A Generalization of the Stone-von Neumann Theorem to Non-Regular Representations of the CCR-Algebra

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Abstract. We give a classification, up to unitary equivalence, of the representations of the C^*-algebra of the Canonical Commutation Relations which generalizes the classical Stone-von Neumann Theorem to the case of representations which are strongly measurable, but not necessarily strongly continuous. The classification includes all the (non-regular) representations which have been considered in physical models.

1. Introduction

The canonical approach to the description of quantum systems is based on the use of the so-called canonical variables $\{\hat{q}, \hat{p}\}$ satisfying the Heisenberg commutation relations, in terms of which all the physical quantities, including the Hamiltonian, are defined. From a mathematical point of view it is more convenient to deal with the so-called *Weyl operators*, i.e. the unitary operators $e^{-i(a\hat{q}+b\hat{p})}$, because they are bounded and can be considered as representatives of elements of an abstract C^*-algebra, the *algebra of the Canonical Commutation Relations* or, briefly, the *CCR-algebra*. Furthermore the Weyl operators have the advantage of being meaningful also in the cases in which the corresponding generators $\hat{q}, \hat{p}$ cannot be recovered as self-adjoint operators in the Hilbert space of the states of the quantum system (non-regular representations of the CCR-algebra) and the use of $\hat{q}$ and $\hat{p}$ can only be justified at an heuristic level, e.g. in order to define the Hamiltonian.

These cases, which were not considered in the foundations of quantum mechanics [19], probably because they were regarded as unphysical pathologies, have recently turned out to be of physical interest to treat quantum systems whose ground state formally corresponds to a non-L^2 function. This phenomenon occurs when there are superselection sectors, in particular when the system lives on a compact manifold (quantum particle on a circle [1]), when the invariance under a "gauge" group characterizes the ground states (quantum particle in a periodic potential, the Bloch electrons [1,12]), when a magnetic field gives rise to non-commutative translations and their eigenstates are considered [22]. This phenomenon also occurs in two-dimensional QFT models (massless fields in $1+1$ dimensions [1]), in the temporal gauge formulations of gauge QFT [4, 16], in the positive gauge quantization of Chern-Simons theory [17] and it is at the basis of the so-called θ-vacua superselection sectors [1, 12].

The above considerations suggest to reconsider the classical Stone-von Neumann (SvN) Theorem [20], which gives the uniqueness (up to unitary equivalence) of the regular irreducible representation of the CCR-algebra, and to look for a classification of non-strongly continuous representations, covering the non-regular ones needed for physically interesting models.

2. Regular Representations and Zak Transform

For a quantum system with one degree of freedom the *algebra of the Canonical Commutation Relations* is, by definition, the C^*-algebra generated by the abstract elements, $W(a,b)$, $(a,b) \in \mathbb{R}^2$, with product rule and involution given by

$$W(a,b)\, W(c,d) \;=\; e^{-\frac{i}{2}(ad-bc)}\, W(a+c, b+d) \qquad\qquad \big(W(a,b)\big)^* \;=\; W(-a,-b)$$

(see [3, 13] for more details). We shall denote this algebra by $\mathcal{A}_W$. A fundamental result in the study and classification of the representations of $\mathcal{A}_W$ is the

STONE-VON NEUMANN UNIQUENESS THEOREM. *Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ satisfying the hypotheses:*
(a) *$(\mathcal{H},\pi)$ is nondegenerate and irreducible*
(b) *the operator-valued function $(a,b) \to \pi(W(a,b))$ is strongly continuous.*
Then $(\mathcal{H},\pi)$ is unitary equivalent to the Schrödinger representation.

A representation of $\mathcal{A}_W$ which satisfies the property (b) of the theorem is called *regular*.

The Schrödinger representation π_S is based on the realization of the maximal abelian subalgebra generated by $W(0,b)$, $b \in \mathbb{R}$, as multiplication operators in $L^2(\mathbb{R})$, where $\mathbb{R}$ arises as the spectrum of the (unbounded) generator of the strongly continuous group $\pi_S(W(0,b))$. More convenient unitarily equivalent realizations (*Zak representations*) are obtained by choosing as maximal abelian C^*-subalgebras those generated by two elements, $W(\lambda,0)$ and $W(0,2\pi/\lambda)$, for a fixed positive λ. The advantage is that such sub-algebras are finitely generated, their Gelfand spectrum is the two-dimensional torus and, under the assumptions of the SvN Theorem, they can be realized as multiplicative operators in the space of square-integrable functions on their spectrum with the Lebesgue measure. Such subalgebras also play a relevant rôle in the above models. Up to automorphisms one can chose $\lambda = 1$; we shall denote by $\mathcal{A}_Z$ the resulting C^*-subalgebra and identify the Gelfand spectrum of $\mathcal{A}_Z$ with the product space $\mathbb{T}^2 = [0,1) \times [0,2\pi)$. The isometric isomorphism $\mathcal{U}_Z$ (*Zak transform* [22,11]) from the Schrödinger representation to the $\lambda = 1$ Zak representation is given by

$$\begin{aligned}
\big(\mathcal{U}_Z\,\phi\big)(\alpha,\beta) &= \sum_{n\in\mathbb{Z}} \phi(n+\alpha)\; e^{-i\,n\beta} && \forall\phi \in L^2(\mathbb{R})\\
\big(\mathcal{U}_Z^{-1}\,\psi\big)(x) &= \frac{1}{2\pi}\int_0^{2\pi} \psi(x \bmod 1,\beta)\; e^{i\,[x]\beta}\,\mathrm{d}\beta && \forall\psi \in L^2(\mathbb{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)
\end{aligned}$$

where $x \in \mathbb{R}$, $(\alpha,\beta) \in [0,1)\times[0,2\pi)$ and $[x]$ is the integer part of x. ¿From these relations it follows that, for each a, b in $\mathbb{R}$ and ψ in $L^2(\mathbb{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$,

$$\begin{aligned}
\big(\mathcal{U}_Z\;\pi_S(W(a,0))\,\mathcal{U}_Z^{-1}\,\psi\big)(\alpha,\beta) &= e^{i\,[\alpha+a]\beta}\;\psi((\alpha+a) \bmod 1,\beta)\\
\big(\mathcal{U}_Z\;\pi_S(W(0,b))\,\mathcal{U}_Z^{-1}\,\psi\big)(\alpha,\beta) &= e^{-i\,b\alpha}\;\psi(\alpha,(\beta+b)\mathrm{mod}\;2\pi) \quad .
\end{aligned} \tag{2.1}$$

In conclusion, a regular irreducible and nondegenerate representation of $\mathcal{A}_W$ is unitarily equivalent to a representation on the space $L^2(\mathbb{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$ such that the elements of the abelian C^*-subalgebra $\mathcal{A}_Z$ act multiplicatively by their Gelfand transforms.

