On a Linear Chaotic Quantum Harmonic Oscillator *

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Abstract

We show that a linear quantum harmonic oscillator is chaotic in the sense of Li-Yorke. We also prove that the weighted backward shift map, used as an infinite dimensional linear chaos model, in a separable Hilbert space is chaotic in the sense of Li-Yorke, in addition to being chaotic in the sense of Devaney.

Key words: infinite dimension, quantum oscillator, linear chaotic system

1 Introduction

We consider an unforced quantum harmonic oscillator, i.e., a very small frictionless mass-spring system whose evolution is modeled by the Schrödinger equation ([8])

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + \frac{k}{2}x^2\psi, \tag{1}$$

with wave function $\psi(x,t)$, displacement x, mass m, stiffness k and Planck number $\hbar$. The nondimensionalized stationary states in the separable Hilbert space $X = L^2(-\infty,\infty)$ form an orthonormal basis

$$\psi_n(x) = e^{-x^2/2}H_n(x)/\sqrt{\sqrt{\pi}2^n n!}, n = 0, 1, \cdots, \tag{2}$$

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where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

is the nth Hermite polynomial. The natural space for the quantum harmonic oscillator is the Schwartz class F of rapidly decreasing functions in $X = L^2(-\infty, \infty)$ as defined in section 2. Gulisashvili and MacCluer ([4]) defined a linear, closed, unbounded operator, namely, the weighted backward shift operator B on the space F by

$$\begin{aligned} B : F &\rightarrow F, && (3)\\ B\psi_n &\equiv \frac{1}{\sqrt{2}}(x + \frac{d}{dx})\psi_n = \sqrt{n}\psi_{n-1}. && (4) \end{aligned}$$

B has no resolvent set since every complex number λ is in the point spectrum of B. By using a result of Godefroy and Shapiro ([3]), Gulisashvili and MacCluer ([4]) have shown that the shift operator B is chaotic in the sense of Devaney ([1]), namely, it has topological transitivity (dense orbits), sensitivity to initial conditions (orbit divergence), and density of periodic points.

There is not a universally accepted definition of "chaos". Although the definition in Devaney [1] (or Wiggins [9]) seems a popular one, other definitions [6], which capture or describe other dynamical behavior of a system, are proposed and used in modern nonlinear dynamics. Sometimes Li-Yorke's definition of chaos does characterize the complexity of dynamical systems. For example, for one-dimensional dynamical systems (the iteration of continuous self-maps on intervals), Li-Yorke's chaos is equivalent to having positive topological entropy. The same conclusion holds for subshifts of finite type.

The relation between various definitions of chaos is not always apparent, yet each definition certainly describes important dynamical behavior, and there is a need to understand different behavior in various systems. This motivates us to study chaos of the above weighted backward shift map B in the sense of Li-Yorke, in terms of asymptotic separation of orbits.

In this paper, we construct a chaotic set for the operator B, and therefore show that the above linear quantum harmonic oscillator is chaotic in the sense of Li-Yorke.

2 Chaotic Set of Operator B

We now show that the above weighted shift operator B is chaotic in the sense of Li-Yorke. We first recall the chaos definition of Li-Yorke [7].

Definition Let M be a metric space with metric ρ and $f : M \rightarrow M$ be a continuous map. The discrete dynamical system (M, f) is called chaotic in the sense of Li-Yorke if there exists an uncountable subset S of nonwandering non-periodic points such that whenever $x, y \in S, x \neq y$, the following conditions hold,

$$\begin{aligned} &(i)\ \limsup_{n\rightarrow+\infty} \rho(f^n(x), f^n(y)) > 0\\ &(ii)\ \liminf_{n\rightarrow+\infty} \rho(f^n(x), f^n(y)) = 0 \end{aligned},$$

The subset S above is called a chaotic set for f.

Remark The original characterization of chaos in Li-Yorke's theorem [7] is via three conditions. The third one is :

$$(iii)\ \limsup_{n\to+\infty} \rho(f^n(x), f^n(p)) > 0, \quad \forall x \in S, \forall p \in P(f)$$

This condition means that no point in S is asymptotically periodic. ¿From conditions (i) and (ii) in the Definition, S contains at most one asymptotically periodic point [11]. So condition (iii) is not essential and can be removed.

In the following, we construct a chaotic set S for the operator $B : F \to F, B\psi_n = \sqrt{n}\psi_{n-1}$.

