin{aligned} \|A^{n+1}\|_* \;&=\; \|c + b(A^n, A^n)\|_* \;\leq\; \|c\|_* + \|b(A^n,A^n)\|_* \\ &\leq\; \frac{1}{2} + \frac{\sqrt{3}}{8}\;(\|A^n\|_*^2 + \|DA^n\|_*^2) \;\leq\; \frac{1}{2} + \frac{\sqrt{3}}{8}\left(\frac{4}{9} + \frac{1}{20^2}\right) \;<\; \frac{2}{3} \\ \|DA^{n+1}\|_* \;&=\; \|Db(A^n,A^n)\|_* \;\leq\; \frac{\sqrt{3}}{16}\;(\|A^n\|_*^2 + \|DA^n\|_*^2) \\ &\leq\; \frac{\sqrt{3}}{16}\left(\frac{4}{9} + \frac{1}{20^2}\right) \;<\; \frac{1}{20}. \end{aligned}$$

It follows that these bounds hold for all n. Let us take the limit $n \to \infty$. Because the convergence of A^n and DA^n is uniform, the sequences $\|A^n\|_*$ and $\|DA^n\|_*$ converge too (taking the integral in (41) as a Riemann integral). Thus the above estimates still hold in the limit, which establishes (39). []

4. Unitarity representations of the Poincaré group

Until now, the representations (13) of the algebra $\mathcal{G}$ were only formal, in the sense that the domains of the operators and of the commutators were not specified. The proposition 4 brings the results which allow us to show that these algebra representations are well defined and can be integrated to give unitary continuous representations of $\mathcal{P}_+^\uparrow$.

Theorem 5 *Let $c \in \mathcal{V}$. Let $\mathcal{O}$ be the interaction operator* (22) *with the kernel h deduced from c in proposition* 3. *Then $\{H, \vec{P}, \vec{J}, \vec{L}\}$, given by* (13), *are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$.*

Proof. According to theorem 5 of [13], H, $\vec{P}$, $\vec{J}$, $\vec{L}$ are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$ if the following three conditions are satisfied: 1) they are self-adjoint, 2) they admit a common invariant dense domain $\mathcal{D}$ on which the commutation rules of the algebra hold, 3) $\mathcal{D}$ is a domain of essential self-adjointness for the operator $\Delta = m^{-2}(H^2 + \vec{P}^2) + \vec{J}^2 + \vec{L}^2$. This scheme gives the three natural steps of the proof.

1st step: proof that H, $\vec{P}$, $\vec{J}$, $\vec{L}$ are self-adjoint. It is clear for $\vec{P}$ and $\vec{J}$. To study H and $\vec{L}$ we need a general estimate of the norm of vectors like

$$\phi_\xi(\vec{P},\vec{Q}) \ := \ \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \ \phi(\vec{P},\vec{Q}') \ \xi(\vec{P},\vec{Q},\vec{Q}')$$

for all $\phi \in \mathcal{H}$ and for kernels ξ such that

$$|||\xi|||^2 \ := \ \sup_{\vec{P}\in I\!R^3} \ \int \frac{d\sigma(\vec{Q})}{\Omega(\vec{P},\vec{Q})} \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')} \ \left|\xi(\vec{P},\vec{Q},\vec{Q}'')\right|^2. \tag{43}$$

is well defined. From the Cauchy-Schwarz inequality and the Fubini theorem we obtain

$$\begin{aligned} ||\phi_\xi||^2 \ &= \ \int \frac{d\vec{P}d\sigma(\vec{Q})}{\Omega(\vec{P},\vec{Q})} \left| \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \ \phi(\vec{P},\vec{Q}') \ \xi(\vec{P},\vec{Q},\vec{Q}') \right|^2 \\ &\leq \ \int \frac{d\vec{P}d\sigma(\vec{Q})}{\Omega(\vec{P},\vec{Q})} \left[\int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \left|\phi(\vec{P},\vec{Q}')\right|^2 \right] \left[\int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')} \left|\xi(\vec{P},\vec{Q},\vec{Q}'')\right|^2 \right] \\ &\leq \ |||\xi|||^2 \ ||\phi||^2. \end{aligned} \tag{44}$$

For H, given by (25), we find the bound on the operator norm $||H - H_0||_{op} \leq |||h|||$. From (39) it follows that $|||h||| \leq m/6$. Thus $H - H_0$ is a bounded operator, and H is self-adjoint on the domain of H_0.

For $\vec{L}$ we consider (26) without the last term (because $\vec{J}^{\vec{Q}}h = \vec{J}^{\vec{Q}'}h = 0$). The estimate (44) leads to

$$||(L_j - L_{0,j})\phi|| \ \leq \ |||h/\Omega||| \ ||L_{0,j}\phi|| \ + \ (|||h/\Omega||| + |||Dh/\Omega|||) \ ||\phi|| \tag{45}$$

for all $j \in \{1,2,3\}$. From the Kato-Rellich theorem (sect. X.2 of [14]) it follows that $\vec{L}$ is self-adjoint on the domain of $\vec{L}_0$ provided $\|h/\Omega\| < 1$. This condition is indeed satisfied because from (39): $\|h/\Omega\| \leq \|h\|/2m < 1/12$. (Note that the condition $\|h\|/2m < 1$ implies the operator norm bound $||H - H_0||_{op} < 2m$, from which the positivity of H follows).

2nd step: the invariant domain and the commutation rules. Let $\mathcal{D}$ be the following domain

$$\begin{aligned} \mathcal{D} \; := \; \Big\{ \phi \in \mathcal{H} \;\Big|\; & \Omega(\vec{P},\vec{Q})^\ell \; P_1^{n_1} P_2^{n_2} P_3^{n_3} \; L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} \; J_1^{j_1} J_2^{j_2} J_3^{j_3} \;\; \phi(\vec{P},\vec{Q}) \in \mathcal{H} \\ & \text{for all} \;\; \ell, n_1, n_2, n_3, m_1, m_2, m_3, j_1, j_2, j_3 \in I\!N \Big\}. \end{aligned} \tag{46}$$

$\mathcal{D}$ is dense (because it contains $C_0^\infty(I\!R^6)$) and is clearly left invariant by H_0, $\vec{P}$, $\vec{J}$ and $\vec{L}_0$. To show that it is also invariant under H and $\vec{L}$ it is sufficient to show that it is invariant under $\mathcal{O}$. Let $\phi \in \mathcal{D}$ and let us apply an operator $\Omega(\vec{P},\vec{Q})^\ell P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} J_1^{j_1} J_2^{j_2} J_3^{j_3}$ to the vector $\mathcal{O}\phi$. From 2), proposition 4, we get 0 unless $j_1 = j_2 = j_3 = 0$. In that case, from the Leibniz rule we find

$$\begin{aligned} & \Omega(\vec{P},\vec{Q})^\ell \; P_1^{n_1} P_2^{n_2} P_3^{n_3} \; L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} \; \mathcal{O}\phi(\vec{P},\vec{Q}) \;\; = \\ & \sum_{\vec{\alpha}+\vec{\beta}=\vec{m}} C_{\vec{\alpha},\vec{\beta}} \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \; \Big[\Omega(\vec{P},\vec{Q}')^{\ell+\beta_1+\beta_2+\beta_3+1} P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{\alpha_1} L_{0,2}^{\alpha_2} L_{0,3}^{\alpha_3} \; \phi(\vec{P},\vec{Q}') \Big] \\ & \times \; \left\{ \left(\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \right)^{\ell+\beta_1+\beta_2+\beta_3} \partial_{P_1}^{\beta_1} \partial_{P_2}^{\beta_2} \partial_{P_3}^{\beta_3} \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}'))} \right\} \end{aligned} \tag{47}$$

for some coefficients $C_{\vec{\alpha},\vec{\beta}}$. Because $\phi \in \mathcal{D}$ the expression in [ ] is a vector of $\mathcal{H}$. By using the estimate (44) we find that (47) belongs to $\mathcal{H}$ if the expression in { } has a well defined $\|.\|$ norm. This is the case because of 3), 4) and 5) of proposition 4, because the derivations of $\Omega(\vec{P},\vec{Q}')^{-1}(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}'))^{-1}$ w.r.t. $\vec{P}$ give bounded functions, and because

$$\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \;\leq\; \frac{||\vec{P}||}{\Omega(\vec{P},\vec{Q}')} \;+\; \frac{2\omega(\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \;\leq\; 1 \;+\; \frac{\omega(\vec{Q})}{m} \tag{48}$$

gives a bounded contribution, due to the compact support of $\vec{Q} \mapsto h(\vec{P},\vec{Q},\vec{Q}')$. Thus $\mathcal{D}$ is an invariant domain for all operators H, $\vec{P}$, $\vec{J}$ and $\vec{L}$.

By working on $\mathcal{D}$ the commutation rules for $\{H,\vec{P},\vec{J},\vec{L}\}$ can be deduced from those for $\{H_0,\vec{P},\vec{J},\vec{L}_0\}$ without having to take care of their domain. Thus we have only to do formal calculations, and thus to apply proposition 2.