3. Spectrally Multiplicity-Free Representations

In the following our strategy will be to characterize the representations of $\mathcal{A}_W$ in terms of representations of its maximal abelian C^*-subalgebra $\mathcal{A}_Z$. The first problem is to understand under which condition the representation of $\mathcal{A}_Z$ can be described in terms of a space $L^2(\mathbb{T}^2,\mu)$, where μ is a positive measure on the Borel (equivalently Baire) sets of $\mathbb{T}^2$, and the elements of $\mathcal{A}_Z$ act as multiplicative operators by their Gelfand transforms. This condition is general enough to cover all the above mentioned models and, in our opinion, it represents a mild restriction which allows a measure theoretical discussion of the problem. In the separable case such a property amounts to the standard property of multiplicity-free, which in turn follows from the irreducibility and regularity of the representation of $\mathcal{A}_W$. However, in the non-separable case, which is typical of non-regular representations of $\mathcal{A}_W$, a multiplicity-free representation of an abelian unital C^*-algebra $\mathcal{B}$ cannot, in general, be diagonalized on a single copy of its spectrum (see [5, 6] for a counterexample). More precisely, in the following, we denote by $\mathcal{B}$ an abelian unital C^*-algebra and by $(\mathcal{H}, \pi)$ a representation of $\mathcal{B}$; for every vector x in $\mathcal{H}$, we denote by $\mathcal{H}_x$ the cyclic $\pi(\mathcal{B})$-invariant subspace of $\mathcal{H}$ given by the closure of the set $\{\pi(A)x \mid A \in \mathcal{B}\}$ and by μ_x the positive Baire measure on the Gelfand spectrum $\widehat{\mathcal{B}}$ of $\mathcal{B}$ defined by x via the Riesz–Markov Representation Theorem

$$(x\,,\,\pi(A)x) = \int_{\widehat{\mathcal{B}}} \widehat{A}\,\mathrm{d}\mu_x \qquad \forall\, A \in \mathcal{B}$$

(with $\widehat{A}$ the Gelfand transform of A).

DEFINITION 3.1. A nondegenerate representation $(\mathcal{H}, \pi)$ of $\mathcal{B}$ is *multiplicity-free* if it verifies one (and therefore all) of the following equivalent [5,6] conditions

(i) for each vector x in $\mathcal{H}$, the projection on the cyclic subspace $\mathcal{H}_x$ belongs to the von Neumann algebra, $\pi(\mathcal{B})''$, generated by $\pi(\mathcal{B})$

(ii) $\pi(\mathcal{B})''$ is maximal abelian

(iii) if $x, y \in \mathcal{H}$ and $\mathcal{H}_x \perp \mathcal{H}_y$, then μ_x and μ_y are mutually orthogonal, i.e., for every x in $\mathcal{H}$ and every vector $y \perp \mathcal{H}_x$, there exists a Baire set, S_x^y of $\widehat{\mathcal{B}}$ such that $\mu_x(\widehat{\mathcal{B}} \setminus S_x^y) = 0$ and $\mu_y(S_x^y) = 0$.

In separable Hilbert spaces we have [14]:

PROPOSITION 3.2. *Let $(\mathcal{H}, \pi)$ be a multiplicity-free nondegenerate representation of $\mathcal{B}$, with $\mathcal{H}$ a separable Hilbert space. Then there exists a positive measure μ on the Baire sets of $\widehat{\mathcal{B}}$ and an isometric isomorphism, U, from $\mathcal{H}$ onto $L^2(\widehat{\mathcal{B}}, \mu)$ such that, for each element A of $\mathcal{B}$, $U\pi(A)\,U^{-1}$ is the operator of multiplication by the Gelfand transform of A.*

In the non-separable case, the rôle of the von Neumann algebra $\pi(\mathcal{B})''$ is played [5, 6] by the Baire*-algebra generated by $\pi(\mathcal{B})$ (i.e. the smallest C^*-algebra of operators in $\mathcal{H}$ which contains $\pi(\mathcal{B})$ and the limit of each of its weakly convergent monotone sequences [10]):

THEOREM 3.3. *Let $(\mathcal{H}, \pi)$ be a representation of a unital abelian C^*-algebra $\mathcal{B}$. Then the following statements are equivalent:*

(i) *for each vector x in $\mathcal{H}$, the projection on the cyclic subspace $\mathcal{H}_x$ belongs to the Baire*-algebra generated by $\pi(\mathcal{B})$*

(ii) *for every x in $\mathcal{H}$, there exists a Baire subset, S_x, of $\widehat{\mathcal{B}}$ such that $\mu_x(\widehat{\mathcal{B}} \setminus S_x) = 0$ and $\mu_y(S_x) = 0$ for each vector $y \perp \mathcal{H}_x$*

(iii) *there exist a positive measure μ on the Baire sets of $\widehat{\mathcal{B}}$ and an isometric isomorphism U from $\mathcal{H}$ onto $L^2(\widehat{\mathcal{B}}, \mu)$ such that, for each element A of $\mathcal{B}$, $U\pi(A)U^{-1}$ is the operator of multiplication by the Gelfand transform of A.*

DEFINITION 3.4. We say that a representation of $\mathcal{B}$ is *spectrally multiplicity-free* if it verifies one (and therefore all) of the statements of Theorem 3.3.

Finally, if $(\mathcal{H}, \pi)$ is spectrally multiplicity-free, the measure μ, in the statement (iii) of Theorem 3.3, can be explicitly constructed as the sum

$$\mu = \sum_{i \in I} \mu_{x_i}$$

where $\{x_i\}_{i \in I}$ is a family of orthogonal vectors such that $\mathcal{H} = \oplus_{i \in I} \mathcal{H}_{x_i}$.

According to the above discussion, in the following we shall consider those representations of $\mathcal{A}_W$ whose restriction to the commutative subalgebra $\mathcal{A}_Z$ is spectrally multiplicity-free. This property is related to the irreducibility of the representation of the full algebra $\mathcal{A}_W$; it is in fact equivalent to it in the regular case and it implies irreducibility under measurability conditions discussed below.

