Suppression of Critical Fluctuations by Strong Quantum Effects in Quantum Lattice Systems

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Abstract

Translation invariant models of quantum anharmonic oscillators with a polynomial anharmonicity and a ferroelectric interaction are considered. For these models, it is proved that the critical fluctuations of the position operator, peculiar to the critical point, are suppressed at all temperatures provided the oscillators are strongly quantum. The latter is shown to occur in particular if the oscillator's mass is less than some threshold value depending on the anharmonicity parameters.

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1 Posing of the problem and results.

The object of our investigation in this paper is a countable system of quantum particles (oscillators) performing one-dimensional oscillations around their equilibrium positions and interacting among themselves. The set of all equilibrium positions is assumed to be a lattice $\mathbb{L}$. For further simplicity we choose this lattice to be simple cubic, that means $\mathbb{L} = \mathbb{Z}^\nu$ with certain positive integer ν. For every $l \in \mathbb{L}$, let a quantum particle with the mass m and one internal degree of freedom be given. The dynamics of its oscillations around the equilibrium position l is described by means of canonical momentum and displacement operators $\{p_l, q_l\}$ defined on (a dense subset of) the complex Hilbert space $\mathcal{H}_l = L^2(\mathbb{R}_l, dx_l)$. For every finite subset $\Lambda \subset \mathbb{L}$, we put $\mathcal{H}_\Lambda = \bigotimes_{l\in\Lambda} \mathcal{H}_l$. The dynamics of the whole system of particles can be described by means of the model Hamiltonian formally given by

$$H = \frac{1}{2} \sum_{l\neq l';l,l'\in\mathbb{L}} d_{ll'} q_l q_{l'} + \sum_{l\in\mathbb{L}} H_l; \tag{1}$$

$$H_l = \frac{{p_l}^2}{2m} + V(q_l), \tag{2}$$

where the coefficients $d_{ll'}$ form the dynamical matrix, m is the oscillator's mass, and V is the one-particle potential. The first term in (1) describes the interaction between the particles, the second one describes their individual properties by means of the one-particle Hamiltonian H_l, $l \in \mathbb{L}$. The interaction is assumed to be translation invariant and ferroelectric. The simplest example of such type can be given as follows. Let $|l - l'|$ be the Euclidean distance between the points l and l'. We set for $l \neq l'$

$$d_{ll'} \;=\; -\phi(|l - l'|), \tag{3}$$

where ϕ is some nonnegative monotone function with the sufficiently fast falloff, such that

$$\sum_{l\in\mathbb{L}} \phi(|l|) < \infty.$$

For the further convenience, we include the case $l = l'$ in the sum in (1) putting $d_{ll} = 0$. The formal Hamiltonian cannot be defined directly as an operator and used is "represented" by its local versions – indexed by finite subsets $\Lambda \subset \mathbb{L}$ local Hamiltonians, each of which is defined as essentially self-adjoint lower bounded operator acting on the corresponding Hilbert space $\mathcal{H}_\Lambda$. For our purposes, it is enough to consider these Hamiltonians indexed by boxes $\Lambda = \{l = (l_1, \ldots, l_\nu) \mid l_j^0 \leq l_j \leq l_j^1, \ j = 1, \ldots, \nu; \ l_j^0 < l_j^1, \ l_j^0, l_j^1 \in \mathbb{Z} \ \}$. To introduce the local Hamiltonian in the box Λ which corresponds to the periodic boundary conditions, we set

$$d_{ll'}^\Lambda \ = \ -\phi(|l - l'|_\Lambda), \ \ l \neq l'; \ d_{ll}^\Lambda \ = \ 0; \tag{4}$$

where the function ϕ is the same as in (3) and the distance $|l-l'|_\Lambda$ is assumed to be on the torus $T(\Lambda)$ which is obtained from Λ by suitable identification of boundary components. For the box Λ described above, it can be defined by the expression

$$|l - l'|_\Lambda^2 = \sum_{j=1}^{\nu} |l_j - l'_j|_\Lambda^2,$$

where

$$|l_j - l'_j|_\Lambda = \min\{|l_j - l'_j|; l_j^1 - l_j^0 + 1 - |l_j - l'_j|\}. \tag{5}$$

This yields

$$|l - l'|_\Lambda \leq |l - l'|$$

for all pairs l, l' in Λ. Taking into account the monotonocity of ϕ, one gets

$$d_{ll'}^\Lambda \leq d_{ll'} \tag{6}$$

for all pairs l, l' in Λ. Thus we set

$$H_\Lambda = \frac{1}{2} \sum_{l,l' \in \Lambda} d_{ll'}^\Lambda q_l q_{l'} + \sum_{l \in \Lambda} H_l, \tag{7}$$

$$H_\Lambda^0 = \frac{1}{2} \sum_{l,l' \in \Lambda} d_{ll'} q_l q_{l'} + \sum_{l \in \Lambda} H_l, \tag{8}$$

where H_l is given by (2). The latter local Hamiltonian corresponds to zero boundary conditions. These ones will also be considered.

As it was mentioned above, the dynamical matrices introduced by (3) – (5) are the simplest ones having the properties we will utilize in this paper. Let us describe them more systematically. Denote $\mathcal{L}$, a family of all boxes. For a given box Λ, let $\mathcal{L}(\Lambda)$ denote a partition of $\mathbb{L}$ by boxes that can be obtained as corresponding translations of this Λ. Denote by $\mathcal{G}$ the group of all translations of $\mathbb{L}$. Let $\mathcal{G}(\Lambda)$ be the subgroup of $\mathcal{G}$ consisting of translations which generate the partition $\mathcal{L}(\Lambda)$. This means $\mathcal{L}(\Lambda) = \{t(\Lambda),\ t \in \mathcal{G}(\Lambda)\}$, where $t(\Lambda) = \{t(l),\ l \in \Lambda\}$. Thereafter, the dynamical matrices $(d_{ll'})_{l,l' \in \mathbb{L}}$ and $(d^{\Lambda}_{ll'})_{l,l' \in \Lambda}$ have the following properties

(D1) $d_{ll'}$ is invariant under translations on $\mathbb{L}$;

(D2) $d^{\Lambda}_{ll'} \leq 0$, $d_{ll'} \leq 0$ (ferroelectricity); $\quad d^{\Lambda}_{ll} = d_{ll} = 0$;

(D3) the series $\sum_{l' \in \mathbb{L}} d_{ll'}$ converges for all $l \in \mathbb{L}$;

(D4)

$$d^{\Lambda}_{ll'} = \min\{d_{lt(l')}\ :\ t \in \mathcal{G}(\Lambda)\}.$$

Now let us describe the one-particle potentials V that will be considered. We assume:

(V1) V is an even polynomial, $\deg V = 2M$ with $M \geq 2$;

(V2) the polynomial v such that $v(q^2) = V(q)$ is convex on $\mathbb{R}_+ = [0, +\infty)$.

An example of V satisfying these conditions is

$$V(q) = aq^2 + q^4 \tag{9}$$

with arbitrary real a. For $a < 0$, the potential (9) has a double-well shape.

In the sequel, when speaking about a model of quantum anharmonic oscillators, we will mean the model possessing the properties (D1)–(D4), (V1),(V2). The simplest example of such a model was described above.

Remark 1.1 *Let V satisfy (V1), (V2). Then the probability measure*

$$d\mu_V(x) = C \exp(-V(x))dx$$

(C is a positive normalizing constant) belongs to the BFS class of measures [6], and for every $\alpha > 0$, the function $\exp(\alpha x^2)$ is μ_V-integrable on $\mathbb{R}$.

