Analyticity properties and thermal effects for general quantum field theory on de Sitter space-time.

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Abstract

We propose a general framework for quantum field theory on the de Sitter space-time (i.e. for local field theories whose truncated Wightman functions are not required to vanish). By requiring that the fields satisfy a *weak spectral condition*, formulated in terms of the analytic continuation properties of their Wightman functions, we show that a geodesical observer will detect in the corresponding "vacuum" a blackbody radiation at temperature $T = 1/2\pi R$. We also prove the analogues of the PCT, Reeh-Schlieder and Bisognano-Wichmann theorems.

1 Introduction

It is known that, when quantizing fields on a gravitational background, it is generally impossible to characterize the physically relevant vacuum states as the fundamental states for the energy in the usual sense, since there is no such thing as a global energy operator.

In the absence of the analogue of an energy-momentum spectrum condition [25, 32], several authors have formulated various alternative prescriptions to select, among the possible representations (vacua) of a quantum field theory, those which can have a meaningful physical interpretation; *the adiabatic prescription, the local Hadamard condition*, and *the conformal criterion*, (see [2, 26] and references therein) have proven to be useful for characterizing linear field theories with vanishing truncated n-point functions (i.e. *free field theories*) on various kinds of space-times.

It is worthwhile to stress immediately that the relevant choice of the vacuum of a quantum field theory on a curved space-time has striking consequences even in the case of free fields: the most celebrated examples are the Hawking thermal radiation on a black-hole background [21, 22, 30], the Unruh effect [33] and the Gibbons and Hawking thermal effect on a de Sitter space-time [16].

As regards interacting field theories on a gravitational background, (i.e. field theories with non-vanishing truncated n-point functions), much less is known. While the property of *locality* (or *local commutativity*) of the field observable algebra (i.e. the commutativity of any couple of field observables localized in space-like separated regions) remains a reasonable postulate for all space-time manifolds which are globally hyperbolic, the problem of specifying a representation of the field algebra becomes still more undetermined than in the free-field case. In the latter, the indeterminacy is confined in the two-point functions of the fields, namely in the splitting of the given (c-number) commutators into the correlators at permuted couples of points. In the general interacting case, the indeterminacy of the possible representations is now encoded (in an unknown way) in the properties of the whole sequence of n-point functions of the fields. If the gravitational background is only considered as a general (globally hyperbolic and pseudo-Riemannian) differentiable manifold, this huge indeterminacy cannot be completely reduced by imposing a (well-justified) *principle of stability* which postulates the existence at each point of the manifold of a Minkowskian scaling limit of the theory satisfying the spectral condition (see [19] and references therein); nor can it be reduced in an operational way by adding the general requirement of a *local definiteness criterion* (based on the principles of local quantum physics [19]). In such a general context, one should however mention the recent use of microlocal analysis which has allowed the introduction of a *wave front set approach to the spectral condition* [29, 12]; after having supplied a simple characterization of the free-field Hadamard states, this promising approach has in its program to give information on the n-point functions of interacting fields *in perturbation theory*.

On the other side, starting from the remark that in Minkowskian theories the spectral condition can be expressed in terms of analyticity properties of the n-point functions in the complexified space-time manifold [25, 32], one can defend the viewpoint that it may be of particular interest to study quantum field theory on an *analytic* gravitational background. As a matter of fact, there is one model of analytic curved universe, and actually the simplest one, that offers the unique possibility of formulating *a global spectral condition for interacting fields* which is very close to the usual spectral condition of Minkowski QFT: this is the de Sitter space-time.

The de Sitter space-time can be represented as a d-dimensional one-sheeted hyperboloid embedded in a Minkowski ambient space $\mathbf{R}^{d+1}$ and it can also be seen as a one-parameter deformation of a d-dimensional Minkowski space-time involving a length R. The Lorentz group of the ambient space acts as a relativity group for this space-time, and the very existence of this (maximal) symmetry group explains the popularity of the de Sitter universe as a convenient simple model for developing techniques of QFT on a gravitational background. Moreover, there has been a regain of interest in the de Sitter metric in the last years, since it has been considered to play a central role in the inflationary cosmologies (see [27] and references therein): a possible explanation of phenomena occurring in the very early universe then relies on an interplay between space-time curvature and thermodynamics and a prominent role is played by the mechanisms of symmetry breaking and restoration in a de Sitter QFT.

The geometrical properties of de Sitter space-time and of its complexification actually make it possible to formulate a general approach to QFT on this universe which closely parallels the Wightman approach [25, 32] to the Minkowskian QFT. In fact, it is not only the existence of a simple causal structure (inherited from the ambient Minkowski space) and of a global symmetry group (playing the same role as the Poincaré group) on the real space-time manifold which are similar; but the complexified manifold itself is equipped with domains which are closely similar to the tube domains of the complex Minkowski space. Since these Minkowskian tubes play a crucial role for expressing the spectral condition in terms of analyticity properties of the n-point functions of the theory, the previous geometrical remark strongly suggests that analogous complex domains might be used for a global formulation of the spectral condition in de Sitter quantum field theory. This approach has been in fact introduced and used successfully in a study of *general two-point functions* on de Sitter space-time [6, 8, 9, 28]. As a by-product, it has been shown [9] that a satisfactory characterization of *generalized free fields (GFF) on de Sitter space-time*, including the preferred family of de Sitter invariant Klein-Gordon field theories (known as Euclidean [16] or Bunch-Davies [11] vacua) can be given in terms of the global analytic structure of their two-point functions in the complexified de Sitter manifold. Moreover, all these theories of GFF were shown to be equivalently characterized by the existence of thermal properties of Gibbons-Hawking type, the temperature $T = (2\pi R)^{-1}$ being induced by the curvature of the space.

In this paper, we will show that the same ideas and methods can be applied with similar results to a general approach to the theory of interacting quantum fields in de Sitter space-time. In fact, we shall work out an axiomatic program (already sketched at the end of [9]) in which the "spectral condition" is replaced by appropriate *global analyticity properties of the n-point vacuum expectation values of the fields* (or "Wightman distributions") in the complexified de Sitter manifold. These postulated analyticity properties are similar to those implied by the usual spectral condition in the Minkowskian case, according to the standard Wightman axiomatic framework. For simplicity, we shall refer to them as to the "weak spectral condition".

As a physical support to our weak spectral condition, we shall establish that all interacting fields which belong to this general framework admit a Gibbons-Hawking-type *thermal interpretation* with the same specifications as the one obtained for GFF's in [9]. In spite of this remarkable interpretative discrepancy with respect to the Minkowskian quantum fields satisfying the usual spectral condition, we shall see that such basic structural properties as the PCT and Reeh-Schlieder theorems are still valid in this general approach to de Sitter QFT. Furthermore, our global analytic framework also supplies an analytic continuation of the theory to the "Euclidean sphere" of the complexified de Sitter space-time, which is the analogue of the (purely imaginary time) "Euclidean subspace" of the complexified Minkowskian space-time. We will also show that the Wick powers of generalized free fields fit within the framework and we have indication that our approach is relevant for the study of perturbation theory. The latter will be developped elsewhere.

From a methodological viewpoint, one can distinguish (as in the Minkowskian case) two types of developments which can be called according to a traditional terminology the "linear" and "non-linear programs".

The *linear program*, which deals exclusively with the exploitation of the postulates of locality, de Sitter covariance and spectral condition (expressed by linear relations between the various permuted n-point functions, for each fixed value of n) results in the definition of primitive analyticity domains D_n for all the n-point (holomorphic) functions $\mathfrak{W}_n(z_1, ..., z_n)$ of the theory. Each domain D_n is an open connected subset of the topological product of n copies of the complexified de Sitter hyperboloid. As in the Minkowskian case, each primitive domain D_n is not a "natural holomorphy domain", but it turns out that new regions of analyticity of the functions $\mathfrak{W}_n$ (contained in the respective holomorphy envelopes of the domains D_n and obtained by geometrical techniques of analytic completion) yield important consequences for the corresponding field theories. A specially interesting example is the derivation of analyticity properties of the functions $\mathfrak{W}_n$ with respect to any subset of points $z_i = z_i(t)$ varying simultaneously on complex hyperbolae interpreted as the (complexified) trajectories of a given time-like Killing vector field on the de Sitter universe. The periodicity with respect to the imaginary part of the corresponding time-parameter t directly implies the interpretation of the obtained analyticity properties of the functions $\mathfrak{W}_n$ as a *KMS-type condition*; in view of the general analysis of [20], this gives a thermal interpretation to all the de Sitter field theories considered. Since the above mentioned analyticity property is completely similar to the one which emerges from the Bisognano-Wichmann results in the Minkowskian theory [3] (see our comments below), we shall call the previous result "Bisognano-Wichmann analyticity property of the n-point functions".

The *non-linear program*, which exploits the Hilbert-space structure of the theory, relies in an essential way on the (quadratic) "positivity inequalities" to be satisfied by the whole sequence of n-point Wightman distributions of the fields; these inequalities just express the existence of the vector-valued distributions defined by the action of field operator products on the "GNS-vacuum state" of the theory. An important issue to be recovered is the fact that these distributions are themselves the boundary values of *vector-valued holomorphic functions* from certain complex domains; it is this mathematical fact which is directly responsible for such important features of the theory as the Reeh-Schlieder property. In the Minkowskian case, this vectorial analyticity is readily obtained from the spectral condition by an argument based on the Laplace transformation. Here, we shall apply an alternative method for establishing vectorial analyticity which directly makes use of the analyticity and positivity properties of the n-point functions. It is based on a general study by V. Glaser [17, 18] of *positive-type sequences of holomorphic kernels in domains of* $\mathbf{C}^m \times \mathbf{C}^n$, whereby the analyticity of the Wightman n-point functions "propagates" their positivity properties to the complex domain. This method is therefore applicable not only to the Minkowskian and de Sitter QFT but also, in principle, to QFT on more general *holomorphic (or real-analytic) space-time manifolds* for which the spectral condition would be replaced by an appropriate (possibly local) version of the analyticity properties of the Wightman functions.

The structure of the paper is the following: in Section 2, we introduce the notations and recall some properties of the de Sitter spacetime and of its complexification; we then formulate our general principles for the interacting fields on this universe, giving a special emphasis on the spectral condition which we propose.

In Section 3 we explore various consequences of our general principles which are the analogues of standard results of the Minkowskian QFT in the Wightman framework. In particular, we establish the existence of an analytic continuation of the Wightman n-point functions to corresponding primitive domains of analyticity. The PCT property is also shown.

In Section 4 we come to the physical interpretation of the spectral condition. We first extend the analytic aspect of the Bisognano-Wichmann theorem [3] to the de Sitter case. Then we show that the thermal interpretation, already known for free field theories [16, 9], is still valid in this more general case.

In section 5 we prove the validity of the Reeh and Schlieder property. The proof of the relevant vectorial analyticity is given as an application of the above mentioned theorem of Glaser.

The paper is ended by three appendices where we discuss some more technical results.

2 QFT on the de Sitter spacetime: the spectral condition

We start with some notations and some well-known facts. The $(d+1)$-dimensional real (resp. complex) Minkowski space is $\mathbf{R}^{d+1}$ (resp. $\mathbf{C}^{d+1}$) equipped with the scalar product $x \cdot y = x^{(0)}y^{(0)} - x^{(1)}y^{(1)} - \ldots - x^{(d)}y^{(d)}$ with, as usual, $x^2 = x \cdot x$. We thus distinguish a particular Lorentz frame and denote e_μ the $(d+1)$-vector with $e_\mu^{(\nu)} = \delta_{\mu\,\nu}$. In this special Lorentz frame, we also distinguish the (e_0, e_d)-plane and the corresponding light-like coordinates u and v, namely we put:

$$x = (x^{(0)}, \underline{\vec{x}}, x^{(d)}), \qquad \underline{\vec{x}} = (x^{(1)}, \ldots, x^{(d-1)}), \tag{1}$$

$$u = u(x) = x^{(0)} + x^{(d)}, \qquad v = v(x) = x^{(0)} - x^{(d)}, \tag{2}$$

and we introduce, for each $\lambda = \mathrm{e}^{\zeta} \in \mathbf{C} \setminus \{0\}$, the special Lorentz transformation $[\lambda]$ such that

$$u([\lambda]x) = \lambda u(x), \quad v([\lambda]x) = \lambda^{-1} v(x), \quad \underline{\vec{x}}([\lambda]x) = \underline{\vec{x}}(x), \quad [\mathrm{e}^{\zeta}] = \exp \zeta\, e_0 \wedge e_d. \tag{3}$$

The future cone is defined in the real Minkowski space $\mathbf{R}^{d+1}$ as the subset

$$V_+ = -V_- = \{x \in \mathbf{R}^{d+1} \ : \ x^{(0)} > 0, \quad x \cdot x > 0\}$$

and the future light cone as $C_+ = \partial V_+ = -C_-$. We denote $x \leq y$ the partial order (called causal order) defined by $\overline{V_+}$, i.e. $x \leq y \Leftrightarrow y - x \in \overline{V_+}$. The d-dimensional real (resp. complex) de Sitter space-time with radius R is identified with the subset of the real (resp. complex) Minkowski space consisting of the points x such that $x^2 = -R^2$ and is denoted $X_d(R)$ or simply X_d (resp. $X_d^{(c)}$). Thus X_d is the one-sheeted hyperboloid

$$X_d = X_d(R) = \{x \in \mathbf{R}^{d+1} : {x^{(0)}}^2 - {x^{(1)}}^2 - \ldots - {x^{(d)}}^2 = -R^2\} \tag{4}$$

The causal order on $\mathbf{R}^{d+1}$ induces the causal order on X_d. The future and past shadows of a given event x in X_d are given by $\Gamma^+(x) = \{y \in X_d : y \geq x\}$, $\Gamma^-(x) = \{y \in X_d : y \leq x\}$. Note that if $x^2 = -R^2$ and $r^2 = 0$, then $(x+r)^2 = -R^2$ is equivalent to $x \cdot r = 0$, and remains true if r is replaced with $t\,r$ for any real t (the same holds in the complex domain.) Hence the boundary set

$$\partial\Gamma(x) = \{y \in X_d : (x-y)^2 = 0\} \tag{5}$$

of $\Gamma^+(x) \cup \Gamma^-(x)$ is a cone ("light-cone") with apex x, the union of all linear generators of X_d containing the point x. Two events x and y of X_d are in "acausal relation", or "space-like separated" if $y \notin \Gamma^+(x) \cup \Gamma^-(x)$, i.e. if $x \cdot y > -R^2$. The relativity group of the de Sitter space-time, called "de Sitter group" in the following, is the connected Lorentz group of the ambient Minkowski space, i.e. $L_+^{\uparrow} = SO_0(1,d)$ leaving invariant each of the sheets of the cone $C = C_+ \cup C_-$. The connected complex Lorentz group in $d+1$ dimensions is denoted $L_+(\mathbf{C})$. We denote σ the $L_+^{\uparrow}$-invariant volume form on X_d given by

$$\int f(x)\, d\sigma(x) = \int f(x)\, \delta(x^2 + R^2)\, dx^{(0)} \wedge \ldots \wedge dx^{(d)}. \tag{6}$$

$L_+^\uparrow$ acts transitively on X_d and $L_+(\mathbf{C})$ on $X_d^{(c)}$.