4. A Classification of Strongly Measurable Representations of $\mathcal{A}_W$

Also within the class of spectrally multiplicity-free representations of $\mathcal{A}_Z$ pathologies may arise if no "regularity" condition is required on the representation of the full algebra $\mathcal{A}_W$. For instance, even in the separable case, one may construct (multiplicity free) representations π which are not weakly measurable in the parameters of the Weyl operators $\pi(W(a, b))$ [5; Ex. V.3.2], so that even the Borel structure of $\mathbb{R}^2$ may be lost in the representation, excluding a discussion in terms of the Borel measures of Theorem 3.3. We recall that, for representations in separable Hilbert spaces, the strong continuity of the Weyl operators (as function from $\mathbb{R}^2$ into $\mathcal{L}(\mathcal{H})$) is equivalent to their strong, or weak, measurability [21]. The SvN Theorem thus amounts to a classification of strongly measurable representations in separable spaces. To obtain a generalization to non-regular representations we are led to discuss measurability properties in non-separable spaces.

Let $(X, \mathcal{M}, \mu)$ be a positive σ-finite measure space and $\mathcal{H}$ be an Hilbert space.

DEFINITION 4.1. A function from X into $\mathcal{H}$ is said to be *countably-valued* if it assumes at most a countable set of values in $\mathcal{H}$, each value being taken on a measurable set. A function from X into $\mathcal{H}$ is called *measurable w.r.t.* μ if there exists a sequence of countably-valued functions converging μ-almost everywhere to it.

DEFINITION 4.2. An operator-valued function V from X into the set $\mathcal{L}(\mathcal{H})$ of bounded operators in $\mathcal{H}$ is called *strongly measurable w.r.t.* μ if, for every x in $\mathcal{H}$, the vector-valued function $X \ni a \to V(a)x \in \mathcal{H}$ is measurable w.r.t. μ.

PROPOSITION 4.3. [9; Thm. 3.5.5] *An operator-valued function $V : X \to \mathcal{L}(\mathcal{H})$ is strongly measurable w.r.t. μ iff:*

(i) *it is weakly measurable (i.e. $\forall x, y \in \mathcal{H}$, the function $X \ni a \to (V(a)x \,|\, y) \in \mathbb{C}$ is measurable)*

(ii) *$V(a)x$ is μ-almost separably-valued for every x in $\mathcal{H}$ (i.e. for every x in $\mathcal{H}$, there exists a μ-null set N such that $\{V(a)x \,|\, a \in X \backslash N\}$ is a separable subset of $\mathcal{H}$).*

By the discussion at the beginning of the section, we have to require some measurability property of the operator-valued function $\mathbb{R}^2 \ni (a, b) \to \pi(W(a, b))$. Actually, since in our analysis the Zak algebra is already represented by multiplication operators, it is enough to discuss the measurability of $\pi(W(a, b))$ with $(a, b) \in [0, 1)\times[0, 2\pi)$. Since the adjoint action of $W(a, b)$ on $\mathcal{A}_Z$ induces a translation on the spectrum of $\mathcal{A}_Z$, it is natural to identify the parameter space $[0, 1) \times [0, 2\pi)$ with the spectrum $\mathbb{T}^2$ of $\mathcal{A}_Z$ and to require strong measurability w.r.t. every measure μ_x, $x \in \mathcal{H}$.

For each vector y in $\mathcal{H}$, the set $\{\pi(W(a, b))y \,|\, (a, b) \in \mathbb{T}^2\}$ is, in general, non-separable, but strong measurability guarantees a sort of "local separability" of the representation; in fact, by Proposition 4.3, it implies that, for every measure μ_x, there exists a Borel subset N of $\mathbb{T}^2$ such that $\mu_x(N) = 0$ and $\{\pi(W(a, b))y \,|\, (a, b) \in \mathbb{T}^2 \setminus N\}$ is separable. This will allow to make a "local" use of standard results which hold only in separable Hilbert spaces (or for σ-finite measures), even if the representation is non-separable. We can now state our theorem.

THEOREM 4.4. *Let $(\mathcal{H}, \pi)$ be a representation of the CCR-algebra $\mathcal{A}_W$ satisfying the following hypotheses:*

(A) *spectrally multiplicity-free condition for the subalgebra $\mathcal{A}_Z$*

(B) *strong measurability w.r.t. every measure μ_x, $x \in \mathcal{H}$, i.e.*

$$[0,1)\times[0,2\pi) \ni (a,b) \longrightarrow \pi(W(a,b))y \in \mathcal{H}$$

is μ_x-measurable for all x and y in $\mathcal{H}$.

Then:

(1) *$(\mathcal{H}, \pi)$ is an irreducible representation of $\mathcal{A}_W$*

(2) *there exist a positive translation-invariant measure μ on the Borel σ-algebra of $\mathbb{T}^2$ and an isometric isomorphism $\mathcal{U}$ from $\mathcal{H}$ onto $L^2(\mathbb{T}^2, \mu)$ such that, for every a, b in $\mathbb{R}$ and every ψ in $L^2(\mathbb{T}^2, \mu)$,*

$$\begin{aligned} \big(\mathcal{U}\,\pi(W(a,0))\,\mathcal{U}^{-1}\,\psi\big)(\alpha,\beta) &= e^{i\,[\alpha+a]\beta}\;\psi((\alpha+a) \bmod 1, \beta) \\ \big(\mathcal{U}\,\pi(W(0,b))\,\mathcal{U}^{-1}\,\psi\big)(\alpha,\beta) &= e^{-i\,b\alpha}\;\psi(\alpha, (\beta+b) \bmod 2\pi) \end{aligned} \tag{4.1}$$

where $[\alpha+a]$ denotes the integer part of $\alpha+a$

(3) *the measure μ can be written as the sum of a family of finite positive Borel measures concentrated on disjoint sets.*

Conditions (A) and (B) allow to characterize the representations, up to unitary equivalence: inequivalent representations correspond to inequivalent translation-invariant measures. This classification covers the known examples:

– the Schrödinger representation corresponds to the two-dimensional Lebesgue measure on $\mathbb{T}^2$; hence $\mathcal{H} \cong L^2(\mathbb{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$

– the (equivalent) representations defined, via the GNS construction, by the "momentum states" ω_p , $p \in \mathbb{R}$,

$$\omega_p(W(a,b)) = \begin{cases} 0\,, & \text{if } b \neq 0 \\ e^{i\,pa}\,, & \text{if } b = 0 \end{cases}$$

(see [1, 2, 8]), correspond to the measure $\mu = \sum_{j\in[0,2\pi)} \mathrm{d}\alpha_j$ where $\mathrm{d}\alpha_j$, $j \in [0, 2\pi)$, denotes the one-dimensional Lebesgue measure concentrated on the interval $\{(\alpha, j) \mid \alpha \in [0,1)\} \subseteq \mathbb{T}^2$

– the (equivalent) representations defined by the "Zak states" $\omega_{\zeta\gamma}$, $\zeta \in [0, 2\pi)$, $\gamma \in [0, 1)$,

$$\omega_{\zeta\gamma}(W(a,b)) = \begin{cases} 0\,, & \text{if } (a,b) \notin \mathbb{Z} \times 2\pi\mathbb{Z} \\ e^{i\pi m\, n}\; e^{i\, n\zeta}\; e^{i2\pi m\,\gamma}\,, & \text{if } (a,b) = (n, 2\pi m) \end{cases}$$

(see [2,22]), correspond to the counting measure on $\mathbb{T}^2$; hence $\mathcal{H} \cong \ell^2(\mathbb{T}^2)$.