For every $\Lambda \in \mathcal{L}$ and given inverse temperature $\beta = T^{-1}$, the Gibbs states in Λ are defined as functionals on appropriate C^*-algebra of observables by means of the local Hamiltonians H_Λ, H^0_Λ given by (7), (8) respectively as follows:

$$\gamma_{\beta,\Lambda}(A) = \frac{\mathrm{trace}(Ae^{-\beta H_\Lambda})}{\mathrm{trace}(e^{-\beta H_\Lambda})}, \quad \gamma^0_{\beta,\Lambda}(A) = \frac{\mathrm{trace}(Ae^{-\beta H^0_\Lambda})}{\mathrm{trace}(e^{-\beta H^0_\Lambda})} \tag{10}$$

see e.g. [4] for the definition and related discussions. The state γ is periodic, the state γ^0 corresponds to zero boundary conditions. These states can be fully determined by means of temperature Green functions. Let S_β be a circle of length β (isometric to the segment $[0,\beta]$ with identified ends). For each ordered set $\{\tau_0, \tau_1, \ldots, \tau_n\}$, $\tau_i \leq \tau_{i+1}$, $\tau_i \in S_\beta$, and appropriate operators $\{A_0, A_1, \ldots, A_n\}$, the corresponding temperature Green function is determined as follows

$$\Gamma^{\beta,\Lambda}_{A_0;A_1;\ldots;A_n}(\tau_0, \tau_1, \ldots, \tau_n)$$
$$= \frac{1}{Z_\Lambda}\mathrm{trace}\left(A_0 e^{-(\tau_1-\tau_0)H_\Lambda} A_1 e^{-(\tau_2-\tau_1)H_\Lambda} \ldots A_n e^{-(\beta-\tau_n+\tau_0)H_\Lambda}\right), \tag{11}$$

where

$$Z_\Lambda = \mathrm{trace}\left(e^{-\beta H_\Lambda}\right). \tag{12}$$

The Green function defined by (11), (12) with H^0_Λ instead of H_Λ will be denoted by $\Gamma^{0,\beta,\Lambda}_{A_0;A_1;\ldots;A_n}(\tau_0, \tau_1, \ldots, \tau_n)$. These functions can be obtained by using the analytic continuation of the usual Green functions corresponding to the unitary time evolution, see [1],[7]. In the sequel, all properties of such Green functions, including the convergence of the sequences of these functions, are described as those that occur almost everywhere or for almost all values of their arguments. This means, they may not occur for values of the corresponding arguments which form a set of a zero Lebesgue measure. Thermodynamic properties of the model can be described by passing to the limit $\Lambda \nearrow \mathbb{L}$. To do it, we introduce the collections of boxes $\mathcal{C}$ as sets completely ordered by inclusion. Each such a set is countable thus can be considered as a sequence of boxes. For the models we consider, the existence of limiting periodic Gibbs states can be proved (see e.g. [2], where it was done for systems satisfying a reflection positivity condition). Our intention in this paper is to show that the large fluctuations of displacements of oscillators described by the Hamiltonians (1)–(7), (8) are absent (suppressed) at any temperature if the model parameters satisfy certain conditions that can be characterized as conditions of "strong quantumness". Thus we restrict ourselves to the consideration of those operators that describe such fluctuations. Due to the Z_2-symmetry of the local Hamiltonians (7), (8) one has $\gamma_{\beta,\Lambda}(q_l) = \gamma^0_{\beta\Lambda}(q_l) = 0$. For some real δ, we define

$$A_\delta(\Lambda) = \frac{1}{|\Lambda|^{\frac{1}{2}+\delta}} Q_\Lambda = \frac{1}{|\Lambda|^{\frac{1}{2}+\delta}} \sum_{l\in\Lambda} q_l. \tag{13}$$

This operator describes the fluctuations of displacements of particles in Λ. For some $\mathcal{C}$, let us consider the sequence $\{A_\delta(\Lambda), \Lambda \in \mathcal{C}\}$ from the point of view of the central limit theorem. In the case of a weak dependence between oscillations, one can expect the asymptotic "normality" of this sequence with $\delta = 0$. But if this "normality" occurs for some $\delta > 0$, this will correspond to the appearance of a strong dependence between oscillations which takes place at the critical temperature. In a quantum case, the normality of these sequences can be understood in terms of corresponding Green functions. Accordingly, we give the following definition of the critical temperature.

Definition 1.1 *For given temperature T, some $\delta > 0$, and a sequence $\mathcal{C}$, let the sequence of temperature Green functions $\{\Gamma^{\beta,\Lambda}_{A_\delta(\Lambda);A_\delta(\Lambda)}(\tau,\tau'), \Lambda \in \mathcal{C}\}$ given by (11)–(13), or the sequence of $\Gamma^{0,\beta,\Lambda}_{A_\delta(\Lambda);A_\delta(\Lambda)}(\tau,\tau')$ functions, converge to some function $\Gamma^\delta(\tau,\tau')$ different from zero on an S^2_β subset of a positive Lebesgue measure. Then one say that the critical fluctuations of displacements of particles occur at this temperature. The latter is said to be the critical temperature. If for every sequence $\mathcal{C}$ and arbitrary positive δ, these sequences of Green functions converge to zero, we say that the critical fluctuations are absent at this temperature.*

Now we can formulate our main result.

Theorem 1.1 *Let the model of quantum anharmonic oscillators which possesses the properties (D1)–(D4), (V1), (V2) be considered. For this model, there exists a positive m_* such that for all values of the oscillator's mass $m \in (0, m_*)$, the critical fluctuations are absent at all temperatures.*

The proof of this statement is preceded by a number of lemmas, other assertions, and remarks.

Remark 1.2 *The model with the small values of oscillator's mass can be characterized as "strongly quantum". Below we analize this and some other conditions of strong or large quantumness.*

Now we simplify our notations by setting

$$\Gamma^{\delta}_{\Lambda}(\tau,\tau') = \Gamma^{\beta,\Lambda}_{A_\delta(\Lambda);A_\delta(\Lambda)}(\tau,\tau') \ , \ \Gamma^{\delta}_{0,\Lambda}(\tau,\tau') = \Gamma^{0,\beta,\Lambda}_{A_\delta(\Lambda);A_\delta(\Lambda)}(\tau,\tau'). \tag{14}$$

¿From the definitions (11)–(13), one has:

$$\Gamma^{\delta}_{\Lambda}(\tau+\theta,\tau'+\theta) = \Gamma^{\delta}_{\Lambda}(\tau,\tau') \tag{15}$$

where the addition of the type of $\tau+\theta$ is understood modulo β. The same periodicity can be shown to hold also for $\Gamma^{\delta}_{0,\Lambda}$.

Proposition 1.1 *The following estimates hold with arbitrary $\delta \geq 0$, $\Lambda \in \mathcal{L}$, almost everywhere on S^2_β:*

$$\Gamma^{\delta}_{\Lambda}(\tau,\tau') \geq \Gamma^{\delta}_{0,\Lambda}(\tau,\tau') > 0.$$

The proof of this variant of the Lebowitz and Griffiths inequalities for our model will be given below.
We set

$$U^{\delta}_{\Lambda} = \frac{1}{\beta}\int\int_{S^2_\beta} \Gamma^{\delta}_{\Lambda}(\tau,\tau') d\tau d\tau' \ . \tag{16}$$

Taking into account (15), we obtain

$$U^{\delta}_{\Lambda} = \int_{S_\beta} \Gamma^{\delta}_{\Lambda}(\tau,\tau') d\tau' = \int_{S_\beta} \Gamma^{\delta}_{\Lambda}(\tau,\tau') d\tau, \tag{17}$$

Proposition 1.2 *For given $\delta > 0$, let the sequence $\{U^{\delta}_{\Lambda}, \Lambda \in \mathcal{C}\}$ converge to zero. Then the sequences $\{\Gamma^{\delta}_{\Lambda}(\tau,\tau'),\ \Lambda \in \mathcal{C}\}$, $\{\Gamma^{\delta}_{0,\Lambda}(\tau,\tau'),\ \Lambda \in \mathcal{C}\}$ converge to zero almost everywhere on S^2_β.*

The proof of this assertion immediately follows from Proposition 1.1 and (16).
Taking into account the definition (13), one has

$$\Gamma^{\delta}_{\Lambda}(\tau,\tau') = |\Lambda|^{-2\delta}\Gamma^{0}_{\Lambda}(\tau,\tau') \ ; \ U^{\delta}_{\Lambda} = |\Lambda|^{-2\delta} U^{0}_{\Lambda}. \tag{18}$$

Let us return to the dynamical matrix $D = (d_{ll'})$ that can naturally be defined on the Hilbert space $l^2(\mathbb{L})$ as a linear operator. Having in mind the translation invariance (D1) and the "ferroelectricity" property (D2), we deduce that the series mentioned in (D3) does not depend on l. Thus the norm of the mentioned operator is

$$\|D\| = -\sum_{l' \in \mathbb{L}} d_{ll'}. \tag{19}$$

Write

$$\tilde{H}_l = H_l + \|D\| q_l^2, \quad l \in \mathbb{L}, \tag{20}$$

where H_l is given by (2). Each Hamiltonian $\tilde{H}_l$ with V obeying (V1), (V2) has purely discrete spectrum. We set

$$\tilde{H}_l \psi_s = E_s \psi_s, \quad s \in \mathbb{Z}_+, \tag{21}$$

where the eigenvalues E_s are numbered in increasing order. Denote

$$\Delta = \min\{E_{s+1} - E_s \mid s \in \mathbb{Z}_+\}. \tag{22}$$

Lemma 1.1 *(Main estimate). For the model of quantum oscillators, let the parameters of the one-particle Hamiltonian m and Δ given by (22), and the interaction parameter $\|D\|$ satisfy the following condition*

$$m\Delta^2 > 2\|D\|. \tag{23}$$

Then for every $\beta > 0$ and arbitrary $\Lambda \in \mathcal{L}$,

$$U_\Lambda^0 \leq \frac{1}{m\Delta^2 - 2\|D\|}. \tag{24}$$

The proof will be given below.