The familiar forward and backward tubes are defined in complex Minkowski space as $\mathrm{T}_\pm = \mathbf{R}^{d+1} \pm iV_+$, and we denote

$$\mathcal{T}_+ = \mathrm{T}_+ \cap X_d^{(c)}, \qquad \mathcal{T}_- = \mathrm{T}_- \cap X_d^{(c)}. \tag{7}$$

Since $\overline{\mathrm{T}_+} \cup \overline{\mathrm{T}_-}$ contains the "Euclidean subspace" $\mathcal{E}_{d+1} = \{z = (iy^{(0)}, x^{(1)}, \ldots, x^{(d)}) : (y^{(0)}, x^{(1)}, \ldots, x^{(d)}) \in \mathbf{R}^{d+1}\}$ of the complex Minkowski space-time $\mathbf{C}^{d+1}$, the subset $\overline{\mathcal{T}_+} \cup \overline{\mathcal{T}_-}$ of $X_d^{(c)}$ contains the "Euclidean sphere" $S_d = \{z = (iy^{(0)}, x^{(1)}, \ldots x^{(d)}) : \ {y^{(0)}}^2 + {x^{(1)}}^2 + \ldots + {x^{(d)}}^2 = R^2\}$.

We denote $\mathcal{D}(X_d^n)$ (resp. $\mathcal{S}(X_d^n)$) the space of functions on X_d^n which are restrictions to X_d^n of functions belonging to $\mathcal{D}(\mathbf{R}^{n(d+1)})$ (resp. $\mathcal{S}(\mathbf{R}^{n(d+1)})$). As in the Minkowskian case, the Borchers algebra $\mathcal{B}$ is defined as the tensor algebra over $\mathcal{D}(X_d)$. Its elements are terminating sequences of test-functions $f = (f_0, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n), \ \ldots)$, where $f_0 \in \mathbf{C}$ and $f_n \in \mathcal{D}(X_d^n)$ for all $n \geq 1$, the product and $\star$ operations being given by

$$(fg)_n = \sum_{\substack{p,\ q \in \mathbf{N} \\ p+q=n}} f_p \otimes g_q, \qquad (f^\star)_n(x_1, \ldots,\ x_n) = \overline{f_n(x_n, \ldots,\ x_1)}.$$

The action of the de Sitter group $L_+^\uparrow$ on $\mathcal{B}$ is defined by $f \mapsto f_{\{\Lambda_r\}}$, where

$$f_{\{\Lambda_r\}} = (f_0, f_{1\{\Lambda_r\}}, \ldots, f_{n\{\Lambda_r\}}, \ldots), \qquad f_{n\{\Lambda_r\}}(x_1, \ldots, x_n) = f_n({\Lambda_r}^{-1} x_1, \ldots, {\Lambda_r}^{-1} x_n), \tag{8}$$

Λ_r denoting any (real) de Sitter transformation.

A quantum field theory (we consider a single scalar field for simplicity) is specified by a continuous linear functional $\mathcal{W}$ on $\mathcal{B}$, i.e. a sequence $\{\mathcal{W}_n \in \mathcal{D}'(X_d^n)\}_{n \in \mathbf{N}}$ where $\mathcal{W}_0 = 1$ and the $\{\mathcal{W}_n\}_{n>0}$ are distributions (Wightman functions) required to possess the following properties:

1. (Covariance). Each $\mathcal{W}_n$ is de Sitter invariant, i.e.

$$\langle \mathcal{W}_n,\ f_{n\{\Lambda_r\}} \rangle = \langle \mathcal{W}_n,\ f_n \rangle \tag{9}$$

for all de Sitter transformations Λ_r. (This is equivalent to saying that the functional $\mathcal{W}$ itself is invariant, i.e. $\mathcal{W}(f) = \mathcal{W}(f_{\{\Lambda_r\}})$ for all Λ_r).

2. (Locality)

$$\mathcal{W}_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathcal{W}_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n) \tag{10}$$

if $(x_j - x_{j+1})^2 < 0$.

3. (Positive Definiteness). For each $f \in \mathcal{B}$, $\mathcal{W}(f^\star f) \geq 0$. Explicitly, given $f_0 \in \mathbf{C}, f_1 \in \mathcal{D}(X_d), \ldots, f_k \in \mathcal{D}(X_d^k)$, then

$$\sum_{n,m=0}^{k} \langle \mathcal{W}_{n+m},\ f_n^\star \otimes f_m \rangle \geq 0. \tag{11}$$

As in the Minkowskian case [34, 5, 25], the GNS construction yields a Hilbert space $\mathcal{H}$, a unitary representation $\Lambda_r \mapsto U(\Lambda_r)$ of $L_+^\uparrow$, a vacuum vector $\Omega \in \mathcal{H}$ invariant under U, and an operator valued distribution ϕ such that

$$\mathcal{W}_n(x_1, \ldots,\ x_n) = (\Omega,\ \phi(x_1) \ldots \phi(x_n)\, \Omega). \tag{12}$$

The GNS construction also provides the vector valued distributions $\Phi_n^{(b)}$ such that

$$\langle \Phi_n^{(b)},\ f_n \rangle = \int f_n(x_1, \ldots,\ x_n)\, \phi(x_1) \ldots \phi(x_n)\, \Omega\, d\sigma(x_1) \ldots d\sigma(x_n) \tag{13}$$

and a representation $f \to \mathbf{\Phi}(f)$ (by unbounded operators) of $\mathcal{B}$ of which the field ϕ is a special case: $\phi(f_1) = \int \phi(x) f_1(x) d\sigma(x) = \mathbf{\Phi}\left((0, f_1, 0, \ldots)\right)$. For every open set $\mathcal{O}$ of X_d the corresponding polynomial algebra $\mathcal{P}(\mathcal{O})$ of the field ϕ is then defined as the subalgebra of $\mathbf{\Phi}(\mathcal{B})$ whose elements $\mathbf{\Phi}(f_0, f_1, \ldots, f_n, \ldots)$ are such that for all $n \geq 1$ $\mathrm{supp} f_n(x_1, \ldots, x_n) \subset \mathcal{O}^n$. The set $\mathrm{D} = \mathcal{P}(X_d)\Omega$ is a dense subset of $\mathcal{H}$ and one has (for all elements $\mathbf{\Phi}(f), \mathbf{\Phi}(g) \in \mathcal{P}(X_d)$):

$$\mathcal{W}(f^\star g) = (\mathbf{\Phi}(f)\Omega,\ \mathbf{\Phi}(g)\Omega). \tag{14}$$

The properties 1-3 are literally carried over from the Minkowskian case, but no literal or unique adaptation exists for the usual spectral property. In the $(d+1)$-dimensional Minkowskian case, the latter is equivalent to the following: for each $n \geq 2$, $\mathcal{W}_n$ is the boundary value in the sense of distributions of a function holomorphic in the tube

$$\mathrm{T}_n = \{z = (z_1, \ldots,\ z_n) \in \mathbf{C}^{n(d+1)}\ :\ \mathrm{Im}\,(z_{j+1} - z_j) \in V_+,\ 1 \leq j \leq n-1\}. \tag{15}$$

In the case of the de Sitter space X_d (embedded in $\mathbf{R}^{d+1}$), a natural substitute for this property is to assume that $\mathcal{W}_n$ is the boundary value in the sense of distributions of a function holomorphic in

$$\mathcal{T}_n = X_d^{(c)n} \cap \mathrm{T}_n. \tag{16}$$

It will be shown below that $\mathcal{T}_n$ is a domain and a tuboid in the sense of [9], namely a domain which is bordered by the reals in such a way that the notion of "distribution boundary value of a holomorphic function from this domain" remains meaningful. It is thus possible to impose:

4. (Weak spectral condition). For each $n > 1$, the distribution $\mathcal{W}_n$ is the boundary value of a function W_n holomorphic in the subdomain $\mathcal{T}_n$ of $X_d^{(c)n}$.

It may seem unnatural, in the absence of translational invariance, to postulate analyticity properties in terms of the difference variables $(z_j - z_k)$. Note however that a Lorentz invariant holomorphic function on a subdomain of $X_d^{(c)n}$ depends only on the invariants $z_j \cdot z_k$. Among these the $z_j \cdot z_j$ are fixed and equal to $-R^2$. Such a function therefore depends only on the $(z_j - z_k)^2$. In the same way as in the Minkowskian case, it may be useful to relax some of the hypotheses 1-3. One may also want to impose:

5. (Temperedness Condition). For each $n > 1$, there are constants $M(n) \geq 0$ and $L(n) \geq 0$ such that the distribution $\mathcal{W}_n$ is the boundary value of a function W_n holomorphic in the subdomain $\mathcal{T}_n$ of $X_d^{(c)n}$ satisfying

$$|\mathrm{W}_n(x+iy)| \leq M(n)(1 + \|x+iy\| + \mathrm{dist}(z, \partial\mathrm{T}_n)^{-1})^{L(n)}. \tag{17}$$

This global bound (which includes the behaviour of W_n at infinity) will not be indispensable in this paper, but the local part of it (indicating a power behaviour near each point x for y tending to zero) is in fact equivalent to the distribution character of the boundary value of W_n postulated in 4 (see our remark 1 below).

For completeness, we now recall the definition of tuboids on manifolds (given in [9]). Let $\mathcal{M}$ be a real n-dimensional analytic manifold, $T\mathcal{M} = \bigcup_{x \in \mathcal{M}} (x, T_x\mathcal{M})$ the tangent bundle to $\mathcal{M}$ and $\mathcal{M}^{(c)}$ a complexification of $\mathcal{M}$. If x_0 is any point in $\mathcal{M}$, $\mathcal{U}_{x_0}$ and $\mathcal{U}_{x_0}^{(c)}$ will denote open neighborhoods of x_0, respectively in $\mathcal{M}$ and $\mathcal{M}^{(c)}$ such that $\mathcal{U}_{x_0} = \mathcal{U}_{x_0}^{(c)} \cap \mathcal{M}$; a corresponding neighborhood of $(x_0, 0)$ with basis $\mathcal{U}_{x_0}$ in $T\mathcal{M}$ will be denoted $T_{\mathrm{loc}}\mathcal{U}_{x_0}$.

Definition 1 *We call admissible local diffeomorphism at a point x_0 any diffeomorphism δ which maps some neighborhood $T_{\mathrm{loc}}\mathcal{U}_{x_0}$ of $(x_0, 0)$ in $T\mathcal{M}$ onto a corresponding neighborhood $\mathcal{U}_{x_0}^{(c)}$ of x_0 in $\mathcal{M}^{(c)}$ (considered as a $2n$-dimensional $\mathcal{C}^\infty$ manifold) in such a way that the following properties hold:*

a) $\forall x \in \mathcal{U}_{x_0}$, $\delta[(x, 0)] = x$;

b) $\forall (x, y) \in T_{\mathrm{loc}}\mathcal{U}_{x_0}$, with $y \neq 0$, $(y \in T_x\mathcal{M})$, the differentiable function $t \to z(t) = \delta[(x, ty)]$ is such that

$$\frac{1}{i}\frac{dz}{dt}(t)|_{t=0} = \alpha y, \quad \textit{with}\ \alpha > 0. \tag{18}$$

A tuboid can now be described as a domain in $\mathcal{M}^{(c)}$ which is bordered by the real manifold $\mathcal{M}$ and whose "shape" near each point of $\mathcal{M}$ is (in the space of $\mathrm{Im}\, z$ and for $\mathrm{Im}\, z \to 0$) very close to a given cone Λ_x of the tangent space $T_x\mathcal{M}$ to $\mathcal{M}$ at the point x. The following more precise definitions are needed.