5. Proof of Theorem 4.4

The logic of the proof is the following. By exploiting the spectrally multiplicity-free condition and the strong measurability we first show that the representation is irreducible (Lemma 5.3). Then we reduce the analysis of the representation to that of a unitary representation T of the commutative group $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ (Lemma 5.5). Such representation is shown to be of the form

$$\big(T(\overline{a})\psi\big)(\overline{\alpha}) \;=\; f(\overline{a}, \overline{\alpha})\; \psi(\overline{\alpha} + \overline{a})$$

(Lemma 5.6). By exploiting the measurability of the function f, which follows from the hypothesis (B), and the group law, f is shown to be of the form

$$f(\overline{a}, \overline{\alpha}) \;=\; \frac{\xi(\overline{a} + \overline{\alpha})}{\xi(\overline{\alpha})}$$

(Lemma 5.7). Hence f can be removed by a unitary transformation and the relations (4.1) are easily obtained.

Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ satisfying the hypotheses (A) and (B) of the theorem. According to the spectrally multiplicity-free property (see Section 3), there exist a positive measure μ on the Borel σ-algebra $\mathcal{B}(\mathbb{T}^2)$ of $\mathbb{T}^2$ and an isometric isomorphism $\mathcal{U}$ from the representation space $\mathcal{H}$ onto $L^2(\mathbb{T}^2, \mu)$ such that, for each A in $\mathcal{A}_Z$, $\mathcal{U}\,\pi(A)\,\mathcal{U}^{-1}$ is the operator of multiplication by the Gelfand transform of A. The measure μ is of the form $\mu = \sum_{i\in I} \mu_{x_i}$ where $\{x_i\}_{i\in I}$ is a family of orthogonal vectors such that $\mathcal{H} = \oplus_{i\in I}\overline{\{\pi(\mathcal{A}_Z)x_i\}}$ and μ_{x_i} is the measure associated to x_i via the Riesz-Markov Theorem. Moreover, for each i in I, there is a Borel set S_i of $\mathbb{T}^2$ such that $\mu_{x_i}(S_i) = \mu_{x_i}(\mathbb{T}^2)$ and $\mu_y(S_i) = 0$ for all $y \perp \mathcal{H}_{x_i}$.
If f is a function on $\mathbb{T}^2$ and $(a,b) \in \mathbb{R}^2$, denote by $f^{(a,b)}$ the translate of f, i.e. $f^{(a,b)}(\alpha, \beta) = f((\alpha - a) \bmod 1, (\beta - b) \bmod 2\pi)$. If μ is a positive measure on $\mathcal{B}(\mathbb{T}^2)$, its

translate, $\mu^{(a,b)}$, is the measure defined by $\mu^{(a,b)}(f) = \mu(f^{(-a,-b)})$, for every bounded Borel-measurable function f on $\mathbb{T}^2$.

LEMMA 5.1. *For every (a,b) in $\mathbb{R}^2$ and x in $\mathcal{H}$, the measure associated to the vector $\pi(W(a,b))x$ is the translated measure $\mu_x^{(-a,-b)}$.*

Proof. The relation $\tau_{ab}(A) = W(a,b)^{-1} A\, W(a,b)$, $A \in \mathcal{A}_Z$, defines a inner automorphism of $\mathcal{A}_Z$. This map induces, by the Gelfand transform, an automorphism $\widehat{\tau_{ab}}$ on $\mathcal{C}(\mathbb{T}^2)$: $\widehat{\tau_{ab}}(\widehat{A}) = \widehat{A}^{(a,b)}$. On the other hand, from the definition of μ_x one has, for every $\widehat{A}$ in $\mathcal{C}(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} \widehat{\tau_{ab}}(\widehat{A})\, \mathrm{d}\mu_x = \big(x\,,\, \pi\big(W(a,b)^{-1} A\, W(a,b)\big)x\big) = \int_{\mathbb{T}^2} \widehat{A}\, \mathrm{d}\mu_{\pi(W(a,b))x} \quad .$$

□

In the following, for simplicity, a point of $\mathbb{T}^2$ will be denoted by an overlined (Greek or Latin) letter; $\overline{\alpha} + \overline{a}$ will indicate a translation of $\overline{\alpha} \in \mathbb{T}^2$ by $\overline{a} \in \mathbb{T}^2$. For each pair (i,j) of indices in I one can consider the product measure $\mu_{x_i} \otimes \mu_{x_j}$ defined on the Borel σ-algebra $\mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)$ of the product space $\mathbb{T}^2 \times \mathbb{T}^2$.

LEMMA 5.2. *Let (i,j) be a pair of indices in I and let A be an element of the σ-algebra $\mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that $(\mu_{x_i} \otimes \mu_{x_j})(A) > 0$. Then there exists k in I such that*

$$\big(\mu_{x_i} \otimes \mu_{x_j}\big)(A \cap \Im_k) > 0\ , \qquad \textit{with} \quad \Im_k = \big\{(\overline{a}, \overline{\alpha}) \in \mathbb{T}^2 \times \mathbb{T}^2 \,\big|\, \chi_{S_k}(\overline{\alpha} + \overline{a}) = 1\big\} \quad .$$

Proof. Note firstly that $\Im_k \in \mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)$ (in fact, if η is the homeomorphism of $\mathbb{T}^2 \times \mathbb{T}^2$ such that $(\overline{a}, \overline{\alpha}) \xrightarrow{\eta} (\overline{a}, \overline{\alpha} - \overline{a})$, then $\Im_k = \eta(\mathbb{T}^2 \times S_k)$). Let A be an element of $\mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that $(\mu_{x_i} \otimes \mu_{x_j})(A) > 0$. We can always assume that $A \subseteq S_i \times S_j$. By Fubini Theorem there exists a measurable $Y_A \subseteq S_i$, with $\mu_{x_i}(Y_A) > 0$, so that, for every $\overline{a}$ in Y_A, the set

$$A|_{\overline{a}} = \{\overline{\alpha} \in \mathbb{T}^2 \,|\, (\overline{a}, \overline{\alpha}) \in A\}$$