Lemma 1.2 *For the model of quantum anharmonic oscillators, let the parameters m, Δ, and $\|D\|$ satisfy the condition (23). Then the critical fluctuations are absent for all $\beta > 0$.*

Proof: Proposition 1.1 implies $U_\Lambda^0 > 0$, for all $\beta > 0$ and $\Lambda \in \mathcal{L}$. Therefore, for m, Δ, and $\|D\|$ satisfying (23), the estimate (24) holds, thus the sequence $\{U_\Lambda^0,\ \Lambda \in \mathcal{C}\}$ is bounded. This yields, for arbitrary $\delta > 0$, that the sequence $\{U_\Lambda^\delta,\ \Lambda \in \mathcal{C}\}$ converges to zero . The latter and Proposition 1.2 yield in their turn that the sequences $\{\Gamma_\Lambda^\delta(\tau,\tau'),\ \Lambda \in \mathcal{C}\}$, $\{\Gamma_{0,\Lambda}^\delta(\tau,\tau'),\ \Lambda \in \mathcal{C}\}$ converge to zero almost everywhere on S_β^2. Thus the critical fluctuations are absent for all $\beta > 0$. ■

For the system of harmonic oscillators, we obtain an equality in (24) (this follows from the proof of Lemma 1.1 given below). This agrees with the corresponding results obtained for such systems (see e.g. expression (7) in [7]). For a harmonic oscillator ($V(q) = \frac{1}{2}a^2q^2$) , we have $m\Delta^2 = \|D\| + a^2$, where the latter coefficient describes the "rigidity" of the oscillator. In this case, the condition (23) does not depend on m. But in the case of anharmonic oscillators where the one-particle potential V satisfies the conditions (V1), (V2), $m\Delta^2$ may infinitely grow with m tending to zero, which implies that the condition (23) can be satisfied for given $\|D\|$ by putting m to be less than some threshold value m_*. Such a growth of $m\Delta^2$ is proven in the following assertion.

Lemma 1.3 *Let V satisfy the conditions (V1), (V2). Then there exists a positive m_* such that for all values of the oscillator's mass m less than m_*, the condition (23) is satisfied.*

Proof: The one-particle potential $V(q) = v(q^2)$ in the Hamiltonian (7) has the form

$$V(q) = b_0 q^{2M} + b_1 q^{2M-2} + \ldots + b_{M-1} q^2 + b_M, \tag{25}$$

where $b_0 > 0$, $M = \deg v \geq 2$, due to the condition (V1). For some $\sigma > 0$, we consider the unitary operator W_σ in $L^2(I\!R, dx)$ (Symanzik scaling) given by the formula

$$(W_\sigma \varphi)(x) = \sigma^{\frac{1}{2}} \varphi(\sigma x). \tag{26}$$

Then from (26), we have

$$W_\sigma p W_\sigma^{-1} = \sigma^{-1} p \ , \ W_\sigma q W_\sigma^{-1} = \sigma q. \tag{27}$$

Taking $\sigma = \sigma_m := m^{-\frac{1}{2M+2}}$ and using (27), we have that $\tilde{H}_l$ is unitary equivalent to the operator

$$m^{-\frac{M}{M+1}} \left[L_0 + m^{\frac{1}{M+1}} R_m(q) \right], \tag{28}$$

where

$$L_0 = \frac{1}{2}p^2 + b_0 q^{2M} \tag{29}$$

and

$$R_m(q) = b_1 q^{2M-2} + b_2 m^{\frac{1}{M+1}} q^{2M-4} + \ldots + (b_{M-1} + \|D\|) m^{\frac{M-2}{M+1}} q^2 + b_M m^{\frac{M-1}{M+1}}.$$

Let $\Delta^{(0)}$ be defined by (22) but with the eigenvalues E_s of the operator L_0 instead of $\tilde{H}_l$. Then using the analytic perturbation theory for the operator

$$L_0 + m^{\frac{1}{M+1}} R_m(q)$$

(see e.g. [8]), we deduce the asymptotic equivalence

$$m\Delta^2 \sim m^{-\frac{M-1}{M+1}} (\Delta^{(0)})^2, \quad m \to 0+.$$

Therefore, we can choose m_* such that $m\Delta^2 > 2\|D\|$ for all $m \in (0, m_*)$. ■

Remark 1.3 *For the of anharmonic oscillator with V meeting (V1), (V2), $m\Delta^2$ can be considered as a parameter describing the quantum character of this oscillator. It may infinitely grow with m tending to zero, as it was established by Lemma 1.3, or with the growth of Δ. The latter may occur for the double-well potential, when its minima are coming close (e.g. by means of the external pressure [12]), increasing the tunnelling between the wells. Therefore, for these systems, the condition (23) may serve as a mathematical realization of the notion of "strong quantumness".*

Proof of Theorem 1.1 It follows directly from Lemmas 1.2 and 1.3. ■

In fact, we can prove stronger statements than that of Theorem 1.1. Let us consider a sequence of operators $\{B(\Lambda),\ \Lambda \in \mathcal{C}\}$.

Definition 1.2 *The sequence $\{B(\Lambda),\ \Lambda \in \mathcal{C}\}$ is said to be degenerate at zero if the sequences of temperature Green functions $\{\Gamma^{\beta,\Lambda}_{B(\Lambda);\ldots;B(\Lambda)}(\tau_1, \ldots, \tau_n)$, $\Lambda \in \mathcal{C}\}$ converge to zero for all $n \in I\!N$ and almost all values of corresponding arguments.*

Theorem 1.2 *Let the conditions of Lemma 1.2 be satisfied. Then for all $\beta > 0$ and arbitrary $\delta > 0$,the sequence of fluctuation operators $\{A_\delta(\Lambda),\ \Lambda \in \mathcal{C}\}$ is asymptotically degenerate at zero.*

The proof of this and the following theorem will be given in the next section. For given sequence $\{\lambda(\Lambda), \Lambda \in \mathcal{C} \mid \lambda(\Lambda) \in I\!\!R\}$, we set

$$B_\lambda(\Lambda) = \lambda(\Lambda) A_0(\Lambda). \tag{30}$$

Theorem 1.3 *Let the conditions of Lemma 1.2 be satisfied. Then for all $\beta > 0$ and an arbitrary sequence $\{\lambda(\Lambda),\ \Lambda \in \mathcal{C}\}$ converging to zero, the sequence of operators $\{B_\lambda(\Lambda),\ \Lambda \in \mathcal{C}\}$ given by (30) is asymptotically degenerate at zero.*

Remark 1.4 *The suppression of the long range order by strong quantum fluctuations in the physical systems described by the above models was experimentally observed (see e.g. J.E. Tibballs et al [12]) and discussed long ago from the physical point of view (see e.g. T. Schneider et al [9] or the book [5], chapter 2.5.4.3). A rigorous justification of this phenomenon was given by A. Verbeure and V.A. Zagrebnov [14]. Our theorems describe a quantum effect which is essentially stronger than the one described in [14] – they imply the suppression not only of the long-range order but also of any critical anomalies. The suppression of the long-range order would correspond to the value of $\delta = \frac{1}{2}$ in (13).*