3th step: proof that $\mathcal{D}$ is a domain of essential self-adjointness for the operator Δ. We show that Δ is self-adjoint, using theorems by Nelson and Kato-Rellich. Because $\{\vec{P},H_0,\vec{J},\vec{L}_0\}$ are the infinitesimal generators of a unitary continuous representation of a Lie group it follows from a theorem by Nelson (see [13], theorem 3) that all these operators and $\Delta_0 = m^{-2}(H_0^2 + \vec{P}^2) + \vec{J}^2 + \vec{L}_0^2$ are self-adjoint and that there exists a domain $\mathcal{D}_0$ which is a common invariant domain and a common core for all of them. However the largest common invariant domain is given by (46), thus $\mathcal{D}$ is also a common core. By the Kato-Rellich theorem (theorem X.12 of [14]), Δ is essentially self-adjoint on $\mathcal{D}$ if

$$\|(\Delta - \Delta_0)\phi\| \;\; \leq \;\; k_1\|\Delta_0\phi\| \; + \; k_2\|\phi\| \tag{49}$$

for all $\phi \in \mathcal{D}$, for some $0 < k_1 < 1$ and $0 < k_2 < \infty$.

The bound (49) concerns the interaction terms of H and $\vec{L}$ because

$$\Delta - \Delta_0 \;=\; m^{-2}(H^2 \;-\; H_0^2) \;+\; \vec{L}^2 \;-\; \vec{L}_0^2.$$

Now, for all $\phi \in \mathcal{D}$

$$(H^2 \;-\; H_0^2)\;\phi(\vec{P},\vec{Q}) \;=\; \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\;\phi(\vec{P},\vec{Q}')\;\Xi(\vec{P},\vec{Q},\vec{Q}') \tag{50}$$

where

$$\begin{aligned} \Xi(\vec{P},\vec{Q},\vec{Q}') \;&:=\; \Big(\Omega(\vec{P},\vec{Q}) \;+\; \Omega(\vec{P},\vec{Q}')\Big)\; h(\vec{P},\vec{Q},\vec{Q}') \\ &+\; \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')}\; h(\vec{P},\vec{Q}',\vec{Q}'')\; h(\vec{P},\vec{Q}'',\vec{Q}') \end{aligned} \tag{51}$$

¿From (44) it follows that $\|(H^2 - H_0^2)\phi\| \leq \|\!|\Xi|\!\|\;\|\phi\|$, provided $\|\!|\Xi|\!\|$ is defined, which gives a contribution to the second term of (49) only. To see the existence of $\|\!|\Xi|\!\|$ we use first (48) to obtain

$$\frac{\Big(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}')\Big)^2}{2\omega(\vec{Q})\Omega(\vec{P},\vec{Q})\;2\omega(\vec{Q}')\Omega(\vec{P},\vec{Q}')} \;\; < \;\; \frac{3}{2m^2}$$

which gives $\|\!|(\Omega + \Omega')h|\!\| < m^2\sqrt{3/2}\|h\|_*$ (which is bounded by $m^2\sqrt{2/3}$ from (39)) and then the Cauchy-Schwarz inequality to get $\|\!|\int d\sigma'' hh/\Omega''|\!\| \;\leq\; \|\!|h|\!\|^2 \;<\; m^2/36$, from (39) again.

Because in (49) the coefficient of $\|\phi\|$ has not to be bounded, we omit to estimate it explicitly henceforth. Now let us put $\vec{L}_h := \vec{L} - \vec{L}_0$. We have to control the action of $\vec{L}^2 - \vec{L}_0^2 = \vec{L}_0 \cdot \vec{L}_h + \vec{L}_h \cdot \vec{L}_0 + \vec{L}_h^2$ on vectors $\phi \in \mathcal{D}$. Replacing ϕ by $\vec{L}_0\phi$ in (45) leads to the estimate

$$\|\vec{L}_h \cdot \vec{L}_0\phi\| \;<\; \|\!|h/\Omega|\!\|\;\|\vec{L}_0^2\phi\| \;+\; (\|\!|h/\Omega|\!\| + \|\!|Dh/\Omega|\!\|)\;\sum_{j=1}^{3}\|L_{0,j}\phi\| \tag{52}$$

On the other hand, applying $\vec{L}_0$ to (26) and neglecting the contribution of $\vec{J}^{\vec{Q}}$ gives

$$\vec{L}_0 \cdot \vec{L}_h\phi(\vec{P},\vec{Q}) \qquad = \qquad \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\;\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')}\;\left[\vec{L}_0^2\phi(\vec{P},\vec{Q}')\right]\;\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}$$

$$
\begin{aligned}
- \quad & i \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \, \frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \, \left[\vec{P}\cdot\vec{L}_0\phi(\vec{P},\vec{Q}')\right] \times \\
& \left[-3\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')^2} + \left(\frac{2}{\Omega(\vec{P},\vec{Q}')} + \frac{1}{\Omega(\vec{P},\vec{Q})}\right) \frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|} \right] \\
- \, \Omega(\vec{P},\vec{Q}) \int \; & d\sigma(\vec{Q}') \;\; \phi(\vec{P},\vec{Q}') \, \left(2 + \|\vec{P}\| \, \partial_{\|\vec{P}\|}\right) \left[\frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|\Omega(\vec{P},\vec{Q}')^2} - \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(P,q')^3} \right]
\end{aligned}
$$

for all $\phi \in \mathcal{D}$. Note that in the last term the factor $\|\vec{P}\|\partial_{\|\vec{P}\|} Dh \|\vec{P}\|^{-1}$ which is well defined, in norm $\|.\|_*$, because h is an analytic function of $\vec{P}^2$ with all derivatives in $\mathcal{B}$ (points 3) and 5) of proposition 4). From the estimate (44) we get

$$
\begin{aligned}
\|\vec{L}_0 \cdot \vec{L}_h \phi\| \;\; \leq \;\; & \|\!|\Omega h/\Omega'^2|\!\| \; \|\vec{L}_0^2\phi\| \; + \; 3(\|\!|\Omega h/\Omega'^2|\!\| + \|\!|\Omega Dh/\Omega'^2|\!\|) \, \sum_{j=1}^{3} \|L_{0,j}\phi\| \\
& + \; k_3\|\phi\|
\end{aligned} \tag{53}
$$

for some $k_3 \in (0,\infty)$. By applying $\vec{L}_h$ to (26) and by using (53) we get an estimate of the last term

$$
\begin{aligned}
\|\vec{L}_h^2\phi\| \;\; \leq \;\; & \|\!|h/\Omega|\!\| \; \|\vec{L}_0 \cdot \vec{L}_h\phi\| \; + \; (\|\!|h/\Omega|\!\| + \|\!|Dh/\Omega|\!\|) \, \sum_{j=1}^{3} \|L_{h,j}\phi\| \\
\leq \;\; & \|\!|\Omega h/\Omega'^2|\!\| \; \|\vec{L}_0^2\phi\| \; + \; \Big[3\|\!|h/\Omega|\!\| \; \left(\|\!|\Omega h/\Omega'^2|\!\| + \|\!|\Omega Dh/\Omega'^2|\!\| \, \right) \\
& + \|\!|h/\Omega|\!\| + \|\!|Dh/\Omega|\!\| \Big] \; \sum_{j=1}^{3} \|L_{0,j}\phi\| \; + \; k_4\|\phi\|
\end{aligned} \tag{54}
$$

for some $k_4 \in (0,\infty)$. Finally by collecting the estimates (52), (53), (54) we obtain

$$
\begin{aligned}
\|(\Delta - \Delta_0)\phi\| \;\; \leq \;\; & \left(\left\|\!\left|\frac{h}{\Omega}\right|\!\right\| + \left\|\!\left|\frac{Dh}{\Omega}\right|\!\right\| \right) \|\vec{L}_0^2\phi\| \; + \; \Bigg[2\left\|\!\left|\frac{h}{\Omega}\right|\!\right\| + 2\left\|\!\left|\frac{Dh}{\Omega}\right|\!\right\| \\
& + \; 3\left(1 + \left\|\!\left|\frac{h}{\Omega}\right|\!\right\|\right) \left(\left\|\!\left|\frac{\Omega h}{\Omega'^2}\right|\!\right\| + \left\|\!\left|\frac{\Omega Dh}{\Omega'^2}\right|\!\right\| \right) \Bigg] \;\; \sum_{j=1}^{3} \|L_{0,j}\phi\| \\
& + \;\; k_5 \; \|\phi\|
\end{aligned} \tag{55}
$$

for some $k_5 \in (0,\infty)$ and all $\phi \in \mathcal{D}$. It remains to find a bound on $\|L_{0,j}\phi\|$, $\|\vec{L}_0^2\phi\|$ of the type $k_6\|\Delta_0\phi\| + k_7\|\phi\|$ for constants k_6 and k_7. Following Nelson ([13], proof of lemma 6.1) we find

$$
\begin{aligned}
\sum_{j=1}^{3} \|L_{0,j}\phi\| \;\; \leq \;\; & \sqrt{3} \left[\sum_{j=1}^{3} \left(\|L_{0,j}\phi\|^2 + m^{-2}\|P_j\phi\|^2 + \|J_j\phi\|^2 \right) + m^{-2} \, \|H_0\phi\|^2 \right]^{1/2} \\
= \;\; & \sqrt{3} \;\; (\phi, \Delta_0\phi)^{1/2} \;\; < \;\; \sqrt{3} \;\; \left(\phi, (\tfrac{1}{2}\Delta_0^2 + \Delta_0 + \tfrac{1}{2})\phi\right)^{1/2} \\
= \;\; & \sqrt{3/2} \; \|(\Delta_0 + 1)\phi\| \;\; \leq \;\; \sqrt{3/2} \;\; (\|\Delta_0\phi\| + \|\phi\|) \, .
\end{aligned}
$$