(the $\overline{a}$-section of A) is Borel measurable and $\mu_{x_j}(A|_{\overline{a}}) > 0$. Thus, if k is an index in I for which $(\mu_{x_i} \otimes \mu_{x_j})(A \cap \Im_k) = 0$, since $A|_{\overline{a}} \subseteq S_j$,

$$\int_{\mathbb{T}^2} \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha} + \overline{a})\ \mathrm{d}\mu(\overline{\alpha}) = 0 \qquad\qquad \mu_{x_i}(\overline{a})\text{-a.e.}\ .$$

Using the translation property of the measures μ_x (Lemma 5.1), for every $\widehat{B}$ in $\mathcal{C}(\mathbb{T}^2)$,

$$\begin{aligned}&(\, \chi_{A|_{\overline{a}}}\,,\ \mathcal{U}\pi(W(\overline{a}))\mathcal{U}^{-1}\widehat{B}\chi_{S_k}\,)\\ &= \int_{\mathbb{T}^2} (\mathcal{U}\pi(W(\overline{a}))\mathcal{U}^{-1}\,\widehat{B}\chi_{S_k})(\overline{\alpha})\ \ \chi_{S_k}(\overline{\alpha}+\overline{a})\ \ \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \mathrm{d}\mu(\overline{\alpha}) \ = \ 0 \qquad \mu_{x_i}(\overline{a})\text{-a.e. .}\end{aligned}$$

Therefore $(\mu_{x_i}\otimes\mu_{x_j})(A\cap\Im_k)=0$ implies $\pi(W(-\overline{a}))\,\mathcal{U}^{-1}\chi_{A|_{\overline{a}}}\perp\mathcal{H}_{x_k}$ almost everywhere w.r.t. $\mu_{x_i}(\overline{a})$. By the measurability property (B), one has that, removing at most a μ_{x_i}-null set from Y_A, the set $\{\pi(W(-\overline{a}))\,\mathcal{U}^{-1}\chi_{A|_{\overline{a}}}\,|\,\overline{a}\in Y_A\}$ is separable. Since the subspaces $\mathcal{H}_{x_k}$ are mutually orthogonal, this implies that the family

$$\widetilde{I} = \left\{k\in I \,\middle|\, \mathcal{H}_{x_k}\ \not\perp\ \pi(W(-\overline{a}))\,\mathcal{U}^{-1}\chi_{A|_{\overline{a}}}\quad \text{for some } \overline{a}\in Y_A\right\}$$

must be countable. In conclusion, if $(\mu_{x_i}\otimes\mu_{x_j})(A\cap\Im_k)=0$, for every k in I, then, since $\widetilde{I}$ is countable,

$$\pi(W(-\overline{a}))\,\mathcal{U}^{-1}\chi_{A|_{\overline{a}}}\ \perp\ \oplus_{k\in\widetilde{I}}\,\mathcal{H}_{x_k} \qquad\qquad \mu_{x_i}(\overline{a})\text{-a.e. .}$$

So $\pi(W(-\overline{a}))\,\mathcal{U}^{-1}\chi_{A|_{\overline{a}}}$ is zero $\mu_{x_i}(\overline{a})$-a.e.. Since $\pi(W(-\overline{a}))\,\mathcal{U}^{-1}$ is norm-preserving, this means that $\mu_{x_j}(A|_{\overline{a}})=0$ for μ_{x_i}-almost all $\overline{a}$ in Y_A, which contradicts the property previously obtained for Y_A. □

LEMMA 5.3. *The representation* $(\mathcal{H},\pi)$ *is irreducible.*

Proof. Let x_1 be a non-null vector of $\mathcal{H}$ and $\mathcal{H}_1=\overline{\{\pi(\mathcal{A}_W)x_1\}}$. Let $x_2\perp\mathcal{H}_1$. Consider $\mathcal{H}_2=\overline{\{\pi(\mathcal{A}_W)x_2\}}$ and the subrepresentation $(\mathcal{H}_1\oplus\mathcal{H}_2,\pi_1\oplus\pi_2)$. Writing

$$\mathcal{H}_1=\oplus_{l\in I_1}\overline{\{\pi(\mathcal{A}_Z)x_{1,l}\}}\quad ,\qquad\qquad \mathcal{H}_2=\oplus_{m\in I_2}\overline{\{\pi(\mathcal{A}_Z)x_{2,m}\}}$$

we obtain the decomposition $\mathcal{H}_1\oplus\mathcal{H}_2=\oplus_{j=1,2\ m\in I_j}\mathcal{H}_{x_{j,m}}$. Now for every product measure $\mu_{x_{1,l}}\otimes\mu_{x_{2,m}}$, with l in I_1 and m in I_2, and every k in I_1,

$$(\mu_{x_{1,l}}\otimes\mu_{x_{2,m}})(\Im_{1,k}) \ = \ \int_{\mathbb{T}^2}\mu_{x_{2,m}}({S_{1,k}}^{(-\overline{a})})\ \ \mathrm{d}\mu_{x_{1,l}}(\overline{a}) \ = \ 0$$

(${S_{1,k}}^{(-\overline{a})}$ is the $(-\overline{a})$-translated of $S_{1,k}$), since $x_{1,k}\in\mathcal{H}_1\ \perp\ \mathcal{H}_2\ni\pi(W(\overline{a}))x_{2,m}$ and therefore, for each $\overline{a}$, $\mu_{x_{2,m}}({S_{1,k}}^{(\overline{a})})=0$. Similarly $(\mu_{x_{1,l}}\otimes\mu_{x_{2,m}})(\Im_{2,k})=0$ for each k in I_2. From Lemma 5.2 it then follows that $x_2=0$. □

The notion of localizability, as introduced by Segal [18], plays an important rôle in the following: a positive measure space (Ω, Σ, μ) is said to be *localizable* if for every (not necessarily countable) collection $\mathcal{G} \subset \Sigma$ of sets of finite measure there exists a set B in Σ, called the *supremum*, such that: (1) for each E in $\mathcal{G}$, $\mu(E \backslash B) = 0$ and (2) if $\widetilde{B}$ is another element of Σ such that $\mu(E \backslash \widetilde{B}) = 0$ for every E in $\mathcal{G}$, then $\mu(B \backslash \widetilde{B}) = 0$.

LEMMA 5.4. *The measure μ is localizable and can be written in the form*

$$\mu = \sum_{i \in I} \chi_{S_i} \, \mu_{x_0}^{(\overline{r}_i)} \equiv \sum_{i \in I} \mu_i$$

where: $\overline{r}_i$'s are points of $\mathbb{T}^2$, x_0 is an arbitrary non-null vector in $\mathcal{H}$ and $\{S_i\}_{i \in I}$ is a disjoint collection of Borel subsets of $\mathbb{T}^2$.