2 Functional Integral Representation.

For given inverse temperature β and $\Lambda \in \mathcal{L}$, let us consider the measure space $(\Omega_{\beta,\Lambda}; \Sigma_{\beta,\Lambda})$, where $\Omega_{\beta,\Lambda} = \{\omega_\Lambda(\cdot) \mid \omega_\Lambda : S_\beta \to I\!\!R^\Lambda\}$, $\omega_\Lambda(\cdot) = \{\omega_l(\cdot), l \in \Lambda \mid \omega_l \in \Omega := C(S_\beta \to I\!\!R)\}$. The set $\Sigma_{\beta,\Lambda}$ is the standard σ-algebra of $\Omega_{\beta,\Lambda}$ subsets generated by cylinder subsets, see e.g. [2]. In view of Definitions 1.1, 1.2 , our main objects are the Green functions defined by (11) with all A_j chosen to be $A_\delta(\Lambda)$. But in the case where A_j are the multiplication operators by measurable functions $A_j(q_\Lambda)$ with $q_\Lambda = \{q_l,\ l \in \Lambda\}$, we have, in fact [7]:

$$\Gamma^{\beta,\Lambda}_{A_0;\ldots;A_n}(\tau_0, \ldots, \tau_n) = \int_{\Omega_{\beta,\Lambda}} \prod_{j=0}^{n} A_j(\omega_\Lambda(\tau_j)) d\nu_{\beta,\Lambda}(\omega_\Lambda(\cdot)) \tag{31}$$

Here $\nu_{\beta,\Lambda}$ is determined as a measure on $\Omega_{\beta,\Lambda}$ by the Hamiltonian H_Λ and has been rigorously defined in [1], [7]. In the sequel, we will use the local

Hamiltonian H_Λ presented as follows

$$H_\Lambda = -\frac{1}{2}\sum_{l,l'\in\Lambda} J^\Lambda_{ll'} q_l q_{l'} + \sum_{l\in\Lambda} \tilde{H}_l, \tag{32}$$

where $\tilde{H}_l$ is given by (20) and

$$J^\Lambda_{ll'} = -d^\Lambda_{ll'} + \|D\|\delta_{ll'}, \tag{33}$$

with $d^\Lambda_{ll'}$ defined by (D4). For the one-particle Hamiltonian , we will use the following representation

$$\tilde{H}_l = \frac{1}{2m}p_l^2 + \tilde{V}(q_l)\ ,\ \tilde{V}(q_l) = V(q_l) + \|D\|q_l^2. \tag{34}$$

By means of the matrix $(J^\Lambda_{ll'})$, one can define on the Euclidean space $I\!R^\Lambda$ the operator J^Λ that has the norm

$$\|J^\Lambda\| = \|D\| - \sum_{l'\in\Lambda} d^\Lambda_{ll'}. \tag{35}$$

Therefore, the smallest eigenvalue of J^Λ is not less than

$$\|D\| + \sum_{l'\in\Lambda} d^\Lambda_{ll'} > 0,$$

where (D2) and (D4) were taken into account. Thus this operator is symmetric and positive. The latter means the scalar product in $I\!R^\Lambda$, $(J^\Lambda x, x)$ is strictly positive for all nonzero $x \in I\!R^\Lambda$. Thus we can write $\nu_{\beta,\Lambda}$ as follows:

$$d\nu_{\beta,\Lambda}(\omega_\Lambda(\cdot)) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(\frac{1}{2}\sum_{l,l'\in\Lambda} J^\Lambda_{ll'} \int_{S_\beta} \omega_l(\tau)\omega_{l'}(\tau)d\tau\right) \underset{l\in\Lambda}{X}\, d\chi_\beta(\omega_l(\cdot)) \tag{36}$$

The measure χ_β is determined by the one-particle Hamiltonian (34) and has also been defined in [1], [7]. For our purpose, it is convenient to use its heuristic representation [1], [7].

$$d\chi_\beta(\omega_l(\cdot)) = \frac{1}{X}\exp\left(-\frac{m}{2}\int_{S_\beta}(\dot{\omega}_l(\tau))^2 d\tau - \int_{S_\beta}\tilde{V}(\omega_l(\tau)d\tau\right) d\omega_l(\cdot). \tag{37}$$

It is possible to give a rigorous meaning to (37) by means of a lattice approximation , as in constructive quantum field theory. Within this approximation, the derivative $\dot{\omega}_l(\tau) := d\omega_l(\tau)/d\tau$ is replaced by a difference relation, whereas the integrals over S_β are put to be equal to corresponding Riemann integral sums [10]. This approach will be used in the next section.

For given $k \in I\!N$, let π be a partition of the set $\{1, 2, \ldots, 2k\}$ onto unordered pairs, i.e. $\pi = \{(\pi(1), \pi(2)); (\pi(3), \pi(4)); \ldots; (\pi(2k-1), \pi(2k))\}$, and $\mathcal{P}(2k)$ be the collection of all such partitions. We set

$$\Gamma^{\delta}_{2k,\Lambda}(\tau_1, \ldots, \tau_{2k}) = \Gamma^{\beta,\Lambda}_{A_\delta(\Lambda);\ldots;A_\delta(\Lambda)}(\tau_1, \ldots, \tau_{2k}), \tag{38}$$

where the right-hand-side of (38) is defined by (11) with $A_\delta(\Lambda)$ given by (13). Then (see (14))

$$\Gamma^{\delta}_{2,\Lambda}(\tau, \tau') = \Gamma^{\delta}_{\Lambda}(\tau, \tau').$$

Lemma 2.1 *For all $\delta \geq 0$, $k \in I\!N$, the following estimate holds for all $\Lambda \in \mathcal{L}$ and almost all values of corresponding variables*

$$\Gamma^{\delta}_{2k,\Lambda}(\tau_1, \ldots, \tau_{2k}) \leq \sum_{\pi \in \mathcal{P}(2k)} \prod_{i=1}^{k} \Gamma^{\delta}_{\Lambda}\left(\tau_{\pi(2i-1)}, \tau_{\pi(2i)}\right). \tag{39}$$

The proof of this Gaussian-upper-bound-like inequality is done in the next section.
For Λ consisting of only one element, we denote the corresponding U^{δ}_{Λ} given by (17) simply by U. Clearly, this U does not depend on δ and is determined by the one-particle Hamiltonian (34). The proof of Lemma 1.1 is based on the following Gaussian-upper-bound-like estimate, which for our model is proved in the next section by means of the lattice approximation.

Lemma 2.2 *For given β, let U and $\|D\|$ satisfy the condition*

$$U^{-1} > 2\|D\|. \tag{40}$$

Then for arbitrary $\Lambda \in \mathcal{L}$,

$$U^{0}_{\Lambda} \leq \frac{U}{1 - 2U\|D\|}. \tag{41}$$

Lemma 2.3 *Let Δ be defined by (22), then for all $\beta > 0$, the following estimate holds*

$$U \leq \frac{1}{m\Delta^2}. \tag{42}$$

Proof: Taking into account (17), (14), (12), (11), (10) and (21), one gets

$$U = \frac{1}{Z} \int_{S_\beta} \mathrm{trace}\left(q e^{-\tau \tilde{H}_l} q e^{-(\beta-\tau)\tilde{H}_l} \right) d\tau, \tag{43}$$

where

$$Z = \mathrm{trace}\left(e^{-\beta \tilde{H}_l} \right).$$

We set $q_{ss'} = (\psi_s, q\psi_{s'})$ and obtain from (43)

$$U = \frac{1}{Z} \sum_{s,s' \in \mathbb{Z}_+} (q_{ss'})^2 \frac{(E_s - E_{s'})(e^{-\beta E_{s'}} - e^{-\beta E_s})}{(E_s - E_{s'})^2}. \tag{44}$$

For symmetry reasons, we have $q_{ss} = 0$, thus the case $s = s'$ in the sum (44) can be excluded. Therewith, taking into account the definition (22), one can estimate the denominator in (44) and obtain

$$\begin{aligned} U &\leq \frac{1}{Z\Delta^2} \sum_{s,s'} (q_{ss'})^2 (E_s - E_{s'})(e^{-\beta E_{s'}} - e^{-\beta E_s}) \\ &= \frac{1}{\Delta^2} \cdot \frac{1}{Z} \mathrm{trace}\left([q, [\tilde{H}_l, q]] e^{-\beta \tilde{H}_l} \right) = \frac{1}{m\Delta^2} \end{aligned}$$

where $[\cdot,\cdot]$ means the commutator. This gives (42). ■

Proof of Lemma 1.1: Follows immediately from Lemmas 2.2 and 2.3.