The bound on $\|\vec{L}_0^2\phi\|$ needs more development. Following [13], proof of lemma 6.1 we try to write $\Delta_0^2 - (\vec{L}_0^2)^2$ as a sum of positive and negative operators. For that we use the commutation rules (1) to (4), which imply

$$\begin{aligned}
\vec{P}^2\vec{L}^2 + \vec{L}^2\vec{P}^2 &= 2\sum_{j=1}^{3} P_j\vec{L}^2P_j \; - \; 6H^2 \\
\vec{P}^2\vec{J}^2 + \vec{J}^2\vec{P}^2 &= 2\sum_{j=1}^{3} P_j\vec{J}^2P_j \; - \; 6\vec{P}^2 \\
\vec{J}^2\vec{L}^2 + \vec{L}^2\vec{J}^2 &= 2\sum_{j=1}^{3} J_j\vec{L}^2J_j \\
H^2\vec{L}^2 + \vec{L}^2H^2 &= 2H\vec{L}^2H \; - \; 2\vec{P}^2.
\end{aligned}$$

These relations imply that $\Delta_0^2 - (\vec{L}_0^2)^2 + 2m^{-2}(2\vec{P}^2 + 3H_0^2)$ is positive. Then

$$(\vec{L}_0^2)^2 \; < \; \Delta_0^2 + 2m^{-2}(2\vec{P}^2 + 3H_0^2) \; < \; \Delta_0^2 + 6\Delta_0 + 9 \; = \; (\Delta_0 + 3)^2$$

and finally $\|\vec{L}_0^2\phi\| < \|(\Delta_0 + 3)\phi\| \leq \|\Delta_0\phi\| + 3\|\phi\|$. By collecting all these results the estimate (55) can be written as (49) with

$$\begin{aligned}
k_1 \; &= \; \left(\left\| \frac{h}{\Omega} \right\| + \left\| \frac{Dh}{\Omega} \right\| \right) \\
&+ \; \sqrt{\frac{3}{2}} \left[2\left\| \frac{h}{\Omega} \right\| + 2\left\| \frac{Dh}{\Omega} \right\| + 3\left(1 + \left\| \frac{h}{\Omega} \right\|\right) \left(\left\| \frac{\Omega h}{\Omega'^2} \right\| + \left\| \frac{\Omega Dh}{\Omega'^2} \right\| \right) \right] \qquad (56)
\end{aligned}$$

Now from (39) it follows that

$$\left\| \frac{h}{\Omega} \right\| \; \leq \; \frac{\|h\|}{2m} \; \leq \; \frac{\|h\|_*}{16} \; \leq \; \frac{1}{24} \qquad\qquad \left\| \frac{Dh}{\Omega} \right\| \; \leq \; \frac{\|Dh\|_*}{16} \; \leq \; \frac{1}{320} \qquad (57)$$

where $\| \, \|_*$ is the norm (41), and, with $\Omega(\vec{P},\vec{Q})/\Omega(\vec{P},\vec{Q}') \leq 2\omega(\vec{Q})\omega(\vec{Q}')/m^{-2}$ from (52), we get

$$\left\| \frac{\Omega h}{\Omega'^2} \right\| \; \leq \; \frac{\|\omega\omega' h\|}{m^3} \; \leq \; \frac{\|h\|_*}{8} \; \leq \; \frac{1}{12} \qquad\qquad \left\| \frac{\Omega Dh}{\Omega'^2} \right\| \; \leq \; \frac{\|Dh\|_*}{8} \; \leq \; \frac{1}{160} \qquad (58)$$

The substitution of (57) and (58) in (56) gives $k_1 < 1/2 < 1$. []

5. Bound states and relativistic Schrödinger equation

We have obtained a family of unitary representations of the Poincaré group by doing perturbations of the two free boson model. We have now to investigate the physics of these models, in order to see the effective differences between them. In a quantum theory, the physics of a model reduces to two problems, the search for the bound states and the construction of the scattering matrix.

We begin with the first problem. In a relativistic theory, the bound states are related to the discrete part of the spectrum of the mass operator M. More precisely a bound state has a mass m_B solution to the eigenvalue problem $M\phi = m_B\phi$, with $\phi \in D(M)$, and $\phi \neq 0$. In our models this equation is better written for M^2 and from (50) it takes the form

$$\begin{aligned} M^2\phi(\vec{P},\vec{Q}) &= 4(\vec{Q}^2+m^2)\ \phi(\vec{P},\vec{Q}) + \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\ \Xi(\vec{P},\vec{Q},\vec{Q}') \\ &= m_B^2\ \phi(\vec{P},\vec{Q}) \end{aligned} \tag{59}$$

with $\phi \in D(M^2)$, $\phi \neq 0$, where the kernel Ξ is given by (51). Here $\vec{P}$ plays the role of a parameter. The function ϕ, when it exists, is continuous on $\vec{P}$ (see the proof of proposition 6 below). So one can take $\vec{P} = 0$ in (59) (choosing the centre-of-mass frame) and this equation reduces to

$$4(\vec{Q}^2+m^2)\ \varphi(\vec{Q}) + \int \frac{d\vec{Q}'}{4(\vec{Q}'^2+m^2)}\ \varphi(\vec{Q}')\ \Xi(0,\vec{Q},\vec{Q}') = m_B^2\ \varphi(\vec{Q}) \tag{60}$$

where $\varphi(\vec{Q}) = \phi(0,\vec{Q})$. Now for $\vec{P} = 0$, M^2 is just H^2. Thus the square root of the operator involved in (60) exists and leads to

$$2\sqrt{\vec{Q}^2+m^2}\ \varphi(\vec{Q}) + \int \frac{d\vec{Q}'}{4(\vec{Q}'^2+m^2)}\ \varphi(\vec{Q}')\ c(\vec{Q},\vec{Q}') = m_B\ \varphi(\vec{Q}). \tag{61}$$

The operator M, the positive square root of the operator M^2 apearing in (59), satisfies the following properties.

Proposition 6 *For all $c \in \mathcal{V}$ the operator M is essentially self-adjoint on the domain $\mathcal{D}$ given by* (46). *Moreover, the spectrum of M is contained in* $(0,\infty)$ *where* $(0,2m)$ *contains at most a finite number of eigenvalues m_B, which are solutions to the eigenvalue equation* (61).

Remark. The equation (59) (or its reduced forms (60), (61)) can be considered as a *relativistic Schrödinger equation* because it plays the role of the Schrödinger equation in Quantum Mechanics: it generates the discrete structure of the bound states.

Historical remark. Looking for a relativistic equation for the Hydrogen atom, Dirac rejected the equation: $(\sqrt{-\Delta+m^2} - e^2/r)f = Ef$, which is the first natural attempt to a relativistic generalization of the Schrödinger equation. The reason was that it is not invariant under Lorentz transformations (see the discussion in [16]). Now, the analogue of this equation in our models is (61), which indeed cannot satisfy such an invariance, because it has been obtained in a given referential frame (what is more, $\vec{Q}$ is not an ordinary momentum, because $\|\vec{Q}\|$ is Lorentz invariant). In fact the relativistic equation is (59) and however, its invariance is not obvious: it is invariant under the transformations of the variables $(\vec{P},\vec{Q})$ deduced from the generators $\vec{L}$ given by (26).

Proof of proposition 6. Note that the spectrum of M_0^2, given by (20), covers the interval $[4m^2,\infty)$ and is absolutely continuous. M_0^2 is clearly essentially self-adjoint on $\mathcal{D}$. From (44) we have the bound on the operator norm $\|M^2 - M_0^2\|_{op} \leq \|\!|\Xi|\!\|$ which is bounded by $\|\!|\Xi|\!\| \leq m^2(\sqrt{2/3}+1/36)$ from the estimates just after (51). It follows that M^2 is essentially self-adjoint on $\mathcal{D}$, and so is M.

Because H and $\vec{P}$ commute they admit a simultaneous spectral measure $d\mathcal{E}(\rho,\vec{P})$ where $\rho>0$ is the spectral variable associated with H. In the spectral representation M becomes the multiplication operator by $(\rho^2-\vec{P}^2)^{1/2}$. Because M commutes with the Poincaré representation, the invariant subspaces are limited in the $(\rho,\vec{P})$ space by half hyperbolas $\rho=(\vec{P}^2+K^2)^{1/2}$ with $K>0$, and so is the support of $d\mathcal{E}$.