In terms of the orthonormal basis $\{\psi_n\}$, the Schwartz class F can be written as ([4])

$$F = \{\phi \in L^2(-\infty, \infty) : \phi = \sum_{n=0}^{\infty} c_n\psi_n, \sum_{n=0}^{\infty} |c_n|^2(n+1)^r < \infty, \forall r \geq 0\}.$$

F is an infinite-dimensional Fréchet space with topology defined by the system of semi-norms $p_r(\cdot)$ of the form ([10])

$$p_r(\phi) = p_r(\sum_{n=0}^{\infty} c_n\psi_n) = (\sum_{n=0}^{\infty} |c_n|^2(n+1)^r)^{1/2}, \quad r \geq 0.$$

This topology on F is also given by the metric ρ

$$\rho(\phi, \psi) = \sum_{m=0}^{\infty} 2^{-m} p_m(\phi - \psi) \cdot (1 + p_m(\phi - \psi))^{-1}.$$

Fix $\theta \in (0,1)$ and define $\phi^\theta = \sum_{n=0}^{\infty} c_n^\theta \psi_n$ by

$$\begin{cases} c_0^\theta = 0 \\ c_n^\theta = \begin{cases} 1/\sqrt{n!} & \text{if } n = k^2, [k\theta] - [(k-1)\theta] = 1 \\ 0 & \text{otherwise ,} \end{cases} \end{cases}$$

where $k = 1, 2, \cdots$, and $[\cdot]$ denotes the integer part of a real number.

Let $S = \{\phi^\theta : \theta \in (0,1)\}$. From

$$\sum_{n=0}^{\infty} |c_n^\theta|^2(n+1)^r \leq \sum_{n=0}^{\infty} \frac{(n+1)^r}{n!} < \infty, \quad \forall r,$$

we have $S \subseteq F$.

Let $V(\phi^\theta, \varepsilon)$ denote $\{\phi \in F : \rho(\phi, \phi^\theta) < \varepsilon\}$. Take

$$\phi_N^\theta = \sum_{n=0}^{N} c_n^\theta \psi_n + \sum_{n=N+1}^{\infty} c_{n-(N+1)}^\theta \cdot (n(n-1)\cdots(n-N))^{-1/2}\psi_n.$$

It is obvious that $\phi_N^\theta \in F$. Moreover, $\forall r \geq 0$, we have

$$
\begin{aligned}
p_r(\phi^\theta_N-\phi^\theta) &= (\sum_{n=N+1}^{\infty}|c^\theta_n-c^\theta_{n-(N+1)}(n(n-1)\cdots(n-N))^{-1/2}|^2(n+1)^r)^{1/2}\\
&\le (\sum_{n=N+1}^{\infty}2(|c^\theta_n|^2+|c^\theta_{n-(N+1)}|^2\cdot(n(n-1)\cdots(n-N))^{-1})(n+1)^r)^{1/2}\\
&\le (\sum_{n=N+1}^{\infty}4(n+1)^r/n!)^{1/2}. \qquad (5)
\end{aligned}
$$

This shows that $p_r(\phi^\theta_N-\phi^\theta)\to 0$ as $N\to\infty$. Therefore, $\forall\varepsilon>0$, there exists $N=N_\varepsilon$ big enough, such that $\phi^\theta_{N_\varepsilon}\in V(\phi^\theta,\varepsilon)$. Because

$$B^{N_\varepsilon+1}\phi^\theta_{N_\varepsilon}=\phi^\theta,$$

we know that

$$(B^{N_\varepsilon+1}V(\phi^\theta,\varepsilon))\cap V(\phi^\theta,\varepsilon)\neq\emptyset.$$

So all points in S are nonwandering. It is obvious that all points in S are nonperiodic as well.

Denote by $P(\phi^\theta,k)$ the number of c^θ_l's which satisfy $c^\theta_l\neq 0, 0\le l\le k$. Then

$$[\sqrt{k}\theta]\le P(\phi^\theta,k)\le[(\sqrt{k}+1)\theta],$$

which implies that

$$\lim_{k\to\infty}\frac{P(\phi^\theta,k)}{\sqrt{k}}=\theta. \qquad (*)$$

For $\theta_1,\theta_2\in(0,1)$ and $\theta_1\neq\theta_2$, we have $\phi^{\theta_1}\neq\phi^{\theta_2}$ from $(*)$. Hence S is an uncountable subset of F.

¿From the construction of ϕ^θ, only the coordinates of type $c^\theta_{k^2}$ may take a non-zero value. Therefore, for $\theta_1,\theta_2\in(0,1),\theta_1\neq\theta_2$, from $(*)$, there exists an infinite number of positive integers $k_n, n=1,2,\cdots$, such that $c^{\theta_1}_{k_n^2}\neq c^{\theta_2}_{k_n^2}$.

So $\forall r>0$,

$$
\begin{aligned}
p_r(B^{k_n^2}(\phi^{\theta_1}-\phi^{\theta_2})) &= (\sum_{m=0}^{\infty}|c^{\theta_1}_{m+k_n^2}-c^{\theta_2}_{m+k_n^2}|^2(m+1)(m+2)\cdots(m+k_n^2)(m+1)^r)^{1/2}\\
&\ge |c^{\theta_1}_{k_n^2}-c^{\theta_2}_{k_n^2}|^2k_n^2!=1>0, \qquad (6)\\
\liminf_{n\to\infty}p_r(B^{k_n^2}(\phi^{\theta_1}-\phi^{\theta_2})) &\ge 1>0. \qquad (7)
\end{aligned}
$$

Therefore, we obtain

$$\limsup_{k\to\infty}\rho(B^k(\phi^{\theta_1}),B^k(\phi^{\theta_2}))>0,\quad \forall\theta_1\neq\theta_2. \qquad (8)$$

This proves condition (i) in the chaos definition of Li-Yorke.