Proof of Theorem 1.2 and 1.3: Follows immediately from Theorem 1.1 and the estimate (39).

3 Proof of Lemmas 2.1 and 2.2.

In order to prove Lemmas 2.1 and 2.2, we will construct some technical background within the lattice approximation approach. The starting point here is the expression (37). As it was mentioned, the main idea of this approach is to replace the integrals over S_β by corresponding Riemann integral sums.

Let us divide S_β by the points $\tau_n = \frac{n}{N}\beta$, $n = 0, 1, \ldots, N-1$, and introduce the following notations

$$\xi_{ln} = \sqrt{\frac{\beta}{N}}\omega_l(\tau_n);\ n = 0, 1, \ldots, N-1; \tag{45}$$

$$\xi_{l0} = \xi_{lN},\ \xi_l = (\xi_{l0}, \xi_{l1}, \ldots, \xi_{lN-1});$$

$$\Delta\tau = \tau_{n+1} - \tau_n = \frac{\beta}{N};\ n = 0, 1 \ldots N-1;$$

$$\dot{\omega}_l(\tau_n) = \frac{\omega_l(\tau_{n+1}) - \omega_l(\tau_n)}{\Delta\tau}.$$

Now we define a sequence of measures on $I\!R^N$ that converges to the corresponding measure on trajectories, when $N \to \infty$, in the sense of convergence of integrals on cylinder functions.We set

$$d\chi_\beta^{(N)}(\xi_l) = \frac{1}{X_\beta^{(N)}} \exp\left(\alpha_N \sum_{n=0}^{N-1} \xi_{ln}\xi_{ln+1}\right) \prod_{n=0}^{N-1} d\rho_\beta^{(N)}(\xi_{ln}) \tag{46}$$

$$d\rho_\beta^{(N)}(\xi_{ln}) = \frac{1}{Y_\beta^{(N)}} \exp\left(-\alpha_N \xi_{ln}^2 - \frac{\beta}{N}\tilde{V}\left(\sqrt{\frac{N}{\beta}}\xi_{ln}\right)\right) d\xi_{ln}; \tag{47}$$

where $X_\beta^{(N)}$, $Y_\beta^{(N)}$ are normalizing constants, and

$$\alpha_N = m\left(\frac{N}{\beta}\right)^2.$$

We consider

$$\begin{aligned}
& d\nu_{\beta,\Lambda}^{(N)}(\xi) \\
= \ & \frac{1}{Z_{\beta,\Lambda}^{(N)}} \exp\left(\frac{1}{2}\sum_{l,l'\in\Lambda} J_{ll'}^{\Lambda} \sum_{n=0}^{N-1} \xi_{ln}\xi_{l'n}\right) \prod_{l\in\Lambda} d\chi_\beta^{(N)}(\xi_l) \qquad (48) \\
= \ & \frac{1}{Z_{\beta,\Lambda}^{(N)} X_\beta^{(N)}} \exp\left(\frac{1}{2}\sum_{l,l'\in\Lambda}\sum_{n=0}^{N-1} J_\Lambda^{(N)}(ln, l'n')\xi_{ln}\xi_{l'n'}\right) \prod_{l\in\Lambda}\prod_{n=0}^{N-1} d\rho_\beta^{(N)}(\xi_{ln}),
\end{aligned}$$

where

$$J_\Lambda^{(N)}(ln, l'n') = \delta_{nn'} J_{ll'}^{\Lambda} + \delta_{ll'}\delta_{|n-n'|_N,1}\alpha_N, \tag{49}$$

$\delta_{|n-n'|_N,1} = 1$ iff $|n-n'| = 1, N-1$. For a suitable function $F(\xi)$, we define

$$< F(\xi) >_{\Lambda}^{(N)} = \int_{I\!R^{N|\Lambda|}} F(\xi) d\nu_{\beta,\Lambda}^{(N)}(\xi), \tag{50}$$

and

$$< F(\xi) >_{0,\Lambda}^{(N)} = \int_{I\!R^{N|\Lambda|}} F(\xi) d\nu_{\beta,\Lambda}^{(0,N)}(\xi), \tag{51}$$

where $\nu_{\beta,\Lambda}^{(0,N)}$ is defined by (48) with

$$J_{0,\Lambda}^{(N)}(ln, l'n') = \delta_{nn'} J_{ll'} + \delta_{ll'} \delta_{|n-n'|_N,1} \alpha_N, \tag{52}$$

instead of $J_{\Lambda}^{(N)}(ln, l'n')$, and

$$J_{ll'} = -d_{ll'} + \|D\|, \tag{53}$$

instead of $J_{ll'}^{\Lambda}$. Note that (D4) yields

$$J_{ll'} \leq J_{ll'}^{\Lambda}. \tag{54}$$

Introduce

$$\begin{aligned} & S_{2k,\Lambda}^{\delta,N}(n_1, \ldots, n_{2k}) \\ = \quad & |\Lambda|^{-k(1+2\delta)} \left(\frac{N}{\beta}\right)^k \sum_{l_1, \ldots l_{2k} \in \Lambda} < \xi_{l_1 n_1} \ldots \xi_{l_{2k} n_{2k}} >_{\Lambda}^{(N)}, \end{aligned} \tag{55}$$

where $k \in I\!N$ and $n_j = 0, 1, \ldots, N-1$. We also introduce $S_{2k,0,\Lambda}^{\delta,N}(n_1, \ldots, n_{2k})$ as given by (55) with $< \ldots >_{0,\Lambda}^{(N)}$ instead of $< \ldots >_{\Lambda}^{(N)}$. Let us consider a sequence $\{N_s \in I\!N, s \in I\!N\}$ possessing the property $N_s < N_{s+1}$ and, therefore, tending to infinity when $s \to \infty$. By means of this sequence, we can define a sequence $\{n^{(s)}(\tau), s \in I\!N \mid 0 \leq n^{(s)} \leq N_s\}$ such that for given $\tau \in S_\beta$, $\lim_{s\to\infty}(n^{(s)}(\tau)/N_s) = \tau/\beta$.

Lemma 3.1 *For every $k \in I\!N$, $\delta \geq 0$, $\Lambda \in \mathcal{L}$, the sequence*

$$\left\{ S_{2k,\Lambda}^{\delta,N_s}(n_1^{(s)}(\tau_1), n_2^{(s)}(\tau_2), \ldots, n_{2k}^{(s)}(\tau_{2k})), s \in I\!N \right\}$$

converges to the temperature Green function $\Gamma_{2k,\Lambda}^{\delta}(\tau_1, \ldots, \tau_{2k})$, given by (38), (31), when $s \to \infty$, for almost all values of $(\tau_1, \ldots, \tau_{2k}) \in S_\beta^{2k}$. The same convergence of the sequence of $S_{2,0,\Lambda}^{\delta N_s}(n_1^{(s)}(\tau_1), n_2^{(s)}(\tau_2))$ to $\Gamma_{0,\Lambda}^{\delta}(\tau_1, \tau_2)$ given by (14) holds true.