Let us introduce the family of Hilbert spaces

$$\mathcal{H}_{\vec{P}} := L^2(I\!R^3, d\sigma_{\vec{P}}) \qquad \text{where} \qquad d\sigma_{\vec{P}}(\vec{Q}) := d\sigma(\vec{Q})\Omega(\vec{P},\vec{Q})^{-1}. \tag{62}$$

and $\vec{P}\in I\!R^3$ plays the role of three parameters. The original Hilbert space $\mathcal{H}$ is the direct integral

$$\mathcal{H} \; = \; \int_{I\!R^3} \bigoplus \mathcal{H}_{\vec{P}}\; d\vec{P} \tag{63}$$

(see sect. 1.5 of [19]). We also consider the operators $H_{0,\vec{P}}$ and $H_{\vec{P}}$, the restrictions of H_0 and H to $\mathcal{H}_{\vec{P}}$, still given by the same formulas as before, but with fixed $\vec{P}$. That is, $H_{\vec{P}}$ is given by

$$H_{\vec{P}}\varphi(\vec{Q}) \;=\; H_{0,\vec{P}}\varphi(\vec{Q}) \;+\; \int d\sigma_{\vec{P}}(\vec{Q}')\;\varphi(\vec{Q}')\;h(\vec{P},\vec{Q},\vec{Q}')$$

for suitable φ, where $H_{0,\vec{P}}\varphi(\vec{Q}) = \Omega(\vec{P},\vec{Q})\varphi(\vec{Q})$.

¿From (39) the integral $\int |h(\vec{P},\vec{Q},\vec{Q}')|^2 d\sigma_{\vec{P}}(\vec{Q})d\sigma_{\vec{P}}(\vec{Q}')$ is well defined. Thus $H_{\vec{P}}$ is a compact perturbation of $H_{0,\vec{P}}$. By the 'classical Weyl theorem' (sect. XIII.4 of [15]), the essential spectrum of $H_{\vec{P}}$ is that of $H_{0,\vec{P}}$, that is the interval $[\Omega(\vec{P},0),\infty)$, and it may exist a finite number of eigenvalues in $(0,\Omega(\vec{P},0))$ of finite multiplicities.

To see how the possible eigenvalues depend on $\vec{P}$ we note that $\vec{P}^2 \to H_{\vec{P}} - H_{0,\vec{P}}$ is an analytic family of compact operators in a $C\!\!\!\!I^2$-neighbourhood of $I\!R_+$. This property holds because by giving a small imaginary value to $\vec{P}^2$, the integral $\int |h(\vec{P},\vec{Q},\vec{Q}')|^2 d\sigma_{\vec{P}}(\vec{Q})d\sigma_{\vec{P}}(\vec{Q}')$ is still defined, because of 4), proposition 4. Thus $H_{\vec{P}}$ is a 'holomorphic family of type (A)' according to [17], and by theorems 1.7 and 1.8 of sect. VII of [17] the eigenvalues (if any) are continuous functions of $\vec{P}$ and the eigenvectors can be chosen continuous. In the $(\rho,\vec{P})$-space the eigenvalues form continuous curves. Now, from the previous discussion about the support of $d\mathcal{E}(\rho,\vec{P})$, these curves can only be half hyperbolas $\rho = (\vec{P}^2 + m_B^2)^{1/2}$, for a finite number of $m_B \in (0, 2m)$ solutions to the eigenvalue equation for $\vec{P} = 0$, that is to (61). Going back to the operator $M^2_{\vec{P}} = H^2_{\vec{P}} - \vec{P}^2$ we conclude that its spectrum is contained in the interval $[4m^2, \infty)$ possibly completed with a finite number of eigenvalues m_B independant of $\vec{P}$, of finite multiplicities and lying in the interval $(0, 4m^2)$.

Because the spectrum of $M^2_{\vec{P}}$ is independant of $\vec{P}$, it is also the spectrum of M^2 (but now the dimension of the eigenspaces, as subspaces of $\mathcal{H}$, is infinite). []

6. Scattering states and scattering matrix

We continue to investigate the physical content of our models by exploiting now the scattering theory. In Quantum Field Theory the notion of interacting particle is ambiguous. Particles can only be recognized in the asymptotic states, where the interaction is no more efficient. In our models the construction of asymptotic states describing two free particles will confirm that they effectively concern two particle systems. Moreover the asymptotic completeness, when it holds, will imply that only two particle states are present (apart from the bound states) in the Hilbert state space.

A vector $\psi \in \mathcal{H}$ is said to be a 'two particle scattering state' if its time evolution, for very large time (or very small time), is not distinguishable from the two free particle evolution, that is, if there exist ψ_{out} (or ψ_{in}) in $\mathcal{H}$ such that $\psi = U^-\psi_{out}$ (or $\psi = U^+\psi_{in}$), where the 'wave operators' U^+, U^- are given by

$$U^\pm \ := \ s - \lim_{t\to\mp\infty} e^{itH} \ e^{-itH_0}. \tag{64}$$

Here H_0 is the two free particle Hamiltonian (17). When the operators (64) exist, they map any vector of $\mathcal{H}$ on a two-particle scattering state, and so the study of two particle scattering can be undertaken.

Proposition 7 *For all $c \in \mathcal{V}$ the operators $U^\pm$ exist on all $\mathcal{H}$.*

Proof. We apply Cook's method (theorem XI.4 of [18]), with $A = H$ and $B = H_0$. Note that $D(H) = D(H_0)$ because $H - H_0$ is bounded. Let us consider the function

$$F(t,\vec{P},\vec{Q}) \ := \ \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \ \phi(\vec{P},\vec{Q}') \ h(\vec{P},\vec{Q},\vec{Q}') \ e^{-it\Omega(\vec{P},\vec{Q}')}$$

for $\phi \in C_0^\infty(I\!\!R^6)$ satisfying $\phi(\vec{P},(0,Q_2)) = \partial_{Q_1}\phi(\vec{P},(0,Q_2)) = 0$. Such vectors ϕ generate a dense subspace of $\mathcal{H}$. By performing two integrations by parts w.r.t. Q'_1 we get

$$F(t,\vec{P},\vec{Q}) = \\ -\frac{1}{32t^2}\int d\vec{Q}' e^{-it\Omega(\vec{P},\vec{Q}')}\partial_{Q'_1}\left(\frac{\Omega(\vec{P},\vec{Q}')}{Q'_1} \ \partial_{Q'_1}\frac{\phi(\vec{P},\vec{Q}')h(\vec{P},\vec{Q},\vec{Q}')}{Q'_1\omega(\vec{Q}')}\right). \tag{65}$$

We recall that $Q_1 \to h(\vec{P},\vec{Q},\vec{Q}')$ belongs to $C_0^2(I\!R)$ by 4), proposition 4. Thus the r.h.s. of (65) is a bounded function of $\vec{P},\vec{Q}$ with compact support and then

$$t \mapsto \left[\int \frac{d\vec{P}\ d\sigma(\vec{Q})}{\Omega(\vec{P},\vec{Q})}\ \left| F(t,\vec{P},\vec{Q}) \right|^2 \right]^{1/2} = \left\| (H-H_0)e^{-itH_0}\phi \right\|$$

is in $L^1([1,\infty))$, as required by Cook's method. []

Proposition 7 leads to the existence of two particle states, defined as the states contained in the range of the operators $U^{\pm}$. Now two particle scattering processes can be obtained by computing the matrix elements of the scattering operator $S := U^{-*}U^{+}$. However the question of the existence of other states (apart from the bound states) naturally arises, which is related to the asymptotic completness problem. We will give a partial answer to this question under an new condition on $c(\vec{Q},\vec{Q}')$, the interaction kernel in the centre-of-mass frame. In Quantum Mechanics, the asymptotic completeness problem is stated once the separation of the centre-of-mass motion has been done. In the relativistic case, this separation cannot be performed, but we can restrict ourselves to a particular frame, given by the choice of a fixed $\vec{P} \in I\!R^3$. As in the proof of proposition 6 we consider the Hilbert space decomposition (62), (63) and the operators $H_{\vec{P}}$, $H_{0,\vec{P}}$, $\vec{L}_{\vec{P}}$, $\vec{J}_{\vec{P}}$ and $M_{\vec{P}}$, acting on suitable subspaces of $\mathcal{H}_{\vec{P}}$, still given by the same formulas as before, but with fixed $\vec{P}$. Now for each $\vec{P}$ the 'wave operators'

$$U^{\pm}_{\vec{P}} = s - \lim_{t\to\mp\infty} e^{itH_{\vec{P}}}\ e^{-itH_{0,\vec{P}}} \tag{66}$$

can be formed on $\mathcal{H}_{\vec{P}}$ (the existence of the strong limit is easily established by repeating the proof of proposition 7, with fixed $\vec{P}$). The question of the asymptotic completeness will be discussed in $\mathcal{H}_{\vec{P}}$. According to [18], it consists of two statements: 1) the Hamiltonian $H_{\vec{P}}$ has no singular continuous spectrum and 2) the ranges of $U^{\pm}_{\vec{P}}$ coincide with the subspace of $\mathcal{H}_{\vec{P}}$ corresponding to the absolutely continuous part of the spectrum of $H_{\vec{P}}$. Only the second statement, called the 'completeness of the wave operators', will be considered here. We note that it is sufficient to insure the unitarity of the scattering operator $S_{\vec{P}} := U^{-*}_{\vec{P}}U^{+}_{\vec{P}}$.

Let us take an interaction kernel in the centre-of-mass $c \in \mathcal{V}$ satisfying moreover

$$c \in C_0^3(I\!R^6) \tag{67}$$

which implies that the integration operator of kernel c is trace class (see appendix D). We will see that the condition (67) implies the completeness of the wave operators $U^{\pm}_{\vec{P}}$ for all $\vec{P} \in I\!R^3$.