Proof. Let x_0 be a fixed non-null vector of $\mathcal{H}$. For every Borel set Y of $\mathbb{T}^2$ and every point $\overline{r}$ in $\mathbb{T}^2$, define

$$x_{Y, \overline{r}} = \pi(W(-\overline{r})) \, \mathcal{U}^{-1} \chi_Y \, \mathcal{U} \, x_0 \ .$$

By Lemma 5.1, $\mu_{x_{Y, \overline{r}}} = (\chi_Y \, \mu_{x_0})^{(\overline{r})}$. Let $\mathcal{J} = \{x_{Y_i, \overline{r}_i}\}_{i \in I}$ be a maximal family of such vectors, with the properties: (i) $\mathcal{J}$ contains x_0 (ii) if $x_{Y_i, \overline{r}_i}$ and $x_{Y_j, \overline{r}_j}$ are in $\mathcal{J}$, then $\mu_{x_{Y_i, \overline{r}_i}} \perp \mu_{x_{Y_j, \overline{r}_j}}$. ¿From the orthogonality of these measures it follows that the corresponding cyclic subspaces $\mathcal{H}_{x_{Y_i, \overline{r}_i}}$ are mutually orthogonal and the irreducibility of the representation (Lemma 5.3) implies that $\mathcal{H} = \oplus_{i \in I} \mathcal{H}_{x_{Y_i, \overline{r}_i}}$. Then, according to the spectrally multiplicity-free property, we can take

$$\mu = \sum_{i \in I} (\chi_{Y_i} \, \mu_{x_0})^{(\overline{r}_i)} \ .$$

We claim that such a μ, naturally extended to the σ-algebra $\bigcap_{x \in \mathcal{H}} \overline{\mathcal{B}(\mathbb{T}^2)}^{\mu_x}$ (where $\overline{\mathcal{B}(\mathbb{T}^2)}^{\mu_x}$ denotes the μ_x-completion of $\mathcal{B}(\mathbb{T}^2)$) is localizable.

Firstly we observe that, as a consequence of the hypotheses (A) and (B), there exists a Borel set Y_0 in $\mathbb{T}^2$ such that: (1) $\mu_{x_0}(\mathbb{T}^2 \backslash Y_0) = 0$ and $\mu_y(Y_0) = 0$ for every $y \perp \mathcal{H}_{x_0}$ (2) $\{\pi(W(\overline{a} - \overline{b})) \, \mathcal{H}_{x_0} \, | \, \overline{a}, \overline{b} \in Y_0\}$ is a separable subset of $\mathcal{H}$. By the first of these two properties, we can write

$$\mu = \sum_{i \in I} \chi_{(Y_i \cap Y_0)^{(\overline{r}_i)}} \, \mu_{x_0}^{(\overline{r}_i)} \quad .$$

The second property implies that, for each index i in I, the set

$$I' = \left\{ j \in I \,\middle|\, j \neq i \text{ and } (Y_0 \cap Y_i)^{(\overline{r}_i)} \cap (Y_0 \cap Y_j)^{(\overline{r}_j)} \neq \emptyset \right\}$$

is countable. In fact, if $\overline{c} \in ((Y_0 \cap Y_i)^{(\overline{r}_i)} \cap (Y_0 \cap Y_j)^{(\overline{r}_j)})$, then $\overline{a} + \overline{r}_j = \overline{c} = \overline{b} + \overline{r}_i$ for some $\overline{a}$ in $Y_0 \cap Y_j$ and $\overline{b}$ in $Y_0 \cap Y_i$; therefore

$$\left\{ x_{Y_j, \overline{r}_j} \,\middle|\, j \in I' \right\} \subseteq \pi(W(-\overline{r}_i)) \left\{ \pi(W(\overline{a} - \overline{b}))\, \mathcal{H}_{x_0} \,\middle|\, \overline{a}, \overline{b} \in Y_0 \right\}$$

and I' must be countable. In conclusion μ turns out to be the sum of finite measures which are concentrated on a family $\{(Y_0 \cap Y_i)^{(\overline{r}_i)}\}_{i \in I}$ of measurable sets such that

$$\mu \left(\cup_{j \neq i} (Y_0 \cap Y_j)^{(\overline{r}_j)} \cap (Y_0 \cap Y_i)^{(\overline{r}_i)} \right) = \mu \left(\cup_{j \in I'} (Y_0 \cap Y_j)^{(\overline{r}_j)} \cap (Y_0 \cap Y_i)^{(\overline{r}_i)} \right) = 0 \, .$$

Using this result and the fact each finite measure is localizable, the localizability of μ can be easily obtained.

To conclude the proof of the lemma we note that, since the generators of $\mathcal{A}_W$ are indexed by $\mathbb{R}^2$ and the representation is cyclic, the dimension of $\mathcal{H}$ and therefore the cardinality of I cannot exceed that of continuum. Since the measure μ is localizable and admits the family $\{(Y_0 \cap Y_i)^{(\overline{r}_i)} \,|\, i \in I\}$ of μ-a.e. disjoint measurable sets, with $0 < \mu((Y_0 \cap Y_i)^{(\overline{r}_i)}) < +\infty$, by a theorem due to McShane [15], there exists a disjoint collection $\{S_i \,|\, i \in I\}$ of Borel subsets of $\mathbb{T}^2$ such that, for each i in I, $S_i \subseteq Y_i^{(\overline{r}_i)}$ and $\mu_{x_0}^{(\overline{r}_i)}(S_i) = \mu_{x_0}^{(\overline{r}_i)}(Y_i^{(\overline{r}_i)})$; hence μ can be written into the desired form. ◻

LEMMA 5.5. *The family of operators $\{T(a,b)\}$ defined by*

$$\left(T(a,b)\,\psi\right)(\alpha,\beta) = e^{-i\,[\alpha+a]\beta}\; e^{i\,b(\alpha+a) \bmod 1}\; e^{-\frac{i}{2}ab} \left(\mathcal{U}\,\pi(W(a,b))\,\mathcal{U}^{-1}\psi\right)(\alpha,\beta)$$

gives a unitary representation of the commutative group $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$.