Proof: The standard arguments yield that the sequence of measures $\{\nu_{\beta,\Lambda}^{(N)}$, $N \in I\!N\}$ converges to the measure $\nu_{\beta,\Lambda}$ given by (36) in the sense of convergence of integrals of cylinder functions. This yields the convergence of the corresponding moments, which yields in turn the convergence to be proven. Just the same arguments yield the convergence to the Green functions in the case of zero boundary conditions. ■

Now we can fix $\Lambda \in \mathcal{L}$, the integer N, and simplify our notations by putting $\sum_l = \sum_{l\in\Lambda}$, $\sum_n = \sum_{n=0}^{N-1}$, and so on. Let us consider the measure (48) as a Gibbs measure (with corresponding boundary conditions) for a classical ferromagnetic spin model with the single-spin measure $\rho_\beta^{(N)}$ given by (47), which belongs to the BFS class of measures (see Remark 1.1). We set

$$u_2(ln, l'n') =< \xi_{ln}\xi_{l'n'} >_\Lambda^{(N)}; \;\; u_2^{(0)}(ln, l'n') =< \xi_{ln}\xi_{l'n'} >_{0,\Lambda}^{(N)} . \tag{56}$$

$$\begin{aligned} & u_4(l_1n_1, l_2n_2, l_3n_3, l_4n_4) \\ = \;\; & < \xi_{l_1n_1}\xi_{l_2n_2}\xi_{l_3n_3}\xi_{l_4n_4} >_\Lambda^{(N)} - u_2(l_1n_1, l_2n_2)u_2(l_3n_3, l_4n_4) \\ - \;\; & u_2(l_1n_1, l_3n_3)u_2(l_2n_2, l_4n_4) - u_2(l_1n_1, l_4n_4)u_2(l_2n_2, l_3n_3). \end{aligned} \tag{57}$$

Proposition 3.1 *For our model, the following estimates hold*

$$0 < u_2^{(0)}(ln, l'n') \leq u_2(ln, l'n'). \tag{58}$$

$$u_4(l_1n_1, l_2n_2, l_3n_3, l_4n_4) < 0 \tag{59}$$

$$< \xi_{l_1n_1} \dots \xi_{l_{2k}n_{2k}} >_\Lambda^{(N)} \leq \sum_{\pi\in\mathcal{P}(2k)} \prod_{i=1}^{k} u_2(l_{\pi(2i-1)}n_{\pi(2i-1)}, l_{\pi(2i)}n_{\pi(2i)}) \tag{60}$$

Proof: (58) (positivity) and (59) are known as Lebowitz inequalities, the estimate (60) is the Gaussian upper bound inequality, the relationship between $u_2^{(0)}$ and u_2 in (58) is the Griffiths inequality produced by (54). The validity of these inequalities follows from the properties of the measures (48), mentioned above (see Section 12 of [6]). ■

Proof of Proposition 1.1 and Lemma 2.1: It follows immediately from (58), (60), (55) and Lemma 3.1 ■

The rest of this paper is devoted to the proof of Lemma 2.2. To achieve this aim, we will need an additional technique explained below.

Let $\mathcal{F}$ be a set of functions F defined on $\mathbb{R}^M$, $M \in \mathbb{N}$, that can be continued on $\mathbb{C}^M$ as entire functions, such that the norm

$$\|F\|_a = \sup\{|F(z)| \exp(-a\|z\|^2) : z \in \mathbb{C}^M\} \tag{61}$$

is finite for all $a > 0$. Here $\|z\|^2 = |z_1|^2 + \ldots + |z_M|^2$. This set equipped with the pointwise linear operations and the topology defined by the family of norms $\{\|\cdot\|_a, a > 0\}$ becomes a Fréchet space that will also be denoted as $\mathcal{F}$. We will use the following property of this space [11]:

Proposition 3.2 *The set of all polynomials is dense in $\mathcal{F}$.*

With each symmetric bilinear form on $\mathbb{R}^M \times \mathbb{R}^M$, one can associate a symmetric linear operator A such that this form can be written as (y, Ax), where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^M$. The matrix $(A_{jj'})$ consisting of the matrix elements of the operator A in the canonical basis of $\mathbb{R}^M$ will also be denoted by A. We will use the notation $A > 0$, for positive operators of such type. Let A be a symmetric linear positive operator. We set

$$(T(A)F)(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1 \ldots j_k = 1}^{M} \sum_{j'_1 \ldots j'_k = 1}^{M} A_{j_1 j'_1} \ldots A_{j_k j'_k} \frac{\partial^{2k} F(x)}{\partial x_{j_1} \ldots \partial x_{j'_k}} \tag{62}$$

Estimating the derivatives, one can prove that for all such A, $T(A)$ continuously maps $\mathcal{F}$ into itself. $T(A)$ can be expressed in the integral form:

$$\begin{aligned} & (T(A)F)(x) \\ = \ & \frac{1}{X(A)} \int_{\mathbb{R}^M} F(y) \exp\left(-\frac{1}{2}(y - x, A^{-1}(y - x)\right) dy, \end{aligned} \tag{63}$$

where

$$X(A) = [\det(2\pi A)]^{\frac{1}{2}}. \tag{64}$$

The identity of (62) and (63) can be proven for monomials, and then for every $F \in \mathcal{F}$ by means of Proposition 3.2. This identity permits for the operator $T(A)$ to be extended to a wider class of functions. Let A be symmetric linear positive , B be symmetric linear, operators on $\mathbb{R}^M$. The set of all such B satisfying with this A the condition $A^{-1} - B > 0$ will be denoted by $\mathcal{B}(A)$.

For every $B \in \mathcal{B}(A)$, the operator $T(A)$ given by (63) can be applied to $F(x) = \exp(\frac{1}{2}(x, Bx))$, giving

$$\begin{aligned} & (T(A)F)(x) \\ = \; & \frac{1}{X(A)} \int_{I\!R^M} \exp\left(\frac{1}{2}(y, By) - \frac{1}{2}(y - x, A^{-1}(y - x))\right) dy. \end{aligned} \tag{65}$$

We put

$$(y, By) - (y - x, A^{-1}(y - x)) = (x, Cx) - (y - Wx, D(y - Wx)), \tag{66}$$

and obtain

$$C = B(1 - AB)^{-1}; \; D = A^{-1} - B; \; W = (1 - AB)^{-1}. \tag{67}$$

Then we insert the right-hand-side of (66) into the exponent in (65). This yields:

$$\begin{aligned} & (T(A)F)(x) \\ = \; & \frac{X((1 - AB)^{-1}A)}{X(A)} \exp(\frac{1}{2}(x, B(1 - AB)^{-1}x)) \\ & \frac{1}{X((1 - AB)^{-1}A)} \int_{I\!R^M} \exp(-\frac{1}{2}(y - Wx, (A^{-1} - B)(y - Wx)))dy \\ = \; & \frac{X((1 - AB)^{-1}A)}{X(A)} \exp(\frac{1}{2}(x, B(1 - AB)^{-1}x)). \end{aligned} \tag{68}$$

Proceeding in this direction, we can extend the operator $T(A)$ to the following class of functions. For a symmetric linear positive operator A, we set

$$\mathcal{F}(A) = \left\{ F(x) = \exp(\frac{1}{2}(x, Bx))G(x) \mid B \in \mathcal{B}(A), G \in \mathcal{F} \right\}. \tag{69}$$

Proposition 3.3 *For a given symmetric linear positive operator A, the operator $T(A)$ defined on $\mathcal{F}$ by the expression (63) can be continuously extended to $\mathcal{F}(A)$ as follows*

$$\begin{aligned} & (T(A)F)(x) \\ = \; & \Xi(A, B) \exp(\frac{1}{2}(x, B(1 - AB)^{-1}x)) \\ & (T((1 - AB)^{-1}A)G)((1 - AB)^{-1}x), \end{aligned} \tag{70}$$

with

$$\Xi(A,B) = \frac{X((1-AB)^{-1}A)}{X(A)}. \tag{71}$$

Proof: For $G \equiv 1$, (70) reduces to (68). Now let $G(x)$ be a polynomial. Then the integral on the right-hand-side of (63) with $F(x) = \exp(\frac{1}{2}(x, Bx))$ $G(x)$ and $B \in \mathcal{B}(A)$ converges. Applying the representation (66), we obtain

$$\begin{aligned}
& (T(A)F)(x) \\
= \;& \Xi(A,B)\exp(\frac{1}{2}(x, B(1-AB)^{-1}x))\frac{1}{X((1-AB)^{-1}A)} \\
& \int_{I\!R^M} G(y)\exp(-\frac{1}{2}(y-Wx,(A^{-1}-B)(y-Wx)))dy \\
= \;& \Xi(A,B)\exp(\frac{1}{2}(x, B(1-AB)^{-1}x))(T((1-AB)^{-1}A)G)(Wx).
\end{aligned}$$

where the integral form (63) is used as a definition, and W is given by (67). Thus (70) holds for $G(y)$ being polynomials. Now we apply Proposition 3.2 and obtain the stated extension. ■

Let us consider $F \in \mathcal{F}(A)$, then, for every $t \in (0,1]$, $F \in \mathcal{F}(tA)$. Having this in mind, we set

$$F_t(x) = (T(tA)F)(x), t \in (0,1];\; F_0(x) = F(x). \tag{72}$$

Proposition 3.4 *For every $t \in [0,1]$, $F_t(x)$ is differentiable with respect to t, and obeys the following equation*

$$\frac{\partial F_t(x)}{\partial t} = \frac{1}{2}\sum_{j,j'=1}^{M} A_{jj'}\frac{\partial^2 F_t(x)}{\partial x_j \partial x_{j'}} \;; F_0(x) = F(x). \tag{73}$$

The proof follows easely from the definition (62) of $T(A)$.