Proposition 8 *For all $c \in \mathcal{V}$ satisfying (67) and for all $\vec{P} \in I\!R^3$, the ranges of the operators $U^{\pm}_{\vec{P}}$ coincide with the absolutely continuous subspace of $H_{\vec{P}}$ in $\mathcal{H}_{\vec{P}}$ (in other words, the wave operators are complete). Then the scattering operator S is unitary.*

Proof. In appendix D we show that any $c \in \mathcal{V}$ satisfying (67) is the kernel of a trace class operator in $\mathcal{H}_{\vec{P}=0}$, which implies that the ranges of $U^{\pm}_{\vec{P}=0}$ coincide with the absolutely continuous subspace of $H_{\vec{P}=0}$ (Kato-Rosenblum theorem, corollary 2, sect. 6.2, [19]). The proposition affirms that this is also the case for all $\vec{P} \neq 0$. This holds if the operator $V_{\vec{P}} := H_{\vec{P}} - H_{0,\vec{P}}$ is trace class in $\mathcal{H}_{\vec{P}}$ for all $\vec{P} \in I\!R^3$. Now $V_{\vec{P}}$ is an integral operator with kernel $h_{\vec{P}}(\vec{Q}, \vec{Q}') := h(\vec{P}, \vec{Q}, \vec{Q}')$ (for fixed $\vec{P}$). The proof of point 4), proposition 4 shows that $(\vec{Q}, \vec{Q}') \mapsto h(\vec{P}, \vec{Q}, \vec{Q}')$, for fixed $\vec{P}$, lies in the same class $C_0^p(I\!R^6)$ as c. Thus $h_{\vec{P}} \in C_0^3(I\!R^6)$, and appendix D can be used again, to conclude the proof. []

The full asymptotic completeness (with absence of singular continuous spectrum) needs probably more restrictions on c (see [6] for the 2-d case).

7. An example and the non-triviality

To establish the non-triviality we consider an example for which c is the kernel of a one dimensional range operator. This has the advantage of leading to explicit results.

Let $\zeta \in C_0^3([0, \infty))$ with support in $[0, R]$, with uniform norm bounded by

$$\|\zeta\|_\infty \ \leq \ \sqrt{\frac{3}{8\pi}} \, \left(\frac{m}{R}\right)^{3/2} \tag{68}$$

and bounded below by

$$\zeta(r) \ \geq \ \begin{cases} \sqrt{\frac{1}{4\pi}} \, \left(\frac{m}{R}\right)^{3/2} & \text{if } \ r < \frac{R}{2} \\ 0 \qquad \text{if } \ r \geq \frac{R}{2} \end{cases} \tag{69}$$

where R and m are the parameters introduced in (37).

Proposition 9 *Let c be the function given by*

$$c(\vec{Q}, \vec{Q}') \ = \ \lambda \, \zeta(\|\vec{Q}\|) \, \zeta(\|\vec{Q}'\|) \tag{70}$$

for all $(\vec{Q}, \vec{Q}') \in I\!R^6$, ζ as above and $\lambda \in I\!R$. For all $|\lambda| \leq 1$ the function c belongs to $\mathcal{V}$ and thus is the interaction kernel in the centre-of-mass frame of a continuous unitary representation of the Poincaré group. Moreover the two-particle wave operators are defined and are complete. The scattering operator S and the mass operator M satisfy

i) $S \neq Id$

ii) $\lambda = -1$ and $m/R > 2$ imply that M has an eigenvalue $m_B \in (0, 2m)$.

Proof. The function c of (70) belongs to $\mathcal{V}$ because it has the right support and because (68) leads to (37). Thus from theorem 5, c is the interaction kernel in the centre-of-mass frame of a continuous unitary representation of the Poincaré group. Moreover c satisfies (67). Then the existence and the completeness of the wave operators follow from propositions 7 and 8. In particular, the scattering operator S is defined and unitary.

Proof of i). Note that $\mathcal{H}$ carries two different unitary continuous representations of the Poincaré group, the free representation generated by $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ and the interaction representation generated by $\{H, \vec{P}, \vec{J}, \vec{L}\}$. Because H_0 and $\vec{P}$ commute they admit a simultaneous spectral measure $d\mathcal{E}_0(E, \vec{P})$ where $E > 0$ is the spectral variable associated with H_0. In the spectral representation M_0 becomes the multiplication operator by $(E^2 - \vec{P}^2)^{1/2}$. Because $M_0 \geq 2m$ the support of $d\mathcal{E}_0$ is $\{(E, \vec{P}) | E^2 - \vec{P}^2 \geq 4m^2\}$. For such $(E, \vec{P})$ let us consider the space $d\mathcal{E}_0(E, \vec{P})\mathcal{H} =: \mathcal{H}_{E,\vec{P}}$ which appears in the spectral decomposition of $\mathcal{H}_{\vec{P}}$ which diagonalizes $H_{0,\vec{P}}$

$$\mathcal{H}_{\vec{P}} \;=\; \int_{sp(H_{0,\vec{P}})} \bigoplus \mathcal{H}_{E,\vec{P}}\; dE$$

where $sp(H_{0,\vec{P}})$ is the spectrum of $H_{0,\vec{P}}$, that is the interval $[\Omega(\vec{P}, 0), \infty)$. The operator $S_{\vec{P}}$ restricted to $\mathcal{H}_{E,\vec{P}}$ is denoted $S_{E,\vec{P}}$ and is called the 'scattering matrix'. The action of $\vec{L}_0$ on the variables $(E, \vec{P})$ is easily computed. For $\vec{\beta} \in \mathbb{R}^3$, $\|\vec{\beta}\| < 1$, the Lorentz boost $L_0(\vec{\beta})$ of speed $\vec{\beta}$ (from which $\vec{L}_0$ are the infinitesimal generators) acts as follows

$$L_0(\vec{\beta})\; \mathcal{H}_{E,\vec{P}} \;=\; \mathcal{H}_{\Lambda_{\vec{\beta}}(E,\vec{P})} \tag{71}$$

where $\Lambda_{\vec{\beta}}$ is the Lorentz matrix given in (79), appendix B. It follows from (71) that the relation between scattering matrix is

$$L_0(\vec{\beta})\; S_{E,\vec{P}}\; L_0(\vec{\beta})^{-1} \;=\; S_{\Lambda_{\vec{\beta}}(E,\vec{P})} \tag{72}$$

which shows that S is known once $S_{E,0}$ is given. Let us restrict ourselves to $\mathcal{H}_{\vec{P}=0}$ on which the interaction operator $V_0 = H_{\vec{P}=0} - H_{0,\vec{P}=0}$ has one-dimensional range, that is $V_0\varphi = \lambda(\eta, \varphi)_0\eta$ where $(.,.)_0$ is the scalar product of $\mathcal{H}_{\vec{P}=0}$ and $\eta(\vec{Q}) = \zeta(\|\vec{Q}\|)$. According to [19], theorem 3 of sect. 6.7, the scattering operator gives for all $v \in \mathcal{H}_{E,\vec{P}=0}$

$$\left(v, (S_{E,0} - I)v\right)_{E,0} \;=\; -2\pi i\lambda\; D(E + i0)^{-1}\; \left|(\eta_E, v)_{E,0}\right|^2 \tag{73}$$

where

$$D(z) \;=\; 1 \;+\; \lambda\left(\eta, (H_{0,0} - z)^{-1}\eta\right)_0 \tag{74}$$

for all $z \in \mathbb{C}$, $Im z > 0$. In (73), $(.,.)_{E,0}$ is the scalar product of $\mathcal{H}_{E,\vec{P}=0}$ and η_E is the restriction of η to $\mathcal{H}_{E,\vec{P}=0}$. In fact $\eta_E = \zeta(\frac{1}{2}\sqrt{E^2 - 4m^2})$ is a constant in $\mathcal{H}_{E,\vec{P}=0}$ so that $(\eta_E, v)_{E,0} = \zeta(\frac{1}{2}\sqrt{E^2 - 4m^2})(1, v)_{E,0}$, which is not zero for all

$2m < E < \sqrt{R^2+4m^2}$ (see the definition of η) and for all v in a large set of $\mathcal{H}_{E,\vec{P}=0}$. On the other hand, for E as above let us compute the limit needed in (73)

$$
\begin{aligned}
D(E+i0) &= 1 - \lambda i\pi^2 \frac{\sqrt{E^2-4m^2}}{2E} \left|\zeta\left(\frac{1}{2}\sqrt{E^2-4m^2}\right)\right|^2 \\
&\quad + \frac{\pi}{2}\, p.v. \int_{2m}^{2\sqrt{R^2+m^2}} \frac{du}{u} \frac{\sqrt{u^2-4m^2}}{u-E} \left|\zeta\left(\frac{1}{2}\sqrt{u^2-4m^2}\right)\right|^2
\end{aligned}
$$

which has obviously a non zero imaginary part. Then $S_{E,0} - I$ is non-zero for all such E. Because the representation in $\mathcal{H}$ generated by $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ is continuous, if follows from (72) that the scattering operator S differs from identity.