Definition 2 *We call "profile" above $\mathcal{M}$ any open subset Λ of $T\mathcal{M}$ which is of the form $\Lambda = \bigcup_{x\in\mathcal{M}}(x, \Lambda_x)$, where each fiber Λ_x is a non-empty cone with apex at the origin in $T_x\mathcal{M}$ (Λ_x can be the full tangent space $T_x\mathcal{M}$).*

It is convenient to introduce the "projective representation" $\dot{T}\mathcal{M}$ of $T\mathcal{M}$, namely $\dot{T}\mathcal{M} = \bigcup_{x\in\mathcal{M}}(x, \dot{T}_x\mathcal{M})$, with $\dot{T}_x\mathcal{M} = T_x\mathcal{M} \setminus \{0\}/\mathbf{R}^+$. The image of each point $y \in T_x\mathcal{M}$ in $\dot{T}_x\mathcal{M}$ is $\dot{y} = \{\lambda y;\ \lambda > 0\}$. Each profile Λ can then be represented by an open subset $\dot{\Lambda} = \bigcup_{x\in\mathcal{M}}(x, \dot{\Lambda}_x)$ of $\dot{T}_x\mathcal{M}$ (each fiber $\dot{\Lambda}_x = \Lambda_x/\mathbf{R}^+$ being now a relatively compact set). We also introduce the complement of the closure of $\dot{\Lambda}$ in $\dot{T}\mathcal{M}$, namely the open set $\dot{\Lambda}' = \dot{T}\mathcal{M} \setminus \overline{\dot{\Lambda}} = \bigcup_{x\in\mathcal{M}}(x, \dot{\Lambda}'_x)$ (note that $\dot{\Lambda}'_x \subset \dot{T}_x\mathcal{M} \setminus \overline{\dot{\Lambda}}_x$).

Definition 3 *A domain Θ of $\mathcal{M}^c$ is called a tuboid with profile Λ above $\mathcal{M}$ if it satisfies the following property. For every point x_0 in $\mathcal{M}$, there exists an admissible local diffeomorphism δ at x_0 such that:*

a) every point $(x_0, \dot{y}_0)$ in $\dot{\Lambda}$ admits a compact neighborhood $K(x_0, \dot{y}_0)$ in $\dot{\Lambda}$ such that
$\delta\,[\{(x,y);\ (x,\dot{y}) \in K(x_0,\dot{y}_0),\ (x,y) \in T_{\mathrm{loc}}\mathcal{U}_{x_0}\}] \subset \Theta$.

b) every point $(x_0, \dot{y}'_0)$ in $\dot{\Lambda}'$ admits a compact neighborhood $K'(x_0, \dot{y}'_0)$ in $\dot{\Lambda}'$ such that
$\delta\left[\{(x,y);\ (x,\dot{y}) \in K'(x_0,\dot{y}_0),\ (x,y) \in T_{\mathrm{loc}}\mathcal{U}'_{x_0}\}\right] \cap \Theta = \emptyset$

In a) and b), $T_{\mathrm{loc}}\mathcal{U}_{x_0}$ and $T_{\mathrm{loc}}\mathcal{U}'_{x_0}$ denote sufficiently small neighbourhoods of $(x_0, 0)$ in $T\mathcal{M}$ which may depend respectively on y_0 and y'_0, but always satisfy the conditions of Definition 1 with respect to δ.

Each fiber Λ_x of Λ will also be called the profile at x of the tuboid Θ.

Using these notions and the results in appendix A of [9], we will show the following

Proposition 1 *i) The set*

$$\mathcal{T}_n = \{z = (z_1, \dots, z_n);\ z_k = x_k + iy_k \in X_d^{(c)},\ 1 \le k \le n;\ y_{j+1} - y_j \in V^+,\ 1 \le j \le n-1\} \tag{19}$$

is a domain of $X_d^{(c)n}$

ii) $\mathcal{T}_n$ is a tuboid above X_d^n, with profile

$$\Lambda^n = \bigcup_{\underline{x}\in X_d^n} (\underline{x}, \Lambda^n_{\underline{x}}), \tag{20}$$

where, for each $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n) \in X_d^n$, $\Lambda^n_{\underline{x}}$ is a non-empty open convex cone with apex at the origin in $T_{\underline{x}}X_d^n$ defined as follows:

$$\Lambda^n_{\underline{x}} = \{\underline{y} = (\underline{y}_1, \dots, \underline{y}_n);\ \underline{y}_k \in T_{\underline{x}_k}X_d,\ 1 \le k \le n;\ \underline{y}_{j+1} - \underline{y}_j \in V^+,\ 1 \le j \le n-1\}. \tag{21}$$

Proof

a) Let C_n be the open convex cone in $\mathbf{R}^{nd}$ defined by

$$\mathrm{C}_n = \{\underline{y} = (\underline{y}_1, \dots, \underline{y}_n);\ \underline{y}_k \in \mathbf{R}^d,\ 1 \le k \le n;\ \underline{y}_{j+1} - \underline{y}_j \in V^+,\ 1 \le j \le n-1\} \tag{22}$$

The set Λ^n defined in Eqs. (20) and (21) can then be seen as the restriction of the open subset $X_d^n \times \mathrm{C}_n$ of $X_d^n \times \mathbf{R}^{nd}$ to the algebraic set with equations $\underline{x}_j \cdot \underline{y}_j = 0$, $1 \le j \le n$, which represents TX_d^n as a submanifold of $X_d^n \times \mathbf{R}^{nd}$; Λ^n is therefore an open subset of TX_d^n. Moreover, for every point $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n) \in X_d^n$, the set $\Lambda_{\underline{x}}^n$ defined in Eq. (21) is an open convex cone in $T_{\underline{x}} X_d^n$,as being the intersection of the latter with C_n. For every $\underline{x}$, this cone is non-empty since one can determine at least one vector $\underline{y} = (\underline{y}_1, \cdots, \underline{y}_n) \in \Lambda_{\underline{x}}^n$ as follows: $\underline{y}_1$ being chosen arbitrarily in $T_{\underline{x}_1} X_d$, we can always find $\underline{y}_2 \in \{T_{\underline{x}_2} X_d\} \cap \left\{\underline{y}_1 + V^+\right\}$, and then by recursion $\underline{y}_{j+1} \in \left\{T_{\underline{x}_j} X_d\right\} \cap \left\{\underline{y}_j + V^+\right\}$, for $j \leq n-1$, because for every point $\underline{x}_j \in X_d$, $\left\{T_{\underline{x}_j} X_d\right\} \cap V^+$ is a non-empty convex cone.

b) Let

$$\Lambda_R^n = \bigcup_{\underline{x} \in X_d^n} (\underline{x}, \Lambda_{\underline{x},R}^n), \quad \Lambda_{\underline{x},R}^n = \{\underline{y} = (\underline{y}_1, \cdots, \underline{y}_n) \in \Lambda_{\underline{x}}^n,\ \underline{y}_j^2 < R^2,\ 1 \leq j \leq n\}. \tag{23}$$

Λ_R^n is (like Λ^n) an open subset of TX_d^n; each fiber $\Lambda_{\underline{x},R}^n$ is a non-empty domain in $T_{\underline{x}} X_d^n$. This results from the property of $\Lambda_{\underline{x}}^n$ proved in a), since the existence of a point in $\Lambda_{\underline{x},R}^n$ or of an arc connecting two arbitrary points inside $\Lambda_{\underline{x},R}^n$, follows from the corresponding property of $\Lambda_{\underline{x}}^n$ by using the dilatation invariance of the latter. It follows that Λ_R^n is (like Λ^n) a connected set and therefore a domain in TX_d^n.

c) We now show that there exists a continuous mapping μ which is one-to-one from Λ_R^n onto the set $\mathcal{T}_n \setminus Y_R^n$, where Y_R^n denotes the following subset of codimension $(d-1)$ of $X_d^{(c)n}$:

$$Y_R^n = \{z = (z_1, \ldots, z_n;\ z_j = x_j + iy_j \in X_d^{(c)},\ 1 \leq j \leq n;\ \exists \text{ at least one } j_0 : x_{j_0} = 0\ \}. \tag{24}$$

Let us consider the following mapping μ:

$$\mu(\underline{x}, \underline{y}) = z = (z_1, \ldots, z_n), \quad z_j = x_j + iy_j = \frac{\sqrt{R^2 - \underline{y}_j^2}}{R} \underline{x}_j + i\underline{y}_j,\ 1 \leq j \leq n. \tag{25}$$

μ is defined on the subset $\{TX_d^n\}_R$ of all the elements $(\underline{x}, \underline{y})$ of TX_d^n such that $\underline{y}_j^2 < R^2$ for $1 \leq j \leq n$; Eq. (25) implies that (for all j) $z_j^2 = -R^2$ and therefore that μ is a global diffeomorphism from $\{TX_d^n\}_R$ onto the subset $Z_R^n = X_d^{(c)n} \setminus Y_R^n$ of $X_d^{(c)n}$; clearly, this diffeomorphism maps Λ_R^n onto $\mathcal{T}_n \setminus Y_R^n$, and therefore (in view of b)), $\mathcal{T}_n \setminus Y_R^n$ is a domain of $X_d^{(c)n}$. Since all points of $\mathcal{T}_n$ are either interior points or boundary points of $\mathcal{T}_n \setminus Y_R^n$, and since $\mathcal{T}_n = X_d^{(c)n} \cap \mathrm{T}_n$ is an open set, it is a domain of $X_d^{(c)n}$.

d) In order to show that $\mathcal{T}_n$ is a tuboid with profile Λ^n above X_d^n, one just notices that the global diffeomorphism μ provides admissible local diffeomorphisms (by local restrictions) at all points $\underline{x}$ in X_d^n. Properties a) and b) of Definition 3 are then satisfied by $\mathcal{T}_n$ (with respect to all these local diffeomorphisms) as an obvious by-product of Eq. (25).

Remark 1 By an application of theorem A.2. of [9], the weak spectral condition implies that for every $\underline{x}$ there is some local tube $\Omega_{\underline{x}} + i\Gamma_{\underline{x}}$ around $\underline{x}$ in any chosen system of local complex coordinates on $X_d^{(c)n}$ whose image in $X_d^{(c)n}$ is contained in $\mathcal{T}_n$ has a profile very close to the profile of $\mathcal{T}_n$ (restricted to a neighborhood of $\underline{x}$), from which the boundary value equation $\mathcal{W}_n = b.v.\mathrm{W}_n$ can be understood in the usual sense. It implies equivalently that, in a complex neighborhood of each point $x = (x_1, \ldots, x_n) \in X_d^n$, the analytic function $\mathrm{W}_n(z_1, \ldots, z_n)$ is of moderate growth (i.e. bounded by a power of $\|y\|^{-1}$, where $\|y\|$ denotes any local norm of $y = \operatorname{Im} z = (y_1, \ldots, y_n)$) when the point $z = (z_1, \ldots, z_n)$ tends to the reals inside $\mathcal{T}_n$.

Remark 2 An important difference with respect to the Minkowski case is that the reals (i.e. X_d^n) are not a distinguished boundary for the tuboid $\mathcal{T}_n$.

3 Consequences of locality, weak spectral condition and de Sitter covariance.

Most of the well-known properties of the Wightman distributions in the Minkowskian case ([32, 25]) hold without change in the de Sitterian case under our assumptions, and their proofs mostly carry over literally. A few points, however require some attention. For each permutation π of $(1, \dots, n)$, the permuted Wightman distribution

$$\mathcal{W}_n^{(\pi)}(x_1, \dots, x_n) = \mathcal{W}_n(x_{\pi(1)}, \dots, x_{\pi(n)}) \tag{26}$$

is the boundary value of a function $\mathrm{W}_n^{(\pi)}(z_1, \dots, z_n)$ holomorphic in the "permuted tuboid"

$$\mathcal{T}_n^\pi = \{z = (z_1, \dots, z_n);\ z_k = x_k + iy_k \in X_d^{(c)},\ 1 \le k \le n;\ y_{\pi(j+1)} - y_{\pi(j)} \in V^+,\ 1 \le j \le n-1\} \tag{27}$$

If two permutations π and σ differ only by the exchange of the indices j and k, then $\mathcal{W}_\pi$ and $\mathcal{W}_\sigma$ coincide in

$$\mathcal{R}_{jk} = X_d^n \cap \mathrm{R}_{jk}, \quad \mathrm{R}_{jk} = \{x \in \mathbf{R}^{n(d+1)}\ :\ (x_j - x_k)^2 < 0\}. \tag{28}$$

Let $\mathcal{R}$ be a non-empty region which is the intersection of a subset of $\{\mathcal{R}_{jk}\ :\ j \neq k\}$. By the edge-of-the-wedge theorem (in its version for tuboids, see theorem A3 of [9]), any maximal set of permuted Wightman distributions which coincide on this region are the boundary value, in $\mathcal{R}$, of a common function holomorphic in a tuboid above $\mathcal{R}$ whose profile is obtained by taking *at each point* $x \in \mathcal{R}$ the convex hull of the profiles at x of the corresponding permuted tuboids. In particular all the permuted Wightman distributions coincide in the intersection Ω_n of all the $\mathcal{R}_{jk}$, and it follows that they all are boundary values of a common function $\mathfrak{W}_n(z_1, \dots z_n)$, holomorphic in a *primitive analyticity domain* $\mathcal{D}_n$. $\mathfrak{W}_n$ is the common analytic continuation of all the holomorphic functions $\mathrm{W}_n^{(\pi)}$ and the domain $\mathcal{D}_n$ is the union of all the permuted tuboids $\mathcal{T}_n^\pi$ and of the above mentioned local tuboids associated (by the edge-of-the-wedge theorem) with finite intersections of the $\mathcal{R}_{jk}$. In particular $\mathcal{D}_n$ contains a complex neighborhood of Ω_n since the tuboids $\mathcal{T}_n^\pi$ and $\mathcal{T}^{\pi_{inv}}{}_n$ are opposite (where $\pi_{inv} = (\pi(n), \dots,\ \pi(1))$). For each permutation π we denote $\mathcal{T}_n^{\pi\ \mathrm{ext}}$ the extended permuted tuboid

$$\mathcal{T}_n^{\pi\mathrm{ext}} = \bigcup_{\Lambda_c \in L_+(\mathbf{C})} \Lambda_c \mathcal{T}_n^\pi = \bigcup_{\Lambda_c \in L_+(\mathbf{C})} \Lambda_c(\mathrm{T}_n^\pi \cap X_d^{(c)n}) = X_d^{(c)n} \cap \bigcup_{\Lambda_c \in L_+(\mathbf{C})} \Lambda_c \mathrm{T}_n^\pi = X_d^{(c)n} \cap \mathrm{T}_n^{\pi\mathrm{ext}}. \tag{29}$$