Now let us return to the measures (46)–(49). We set

$$\Phi_\Lambda^{(N)}(\eta) = \int_{I\!R^{N|\Lambda|}} \exp\left(\sum_{l,n}\eta_{ln}\xi_{ln}\right) d\nu_{\beta,\Lambda}^{(N)}(\xi), \tag{74}$$

$$F_\Lambda^{(N)}(\eta) = \int_{I\!R^{N|\Lambda|}} \exp\left(\sum_{l,n}\eta_{ln}\xi_{ln}\right) \underset{l}{X}\, d\chi_\beta^{(N)}(\xi_l), \eta \in I\!R^{N|\Lambda|}.$$

Taking into account that the one-particle potential V satisfies the requirements (V1), (V2), we conclude that both $\Phi_\Lambda^{(N)}$ and $F_\Lambda^{(N)}$ belong to $\mathcal{F}$, where M used in the definition of the latter is put to be equal $N|\Lambda|$. Moreover, the expression (48) implies

$$\Phi_\Lambda^{(N)}(\eta) = \frac{1}{Z_{\beta,\Lambda}^{(N)}}(T(I_\Lambda)F_\Lambda^{(N)})(\eta) \tag{75}$$

where the operator $I_\Lambda : I\!R^{|\Lambda|N} \to I\!R^{|\Lambda|N}$ is defined by its matrix elements

$$I_\Lambda(ln, l'n') = J_{ll'}^\Lambda \delta_{nn'} \tag{76}$$

and then is positive, symmetric, and periodic. The linear operator $C : I\!R^{|\Lambda|N} \to I\!R^{|\Lambda|N}$ is said to be periodic if for its matrix elements, one has $C(ln, l'n') = C((l+\lambda)(n+\nu), (l'+\lambda)(n'+\nu))$ with arbitrary $\lambda \in \Lambda$ and $\nu \in \{0, 1, \ldots, N-1\}$, where the additions are modulo $|\Lambda|$ and N respectively. Since Λ and N are fixed, we simplify our notations by omitting these symbols in Φ, F, Z_β and I. Now let C be a periodic linear symmetric operator belonging to $\mathcal{B}(I)$. Write

$$F(\eta) = \exp(\frac{1}{2}(\eta, C\eta))G(\eta). \tag{77}$$

For given C, this can be considered as a definition of $G(\eta)$. It is not difficult to check that for each positive symmetric I, $C \in \mathcal{B}(I)$ implies $-C \in \mathcal{B}((1 - IC)^{-1}I)$. This yields $G(\eta) \in \mathcal{F}((1-IC)^{-1}I)$ (due to $F \in \mathcal{F}$). We set for $t \in [0, 1]$ (see (72)):

$$\exp R(t, \eta) = (T(t(1 - IC)^{-1}I)G)(\eta). \tag{78}$$

Inserting $F(\eta)$ as given by (77) into (75) and applying the identity (70), one obtains

$$\Phi(\eta) = \frac{1}{Z_\beta}\Xi(I, C)\exp\{\frac{1}{2}(\eta, C(1 - IC)^{-1}\eta) + R(1, (1 - IC)^{-1}\eta)\}. \tag{79}$$

Now let us consider the expression (78) having in mind Proposition 3.4. This yields

$$\begin{aligned} \dot{R}(t,\eta) \ &:= \ \frac{\partial R(t,\eta)}{\partial t} = \frac{1}{2}\sum_{ln,l'n'}[(1 - IC)^{-1}I](ln, l'n') \\ & \left\{\frac{\partial^2 R(t,\eta)}{\partial\eta_{ln}\partial\eta_{l'n'}} + \frac{\partial R(t,\eta)}{\partial\eta_{ln}}\frac{\partial R(t,\eta)}{\partial\eta_{l'n'}}\right\}. \end{aligned} \tag{80}$$

It is not difficult to show that for all $t \in [0,1]$, $R(t,\eta)$ and $\dot{R}(t,\eta)$ are even and infinitely often differentiable functions with respect to η at the point $\eta = 0$. We set

$$
\begin{aligned}
R_{2k}(t \mid l_1 n_1, l_2 n_2, \ldots, l_{2k} n_{2k}) &= \left(\frac{\partial^{2k} R(t,\eta)}{\partial \eta_{l_1 n_1} \ldots \partial \eta_{l_{2k} n_{2k}}} \right)_{\eta=0}, \\
\dot{R}_{2k}(t \mid l_1 n_1, \ldots, l_{2k} n_{2k}) &= \frac{\partial}{\partial t} R_{2k}(t \mid l_1 n_1, \ldots, l_{2k} n_{2k}).
\end{aligned}
$$

Then we obtain from (80):

$$
\begin{aligned}
\dot{R}_2(t \mid ln, l'n') &= \frac{1}{2} \sum_{l_1 n_1, l_2 n_2} [(1 - IC)^{-1} I](l_1 n_1, l_2 n_2) \\
&\quad \{ R_4(t \mid l_1 n_1, l_2 n_2, ln, l'n') \\
&\quad + 2 R_2(t \mid l_1 n_1, ln) R_2(t \mid l_2 n_2, l'n') \}. \qquad (81)
\end{aligned}
$$

Taking into account the periodic boundary conditions with respect to $l \in \Lambda$, as well as to $n \in \{0, 1, \ldots, N-1\}$, one deduces that

$$\sum_{ln, l'n'} R_2(t \mid ln, l'n') = |\Lambda| N \sum_{ln} R_2(t \mid ln, l'n'). \qquad (82)$$

We set

$$R_2(t) = \frac{1}{|\Lambda| N} \sum_{ln, l'n'} R_2(t \mid ln, l'n'), \qquad (83)$$

and obtain from (81) taking into account (82).

$$\dot{R}_2(t) = K R_2^2(t) + Q(t), \qquad (84)$$

where

$$K = \frac{1}{|\Lambda| N} \sum_{ln, l'n'} [(1 - IC)^{-1} I](ln, l'n'); \qquad (85)$$

$$Q(t) = \frac{1}{2|\Lambda| N} \sum_{ln, l'n'} [(1 - IC)^{-1} I](ln, l'n') \sum_{l_1 n_1, l_2 n_2} R_4(t \mid ln, l'n', l_1 n_1, l_2 n_2). \qquad (86)$$

The initial condition for $R_2(t)$ follows from (78) and (77):

$$\begin{aligned} R_2(0) &= \frac{1}{|\Lambda|N} \sum_{ln,l'n'} G_2(ln,l'n') \\ &= \frac{1}{|\Lambda|N} \left\{ \sum_{ln,l'n'} F_2(ln,l'n') - \sum_{ln,l'n'} C(ln,l'n') \right\} \end{aligned} \quad (87)$$

where

$$G_2(ln,l'n') = \left(\frac{\partial^2 \log G(\eta)}{\partial \eta_{ln} \partial \eta_{l'n'}} \right)_{\eta=0} \quad (88)$$

$$F_2(ln,l'n') = \left(\frac{\partial^2 \log F(\eta)}{\partial \eta_{ln} \partial \eta_{l'n'}} \right)_{\eta=0} \quad (89)$$

The operator C has been restricted to be periodic and to belong to $B(I)$. Now we impose more essential restrictions as follows:

$$\sum_{ln,l'n'} C(ln,l'n') = \sum_{ln,l'n'} F_2(ln,l'n'), \quad (90)$$

$$C = gI^{-1}, g < 1 \quad (91)$$

where I^{-1} is the inverse operator of I (it exists due to the positivity of J^{Λ}), g is a positive quantity (depending on Λ and N) to be determined from (90). The possibility to choose it being less than one will be shown to be produced by (23).