Proof. Indicate by M_f the operator in $L^2(\mathbb{T}^2, \mu)$ of multiplication by a Borel-measurable function f on $\mathbb{T}^2$. If $\widehat{A}$ is a continuous function, from the properties of $\widehat{\tau_{\overline{a}}}$ (see the proof of Lemma 5.1) we have

$$\left(\mathcal{U}\,\pi(W(-\overline{a}))\,\mathcal{U}^{-1}\right) M_{\widehat{A}} \left(\mathcal{U}\,\pi(W(\overline{a}))\,\mathcal{U}^{-1}\right) = M_{\widehat{A}(\overline{a})} \, .$$

By weak density this relation extends to each operator M_f, with f bounded Borel-measurable function on $\mathbb{T}^2$; the commutation relations of the algebra $\mathcal{A}_W$ imply the desired result on the family $\{T(a,b)\}$ by a straightforward computation. ◻

LEMMA 5.6. *There exists a function $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{C}$ such that, for every $\overline{a}$ in $\mathbb{T}^2$,*

$$\left(T(\overline{a})\psi\right)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\; \psi(\overline{\alpha} + \overline{a}) \qquad \forall\, \psi(\overline{\alpha}) \in L^2(\mathbb{T}^2, \mu) \, .$$

Moreover f is measurable w.r.t. the σ-algebra $\bigcap_{i,j\in I} \overline{\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)}^{\mu_i\otimes\mu_j}$.

Proof. For each j,k in I, define

$$\mathbb{T}^2 \ni \overline{a} \longrightarrow G_j^k(\overline{a}) \equiv \chi_{S_j}\, T(\overline{a})\chi_{S_k} \in L^2(\mathbb{T}^2,\mu_j) \quad .$$

The strong measurability of the representation and the boundedness of operators $T(\overline{a})$'s imply (see [7; Thm. III.11.17]) that, for all i in I, there exist complex functions g_{ij}^k on $\mathbb{T}^2\times\mathbb{T}^2$ which are $\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)$-measurable and such that $g_{ij}^k(\overline{a},\cdot)=G_j^k(\overline{a})$ almost everywhere w.r.t. μ_i. Redefining g_{ij}^k on a $\mu_i\otimes\mu_j$-null set we obtain a new function, say f_{ij}^k, measurable w.r.t. the completed σ-algebra $\overline{\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)}^{\mu_i\otimes\mu_j}$ and verifying the equality $f_{ij}^k(\overline{a},\cdot)=G_j^k(\overline{a})$ for each $\overline{a}$ in $\mathbb{T}^2$. Since the sets $\{S_i\times S_j\}_{i,j\in I}$ are disjoint, we can define, for each k in I,

$$f^{(k)}(\overline{a},\overline{\alpha}) = \begin{cases} f_{ij}^k(\overline{a},\overline{\alpha})\,, & \text{if } (\overline{a},\overline{\alpha})\in S_i\times S_j\,,\quad i,j\in I \\ \big(T(\overline{a})\chi_{S_k}\big)(\overline{\alpha})\,, & \text{otherwise} \end{cases} \quad .$$

Such $f^{(k)}$ turns out to be $\bigcap_{i,j\in I}\overline{\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)}^{\mu_i\otimes\mu_j}$-measurable and $f^{(k)}(\overline{a},\cdot)=T(\overline{a})\chi_{S_k}$ for each $\overline{a}$ in $\mathbb{T}^2$; moreover we can always assume $f^{(k)}(\overline{a},\overline{\alpha})$ to be zero if $(\overline{a},\overline{\alpha})$ does not belong to the set $\Im_k$ (see Lemma 5.2). The function f of the lemma can now be obtained setting

$$f(\overline{a},\overline{\alpha}) = \begin{cases} f^{(k)}(\overline{a},\overline{\alpha})\,, & \text{if } (\overline{a},\overline{\alpha})\in \Im_k\,,\quad k\in I \\ 0\,, & \text{otherwise} \end{cases} \quad .$$

To verify that f is measurable w.r.t. the σ-algebra $\bigcap_{i,j\in I}\overline{\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)}^{\mu_i\otimes\mu_j}$ consider a Borel subset Y of $\mathbb{C}$ and a pair i,j in I. Since

$$f^{-1}(Y)=[\,f^{-1}(Y)\cap(S_i\times S_j)\,]\cup[\,f^{-1}(Y)\cap(S_i\times S_j)^{\mathrm{c}}\,]$$

and $f^{-1}(Y)\cap(S_i\times S_j)^{\mathrm{c}}$ is in $\overline{\mathcal{B}(\mathbb{T}^2\times\mathbb{T}^2)}^{\mu_i\otimes\mu_j}$, it remains to discuss $f^{-1}(Y)\cap(S_i\times S_j)$. The family $I'=\{k\in I\,|\,(\mu_i\otimes\mu_j)(\Im_k)>0\}$ must be countable (since $\mu_i\otimes\mu_j$ is a finite measure and the sets $\Im_k$ are disjoint). Then, if $0\notin Y$, we write

$$f^{-1}(Y)\,\cap\,(S_i\times S_j) \;=\; A\cup B$$

with $A=\cup_{k\in I'}\big(f^{(k)}\big)^{-1}(Y)\,\cap\,(S_i\times S_j)$ and $B=\cup_{k\notin I'}\big(f^{(k)}\big)^{-1}(Y)\,\cap\,(S_i\times S_j)$. The set A is a countable union of measurable sets. The set B is contained in $\left(\cup_{k\in I'}\Im_k\right)^{\mathrm{c}}$ which is $\mu_i\otimes\mu_j$-null. (In fact $(\mu_i\otimes\mu_j)\big((\cup_{k\in I'}\Im_k)^{\mathrm{c}}\big)>0$ would imply, according to

Lemma 5.2, the existence of a k_0 such that $(\mu_i \otimes \mu_j)((\cup_{k\in I'} \Im_k)^c \cap \Im_{k_0}) > 0$ and this would imply $k_0 \notin I'$.) Therefore $B \in \overline{\mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)}^{\mu_i\otimes\mu_j}$. If $0 \in Y$, the above argument applies apart from a set contained in $(\cup_{k\in I} \Im_k)^c \cap (S_i \times S_j)$ which is $\mu_i \otimes \mu_j$-null. Finally for every k in I and A in $\mathcal{A}_Z$, using the definition of $T(\overline{a})$ and the translation property of the automorphism $\widehat{\tau_{\overline{a}}}$ (see the proof of Lemma 5.1), one has

$$\begin{aligned}\big(T(\overline{a})\,\widehat{A}\,\chi_{S_k}\big)(\overline{\alpha}) &= \widehat{A}(\overline{\alpha}+\overline{a})\,\,\big(T(\overline{a})\chi_{S_k}\big)(\overline{\alpha}) \\ &= \widehat{A}(\overline{\alpha}+\overline{a})\,\,\chi_{S_k}(\overline{\alpha}+\overline{a})\,\,f^{(k)}(\overline{a},\overline{\alpha}) \;=\; f(\overline{a},\overline{\alpha})\,\,\big(\widehat{A}\,\chi_{S_k}\big)(\overline{\alpha}+\overline{a})\end{aligned}$$

and, by continuity, $\big(T(\overline{a})\psi\big)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\,\psi(\overline{\alpha}+\overline{a})$ for every ψ in $L^2(\mathbb{T}^2,\mu)$. ◻