3.1 The Jost points and the Glaser-Streater theorem

The set of real points of $\mathrm{T}_n^{\mathrm{ext}} = \mathrm{T}_n^{1\ \mathrm{ext}}$ (Jost points in the ambient space) is denoted J_n. Its intersection $\mathcal{J}_n$ with X_d^n will be called *the set of Jost points associated with the tuboid* $\mathcal{T}_n$. The set $\mathcal{J}_n$ is generated (like J_n) by the action of the connected group $L_+^\uparrow$ on a special subset of Jost points associated with a given maximal space-like cone such as the "right-wedge" $W_{(r)}$ of the ambient space:

$$W_{(r)} = -W_{(l)} = \{x \in \mathbf{R}^{d+1}\ :\ u(x) > 0,\ \ v(x) < 0\}, \tag{30}$$

the notations u,v being those of Eq. (2). The corresponding *special Jost subset* ${\mathcal{J}_n}^{(r)}$ is defined by

$${\mathcal{J}_n}^{(r)} = \mathrm{J}_n^{(r)} \cap X_d^n, \tag{31}$$

with

$$\mathrm{J}_n^{(r)} = \{(x_1,\ \dots, x_n) \in \mathbf{R}^{n(d+1)}\ :\ x_1 \in W_{(r)},\ (x_2 - x_1) \in W_{(r)},\ \dots,\ (x_n - x_{n-1}) \in W_{(r)}\}. \tag{32}$$

The fact that $\mathcal{J}_n$ is a non-empty and, if $d > 2$, connected set is then a consequence of the connectedness of $\mathcal{J}_n^{(r)}$. The latter property can be checked as follows. The projection $[\mathcal{J}_n^{(r)}]_{u,v}$ of $\mathcal{J}_n^{(r)}$ onto the space $\mathbf{R}^{2n}$ of the (u, v)-coordinates is the intersection of the convex cone $(u_1 > 0,\ v_1 < 0,\ u_{j+1} - u_j > 0,\ v_{j+1} - v_j <$

$0,\ 1 \leq j \leq n-1)$ (here we have put $u_j = u(x_j), v_j = v(x_j)$) with the set $(u_1 v_1 > -R^2, \ldots, u_n v_n > -R^2)$ which is preserved by the contractions; therefore, any couple of points in $[\mathcal{J}_n^{(r)}]_{u,v}$ can be connected by a broken line contained in this set. Considering now $\mathcal{J}_n^{(r)}$ as a fiberspace over its projection $[\mathcal{J}_n^{(r)}]_{u,v}$, we see that it is locally trivialized with a toroidal fiber of the form $\underline{\vec{x}}_j^2 = constant,\ 1 \leq j \leq n$ which is connected provided d is larger than 2; the connectedness of $\mathcal{J}_n^{(r)}$ follows correspondingly.

As in the Minkowskian case, one can then state a de Sitterian version of the Glaser-Streater property, according to which any function holomorphic in $\mathcal{T}_n \cup -\mathcal{T}_n \cup \mathcal{J}_n$ has a single-valued analytic continuation in $\mathcal{T}_n^{\text{ext}} = \mathcal{T}_n^{1\ \text{ext}}$. (see e.g. [7, 25, 31]). Hence every permuted Wightman distribution is the boundary value of a function holomorphic in the corresponding extended permuted tuboid $\mathcal{T}_n^{\pi\ \text{ext}}$; this function is in fact an analytic continuation of $\mathrm{W}_n^{(\pi)}$ and thereby of the common holomorphic n-point function $\mathfrak{W}_n$.

Remark. The proof of the Glaser-Streater property is based on a lemma of analytic completion in the orbits of the complex Lorentz group and this is why it holds for the complexified de Sitter space (since $X_d^{(c)n}$ is a union of such orbits), the connectedness of the set of orbits generated by the Jost points being of course crucial. To be complete, one must also point out that it requires the following strong form of the Bargmann-Hall-Wightman lemma, ([23], pp. 95-97, [32] pp. 67-70) proved for $d+1 \leq 4$ in these references, and extended to all dimensions in [24]. An alternative proof of the latter is given in Appendix B.

Lemma 1 *(Bargmann-Hall-Wightman-Jost) Let $M \in L_+(\mathbf{C})$ be such that $\mathrm{T}_+ \cap M^{-1}\mathrm{T}_+ \neq \emptyset$. There exists a continuous map $t \mapsto M(t)$ of $[0,\ 1]$ into $L_+(\mathbf{C})$ such that $M(0) = 1$, $M(1) = M$, and that, for every $z \in \mathrm{T}_+ \cap M^{-1}\mathrm{T}_+$ and $t \in [0,\ 1]$, $M(t)z \in \mathrm{T}_+$.*

3.2 The PCT-property

The standard proof of the PCT theorem (see [25, 32] and references therein) extends in a straightforward way to the de Sitterian case under the assumptions of covariance, weak spectral condition, and locality. The latter can be relaxed to the condition of weak locality [13, 25, 32], namely:
Weak locality condition: *For every Jost point $(r_1, \ldots,\ r_n) \in \mathcal{J}_n$,*

$$\mathcal{W}_n(r_1, \ldots,\ r_n) = \mathcal{W}_n(r_n, \ldots,\ r_1) \tag{33}$$

which obviously follows from locality.

Proposition 2 *(PCT invariance) From the weak spectral condition, the covariance condition, and the weak locality condition, it follows that*

$$\mathcal{W}_n(x_1, \ldots, x_n) = \mathcal{W}_n(\mathrm{I}_0 x_n, \ldots, \mathrm{I}_0 x_1) \tag{34}$$

holds at every real $x \in X_d^n$ (in the sense of distributions), where $\mathrm{I}_0 = -1$ if d is odd, and, if d is even, for every $z \in \mathbf{C}^{d+1}$,

$$(\mathrm{I}_0\, z)^{(\mu)} = -z^{(\mu)} \quad \text{for } 0 \leq \mu < d, \quad (\mathrm{I}_0\, z)^{(d)} = z^{(d)}. \tag{35}$$

If moreover the positivity condition holds, there exists an antiunitary operator $\Theta:\ \mathcal{H} \to \mathcal{H}$ such that

$$\Theta\Omega = \Omega, \quad \Theta\langle \Phi_n^{(b)},\ f\rangle = \langle \Phi_n^{(b)},\ f_{\mathrm{I}_0}^{\star}\rangle, \tag{36}$$

where $f_{\mathrm{I}_0}^{\star}(x_1, \ldots,\ x_n) = \bar{f}(\mathrm{I}_0 x_n, \ldots, \mathrm{I}_0 x_1)$.

One notices that, in this statement, it is the symmetry I_0 (which depends on the parity of the dimension) which has to be used (as it is also the case for $d+1$-dimensional Minkowskian theories). This is due to the fact that I_0 always belongs to the corresponding complex connected group $L_+(\mathbf{C})$ under which the functions $\mathfrak{W}_n$ are invariant. Since the mapping $(z_1,\dots,z_n) \to (\mathrm{I}_0 z_n,\dots,\mathrm{I}_0 z_1)$ is always (for every n) an automorphism of the tuboid $\mathcal{T}_n$, the standard analytic continuation argument [25, 32] applies to the proof of Eq. (34). Now, it is interesting to note that for d even (in particular in the "physical case" $d=4$) I_0 does have the interpretation of a *space-time inversion* in a local region of the de Sitter universe around the *base point* x_0 with coordinates $(0,\dots,0,R)$, considered as playing the role of the origin in Minkowski space. In fact, the stabilizer of x_0 (inside the de Sitter group) is the analogue of the Lorentz group (inside the Poincaré group) and indeed it acts as the latter in the (Minkowskian) tangent space to X_d at x_0; I_0 then appears as the corresponding space-time inversion (contained in the complexified stabilizer of x_0). This means that (for d even) the previous proposition can be seen as introducing a *PCT-symmetry relative to the point* x_0; analogous symmetry operators could be associated with all points of the de Sitter manifold.

3.3 Euclidean points

In the ambient complex n-point Minkowskian space-time $\mathbf{C}^{n(d+1)}$, the union of the permuted extended tubes $\bigcup_\pi \mathrm{T}_n^{\pi\mathrm{ext}}$ contains all non-coinciding Euclidean points. Since the intersection of this union with $X_d^{(c)n}$ is the union of all permuted extended tuboids $\mathcal{T}_n^{\pi\mathrm{ext}}$, it follows that the domain of analyticity of $\mathfrak{W}_n$ contains the set of all non-coinciding points of the product of n Euclidean spheres.

3.4 The case $n=2$. Generalized free fields and their Wick powers

The extended tube $\mathrm{T}_2^{\mathrm{ext}}$ is equal to $\{(z_1,\ z_2) \in \mathbf{C}^{2(d+1)}\ :\ (z_1-z_2)^2 \notin \mathbf{R}_+\}$. Hence

$$\mathcal{T}_2^{\mathrm{ext}} = \{(z_1,\ z_2) \in X_d^{(c)2}\ :\ (z_1-z_2)^2 \notin \mathbf{R}_+\}. \tag{37}$$

In particular $\mathfrak{W}_2(z_1,z_2) - \mathfrak{W}_2(z_2,z_1)$ is analytic, odd, and Lorentz invariant at real space-like separations, hence vanishes there even without the locality assumption. Thus under the assumptions of weak spectral condition and covariance, $\mathrm{W}_2(z_1,\ z_2)$ defines an "invariant perikernel" in the sense of [10] which can be represented by a function $w(\zeta)$ of the single complex variable $\zeta = 1+(z_1-z_2)^2/2R^2 = -z_1\cdot z_2/R^2$, holomorphic in the cut-plane $\mathbf{C}\setminus[1,\ \infty)$. Any such two-point function completely determines a generalized free field A whose Wightman functions are obtained by the same formulae as in the Minkowskian case. (see [9] for a detailed study of all that). A can also be seen as the restriction of a generalized free field on the ambient Minkowski space, in general with an indefinite metric (see also in this connection subsection 5.4 of [9]). Wick monomials in A have well-defined Wightman functions, again given by the same formulae as in the Minkowskian case, i.e. as sums of products of two-point functions. Since these Wightman functions can be obtained as limits of Wightman functions of Wick monomials of group-regularizations of A, they satisfy all the conditions 1-5 (in particular positivity) provided A does. In particular the Wick monomials in A are unbounded distribution valued operators in the Fock space of A, and provide examples of theories satisfying all the axioms.

4 Physical interpretation of the weak spectral condition

In this section, we are still in the Lorentz coordinate frame $\{e_0,\dots,\ e_d\}$ in the ambient real Minkowski space, the notations u, v, $[\lambda]$ are as in Eqs. (2) and (3).

Let us now discuss the physical interpretation of the spectral condition we have introduced. Following the pioneering approach of Unruh [33], Gibbons and Hawking [16] we adopt the viewpoint of a geodesical observer and namely the one moving on the geodesic $h(x_0)$ of the base point x_0 contained in the $(x^{(0)},x^{(d)})$-plane, which we parametrize as follows:

$$h(x_0) = \{x = x(\tau);\quad x^{(0)} = R\sinh\frac{\tau}{R},\ x^{(1)} = \cdots = x^{(d-1)} = 0,\ x^{(d)} = R\cosh\frac{\tau}{R}\} \tag{38}$$

The parameter τ of the representation (38) is the proper time of the observer and the base point x_0 is the event for which $\tau = 0$.

The set of all events of X_d which can be connected with the observer by the reception and the emission of light-signals is the region:

$$\mathcal{U}_{h(x_0)} = \{x \in X_d : \quad x^{(d)} > |x^{(0)}|\} = W_{(r)} \cap X_d. \tag{39}$$

Points in $\mathcal{U}_{h(x_0)}$ can be parametrized by $(\tau, \underline{\vec{x}})$ as follows:

$$x(\tau, \underline{\vec{x}}) = \left[\begin{array}{l} x^{(0)} = \sqrt{R^2 - \underline{\vec{x}}^2}\, \sinh \frac{\tau}{R} \\ (x^{(1)}, \ldots, x^{(d-1)}) = \underline{\vec{x}} \\ x^{(d)} = \sqrt{R^2 - \underline{\vec{x}}^2}\, \cosh \frac{\tau}{R} \end{array} \right. , \qquad \tau \in \mathbf{R}, \ \ \underline{\vec{x}}^2 < R^2 \tag{40}$$

$\mathcal{U}_{h(x_0)}$ is the intersection of the hyperboloid with the wedge $W_{(r)}$ of the ambient space and admits two boundary parts $H^+_{h(x_0)}$ and $H^-_{h(x_0)}$, respectively called the "future" and "past horizons" of the geodesical observer:

$$H^{\pm}_{h(x_0)} = \{x \in X_d : \quad x^{(0)} = \pm x^{(d)}, \ \ x^{(d)} \geq 0\}. \tag{41}$$

$\mathcal{U}_{h(x_0)}$ is stable under the transformation (3), for $\lambda = e^{\frac{t}{R}} > 0$. These transformations constitute a subgroup $T_{h(x_0)}$ of $L^{\uparrow}_{+}$. The action of $T_{h(x_0)}(t)$ on $\mathcal{U}_{h(x_0)}$ written in terms of the parameters t and τ can be interpreted as a "time-translation":

$$T_{h(x_0)}(t)[x(\tau, \underline{\vec{x}})] = x(t + \tau, \underline{\vec{x}}) \equiv x^t. \tag{42}$$

$T_{h(x_0)}$ thus defines a group of isometric automorphisms of $\mathcal{U}_{h(x_0)}$ whose orbits are all branches of hyperbolae of $\mathcal{U}_{h(x_0)}$ in two-dimensional plane sections parallel to the $(x^{(0)}, x^{(d)})$-plane (see [26] for a general discussion of this kind of structure).