Proof of ii). From proposition 6 with (70) and $\lambda = -1$ the equation (61) can be written as follows

$$
\varphi(\vec{Q}) = K \frac{\zeta(\|\vec{Q}\|)}{2\omega(\vec{Q}) - m_B}
$$

where K is a constant. This function belongs to $D(M_{\vec{P}=0}) \subset \mathcal{H}_{\vec{P}=0}$ and is an eigenfunction of $M_{\vec{P}=0}$ provided the implicit equation

$$
1 = \int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2} - m_B}. \tag{75}
$$

admits a solution $0 < m_B < 2m$. To study this last question we consider the r.h.s. of (75) as a function of the variable m_B. For $m_B = 0$ we get from (68)

$$
\begin{aligned}
\int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2}} &\le \|\zeta\|_\infty^2 \int_{\mathcal{K}} \frac{d\vec{Q}}{8(\vec{Q}^2+m^2)^{3/2}} \\
&\le \frac{m^3}{2|\mathcal{K}|} \frac{|\mathcal{K}|}{8m^3} = \frac{1}{16} < 1.
\end{aligned}
$$

Now it follows from (69) that for $m_B = 2m$ the r.h.s. of (75) is majorated as follows

$$
\begin{aligned}
\int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2} - 2m} &= \frac{\pi}{2} \int_0^R dr\, d(r)^2 \frac{\sqrt{r^2+m^2}+m}{r^2+m^2} \\
\ge \frac{m^3}{8R^3} \int_0^{R/2m} dr \frac{1+\sqrt{r^2+1}}{r^2+1} &= \left(\frac{m}{2R}\right)^3 \left(\arctan\frac{R}{2m} + \operatorname{arg\,sinh}\frac{R}{2m}\right)
\end{aligned}
$$

which is > 1 for $m/R > 2$. Thus (75) has a (unique) solution $0 < m_B < 2m$, which is an eigenvalue of (61) and then, from proposition 6, an eigenvalue of M. []

8. Conclusion

Let us sum up what we have found. We have constructed a family of unitary, continuous representations of the Poincaré group in four space-time dimensions, as perturbations of the two free boson model. The scattering operator S can be performed and is unitary and non trivial in some cases.

The physical content of this mathematical construction is based on Wigner's famous interpretation [21], according to which an elementary particle is described by an irreducible, unitary, and continuous representation of $\mathcal{P}_+^\uparrow$. In the construction of S we have verified the existence of scattering states, i.e. states which for large (or small) time approach the tensor product of two irreducible unitary representations of $\mathcal{P}_+^\uparrow$, interpreted as describing two free particles. This permits us to affirm that our models really describe relativistic quantum systems of two particles. Moreover the non triviality of S allows us to claim that the interaction between the particles can be effective.

The absence of particle creation or anhilation makes these models interesting for low energy physics, in particular for the bound state problem. The existence of a bound state equation similar to the ordinary Schrödinger equation of Quantum Mechanics (and which can be called a 'relativistic Schrödinger equation') is probably the strongest result of the paper.

However these results have been obtained for interaction operators $\mathcal{O}$ with ranges contained in the subspace of zero angular momentum (the so-called 's-wave subspace'). To get a better characterization of these interactions, let us compute the change of the second Casimir operator, the Pauli-Lubanski operator, given by (6). In the representation (13) it becomes

$$W^2 \;\;=\;\; W_0^2 \;+\; \{\{\mathcal{O},\vec{W}_0\},\vec{W}_0\} \;\;+\;\; \{\mathcal{O},\vec{W}_0\}^2$$

where W_0^2 is given by (21) and $\vec{W}_0 = -H_0\vec{J} + \vec{P}\wedge\vec{L}_0$. Now condition (27) implies $\{\mathcal{O},\vec{W}_0\} = 0$ and thus

$$W^2 \;\;=\;\; W_0^2.$$

In contradiction with what happens to the Casimir operator M, the introduction of these interactions has no effect on the second Casimir operator, the Pauli-Lubanski operator W^2.

Appendix A. One-particle representation

We show that the operators (7) are the infinitesimal generators of a unitary, continuous and irreducible representation of the universal covering of the Poincaré group $\mathcal{P}_+^\uparrow$. Such a representation is a good candidate for a theory describing a quantum relativistic spinless particle of mass $m > 0$. The Poincaré representation describing one particle of mass $m > 0$ without spin in given by the following lemma (which is essentially given in [21]).

Lemma 10 *The operators* (7) *are the infinitesimal generators of a unitary, continuous and irreducible representation of the universal covering of the Poincaré group* $\mathcal{P}_+^\uparrow$ *in the space* $L^2(\mathbb{R}^3,\sigma)$.

We recall that $d\sigma(\vec{p}) = d\vec{p}\,[2\omega(\vec{p})]^{-1}$ where $\omega(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$.

Proof. Let us consider the following transformations of functions $\phi \in L^2(I\!R^3, \sigma)$:

$$(\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta}) \cdot \ \phi(\vec{p}) \ := \ e^{i(\tau\omega(\vec{p}) + \vec{\xi} \cdot \vec{p})} \ \phi(R_{\vec{\alpha}}\lambda_{\vec{\beta}}\vec{p}) \tag{76}$$

for all $\tau \in I\!R$, $\vec{\xi}, \vec{\alpha}, \vec{\beta} \in I\!R^3$, $\|\vec{\beta}\| < 1$, and almost all $\vec{p} \in I\!R^3$, where $R_{\vec{\alpha}}$ is the matrix of the rotation of $I\!R^3$ of the Euler angles $\vec{\alpha}$ and $\lambda_{\vec{\beta}}\,\vec{p}$ is the 'spatial' component of the pure Lorentz transform of $(\omega(\vec{p}), \vec{p})$ of velocity $\vec{\beta}$ (i.e. the 'spatial' component of (79) (see below) for such four-vector).

Let us show that the transformation (77) defines a unitary, continuous and irreducible representation in $L^2(I\!R^3, \sigma)$ of the Poincaré group $\mathcal{P}_+^\uparrow$, the infinitesimal generators of which are given by (7).

The unitarity and the group law are easily seen by introducing a fourth variable p^0 together with a Dirac-delta function in the measure, that is by replacing $L^2(I\!R^3, \sigma)$ by $L^2(I\!R^4, \Sigma)$ where $d\Sigma(p^0, \vec{p}) = dp^0 dp_1 dp_2 dp_3 \delta({p^0}^2 - \vec{p}^{\,2} - m^2)\theta(p^0)$, θ being the Heaviside function. Because of the Σ measure, a function $\phi(p^0, \vec{p})$ is not distinguishable from $\phi(\omega(\vec{p}), \vec{p})$. The transformation (78) becomes, after Fourier transformation

$$(\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta}) \cdot \ \tilde{\phi}(t, \vec{x}) \ = \ \tilde{\phi}\left((\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta})^{-1} \cdot (t, \vec{x})\right)$$

(where $\tilde{\phi}$ is the Fourier transform of ϕ) that is, given in terms of the ordinary Poincaré transformation on space-time. The group law is now obvious. The unitarity follows from the Poincaré invariance of the measure Σ.

The continuity of the representation is easily seen in $L^2(I\!R^3, \sigma)$, by standard analysis methods. The irreducibility is obtained as follows. Let us take any $\phi \in L^2(I\!R^3, \sigma)$, $\phi \neq 0$. Let us show that the only vector orthogonal to $\{e^{i\vec{\xi} \cdot \vec{p}}\phi(\lambda_{\vec{\beta}}\,\vec{p}) \mid \vec{\xi}, \vec{\beta} \in I\!R^3\}$ is 0. By the Fourier theory

$$\int_{I\!R^3} d\sigma(\vec{p}) \ \psi(\vec{p})^* \ e^{i\vec{\xi} \cdot \vec{p}}\phi(\lambda_{\vec{\beta}}\,\vec{p}) \ = \ 0 \quad \text{for all} \quad \vec{\xi} \in I\!R^3 \tag{77}$$

implies that $\psi(\vec{p}) = 0$ for almost all $\vec{p}$ such that $\phi(\lambda_{\vec{\beta}}\,\vec{p}) \neq 0$. Without loss of generality we may assume that $\phi(0) \neq 0$. For all $\vec{p} \in I\!R^3$ the vector $\vec{\beta} = -\,\vec{p}/\omega(\vec{p})$ satisfies $\lambda_{\vec{\beta}}\,\vec{p} = 0$ (see appendix B). Thus if moreover we impose that (78) holds for all $\vec{\beta} \in I\!R^2$ we get $\psi = 0$.

After an elementary calculation the operators (7) turn out to be the infinitesimal generators of the representation (77). []

Appendix B. The relative momentum

We explain the change of variable $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ given in (16) and establish somes useful formulas.

The meaning of $\vec{P}=\vec{p}_1+\vec{p}_2$, the total momentum, is clear. The notion of relative momentum is more subtle. Physically it must be related to the momentum of one of the particles in the centre-of-mass frame.