LEMMA 5.7. *There exists a complex function, $\xi(\overline{\alpha})$, on $\mathbb{T}^2$, measurable w.r.t. the σ-algebra $\bigcap_{x\in\mathcal{H}} \overline{\mathcal{B}(\mathbb{T}^2)}^{\mu_x}$, such that $|\xi(\overline{\alpha})| \in (0,+\infty)$ μ-a.e. and, for each $\overline{a}$ in $\mathbb{T}^2$,*

$$f(\overline{a},\overline{\alpha}) \;=\; \frac{\xi(\overline{a}+\overline{\alpha})}{\xi(\overline{\alpha})} \qquad\qquad \mu(\overline{\alpha})\text{-a.e.}\,.$$

Proof. It is useful to introduce the measure

$$\nu \;=\; \sum_{i,j\in I} \mu_i \otimes \mu_j$$

defined on the σ-algebra $\bigcap_{i,j\in I} \overline{\mathcal{B}(\mathbb{T}^2 \times \mathbb{T}^2)}^{\mu_i\otimes\mu_j}$ and the homeomorphism ϑ of $\mathbb{T}^2 \times \mathbb{T}^2$ given by $\vartheta(\overline{a},\overline{\alpha}) = (\overline{a}+\overline{\alpha},\overline{\alpha})$. ¿From the strong measurability of the representation and Lemma 5.2, it follows that, if $N \subset \mathbb{T}^2 \times \mathbb{T}^2$ and $\nu(N) = 0$, then also $\nu(\vartheta(N)) = 0$. The group law $T(\overline{a}+\overline{b}) = T(\overline{a}) \circ T(\overline{b})$ and Lemma 5.6 imply that, for every $\overline{a},\overline{b}$ in $\mathbb{T}^2$,

$$f(\overline{a}+\overline{b},\overline{\alpha}) \;=\; f(\overline{b},\overline{\alpha}+\overline{a})\,\,f(\overline{a},\overline{\alpha}) \qquad\qquad \mu(\overline{\alpha})\text{-a.e.}$$

and, replacing f with $g = f \circ \vartheta^{-1}$, we obtain

$$g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha}) = g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha}+\overline{a})\,\,\,g(\overline{a}+\overline{\alpha},\overline{\alpha}) \qquad\qquad \mu(\overline{\alpha})\text{-a.e.}\,.$$

Thus the function G given by

$$G(\overline{b},\overline{a},\overline{\alpha}) \;=\; g(\overline{a}+\overline{b}+\overline{\alpha}\,,\overline{\alpha}) - g(\overline{a}+\overline{b}+\overline{\alpha}\,,\alpha+\overline{a})\,\,\,g(\overline{a}+\overline{\alpha}\,,\overline{\alpha})$$

is equal to zero $\nu(\overline{a},\overline{\alpha})$-a.e. for each fixed $\overline{b}$. According to the properties of ϑ we have that $G(\overline{b},\cdot,\cdot)\circ\vartheta^{-1}$ is still $\nu(\overline{a},\overline{\alpha})$-a.e. null, i.e.

$$g(\overline{a}+\overline{b}\,,\overline{\alpha}) \;=\; g(\overline{a}+\overline{b}\,,\overline{a})\;\; g(\overline{a}\,,\overline{\alpha}) \qquad\qquad \nu(\overline{a},\overline{\alpha})\text{-a.e. for each } \overline{b} \text{ in } \mathbb{T}^2\ . \tag{5.1}$$

The strong measurability of the representation and the unitarity of $T(\overline{a})$ ensure that (see [5]) there exist at least a Borel set X_0 in $\mathbb{T}^2$ such that the function

$$\xi\left(\overline{b}\right) \;=\; \int_{X_0} g(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha})$$

is well-defined and such that $|\xi\left(\overline{b}\right)| \in (0,+\infty)$ μ-almost everywhere; furthermore ξ is measurable w.r.t. the σ-algebra $\bigcap_{x\in\mathcal{H}}\overline{\mathcal{B}\left(\mathbb{T}^2\right)}^{\mu_x}$. Then, using relation (5.1), we can conclude that, for each $\overline{d}$ in $\mathbb{T}^2$,

$$g(\overline{b}+\overline{d},\overline{b})\;\;\xi\left(\overline{b}\right) \;=\! \int_{X_0}\! g(\overline{b}+\overline{d},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha}) \;=\; \xi\left(\overline{b}+\overline{d}\right)$$

$\mu(\overline{b})$-almost everywhere. So, for every $\overline{d}$ in $\mathbb{T}^2$, we have

$$f(\overline{d},\overline{b}) \;=\; g(\overline{b}+\overline{d},\overline{b}) \;=\; \frac{\xi\left(\overline{b}+\overline{d}\right)}{\xi\left(\overline{b}\right)} \qquad\qquad \mu(\overline{b})\text{-a.e.}$$

and the lemma is proved. $\square$

To complete the proof of the theorem define now a new measure $\widetilde{\mu}$ on $\mathbb{T}^2$ setting

$$\widetilde{\mu} \;=\; |\xi|^{-2}\,\mu\ .$$

It is not difficult to verify that the map $\varXi$, given by $(\varXi\psi)(\alpha,\beta) = \xi(\alpha,\beta)\ \psi(\alpha,\beta)$, is a norm-preserving isomorphism from $L^2(\mathbb{T}^2,\mu)$ onto $L^2(\mathbb{T}^2,\widetilde{\mu})$ and, for each $\widetilde{\psi}$ in $L^2(\mathbb{T}^2,\widetilde{\mu})$ and each (a,b) in $\mathbb{T}^2$,

$$\left(\varXi\ T(a,b)\ \varXi^{-1}\ \widetilde{\psi}\,\right)(\alpha,\beta) \;=\; \widetilde{\psi}((\alpha+a)\mathrm{mod}\,1,(\beta+b)\mathrm{mod}\,2\pi)\ .$$

This equation implies that $\widetilde{\mu}$ is translation-invariant (since the operators $\varXi\ T(a,b)\ \varXi^{-1}$ are norm-preserving) and the isomorphism $\varXi\circ\mathcal{U}$ (from $\mathcal{H}$ onto $L^2(\mathbb{T}^2,\widetilde{\mu})$) gives equations (4.1) for the action of the Weyl operators.

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