Before discussing the physical interpretation of the spectral condition, we need to extend to the de Sitter case one aspect of a well-known result of Bisognano and Wichmann [BW] which concerns analyticity properties in orbits of the complexified group $T^{(c)}_{h(x_0)}$ of $T_{h(x_0)}$.

4.1 Bisognano-Wichmann analyticity.

For every function g_n in $\mathcal{D}(X_d^n)$ or $\mathcal{S}(X_d^n)$ and every $\lambda \in \mathbf{R} \setminus \{0\}$, $[\lambda]$ as in Eq. (3), we denote (with a simplified form of (8))

$$g_{n\lambda}(x_1, \ldots, \ x_n) = g_n([\lambda^{-1}]x_1, \ldots, \ [\lambda^{-1}]x_n). \tag{43}$$

and

$$g_n^{\leftarrow}(x_1, \ldots, \ x_n) = g_n(x_n, \ldots, \ x_1) \tag{44}$$

Then one has:

Theorem 1 *If a set of Wightman distributions satisfies the locality and weak spectral conditions, then for all $m, \ n \in \mathbf{N}$, $f_m \in \mathcal{D}(W^m_{(r)} \cap X^m_d)$ and $g_n \in \mathcal{D}(W^n_{(r)} \cap X^n_d)$, there is a function $G_{(f_m, g_n)}(\lambda)$ holomorphic on $\mathbf{C} \setminus \mathbf{R}_+$ with continuous boundary values $G^{\pm}_{(f_m, g_n)}$ on $(0, \ +\infty)$ from the upper and lower half-planes such that:*

a) for all $\lambda \in (0, \ +\infty)$,

$$G^+_{(f_m, g_n)}(\lambda) = \langle \mathcal{W}_{m+n}, \ f_m \otimes g_{n\lambda} \rangle, \qquad G^-_{(f_m, g_n)}(\lambda) = \langle \mathcal{W}_{m+n}, \ g_{n\lambda} \otimes f_m \rangle. \tag{45}$$

b) for all $\lambda \in (-\infty, \ 0)$,

$$G_{(f_m, g_n)}(\lambda) = \langle \mathcal{W}_{m+n}, \ f_m \otimes g^{\leftarrow}_{n \ \lambda} \rangle = \langle \mathcal{W}_{m+n}, \ g^{\leftarrow}_{n \ \lambda} \otimes f_m \rangle. \tag{46}$$

This theorem neither requires positivity nor Lorentz covariance. It expresses a property of the domain of holomorphy of the Wightman functions, and of the boundary values from this domain. In fact, it states that appropriate boundary values of the $(m+n)-$point holomorphic function $\mathfrak{W}_{m+n}$, taken in the region where all the variables $w_1, \ldots, w_m, z_1, \ldots, z_n$ belong to $W_{(r)} \cap X_d$, are holomorphic with respect to the group variable λ (for $\lambda \in \mathbf{C} \setminus \mathbf{R}_+$) in the orbits $(w,x) \mapsto (w, [\lambda]x)$ of $T^{(c)}_{h(x_0)}$ (with $w = (w_1, \ldots, w_m)$, $x = (x_1, \ldots, x_n), \lambda = e^{\frac{t}{R}}$) and such that:

for $\lambda > 0$,

$$\mathfrak{W}_{m+n}(w,\ [\lambda + i0]x) = \mathrm{W}_{m+n}(w,\ [\lambda]x), \quad \mathfrak{W}_{m+n}(w,\ [\lambda - i0]x) = \mathrm{W}_{m+n}([\lambda]x,\ w) \tag{47}$$

and for $\lambda < 0$, putting $x_{\leftarrow} = (x_n, \ldots, x_1)$,

$$\mathfrak{W}_{m+n}(w,\ [\lambda]x) = \mathrm{W}_{m+n}(w,\ [\lambda]x_{\leftarrow}) = \mathrm{W}_{m+n}([\lambda]x_{\leftarrow},\ w) \tag{48}$$

the latter equality being a direct consequence of locality (since $x \in W^n_{(r)}$ and $\lambda < 0$ imply $[\lambda]x_{\leftarrow} \in W^n_{(l)}$).

The theorem will be proved here under the simplifying assumption that the temperedness condition (17) holds.

Proof

Four permuted branches of the function $\mathfrak{W}_{m+n}$ are involved in the proof. The variables $w = (w_1, \ldots, w_m)$ will always be kept real in $W^m_{(r)} \cap X^m_d$, while the variables $z = (z_1, \ldots, z_n)$ are complex (in $X^{(c)n}_d$) and we denote $y = \mathrm{Im}\, z$. The corresponding analyticity domains in the variables z (described below) are obtained in the boundaries (i.e. in the "face" $\mathrm{Im}\, w = 0$) of four permuted tuboids $\mathcal{T}^{\pi}_{m+n}$ according to the prescription of our weak spectral condition. In view of the distribution boundary value procedure, restricted to the subset of variables w, these analyticity domains are obtained whenever one smears out the permuted functions W^{π}_{m+n} under consideration with a fixed function $f_m \in \mathcal{D}(W^m_{(r)} \cap X^m_d)$. (this function being understood as *the* function named f_m in the statement of the theorem). These four branches are:

i) $\mathrm{W}_{m+n}(w_1, \ldots, w_m, z_1, \ldots, z_n) = \mathrm{W}_{m+n}(w,\ z)$, holomorphic in the tuboid:
$\mathcal{Z}_{n+} = \left\{z \in X^{(c)n}_d;\ y_1 \in V_+, y_j - y_{j-1} \in V_+,\ j = 2, \ldots, n\right\}$;

ii) $\mathrm{W}_{m+n}(z_n, \ldots, z_1, w_1, \ldots, w_m) = \mathrm{W}_{m+n}(z_{\leftarrow},\ w)$, holomorphic in the opposite tuboid:
$\mathcal{Z}_{n-} = \{z \in X^{(c)n}_d;\ y_1 \in V_-, y_j - y_{j-1} \in V_-,\ j = 2, \ldots, n\}$;

iii) $\mathrm{W}_{m+n}(z_1, \ldots, z_n, w_1, \ldots, w_m) = \mathrm{W}_{m+n}(z,\ w)$, holomorphic in the tuboid:
$\mathcal{Z}'_{n+} = \{z \in X^{(c)n}_d;\ y_n \in V_-, y_j - y_{j-1} \in V_+, j = 2, \ldots, n\}$;

iv) $\mathrm{W}_{m+n}(w_1, \ldots, w_m, z_n, \ldots, z_1) = \mathrm{W}_{m+n}(w,\ z_{\leftarrow})$, holomorphic in the opposite tuboid:
$\mathcal{Z}'_{n-} = \{z \in X^{(c)n}_d;\ y_n \in V_+, y_j - y_{j-1} \in V_-, j = 2 \ldots, n\}$.

Correspondingly, with the fixed function $f_m \in \mathcal{D}(W^m_{(r)} \cap X^m_d)$ we associate the following four functions $z \mapsto F_{\pm}(f_m;\ z)$ and $z \mapsto F'_{\pm}(f_m;\ z)$:

$$F_+(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(w,\ z)\ f_m(w)\ d^m\sigma(w), \qquad F_-(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(z_{\leftarrow},\ w)\ f_m(w)\ d^m\sigma(w) \tag{49}$$

$$F'_+(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(z,\ w)\ f_m(w)\ d^m\sigma(w), \qquad F'_-(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(w,\ z_{\leftarrow})\ f_m(w)\ d^m\sigma(w) \tag{50}$$

which are respectively holomorphic in $\mathcal{Z}_{n+}$, $\mathcal{Z}_{n-}$, ${\mathcal{Z}'}_{n+}$ and ${\mathcal{Z}'}_{n-}$. By letting the variables z tend to the reals from the respective tuboids $\mathcal{Z}_{n+}$, $\mathcal{Z}_{n-}$, ${\mathcal{Z}'}_{n+}$ and ${\mathcal{Z}'}_{n-}$, and taking the corresponding boundary values $F_\pm^{(b)}(f_m;\ x)$ and $F'^{(b)}_\pm(f_m;\ x)$ of $F_\pm$ and $F'_\pm$ on X_d^n in the sense of distributions, one then obtains for every $g_n \in \mathcal{D}(X_d^n)$ the following relations which involve the $(m+n)-$point Wightman distributions considered in the statement of the theorem:

$$\int_{X_d^n} F_+^{(b)}(f_m;\ x)\, g_n(x)\, d^n\sigma(x) \ = \ \langle \mathcal{W}_{m+n},\ f_m \otimes g_n \rangle, \tag{51}$$

$$\int_{X_d^n} F_-^{(b)}(f_m;\ x)\, g_n(x)\, d^n\sigma(x) \ = \ \langle \mathcal{W}_{m+n},\ g_n^{\leftarrow} \otimes f_m \rangle. \tag{52}$$

$$\int_{X_d^n} F'^{(b)}_+(f_m;\ x)\, g_n(x)\, d^n\sigma(x) \ = \ \langle \mathcal{W}_{m+n},\ g_n \otimes f_m \rangle. \tag{53}$$

$$\int_{X_d^n} F'^{(b)}_-(f_m;\ x)\, g_n(x)\, d^n\sigma(x) \ = \ \langle \mathcal{W}_{m+n},\ f_m \otimes g_n^{\leftarrow} \rangle. \tag{54}$$

We now notice that, in view of local commutativity, $F_+^{(b)}(f_m;\ x)$ and $F_-^{(b)}(f_m;\ x)$ coincide in the sense of distributions on the set of special Jost points $\mathcal{J}_n^{(l)} = -\mathcal{J}_n^{(r)} = \{(x_1, \dots,\ x_n) \in X_d^n;\ 0 > u_1 > \dots u_{n-1} > u_n,\ \ 0 < v_1 < \dots v_{n-1} < v_n\}$; therefore, in view of the edge-of-the-wedge theorem, the functions $z \mapsto F_+(f_m;\ z)$ and $z \mapsto F_-(f_m;\ z)$ have a common holomorphic extension, denoted $F(f_m;\ z)$, in $\Delta = \mathcal{Z}_{n+} \cup \mathcal{Z}_{n-} \cup \mathcal{V}$, where $\mathcal{V}$ is a complex neighborhood of $\mathcal{J}_n^{(l)}$, such that $[\lambda]\mathcal{V} = \mathcal{V}$ for all $\lambda > 0$ (in particular $F_+^{(b)}(f_m;\ x)$ and $F_-^{(b)}(f_m;\ x)$ are continuous on $\mathcal{J}_n^{(l)}$). By a similar use of local commutativity for $F'^{(b)}_+$ and $F'^{(b)}_-$, which coincide on the set of special Jost points $\mathcal{J}'^{(l)}_n = \{(x_1, \dots,\ x_n) \in X_d^n; 0 > u_n > u_{n-1} \dots > u_1,\ \ \ 0 < v_n < v_{n-1} \dots < v_1\}$, we also notice that the functions $z \mapsto F'_+(f_m;\ z)$ and $z \mapsto F'_-(f_m;\ z)$ have a common holomorphic extension, denoted $F'(f_m;\ z)$, in $\Delta' = \mathcal{Z}'_{n+} \cup \mathcal{Z}'_{n-} \cup \mathcal{V}'$, where $\mathcal{V}'$ is a complex neighborhood of $\mathcal{J}'^{(l)}_n$, such that $[\lambda]\mathcal{V}' = \mathcal{V}'$ for all $\lambda > 0$. Moreover, if the temperedness condition (17) is satisfied by the function W_{m+n}, it can be checked that similar inequalities are satisfied by the holomorphic functions $F(f_m;\ z)$ and $F'(f_m;\ z)$ with respect to the variables z in their respective tuboids $\mathcal{Z}_{n\pm}$ and $\mathcal{Z}'_{n\pm}$.

At this point, we shall rely on the following basic lemma which provides analytic completion in the orbits of the group $\{z \mapsto [\lambda]z\}$ (for $\lambda \in \mathbf{C}_\pm$) and whose proof is given below (after the end of our argument).