Moreover, $\forall k\ge 1$, when $k^2+1\le l\le(k+1)^2-1, c^{\theta_1}_l=c^{\theta_2}_l=0,\forall\theta_1,\theta_2\in(0,1)$. So we have

$$
\begin{aligned}
p_r^2(B^{k^2+1}(\phi^{\theta_1}-\phi^{\theta_2})) &= \sum_{m=0}^{\infty} |c^{\theta_1}_{m+k^2+1}-c^{\theta_2}_{m+k^2+1}|^2(m+1)(m+2)\cdots(m+k^2+1)(m+1)^r \\
&\leq \sum_{N=k+1}^{\infty} |c^{\theta_1}_{N^2}-c^{\theta_2}_{N^2}|^2(N^2-k^2)(N^2-k^2+1)\cdots N^2\cdot(N^2-k^2)^r \\
&\leq \sum_{N=k+1}^{\infty} \frac{(N^2-k^2)^r}{(N^2-k^2-1)!} \\
&\leq \sum_{N=k+1}^{\infty} \frac{(N^2-k^2)^r}{2^{N^2-k^2-2}} \\
&= \sum_{m=1}^{\infty} 4(m^2+2mk)^r/2^{m^2+2mk} \qquad (9)
\end{aligned}
$$

Thus, $\forall r > 0$,

$$0 \leq \lim_{k\to\infty} p_r(B^{k^2+1}(\phi^{\theta_1}-\phi^{\theta_2})) \leq \lim_{k\to\infty} (\sum_{m=1}^{\infty} \frac{4(m^2+2mk)^r}{2^{m^2+2mk}})^{1/2} = 0,$$

which implies that

$$\liminf_{k\to\infty} \rho(B^k(\phi^{\theta_1}), B^k(\phi^{\theta_2})) = 0. \qquad (10)$$

This proves condition (ii) of the Li-Yorke chaos definition. Therefore, S is a chaotic set for B, and (F, B) is a chaotic system in the sense of Li-Yorke.

3 Remarks

Recently, Godefroy and Shapiro ([3]) have shown that the weighted backward shift operator in a separable Hilbert space H, with a complete orthonormal basis $\{\phi_n\}$,

$$
\begin{aligned}
b &: H \to H, \\
b\phi_n &= \mu\phi_{n-1},
\end{aligned}
$$

is chaotic in the sense of Devaney, whenever $|\mu| > 1$.

Following the method given in the last section, we can also discuss the chaos of the operator b in the sense of Li-Yorke. When $|\mu| > 1$, the chaotic set S of b can be constructed as follows:

$$S = \{\phi^\theta = \sum_{n=0}^{\infty} c_n^\theta \phi_n : \theta \in (0,1)\},$$

where

$$
\begin{cases}
c_0^\theta = 0 \\
c_n^\theta = \begin{cases} 1/\mu^n & \text{if } n = k^2, [k\theta]-[(k-1)\theta] = 1, k \geq 1 \\ 0 & \text{otherwise .} \end{cases}
\end{cases}
$$

Hence b is also chaotic in the sense of Li-Yorke whenever $|\mu| > 1$.

When $\mu = 1, b$ is similar to the left shift map σ in symbolic dynamics. However, when $|\mu| \leq 1$, the global attractor of b is the one-point set containing only the zero vector; thus b is not chaotic in the sense of Li-Yorke or Devaney.

On a different phase space, i.e., on the Fréchet space $\Sigma(X) = \{(x_0, x_1, \cdots, x_k, \cdots) : x_k \in X, k \geq 0\}$, where X is a non-trivial Fréchet space, and $\Sigma(X)$ is equipped with product topology, Fu and Duan ([2]) have shown that σ is chaotic on $\Sigma(X)$ in the senses of both Li-Yorke and Devaney (or Wiggins).

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References

[1] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1986.

[2] X.-C. Fu and J. Duan, Infinite-dimensional linear dynamical systems with chaoticity, *J. Nonlinear Sci.*, accepted, 1998.

[3] G. Godefroy and J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds, *J. Funct. Anal.*, **98**(1991), 229-269.

[4] A. Gulisashvili and C. R. MacCluer, Linear chaos in the unforced quantum harmonic oscillator, *J. Dynamic Systems, Measurements and Control*, **118** (1996), 337-338.

[5] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer-Verlag, New York, 1990.

[6] U. Kirchgraber and D. Stoffer, On the definition of chaos, *Z. Angew. Math. Mech.*, **69** (1989), 175-185.

[7] T.-Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly*, **82**(1975), 985-992.

[8] R. L. Liboff, *Introductory Quantum Mechanics*, 2nd ed., Addison-Wesley, 1992.

[9] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.

[10] K. Yosida, *Functional Analysis*, 4th Edition, Springer-Verlag, 1978.

[11] Z.-L. Zhou, Chaos and totally chaos, *Science Bulletin* (China), **32**(1987), 248-250.