Lemma 3.2 *Let the operator C satisfy (90), (91). Then $R_2(t)$ given by (83) is negative for all $t \in (0,1]$. In particular*

$$R_2(1) < 0. \quad (92)$$

Proof: Taking into account (90), one gets $R_2(0) = 0$. The condition (91) yields in turn

$$[(1 - IC)^{-1} I](ln,l'n') = \frac{1}{1-g} J^{\Lambda}_{ll'} \delta_{nn'} \geq 0, \quad (93)$$

due to (33) and (D2). Let us prove the assertion of this Lemma assuming

$$R_4(t \mid l_1 n_1, l_2 n_2, ln, l'n') < 0 \quad (94)$$

for $t \in [0,1]$. Then $Q(t)$ given by (86) is negative for $t \in [0,1]$. Taking into account (93), we obtain that K given by (85) is positive. Thus the function $R_2(t)$ possesses the properties: it is equal to zero at $t = 0$, has a negative derivative at $t = 0$ and a negative derivative at each t such that $R_2(t) = 0$. The latter can be deduced from (84). Clearly, each continuous function of t possesing these properties becomes zero at most once. In our case this occurs at the point $t = 0$. Thus, this function preserves its sign for all $t \in (0,1]$, that means it is negative (it decreases at $t = 0$) for these t, which was to be proved. Now let us prove (94). We return to (78) and insert in it G as given by (77), that is

$$G(\eta) = \exp(-\frac{1}{2}(\eta, C\eta))F(\eta).$$

Taking into account (91), we obtain then

$$\begin{aligned} \exp R(t,\eta) &= \left(\frac{1-g}{1-(1-t)g}\right)^{\frac{N|\Lambda|}{2}} \exp\left(-\frac{1}{2}\frac{g(1-g)}{1-(1-t)g}(\eta, I^{-1}\eta)\right) \\ &\quad \left(T\left(\frac{t}{1-(1-t)g}I\right)F\right)\left(\frac{1-g}{1-(1-t)g}\eta\right). \end{aligned} \tag{95}$$

We set $h_t = (1-g)[1-(1-t)g]^{-1}$ and

$$\exp\Psi(t,\eta) = (T(\frac{th_t}{1-g}I)F)(\eta). \tag{96}$$

Then (95) can be rewritten as

$$\exp(R(t,\eta) = h_t^{\frac{N|\Lambda|}{2}} \exp(-\frac{1}{2}gh_t(\eta, I^{-1}\eta) + \Psi(t, h_t\eta)),$$

hence

$$R_4(t \mid l_1n_1, l_2n_2, ln, l'n') = h_t^{-4}\Psi_4(t \mid l_1n_1, l_2n_2, ln, l'n'), \tag{97}$$

where Ψ_4 is defined in the same way as R_4. Recalling the definition (74) of $F(\eta) = F_\Lambda^{(N)}(\eta)$, we obtain from (96)

$$\begin{aligned} \exp\Psi(t,\eta) &= \int_{I\!R^{N|\Lambda|}} \exp(\sum_{ln}\eta_{ln}\xi_{ln} + \frac{1}{2}th_t\sum_{l,l',n}J^\Lambda_{ll'}\xi_{ln}\xi_{l'n} \\ &\quad +\alpha_N\sum_{ln}\xi_{ln}\xi_{ln+1})\underset{ln}{X}\, d\rho_\beta^{(N)}(\xi_{ln}). \end{aligned} \tag{98}$$

where we have also taken into account (46). Therefore, Ψ_4 is the fourth semiinvariant (of the type (57)) of the ferromagnetic Gibbs measure ($J^\Lambda_{ll'} \geq 0, \alpha_N > 0$) with the initial measure $\rho^{(N)}_\beta$ belonging to the BFS class. Then Ψ_4 is negative (see Proposition 3.1). The latter implies (94), which completes the proof. $\blacksquare$

Now let us return to (79). We set

$$U^{(N)}_\Lambda = \frac{1}{N|\Lambda|}\sum_{ln,l'n'} \Phi_2(ln, l'n'), \tag{99}$$

where

$$\Phi_2(ln, l'n') = \left(\frac{\partial^2 \log \Phi(\eta)}{\partial \eta_{ln}\partial \eta_{l'n'}}\right)_{\eta=0}. \tag{100}$$

Then (79) gives with $C = gI^{-1}$:

$$U^{(N)}_\Lambda = \frac{g}{(1-g)\|J^\Lambda\|} + \frac{1}{(1-g)^2}R_2(1); \tag{101}$$

where

$$\|J^\Lambda\| = \sum_{l\in\Lambda} J^\Lambda_{ll'}. \tag{102}$$

is given by (35). To obtain (101), we have used (76). But Lemma 3.2 yields (92), thus

$$U^{(N)}_\Lambda < \frac{g}{(1-g)\|J^\Lambda\|}. \tag{103}$$

The zeroth initial condition for (84) is produced by the relation (90). But (74) implies

$$F(\eta) = \prod_l f(\eta_l) \tag{104}$$

with

$$f(\eta_l) = \int_{I\!R^N} \exp(\eta_l \xi_l) d\chi^{(N)}_\beta(\xi_l). \tag{105}$$

Taking into account (105), (104), and (89), we get

$$\begin{aligned} F_2(ln, l'n') &= \delta_{ll'} f_2(n, n') = \delta_{ll'}\left(\frac{\partial^2 \log f(\eta_l)}{\partial \eta_{ln}\partial \eta_{ln'}}\right)_{\eta_l=0} \\ &= \delta_{ll'}\int_{I\!R^N} \xi_{ln}\xi_{ln'} d\chi^{(N)}_\beta(\xi_l). \end{aligned} \tag{106}$$

Proof of Lemma 2.2: Comparison of (99), (100) and (50) yields

$$U_\Lambda^{(N)} = \frac{1}{N|\Lambda|} \sum_{ln,l'n'} < \xi_{ln}\xi_{l'n'} >_\Lambda^{(N)} . \tag{107}$$

Then taking into account the definition (55), one gets

$$U_\Lambda^{(N)} = \frac{1}{\beta} \sum_{n,n'} S_{2,\Lambda}^{0,N}(n,n') \left(\frac{\beta}{N}\right)^2 .$$

The latter can be rewritten as follows

$$U_\Lambda^{(N)} = \frac{1}{\beta} \sum_{n,n'} S_{2,\Lambda}^{0,N}(n,n')\Delta\tau\Delta\tau'. \tag{108}$$

The right-hand-side of (108) tends to

$$\frac{1}{\beta}\int\int_{S_\beta^2} \Gamma_{2,\Lambda}^0(\tau,\tau')d\tau d\tau' = \frac{1}{\beta}\int\int_{S_\beta^2} \Gamma_{\Lambda}^0(\tau,\tau')d\tau d\tau' = U_\Lambda^0$$

as $N \to \infty$. Here we have taken into account (16) and Lemma 3.1. This means that the left-hand-side of (103) tends to U_Λ^0 when $N \to \infty$. In order to obtain the limit of the right-hand-side of (103), we use (91), (90), and obtain

$$g = \frac{\sum_{ln,l'n'} F_2(ln,l'n')}{\sum_{ln,l'n'}(I^{-1})(ln,l'n')},$$

and then by means of (106) and (45), we have

$$\begin{aligned}
\lim_{N\to\infty} g = &= \frac{\|J^\Lambda\|}{\beta} \lim_{N\to\infty} \int_{I\!R^N} \left(\sum_n \omega(\tau_n)\Delta\tau\right)^2 d\chi_\beta^{(N)}(\xi) \\
&= \frac{\|J^\Lambda\|}{\beta} \int_\Omega \left(\int_{S_\beta} \omega(\tau)d\tau\right)^2 d\chi_\beta(\omega(\cdot)) \\
&= \frac{\|J^\Lambda\|}{\beta} \int\int_{S_\beta^2} \Gamma(\tau,\tau')d\tau d\tau' = \|J^\Lambda\|U < 2\|D\|U < 1,
\end{aligned}$$

where we have used (40) as well as the estimate $\|J^\Lambda\| < 2\|D\|$ which follows from (35) and (D2), (D4). Meanwhile, $\Gamma(\tau, \tau')$ means $\Gamma^\delta_\Lambda(\tau, \tau')$ with one–point Λ, and U is the same as in Lemma 2.2. Then passing to the limit $N \to \infty$ in (103), we obtain

$$U^0_\Lambda \leq \frac{U}{1 - U\|J^\Lambda\|} \leq \frac{U}{1 - 2U\|D\|}$$

where we have taken is account the definition (35) which yields

$$\lim_{\Lambda \nearrow \mathbb{L}} \|J^\Lambda\| = 2\|D\|,$$

when $\Lambda \nearrow \mathbb{L}$.

$\blacksquare$

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