Lemma 2 *a) Given any function $H(z)$ holomorphic in Δ, the function $(z,\ \lambda) \mapsto H([\lambda]z)$ is holomorphic in $\mathcal{Z}_{n+} \times \mathbf{C}_+$. Moreover, if $H(x+iy)$ satisfies majorizations of the form (17) in the tuboids $\mathcal{Z}_{n+}$ and $\mathcal{Z}_{n-}$ allowing one to define the boundary values $H_+^{(b)}$ and $H_-^{(b)}$ of H from $\mathcal{Z}_{n+}$ and $\mathcal{Z}_{n-}$ as tempered distributions, then the function $(z,\ \lambda) \mapsto H([\lambda]z)$ admits a distribution boundary value on $X_d^n \times \mathbf{C}_+$ (still denoted $H([\lambda]x)$); the latter is a distribution in x with values in the functions of λ which are holomorphic in $\mathbf{C}_+$ and continuous in $\overline{\mathbf{C}_+} \setminus \{0\}$ and one has:*

$$H([\pm\lambda]x) = H_\pm^{(b)}([\pm\lambda]x) \qquad \text{for } \lambda > 0 \tag{55}$$

(the latter being identities between distributions in x with values in the continuous functions of λ).

b) Similarly, given any function $H'(z)$ holomorphic in Δ', the function $(z,\ \lambda) \mapsto H'([\lambda]z)$ is holomorphic in $\mathcal{Z}'_+ \times \mathbf{C}_-$. Moreover, if $H'(x+iy)$ satisfies majorizations of the form (17) in the tuboids $\mathcal{Z}'_+$ and $\mathcal{Z}'_-$ allowing one to define the boundary values $H'^{(b)}_+$ and $H'^{(b)}_-$ of H' from $\mathcal{Z}'_+$ and $\mathcal{Z}'_-$ as tempered distributions, then the function $(z,\ \lambda) \mapsto H'([\lambda]z)$ admits a distribution boundary value on $X_d^n \times \mathbf{C}_-$, holomorphic in $\mathbf{C}_-$ and continuous in $\overline{\mathbf{C}_-} \setminus \{0\}$, and one has:

$$H'([\pm\lambda]x) = H'^{(b)}_\pm([\pm\lambda]x) \qquad \text{for } \lambda > 0. \tag{56}$$

Since the function $F(f_m;\ z)$ satisfies the analyticity and temperedness properties of the function $H(z)$ of Lemma 2 a), it follows that one can take the boundary value onto $X_d^n \times \mathbf{C}_+$ from $\mathcal{Z}_{n+} \times \mathbf{C}_+$ of the holomorphic function $(z,\ \lambda) \mapsto F(f_m;\ [\lambda]z)$ and obtain for every $g_n \in \mathcal{D}(X_d^n)$ the following relations (deduced from Eq. (55) after taking into account Eqs. (51) and (52)):

$$\int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ f_m \otimes g_{n\lambda} \rangle \quad \text{for } \lambda > 0, \tag{57}$$

$$\int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ g^{\leftarrow}_{n\ \lambda} \otimes f_m \rangle \quad \text{for } \lambda < 0. \tag{58}$$

Similarly, one can apply the results of Lemma 2 b) to the function $H'(z) = F'(f_m;\ z)$; one can thus take the boundary value onto $X_d^n \times \mathbf{C}_-$ from $\mathcal{Z}'_{n+} \times \mathbf{C}_-$ of the holomorphic function $(z,\ \lambda) \mapsto F'(f_m;\ [\lambda]z)$ and obtain for every $g_n \in \mathcal{D}(X_d^n)$ the following relations (deduced from Eq. (56) after taking into account Eqs. (53) and (54)):

$$\int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ g_{n\lambda} \otimes f_m \rangle \quad \text{for } \lambda > 0, \tag{59}$$

$$\int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ f_m \otimes g^{\leftarrow}_{n\ \lambda} \rangle \quad \text{for } \lambda < 0. \tag{60}$$

The l.h.s. of Eqs. (57) (or (58)) and (59) (or (60)) are respectively the boundary values of the holomorphic functions

$$G_{(f_m, g_n)}(\lambda) = \int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) \tag{61}$$

defined for $\lambda \in \mathbf{C}_+$ and

$$G'_{(f_m, g_n)}(\lambda) = \int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) \tag{62}$$

defined for $\lambda \in \mathbf{C}_-$. For an arbitrary function $g_n \in \mathcal{D}(X_d^n)$, these two holomorphic functions are distinct from each other. Now, if g_n is taken in $\mathcal{D}(\mathcal{U}^n_{h(x_0)})$, the r.h.s. of Eqs. (58) and (60) coincide in view of local commutativity, and therefore these two holomorphic functions admit a common holomorphic extension $G_{(f_m, g_n)}(\lambda)$ in $\mathbf{C} \setminus \mathbf{R}_+$ whose boundary values on $\mathbf{R} \setminus 0$ satisfy the properties a) and b) of the theorem. (in view of Eqs. (57)—(60)).

Proof of Lemma 2

We concentrate on part a) of the lemma, part b) being completely similar. At first, the fact that the function $(z,\ \lambda) \mapsto H([\lambda]z)$ can be analytically continued in $\mathcal{Z}_{n+} \times \mathbf{C}_+$ is a result of purely geometrical nature (based on the tube theorem) which can be obtained as a direct application of lemma 3 (ii) of Appendix A. In fact, for each point $x \in \mathcal{J}_n^{(r)}$, the set $\{z = [\lambda]x;\ \lambda \in \mathbf{C}_+\}$ is contained in Δ (namely in $\mathcal{Z}_{n+}$, as it directly follows from Eq. (3) and from the definitions of $\mathcal{J}_n^{(r)}$ and $\mathcal{Z}_{n+}$). One can even check that each point $x \in \mathcal{J}_n^{(r)}$ is on the edge of a small open tuboid $\tau(x)$ contained in $\mathcal{Z}_{n+}$ such that $\{z = [\lambda]z';\ z' \in \tau(x),\ \lambda \in \mathbf{C}_+\} \subset \mathcal{Z}_{n+} \cup \mathcal{V} \subset \Delta$. On the other hand, for each point $z \in \mathcal{Z}_{n+}$ there exists a neighbourhood $\delta_+(z)$ of the real positive axis and a neighbourhood $\delta_-(z)$ of the real negative axis in the complex λ-plane, such that the set $\{[\lambda]z;\ \lambda \in \delta^+(z) \cup \delta^-(z)\}$ is contained in Δ: for $\lambda \in \delta_+(z)$ and $\lambda \in \delta_-(z)$ the corresponding subsets are respectively contained in $\mathcal{Z}_{n+}$ and in $\mathcal{Z}_{n-}$. Therefore, the assumptions of lemma 3 (ii) of Appendix A are fulfilled (by choosing the set Q of the latter as a subset of $\tau(x)$ and $D' = \mathcal{Z}_{n+}$ after an appropriate adaptation of the variables). In order to see that the new domain thus obtained (i.e. $\{z = [\lambda]z';\ z' \in \mathcal{Z}_{n+},\ \lambda \in \mathbf{C}_+\}$ yields an enlargement of Δ, it is sufficient to notice that every real point x such that at least one component $x_j - x_{j-1}$ is time-like is transformed by any complex transformation $[\lambda]$ into a point outside $\mathcal{Z}_{n\pm}$ and this is of course also true for all points $z \in \mathcal{Z}_{n+}$ tending to such real (boundary) points (the neighbourhoods $\delta^{\pm}(z)$ becoming arbitrarily thin in such limiting configurations). The second statement of the lemma precisely deals with these limiting real configurations and with the fact that the analyticity of the boundary value $H([\lambda]x)$ in $\{\lambda \in \mathbf{C}_+\}$ is maintained for all $x \in X_d^n$. The boundary value relations (55) then follow from the fact that every point x is a limit of points $z \in \mathcal{Z}_{n+}$ and that the latter are always such that $[\lambda]z \in \mathcal{Z}_{n+}$ for $\lambda > 0$ and $[\lambda]z \in \mathcal{Z}_{n-}$ for $\lambda < 0$. In order to avoid too subtle an argument for justifying the analyticity of the limit $H([\lambda]x)$ in $\{\lambda \in \mathbf{C}_+\}$, we prefer to rely on an assumption of tempered growth (of the form (17)) for H; the latter allows one to give an alternative version of the analytic completion procedure which is based on the Cauchy integral representation, and thereby includes the treatment of the boundary values.

For $z = (z_1, \dots, z_n) \in \mathbf{C}^{nd}$, we adopt the coordinates

$$\zeta_1 = z_1, \quad \zeta_k = z_k - z_{k-1} \text{ for } 1 < k \le n, \quad \mathrm{u}_j = \zeta_j^{(0)} + \zeta_j^{(d)},\ \mathrm{v}_j = \zeta_j^{(0)} - \zeta_j^{(d)}, \text{ for } 1 \le j \le n, \tag{63}$$

$$r_j = (\zeta_j^{(1)}, \dots, \zeta_j^{(d-1)}). \tag{64}$$

For every $z = (z_1, \dots, z_n) \in \mathcal{Z}_{n+}$, we define $G(z,\ \lambda) = H([\lambda]z)$. Easy computations using the tempered growth condition show that $G(z,\ \lambda)$ is a holomorphic function of z and $\lambda = \rho e^{i\theta}$ for $z \in \mathcal{Z}_{n+}$, $\rho \in (0,\ +\infty)$ and

$$|\sin\theta| < \frac{\kappa}{2(1+2M)}, \quad \kappa = \min_j \left(1 - \frac{\|\mathrm{Im}\, r_j\|^2}{\mathrm{Im}\,\mathrm{u}_j\, \mathrm{Im}\,\mathrm{v}_j}\right), \tag{65}$$

with

$$M = \frac{1}{\mu} \max_j\ \max\{|\mathrm{Re}\,\mathrm{u}_j|,\ |\mathrm{Re}\,\mathrm{v}_j|\}, \quad \mu = \min_j\ \min\{\mathrm{Im}\,\mathrm{u}_j,\ \mathrm{Im}\,\mathrm{v}_j\}, \tag{66}$$

which (for such values) satisfies bounds of the following form:

$$|G(z,\ \lambda)| \le K_1(|\lambda|+1/|\lambda|)^L \left(\frac{1}{\mu\kappa} + \max_j |\zeta_j|\right)^L \le K_2(|\lambda|+1/|\lambda|)^L \left(\mathrm{dist}(z, \partial \mathrm{T}_n)^{-1} + \max_j |\zeta_j|\right)^L, \tag{67}$$

where K_1, K_2 are suitable constants.

On the other hand if z is real and $z \in \mathcal{J}_n^{(r)}$, i.e. $\mathrm{u}_j > 0$ and $\mathrm{v}_j < 0$ for all j (with the notations of Eq. (63)), then $[\lambda]z \in \mathcal{Z}_{n+}$ whenever $\mathrm{Im}\,\lambda > 0$ and

$$|H([\lambda]z)| \le K(|\lambda| + 1/|\lambda|)^L \left(\frac{1}{\mathrm{Im}\,\lambda} \max_j (1/\mathrm{Re}\,\mathrm{u}_j - 1/\mathrm{Re}\,\mathrm{v}_j) + \max_j |\zeta_j|\right)^L. \tag{68}$$

This shows that $H([\lambda + i0]z)$ is a tempered distribution in $\lambda \in \mathbf{R}$ with values in the polynomially bounded functions of z on $\mathcal{J}_n^{(r)}$ (actually in the $\mathcal{C}^\infty$ functions of z, as the z derivatives of H and G satisfy similar bounds). When $\lambda < 0$, as already noted, one has $[\lambda]z \in \mathcal{V}$ and $G(z,\ \lambda) = H([\lambda]z)$ is analytic in z and λ. Hence, for $z \in \mathcal{J}_n^{(r)}$, $G(z,\ \lambda + i0)$ is well-defined as a tempered distribution in $\lambda \in \mathbf{R}$ with values in the polynomially bounded functions of z on $\mathcal{J}_n^{(r)}$ and is the boundary value of a function holomorphic in $\mathbf{C}_+$ and bounded by the r.h.s. of Eq. (68). For $\lambda \in \mathbf{C}_+$ this function can be computed by the Cauchy formula:

$$G(z,\ \lambda) = \frac{1}{2\pi i}(i + \lambda - 1/\lambda)^{2L+2} \int_{\mathbf{R}} \frac{G(z,\ \lambda' + i0)}{(i + \lambda' - 1/\lambda')^{2L+2}(\lambda' - \lambda)} d\lambda'. \tag{69}$$

As shown by Eq. (67), the r.h.s. of this formula continues to make sense for $z \in \mathcal{Z}_{n+}$ and defines a function of z holomorphic and of tempered growth in $\mathcal{Z}_{n+}$, with values in the functions of λ holomorphic in $\mathbf{C}_+$ and continuous on $\overline{\mathbf{C}_+} \setminus \{0\}$. Therefore it has a boundary value in the sense of distributions as z tends to the reals. For real $\lambda \neq 0$, this boundary value coincides with $G(z,\ \lambda)$ (in the sense of distributions) when $z \in \mathcal{J}_n^{(r)}$, hence (in view of the analytic continuation principle extended by the edge-of-the-wedge theorem) the rhs of Eq. (69) coincides with $G(z,\ \lambda)$ for all $z \in \mathcal{Z}_{n+}$ and all real $\lambda \neq 0$. The formula (69) thus holds for all $z \in \mathcal{Z}_{n+}$, $\lambda \in \mathbf{C}_+$ and, in the sense of distributions, when z

tends to the reals; moreover, the relations (55) hold in this limit as explained above in the geometrical analysis.

The previous argument could be identically repeated for part b) of the lemma, replacing $\mathcal{Z}_{n\pm}$ by $\mathcal{Z}'_{n\pm}$ etc... and $\mathbf{C}_+$ by $\mathbf{C}_-$, since (as one can check directly) for each point $x \in \mathcal{J}'^{(r)}_n = -\mathcal{J}'^{(l)}_n$, the set $\{z = [\lambda]x;\ \lambda \in \mathbf{C}_-\}$ is contained in $\mathcal{Z}'_{n+}$.

Remark. Using the vector-valued analyticity provided by Lemma 4 below, it is possible to carry over the analysis of Bisognano and Wichmann without change to the de Sitterian case. The above proof (also valid in the Minkowskian case) aims at a clear distinction of the part of this theory which does not depend on positivity.

4.2 Physical interpretation

The following theorem gives a thermal physical interpretation to the weak spectral condition we have introduced.