Let $p_1=(\omega(\vec{p}_1),\vec{p}_1)$ and $p_2=(\omega(\vec{p}_2),\vec{p}_2)$ be the energy-momenta of two free particles, of the same mass to simplify. We define first

$$P=\left(P^0,\vec{P}\right)=p_1+p_2 \qquad\qquad q=\left(q^0,\vec{q}\right)=\frac{1}{2}(p_1-p_2).$$

Note the useful relation $\vec{P}\cdot\vec{q}=P^0q^0$. Let $\Lambda_{\vec{\beta}}$ be a Lorentz transformation such that $\Lambda_{\vec{\beta}}P=(M_0,\vec{0}\ )$, where $M_0=[(P^0)^2-\vec{P}^2]^{1/2}$. Physically $\Lambda_{\vec{\beta}}$ is associated with a change of referential frame, which put the two particles of momenta $\vec{p}_1,\vec{p}_2$ on their centre-of-mass frames. Then we define the relative momentum $\vec{Q}$ as the spatial projection of $\Lambda_{\vec{\beta}}\ q$. Note that the relative momentum just defined depends on the way to reach the centre-of-mass frame, which is not unique (up to rotations of $I\!\!R^3$).

To get more explicit expressions we need the formula of a pure Lorentz transform $\Lambda_{\vec{\beta}}$ of velocity $\vec{\beta}\in I\!\!R^3$, $\|\vec{\beta}\|<1$. It acts on a four-dimensional vector $(p^0,\vec{p})\in I\!\!R\times I\!\!R^3$ as follows

$$\Lambda_{\vec{\beta}}\begin{pmatrix} p^0 \\ \vec{p}\end{pmatrix}=\begin{pmatrix}\gamma(p^0+\vec{\beta}\cdot\vec{p}) \\ \\ \vec{p}+\vec{\beta}\left[\gamma p^0+\frac{\gamma-1}{\beta^2}\vec{p}\cdot\vec{\beta}\right]\end{pmatrix} \tag{78}$$

where we have put $\beta=\|\vec{\beta}\|$ and $\gamma=(1-\beta^2)^{-1/2}$. The condition that the spatial component of $\Lambda_{\vec{\beta}}P$ vanishes gives

$$\vec{P}\ +\ \vec{\beta}\left[\gamma P^0\ +\ \frac{\gamma-1}{\beta^2}\ \vec{P}\cdot\vec{\beta}\right]\ =\ 0.$$

If we choose the direction of $\vec{\beta}$ parallel to $\vec{P}$ (this condition leads to a particular way to reach the centre-of-mass frame), we get the unique solution $\vec{\beta}=-\vec{P}/P^0$. After some calculation one gets $\Lambda_{-\vec{P}/P^0}\ q=(0,\vec{Q})$ where

$$\vec{Q}\ =\ \vec{q}\ -\ \frac{\vec{P}}{P^0}\left[\frac{P^0q^0}{M_0}\ -\ \frac{P^0-M_0}{M_0}\ \frac{(P^0)^2}{\vec{P}^2}\ \frac{\vec{q}\cdot\vec{P}}{P^0}\right]$$

which, after some easy development, gives finally the formula (16).

Let us calculate the operators of the two free boson representation (8) in these variables, to establish (17). The norm $\|\vec{Q}\|$ is obtained by the properties of the Lorentz transform, which gives $\vec{Q}^2=(q^0)^2-\vec{q}^{\ 2}$. From the trivial identity $(P^0)^2-\vec{P}^2=4m^2+4[(q^0)^2-\vec{q}^{\ 2}]$ it follows

$$M_0\ =\ 2\omega(\vec{Q})$$

and then

$$P^0\ =\ \omega(\vec{p}_1)+\omega(\vec{p}_1)\ =\ M_0^2+\vec{P}^2\ =\ \sqrt{\vec{P}^2+4\vec{Q}^2+4m^2}\ =\ \Omega(\vec{P},\vec{Q})$$

in agreement with (20) and (18). The angular momentum $\vec{J}$ is easily obtained because a rotation of the system $\vec{p}_1, \vec{p}_2$ leads to the same rotation of $\vec{P}, \vec{Q}$ (this is an obvious consequence of the formula (16)). The Lorentz generator $L_{0,1}$ is obtained as follows. Under an infinitesimal Lorentz transform of hyperbolic angle γ in the first direction (of unit vector $\vec{e}_1$) the fundamental quantities become

$$\begin{aligned} \vec{P} &\to \vec{P} + \gamma P^0 \vec{e}_1 & \qquad \vec{q} &\to \vec{q} + \gamma q^0 \vec{e}_1 \\ P^0 &\to P^0 + \gamma P_1 & \qquad q^0 &\to q^0 + \gamma q_1 \ . \end{aligned}$$

Under such a transformation $\vec{Q}$ becomes

$$\begin{aligned} \vec{Q} \ &\to \ \vec{q} + \gamma q^0 \vec{e}_1 - \frac{q^0 + \gamma q_1}{M_0 + P^0 + \gamma P_1} \ \left(\vec{P} + \gamma P^0 \vec{e}_1\right) \\ &\approx \ \vec{Q} \ + \ \gamma \ \frac{1}{M_0 + P^0} \ \left((\vec{P}\cdot\vec{Q})\vec{e}_1 - Q_1 \vec{P}\right) \end{aligned}$$

where we have performed a first order development and used the relation $q^0 M_0 = \vec{P}\cdot\vec{Q}$. The formula for $L_{0,1}$ given in (17) follows from the calculation

$$L_{0,1}\phi(\vec{P},\vec{Q}) \ = \ -i\partial_\gamma \ \phi\left(\vec{P} + \gamma P^0 \vec{e}_1, \vec{Q} + \gamma \ \frac{(\vec{P}\cdot\vec{Q})\vec{e}_1 - Q_1\vec{P}}{M_0 + P^0}\right)\Bigg|_{\gamma=0} .$$

$L_{0,2}$ and $L_{0,3}$ are obtained in the same way.

The Jacobian of the transformation $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ is calculated in two steps. First we perform the transformation $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{q}\,)$, with Jacobian 1, and then the transformation $(\vec{P}, \vec{q}\,) \to (\vec{P}, \vec{Q})$ with Jacobian (using the formula (16) for $\vec{Q}$)

$$\left|\det\left(\frac{\partial Q_i}{\partial q_j}\right)\right|^{-1} \ = \ \left[1 \ - \ \vec{P}\cdot\vec{\nabla}_{\vec{q}} \ \frac{q^0}{M_0 + P^0}\right]^{-1}$$

which, after a long and tedious calculation, gives $4\omega(\vec{p}_1)\omega(\vec{p}_2)[M_0 P^0]^{-1}$. Thus the measure σ_2 becomes $d\sigma_2(\vec{p}_1, \vec{p}_2) = d\mu(\vec{P}, \vec{Q}) = d\vec{P} d\sigma(\vec{Q})\Omega(\vec{P}, \vec{Q})^{-1}$, as in (19).

Appendix C. The fundamental equation

The fundamental equation (29) is the condition on a kernel $h(\vec{P}, \vec{Q}, \vec{Q}')$, depending only on the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$, for the relation (14) and (15) to hold. This norm dependence leads to (27) and thus to $L_{0,j}\mathcal{O} = L^{\vec{P}}_{0,j}\mathcal{O}$ and $\mathcal{O}L_{0,j} = \mathcal{O}L^{\vec{P}}_{0,j}$ where $L^{\vec{P}}_{0,j} := -i\Omega(\vec{P}, \vec{Q})\partial_{P_j}$ for all $j \in \{1, 2, 3\}$.

Let us calculate the jth equation of (14). The linear part in $\mathcal{O}$ can be written

$$A_j \ := \ [\{H_0, \mathcal{O}\}, L_{0,j}] + [H_0, \{L_{0,j}, \mathcal{O}\}] \ = \ 2(H_0\mathcal{O}L_{0,j} - L_{0,j}\mathcal{O}H_0 + iP_j\mathcal{O})$$

where we have used $[H_0, L_{0,j}] = iP_j$ and the fact that P_j and $\mathcal{O}$ commute. By applying this operator to a vector $\phi \in \mathcal{D}$ (the domain given by (46)) we obtain, in obvious symbolic notation

$$\begin{aligned} A_j\phi(\vec{P},\vec{Q}) &= -2i\Omega \int d\sigma(\vec{Q}')\frac{\partial_{P_j}\phi h}{\Omega+\Omega'} + 2i\Omega\partial_{P_j}\int d\sigma(\vec{Q}')\frac{\phi h}{\Omega+\Omega'} \\ &+ 2iP_j \int \frac{d\sigma(\vec{Q}')}{\Omega'}\frac{\phi h}{\Omega+\Omega'} = 2i\int d\sigma(\vec{Q}')\phi\left(\Omega\partial_{P_j}\frac{h}{\Omega+\Omega'} + \frac{P_j}{\Omega'}\frac{h}{\Omega+\Omega'}\right) \\ &= 2i\int d\sigma(\vec{Q}')\ \phi(\vec{P},\vec{Q}')\ \frac{\Omega(\vec{P},\vec{Q})\ \partial_{P_j}h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}. \end{aligned}$$

To study the bilinear part, we rewrite (25) and (26) :

$$\begin{aligned} \{H_0,\mathcal{O}\}\ \phi(\vec{P},\vec{Q}) &= \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\ h(\vec{P},\vec{Q},\vec{Q}') \\ \{L_{0,j},\mathcal{O}\}\ \phi(\vec{P},\vec{Q}) &= -i\int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}(\partial_{P_j}\phi(\vec{P},\vec{Q}'))\ h(\vec{P},\vec{Q},\vec{Q}) \\ &- i\Omega(P,q) \int d\sigma(\vec{Q}')\phi(\vec{P},\vec{Q}')\ \partial_{P_j}\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')(\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}'))}\ . \end{aligned}$$

Thus the bilinear part in $\mathcal{O}$ gives, in symbolic notation

$$\begin{aligned} &B_j\phi(\vec{P},\vec{Q}) := [\{H_0,\mathcal{O}\},\{L_0,\mathcal{O}\}]\ \phi(\vec{P},\vec{Q}) = \\ &\int \frac{d\sigma''}{\Omega''}h^{Q,Q''}\left[-i\int\frac{d\sigma'}{\Omega'}(\partial_j\phi')h^{Q'',Q'} - i\Omega''\int d\sigma'\phi'\partial_j\frac{h^{Q'',Q'}}{\Omega'(\Omega''+\Omega')}\right. \\ &\qquad \left. + i\partial_j\int\frac{d\sigma'}{\Omega'}\phi' h^{Q'',Q'}\right] + i\Omega\int d\sigma''\left(\partial_j\frac{h^{Q,Q''}}{\Omega''(\Omega+\Omega'')}\right)\int\frac{dq'}{\Omega'}\phi' h^{Q'',Q'} \\ &= i\int d\sigma''\int d\sigma'\phi'\left(-\ h^{Q,Q''}\partial_j\frac{h^{Q'',Q'}}{\Omega'(\Omega''+\Omega')} + \frac{h^{Q,Q''}}{\Omega''}\partial_j\frac{h^{Q'',Q'}}{\Omega'}\right. \\ &\qquad\qquad\qquad\qquad \left. + \frac{\Omega}{\Omega'}\left(\partial_j\frac{h^{Q,Q''}}{\Omega''(\Omega+\Omega'')}\right)h^{Q'',Q'}\right) \\ &= i\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \phi(\vec{P},\vec{Q}')\int\frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\left\{-\ \frac{P_j}{\Omega(\vec{P},\vec{Q}'')}\ h(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}')\right. \\ &\qquad + \frac{\Omega(\vec{P},\vec{Q})\Omega(\vec{P},\vec{Q}'')\partial_{P_j}h(\vec{P},\vec{Q},\vec{Q}'')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}'')}\ h(\vec{P},\vec{Q}'',\vec{Q}') \\ &\qquad \left. + \ h(\vec{P},\vec{Q},\vec{Q}'')\ \frac{\Omega(\vec{P},\vec{Q}'')\Omega(\vec{P},\vec{Q}')\partial_{P_j}h(\vec{P},\vec{Q}'',\vec{Q}')}{\Omega(\vec{P},\vec{Q}'')+\Omega(\vec{P},\vec{Q}')}\right\}. \end{aligned}$$

Because h depends on $\vec{P}$ only through $\|\vec{P}\|$ we can replace everywhere ∂_{P_j} by $P_j\|\vec{P}\|^{-1}\partial_{\|\vec{P}\|}$. The condition $A_j + B_j = 0$, which must hold for all $\phi \in \mathcal{D}$, leads to the fundamental equation (29), for all $j \in \{1,2,3\}$.

The equations (15) are slightly more difficult to establish. By using $\vec{J}^{\vec{Q}}\mathcal{O} = 0$ and by putting $\vec{J}^{\vec{P}} := \vec{J} - \vec{J}^{\vec{Q}}$ the linear part of the first equation of (15) can be written, in symbolic notations :

$$C\phi(\vec{P},\vec{Q}) \ := \ 2\left(L_{0,1}^{\vec{P}}\mathcal{O}L_{0,2}^{\vec{P}} - L_{0,2}^{\vec{P}}\mathcal{O}L_{0,1}^{\vec{P}} - i\mathcal{O}J_3^{\vec{P}}\right)\phi(\vec{P},\vec{Q}) \ =$$

$$-2\Omega\int d\sigma'\left[(\partial_2\phi')\,\partial_1 - (\partial_1\phi')\,\partial_2\right]\frac{h}{\Omega+\Omega'} \ - \ 2\int\frac{d\sigma'}{\Omega'}\left[(P_1\partial_2 - P_2\partial_1)\,\phi'\right]\frac{h}{\Omega+\Omega'}$$

$$= \ -\frac{2i}{\|\vec{P}\|}\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\ \left(J_3^{\vec{P}}\phi(\vec{P},\vec{Q}')\right)\ Dh(\vec{P},\vec{Q},\vec{Q}')$$

The bilinear part of this equation gives

$$D\phi(\vec{P},\vec{Q}) \ := \ \left(\frac{1}{2}(C\mathcal{O}+\mathcal{O}C) + L_{0,1}^{\vec{P}}\mathcal{O}^2L_{0,2}^{\vec{P}} - L_{0,2}^{\vec{P}}\mathcal{O}^2L_{0,1}^{\vec{P}} + i\mathcal{O}^2J_3^{\vec{P}}\right)\phi(\vec{P},\vec{Q}) =$$

$$= \ -\frac{i}{\|\vec{P}\|}\int\frac{d\sigma'}{\Omega'}J_3^{\vec{P}}\phi'\int\frac{d\sigma''}{\Omega''}\left(Dh^{\vec{Q},\vec{Q}''}\frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''} \ + \ \frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}Dh^{\vec{Q}'',\vec{Q}'}\right)$$

$$(-i)^2\int\frac{d\sigma'}{\Omega'}\,\Omega\Omega'\left[(\partial_2\phi')\partial_1 - (\partial_1\phi')\partial_2\right]\int\frac{d\sigma''}{\Omega''}\,\frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}\,\frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''}$$

$$+ \ \int\frac{d\sigma'}{\Omega'}\ \left[(P_1\partial_2 - P_2\partial_1)\phi'\right]\int\frac{d\sigma''}{\Omega''}\ \frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}\ \frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''}$$

$$= \ -\ \frac{i}{\|\vec{P}\|}\int\frac{d\sigma'}{\Omega'}\ J_3^{\vec{P}}\phi'\int\frac{d\sigma''}{\Omega''^2}\left(Dh^{\vec{Q},\vec{Q}''}h^{\vec{Q}'',\vec{Q}'} \ + \ h^{\vec{Q},\vec{Q}''}Dh^{\vec{Q}'',\vec{Q}'}\right)$$

$$+ \ i\int\frac{d\sigma'}{\Omega'}\ J_3^{\vec{P}}\phi'\int\frac{d\sigma''}{\Omega''^3}\ h^{\vec{Q},\vec{Q}''}h^{\vec{Q}'',\vec{Q}'}$$

The condition $C + D = 0$, which must hold for all $J_3^{\vec{P}}\phi$, leads to the fundamental equation (29) again. The two other equations of (15) lead to the same result.

Appendix D. A criterium for trace-class operators

Let $\mathcal{H}_s = L^2(\mathbb{R}^3, s)$ be a Hilbert space where s is any positive σ-finite measure, and let F be a integral operator on $\mathcal{H}_s$ of kernel $f : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{C}$ that is

$$(Fu)(x) \ = \ \int_{\mathbb{R}^3} f(x,y)\ u(y)\ ds(y) \tag{79}$$

for all suitable $u \in \mathcal{H}_s$.

Lemma 11 *Let K be a compact subset of $\mathbb{R}^3$ and $f \in L^2(\mathbb{R}^3\times\mathbb{R}^3, s\otimes s)$ such that for almost all $x \in \mathbb{R}^3$ the function $y \mapsto f(x,y)$ has support in K. Let us suppose that the three derivatives*

$$\partial_{y_1}f(x,y), \quad \partial_{y_2}\partial_{y_1}f(x,y)\ , \quad \partial_{y_3}\partial_{y_2}\partial_{y_1}f(x,y)$$

exist in $L^2(\mathbb{R}^3\times\mathbb{R}^3, s\otimes s)$. Then F, defined by (80), is trace class.

Proof. We follow [20], sect. XI.9.32. We will use that a product of two Hilbert-Schmidt operators is a trace class operator. Let $k > 0$ be such that $K \subset [-k,k]^3$. Let $u \in \mathcal{H}_s$. From three integrations by parts we get

$$\begin{aligned}(Fu)(x) &= \int_{[-k,k]^3} [\partial_{y_3}\partial_{y_2}\partial_{y_1} f(x,y)] \int_{-k}^{y_1}\int_{-k}^{y_2}\int_{-k}^{y_3} u(t)\ dt\ ds(y) \\ &= \int_{I\!R^3} [\partial_{y_3}\partial_{y_2}\partial_{y_1} f(x,y)] \int_{-k}^{y_1}\int_{-k}^{y_2}\int_{-k}^{y_3} \chi_{[-k,k]^3}(y)\ u(t)\ dt\ ds(y)\end{aligned}$$

We have written F as a product of two Hilbert-Schmidt operators, the first one of kernel $(x,y) \mapsto \partial_{y_3}\partial_{y_2}\partial_{y_1} f(x,y)$ and the second one of kernel $(x,y) \mapsto \chi_{[-k,k]^3}(x)$ $\chi_{[-k,x_1]}(y_1)\chi_{[-k,x_2]}(y_2)\chi_{[-k,x_31]}(y_3)$. Thus F is trace class. []

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