Theorem 2 *(KMS condition)*
For every pair of bounded regions $\mathcal{O}_1$, $\mathcal{O}_2$ of $\mathcal{U}_{h(x_0)}$, the correlation functions between elements of the corresponding polynomial algebras $\mathcal{P}(\mathcal{O}_1)$, $\mathcal{P}(\mathcal{O}_2)$ of a field on X_d satisfying the previous postulates enjoy a KMS condition with respect to the time-translation group $T_{h(x_0)}$ whose temperature is $\mathrm{T} = 1/2\pi R$.

Proof.
Being given any general correlation function $(\Omega, \mathbf{\Phi}(f)\mathbf{\Phi}(g)\Omega)$ between arbitrary elements $\mathbf{\Phi}(f) \in \mathcal{P}(\mathcal{O}_1)$ and $\mathbf{\Phi}(g) \in \mathcal{P}(\mathcal{O}_2)$, with $f = (f_0, f_1, \ldots, f_m, \ldots)$, $g = (g_0, g_1, \ldots, g_n, \ldots)$, $(f_m \in \mathcal{D}(\mathcal{O}_1^m), g_n \in \mathcal{D}(\mathcal{O}_2^n))$, we consider, for each "time-translation" $T_{h(x_0)}(t)$, the transformed quantities

$$\mathcal{W}_{(f,g)}(t) = (\Omega, \mathbf{\Phi}(f)\mathbf{\Phi}(g_{\{e^{t/R}\}})\Omega) \tag{70}$$

and

$$\mathcal{W}'_{(f,g)}(t) = (\Omega, \mathbf{\Phi}(g_{\{e^{t/R}\}})\mathbf{\Phi}(f)\Omega) \tag{71}$$

(the notation $g_{\{e^{t/R}\}}$ being as in Eq. (8), with $\Lambda_r = [\lambda], \lambda = e^{\frac{t}{R}}$).

In view of Theorem 1, one can introduce the function $G_{(f,g)}(\lambda) = \sum_{m,n} G_{(f_m,g_n)}(\lambda)$, which is holomorphic for $\lambda = e^{\frac{t}{R}} \in \mathbf{C} \setminus \mathbf{R}_+$ and admits continuous boundary values $G^{\pm}_{(f,g)}$ on $(0, +\infty)$ from the upper and lower half-planes given respectively (in view of Eqs. (45) and (14)) by:

$$G^{+}_{(f,g)}(\lambda) = \sum_{m,n} \langle \mathcal{W}_{m+n},\ f_m \otimes g_{n\lambda} \rangle\ = (\Omega, \mathbf{\Phi}(f)\mathbf{\Phi}(g_{\{e^{t/R}\}})\Omega), \tag{72}$$

$$G^{-}_{(f,g)}(\lambda) = \sum_{m,n} \langle \mathcal{W}_{m+n},\ g_{n\lambda} \otimes f_m \rangle\ = (\Omega, \mathbf{\Phi}(g_{\{e^{t/R}\}})\mathbf{\Phi}(f)\Omega). \tag{73}$$

This readily implies that the function $\mathrm{W}_{(f,g)}(t) = G_{(f,g)}(e^{\frac{t}{R}})$ is holomorphic in the strip $0 < \mathrm{Im}\, t < 2\pi R$ and that it admits continuous boundary values on the edges of this strip which are:

$$\lim_{\epsilon \to 0^+} \mathrm{W}_{(f,g)}(t + i\epsilon) = \mathcal{W}_{(f,g)}(t), \qquad \lim_{\epsilon \to 0^+} \mathrm{W}_{(f,g)}(t + 2i\pi R - i\epsilon) = \mathcal{W}'_{(f,g)}(t). \tag{74}$$

The latter express the fact that all the field observables localized in $\mathcal{U}_{h(x_0)}$ and submitted to the time-translation group $T_{h(x_0)}$ satisfy a KMS-condition at temperature $\mathrm{T} = (2\pi R)^{-1}$.

The previous property must be completed by the following results:

i) Periodicity in the complex time variable

Since f and g are localized respectively in $\mathcal{O}_1$ and $\mathcal{O}_2$, it follows from local commutativity that the function $\mathrm{W}_{(f,g)}(t)$ can be analytically continued across the part of the line $\mathrm{Im}\, t = 0$ (and therefore $\mathrm{Im}\, t = 2n\pi R,\ n \in \mathbf{Z}$) on which the two matrix elements of Eqs. (70) and (71) are equal. One concludes that the function $\mathrm{W}_{(f,g)}(t)$ is holomorphic and periodic with period $2i\pi R$ in the following cut-plane $\mathbf{C}^{cut}(\mathcal{O}_1, \mathcal{O}_2)$ which is *connected* (in particular) *if* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are space-like separated*:

$$\mathbf{C}^{cut}(\mathcal{O}_1, \mathcal{O}_2) = \bigcap_{x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2} \mathbf{C}^{cut}_{x_1,x_2}, \tag{75}$$

where

$$\mathbf{C}^{cut}_{x_1,x_2} = \{t \in \mathbf{C};\ \mathrm{Im}\, t \neq 2n\pi R,\ n \in \mathbf{Z}\} \cup \{t;\ t - 2in\pi R \in I_{x_1,x_2},\ n \in \mathbf{Z}\}. \tag{76}$$

and for any pair (x_1, x_2) we have set

$$I_{x_1,x_2} = \{t \in \mathbf{R} :\ (x_1 - [e^{-\frac{t}{R}}]x_2)^2 < 0\} \tag{77}$$

ii) The antipodal condition

The following property, relating by analytic continuation the field observables localized in the region $\mathcal{U}_{h(x_0)}$ with those localized in the antipodal region

$$\check{\mathcal{U}}_{h(x_0)} = \{x \in X_d, -x \in \mathcal{U}_{h(x_0)}\} = \{x = (x^{(0)}, \underline{\vec{x}}, x^{(d)}) \in X_d,\ \check{x} = (-x^{(0)}, \underline{\vec{x}}, -x^{(d)}) \in \mathcal{U}_{h(x_0)}\} \tag{78}$$

can also be obtained as a by-product of theorem 1.

With each sequence $g = (g_0, g_1, \dots, g_n, \dots)$ such that $g_n \in \mathcal{D}(\mathcal{U}^n_{h(x_0)})$ let us associate the sequence $\check{g} = (\check{g}_0, \check{g}_1, \dots, \check{g}_n, \dots)$, where $\check{g}_n(x_1, \dots, x_n) = g_n^{\leftarrow}(\check{x}_1, \dots, \check{x}_n) = g_n(\check{x}_n, \dots, \check{x}_1)$. Since (for each n) one has $\check{g}_n \in \mathcal{D}(\check{\mathcal{U}}^n_{h(x_0)})$, it follows that $\mathbf{\Phi}(\check{g})$ belongs to $\mathcal{P}(\check{\mathcal{U}}_{h(x_0)})$.

Let us also note that for the Lorentz transformation $[\lambda] = [-1]$, one has $g^{\leftarrow}_{n\ -1} = \check{g}_n$ and therefore, for all $\lambda > 0$, $g^{\leftarrow}_{n\ -\lambda} = \check{g}_{n\lambda}$.

We then see that the holomorphic function $G_{(f,g)}(\lambda)$ introduced above satisfies (in view of Eq. (46)) the following relations:

for all $\lambda > 0$,

$$G_{(f,g)}(-\lambda) = \sum_{m,n} \langle \mathcal{W}_{m+n},\ f_m \otimes g^{\leftarrow}_{n\ -\lambda} \rangle = \sum_{m,n} \langle \mathcal{W}_{m+n},\ g^{\leftarrow}_{n\ -\lambda} \otimes f_m \rangle. \tag{79}$$

and therefore in view of Eq. (14):

$$G_{(f,g)}(-e^{t/R}) = (\Omega, \mathbf{\Phi}(f)\mathbf{\Phi}(\check{g}_{\{e^{t/R}\}})\Omega). = (\Omega, \mathbf{\Phi}(\check{g}_{\{e^{t/R}\}})\mathbf{\Phi}(f)\Omega). \tag{80}$$

We can then state the following

Proposition 3 *(antipodal condition)*
Being given arbitrary observables $\mathbf{\Phi}(f)$ *and* $\mathbf{\Phi}(g)$ *in* $\mathcal{P}(\mathcal{U}_{h(x_0)})$ *and the corresponding observable* $\mathbf{\Phi}(\check{g})$ *in* $\mathcal{P}(\check{\mathcal{U}}_{h(x_0)})$*, the following identities hold:*

$\forall\ t \in \mathbf{R}$,

$$\mathrm{W}_{(f,g)}(t + i\pi R) = (\Omega, \mathbf{\Phi}(f)\mathbf{\Phi}(\check{g}_{\{e^{t/R}\}})\Omega) = (\Omega, \mathbf{\Phi}(\check{g}_{\{e^{t/R}\}})\mathbf{\Phi}(f)\Omega) \tag{81}$$

The geodesic and antipodal spectral conditions

We can introduce an "energy operator" $\mathcal{E}_{h(x_0)}$ associated with the geodesic $h(x_0)$ by considering in $\mathcal{H}$ the continuous unitary representation $\{U^t_{h(x_0)}; t \in \mathbf{R}\}$ of the time-translation group $T_{h(x_0)}$ and its spectral resolution

$$U^t_{h(x_0)} = \int_{-\infty}^{\infty} e^{i\omega t} dE_{h(x_0)}(\omega); \tag{82}$$

This defines (on a certain dense domain of $\mathcal{H}$ containing $\mathbf{\Phi}(\mathcal{B})\Omega$) the self-adjoint operator

$$\mathcal{E}_{h(x_0)} = \int_{-\infty}^{\infty} \omega dE_{h(x_0)}(\omega). \tag{83}$$

For any pair of vector states $\Psi^{(1)} = \mathbf{\Phi}(f^\star)\Omega$, $\Psi^{(2)} = \mathbf{\Phi}(g)\Omega$, the corresponding correlation function given in Eq. (70) can be written as follows:

$$\mathcal{W}_{(f,g)}(t) = (\mathbf{\Phi}(f^\star)\Omega, U^t_{h(x_0)}\mathbf{\Phi}(g)\Omega), \tag{84}$$

which shows that $\mathcal{W}_{(f,g)}(t)$ is a continuous and bounded function. In view of Eq. (82) it can be expressed as the Fourier transform of the bounded measure

$$\widetilde{\mathcal{W}}_{(f,g)}(\omega) = (\mathbf{\Phi}(f^\star)\Omega, dE_{h(x_0)}(\omega)\mathbf{\Phi}(g)\Omega) \tag{85}$$

Similarly, one has:

$$\mathcal{W}'_{(f,g)}(t) = (\mathbf{\Phi}(g^\star)\Omega, U^{-t}_{h(x_0)}\mathbf{\Phi}(f)\Omega), \tag{86}$$

which is the Fourier transform of

$$\widetilde{\mathcal{W}}'_{(f,g)}(\omega) = (\mathbf{\Phi}(g^\star)\Omega, dE_{h(x_0)}(-\omega)\mathbf{\Phi}(f)\Omega). \tag{87}$$

Eqs. (85) and (87) are valid for arbitrary f and g in $\mathcal{B}$. Now, if f and g have supports in $\mathcal{U}_{h(x_0)}$, the functions $\mathcal{W}_{(f,g)}(t)$ and $\mathcal{W}'_{(f,g)}(t)$ satisfy the KMS relations (74) and their Fourier transforms satisfy (as bounded measures) the following relation which is equivalent to Eq. (74):

$$\widetilde{\mathcal{W}}'_{(f,g)}(\omega) = e^{-2\pi R\omega}\widetilde{\mathcal{W}}_{(f,g)}(\omega). \tag{88}$$

Moreover, if we rewrite the antipodal condition (81) as follows (with notations similar to those of Eqs. (70) and (71)):

$$\mathrm{W}_{(f,g)}(t + i\pi R) = \mathcal{W}_{(f,\check{g})}(t) = \mathcal{W}_{(\check{g},f)}(t) \tag{89}$$

we see that the corresponding Fourier transforms satisfy the following equivalent relations:

$$\widetilde{\mathcal{W}}_{(f,\check{g})}(\omega) = \widetilde{\mathcal{W}}_{(\check{g},f)}(\omega) = e^{-\pi R\omega}\widetilde{\mathcal{W}}_{(f,g)}(\omega). \tag{90}$$

We have thus proved the

Theorem 3 *i) For every pair of states $\Psi^{(1)} = \mathbf{\Phi}(f^\star)\Omega$, $\Psi^{(2)} = \mathbf{\Phi}(g)\Omega$ in $\mathcal{P}(\mathcal{U}_{h(x_0)})\Omega$, the corresponding matrix elements of the spectral measure $dE_{h(x_0)}(\omega)$ satisfy the following* geodesic spectral condition*:*

$$(\mathbf{\Phi}(g^\star)\Omega, dE_{h(x_0)}(-\omega)\mathbf{\Phi}(f)\Omega) = e^{-2\pi R\omega}(\mathbf{\Phi}(f^\star)\Omega, dE_{h(x_0)}(\omega)\mathbf{\Phi}(g)\Omega) \tag{91}$$

ii) Moreover, the previous matrix elements of the spectral measure are also related to a third one which involves the antipodal state $\mathbf{\Phi}(\check{g})\Omega$ in $\mathcal{P}(\check{\mathcal{U}}_{h(x_0)})\Omega$, by the following antipodal spectral condition*:*

$$(\mathbf{\Phi}(f^\star)\Omega, dE_{h(x_0)}(\omega)\mathbf{\Phi}(\check{g})\Omega) = e^{-\pi R\omega}(\mathbf{\Phi}(f^\star)\Omega, dE_{h(x_0)}(\omega)\mathbf{\Phi}(g)\Omega). \tag{92}$$

Remark 3 i) The geodesic spectral condition (91) gives a precise content to the statement that in the region $\mathcal{U}_{h(x_0)}$ corresponding to an observer living on the geodesic $h(x_0)$, the energy measurements (relative to this observer) give exponentially damped expectation values in the range of negative energies. In the limit of flat space-time the l.h.s. of Eq. (91) would be equal to zero for $\omega > 0$, which corresponds to recovering the usual spectral condition of "positivity of the energy".

ii) The antipodal spectral condition (92) asserts that the spectral measure $dE_{h(x_0)}$ has exponentially damped matrix elements, in the high energy limit, between states localized in the mutually antipodal regions $\mathcal{U}_{h(x_0)}$ and $\check{\mathcal{U}}_{h(x_0)}$.

Remark 4 All the features that have been discussed in this section are also naturally interpreted in terms of the existence of an antiunitary involution J relating the algebras $\mathcal{P}(\mathcal{U}_{h(x_0)})$ and $\mathcal{P}(\check{\mathcal{U}}_{h(x_0)})$ and the validity of the corresponding Bisognano-Wichmann duality theorem for the Von Neumann algebras $\mathcal{A}(\mathcal{U}_{h(x_0)})$ and $\mathcal{A}(\check{\mathcal{U}}_{h(x_0)})$ [1, 3].

5 A consequence of positivity and weak spectral condition: the Reeh-Schlieder property

In this section we wish to show that the vector-valued distributions $f_n \mapsto \langle \Phi_n^{(b)},\ f_n \rangle$, (which are provided by the GNS construction, see Eq. (13)), are boundary values of vector-valued functions holomorphic in the tuboids $\mathcal{Z}_n = \mathcal{Z}_{n+} = \mathrm{Z}_{n,\ d+1} \cap X_d^{(c)n}$, where

$$\mathrm{Z}_{n,\,d+1} = \left\{ z \in \mathbf{C}^{n(d+1)};\ y_1 \in V_+,\ y_j - y_{j-1} \in V_+,\ j = 2, \ldots,\ n \right\}, \tag{93}$$

with, in particular, the Reeh-Schlieder property as a consequence. Let us also recall the definition of $\mathcal{Z}'_n = \mathcal{Z}'_{n+} = \left\{ z \in X_d^{(c)n};\ y_n \in V_-,\ y_j - y_{j-1} \in V_+,\ j = 2, \ldots,\ n \right\}$.

In the Minkowskian, flat, d-dimensional case, assuming the temperedness condition, as a consequence of the spectral condition (see e.g. [25]), the vector-valued distribution $\Phi_n^{(b)}$ is the Fourier transform of a vector-valued tempered distribution with support in the cone dual to the base of the tube $\mathrm{Z}_{n,\,d}$. Hence $\Phi_n^{(b)}$ is the boundary value of a function holomorphic in $\mathrm{Z}_{n,\,d}$. This fact can also be seen, in this case, by using the maximum principle and the fact that the distinguished boundary of $\mathrm{Z}_{n,\,d}$ is $\mathbf{R}^{dn}$. These tools are not available in the de Sitterian case, but, as mentioned before, a theorem of V. Glaser, stated below, can be used in conjunction with the positivity and weak spectral conditions, to prove:

Theorem 4 *There exists, for each $n \geq 1$, a function Φ_n holomorphic in $\mathcal{Z}_n$ with values in $\mathcal{H}$ such that $\Phi_n^{(b)}$ is the boundary value of Φ_n in the sense of distributions and of the Hilbert space topology.*

Theorem 4 implies the Reeh-Schlieder property:

Theorem 5 *(Reeh-Schlieder) For every open subset $\mathcal{O}$ of X_d, the vacuum is cyclic for the algebra of all field polynomials localized in $\mathcal{O}$.*

Proof. For every $\Psi \in \mathcal{H}$ and every $n \geq 1$, the distribution $(\Psi,\ \Phi_n^{(b)})$ is the boundary value of the function $z \mapsto (\Psi,\ \Phi_n(z))$, holomorphic in $\mathcal{Z}_n$. If $\mathcal{O}$ is an open subset of X_d such that $(\Psi,\ \langle \Phi_n^{(b)},\ \varphi \rangle)$ vanishes for every $\varphi \in \mathcal{D}(\mathcal{O}^n)$ then it vanishes for all $\varphi \in \mathcal{D}(X^d)$ by analytic continuation, and since the vector space $\mathcal{P}(X_d)\Omega$ is dense in $\mathcal{H}$, this implies that $\Psi = 0$. Therefore the vector space generated by $\{\langle \Phi_n^{(b)},\ \varphi \rangle\ :\ \varphi \in \mathcal{D}(\mathcal{O}^n),\ n \in \mathbf{N}\}$ is dense in $\mathcal{H}$.

For proving theorem 4 we shall make use of the following immediate consequence of the weak spectral condition

Proposition 4 *For each pair of integers (m,n), the function $(w,\ z) \mapsto \mathrm{W}_{m+n}(w,\ z)$, ($w \in X_d^{(c)m}$, $z \in X_d^{(c)n}$), is holomorphic in the corresponding topological product $\mathcal{Z}'_m \times \mathcal{Z}_n$.*

We are now in a position to apply the following theorem proved by V. Glaser in [17] (see also a re-statement in [18]). We suppose given a finite sequence of non-empty domains $U_n \subset \mathbf{C}^{N_n}$, $1 \le n \le M$, where the N_n are integers and $N_n \ge 1$. We set $N_0 = 0$, i.e. U_0 can be considered as consisting of a single point. U_n^* will denote the complex conjugate domain of U_n. For $n \ge 1$, λ_n denotes the Lebesgue measure in $\mathbf{C}^{N_n} \equiv \mathbf{R}^{2N_n}$.

Glaser's theorem 1 *For each pair of integers $(n,\ m)$ with $0 \le n,\ m \le M$, let $(p_n,\ q_m) \mapsto A_{n\,m}(p_n,\ q_m)$ be a holomorphic function on $U_n \times U_m^*$. (In particular $A_{0\,0}$ is just a complex number.) Then the following properties are equivalent:*
(G.0) For each $n \in [1,\ M]$, there is an open neighborhood V_n of 0 in $\mathbf{R}^{N_n}$ and a point $p_n \in U_n$ such that $p_n + V_n \subset U_n$ and, for each sequence $\{f_n\}_{0 \le n \le M}$, $f_0 \in \mathbf{C}$, $f_n \in \mathcal{D}(V_n)$ for $n > 0$,

$$\sum_{0 \le n,\ m \le M} \int_{\mathbf{R}^{N_n} \times \mathbf{R}^{N_m}} A_{n\,m}(p_n + h_n,\ \bar{p}_m + k_m)\, \bar{f}_n(h_n) f_m(k_m) dh_n\, dk_m \ge 0 \tag{94}$$

(with an obvious meaning when n or m is equal to 0).
(G.1) For every sequence $\{g_n\}_{0 \le n \le M}$, $g_0 \in \mathbf{C}$, $g_n \in \mathcal{D}(U_n)$ for $n > 0$,

$$\sum_{0 \le n,\ m \le M} \int_{U_n \times U_m} A_{n\,m}(p_n,\ \bar{q}_m)\, \bar{g}_n(p_n)\, g_m(q_m)\, d\lambda_n(p_n) d\lambda_m(q_m) \ge 0. \tag{95}$$

(G'.1) For each $n \in [1,\ M]$, there is an open subset ω_n of U_n such that for every sequence $\{g_n\}_{0 \le n \le M}$, $g_0 \in \mathbf{C}$, $g_n \in \mathcal{D}(\omega_n)$ for $n > 0$,

$$\sum_{0 \le n,\ m \le M} \int_{\omega_n \times \omega_m} A_{n\,m}(p_n,\ \bar{q}_m)\, \bar{g}_n(p_n)\, g_m(q_m)\, d\lambda_n(p_n) d\lambda_m(q_m) \ge 0. \tag{96}$$

(G.2) There is a sequence $\{f_{\nu,\ 0}\}_{\nu \in \mathbf{N}} \in \mathbf{C}$ and, for each $n \in [1,\ M]$, a sequence $\{f_{\nu,\ n}\}_{\nu \in \mathbf{N}}$ of functions holomorphic in U_n, such that

$$A_{n\,m}(p_n,\ q_m) = \sum_{\nu \in \mathbf{N}} f_{\nu,\ n}(p_n)\, \overline{f_{\nu,\ m}(\bar{q}_m)} \tag{97}$$

holds in the sense of uniform convergence on every compact subset of $U_n \times U_m^*$, again with an obvious meaning when n or m is equal to 0.
(G.3) For every sequence $\{p_n \in U_n\}_{1 \le n \le M}$, and every finite sequence $\{a(n)_\alpha\}$ of complex numbers,

$$Q_p(a,\ a) = \sum_{n,\ m} \sum_{\alpha,\ \beta} \frac{a(n)_\alpha\, \bar{a}(m)_\beta}{\alpha!\, \beta!}\, \partial_{p_n}^{\alpha}\, \partial_{\bar{p}_m}^{\beta}\, A_{n\,m}(p,\ \bar{p}) \ge 0. \tag{98}$$

(G.4) There is a particular sequence $\{p_n \in U_n\}_{1 \le n \le M}$ such that, for every finite sequence $\{a(n)_\alpha\}$ of complex numbers, $Q_p(a,\ a) \ge 0$.

The following striking theorem, also proved in [17] is mentioned here for completeness although it is not used in the proof of theorem 4:

Glaser's theorem 2 *Let U be a non-empty simply connected domain in $\mathbf{C}^N$ (with $N \ge 1$), and F a distribution over U, such that, for every finite sequence $\{a_\alpha\}$ of complex numbers indexed by N-multiindices,*

$$\sum_{\alpha,\ \beta} \frac{a_\alpha\, \bar{a}_\beta}{\alpha!\, \beta!}\, \partial^\alpha\, \bar{\partial}^\beta F \ge 0 \tag{99}$$

(in the sense of distributions). Then there is a function $(p,\ q) \mapsto A(p,\ q)$, holomorphic on $U \times U^*$ and possessing the properties (G.1)-(G.3) of Glaser's theorem 1 (in the case $M = 1$), such that F coincides with $p \mapsto A(p,\ \bar{p})$.

Remarks.
1) The statement of Glaser's theorem 1 does not literally coincide with the original in [17], but it follows from the proofs given there.
2) In the condition (G.1) one can equivalently require the g_n to be arbitrary complex measures with compact support contained in U_n. Since any measure can be weakly approximated by finite linear combinations of Dirac measures, the condition (G.1) is equivalent to
(G".1) For every finite sequence $\{(c_{n,l},\ t_{n,l})\ :\ c_{n,l} \in \mathbf{C},\ t_{n,l} \in U_n,\ 0 \le n \le M,\ 1 \le l \le L\}$,

$$\sum_{n,\ m=0}^{M} \sum_{l,\ k=1}^{L} c_{n,l}\overline{c_{m,k}} A_{n\ m}(t_{n,l},\ \overline{t_{m,k}}) \ge 0. \tag{100}$$

3) Apart from condition (G.0), the properties mentioned in these theorems are essentially invariant under holomorphic self-conjugated coordinate changes and in fact *the various U_n can be replaced by connected complex manifolds which are separable at infinity (i.e. are unions of increasing sequences of compacts)* as it can be seen from the sketch of the proof given in Appendix C.

Proof of Theorem 4
Taking into account the previous remark 3), we shall apply Glaser's theorem 1 to the case when each U_n is the domain $\mathcal{Z}_n$ of the corresponding manifold $X_d^{(c)n}$ and

$$A_{0\,0} = 1, \quad A_{n\,m}(z,\ w) = \mathrm{W}_{m+n}(w_{\leftarrow},\ z) = \mathrm{W}_{m+n}(w_m, \ldots, w_1,\ z_1, \ldots, z_n,) \tag{101}$$

with $n,\ m \in [0,\ M]$, M being any fixed integer.

In fact, in view of proposition 4 and of the remark that

$$\mathcal{Z}_m^* = \{w = (w_1, \ldots,\ w_m) \in X_d^{(c)m};\ w_{\leftarrow} = (w_m, \ldots,\ w_1) \in \mathcal{Z'}_m\}, \tag{102}$$

it follows that for all pairs of integers (n, m) the functions defined by Eq. (101) are holomorphic in the corresponding domains $\mathcal{Z}_n \times \mathcal{Z}_m^*$. Now our aim is to prove that, as a consequence of the positivity property (11), these functions possess the properties (G.0)-(G.4) of Glaser's theorem 1. Let a be a particular point of X_d (e.g. $a = (0, \ldots, 0,\ R)$). It is clear that we can define, for each $n \ge 1$, a holomorphic diffeomorphism σ_n of an open ball centered at 0 in $\mathbf{C}^{nd}$ onto a complex neighborhood $\mathcal{N}_n$ of $a_n = (a,\ a, \ldots,\ a)$ in $X_d^{(c)\,n}$ with the following properties:

1. σ_n is self-conjugate, i.e. $\sigma_n(\bar{z}) = \overline{\sigma_n(z)}$ for all z.
2. $\sigma_n(0) = a_n$
3. σ_n maps the "local tube"

$$\{z = x + iy \in \mathbf{C}^{nd}\ :\ |z_j| < 1,\ 0 < y_j,\ 1 \le j \le nd\} \tag{103}$$

into $\mathcal{Z}_n \cap \mathcal{N}_n$.

In $\sigma_n^{-1}(\mathcal{N}_n \cap \mathcal{Z}_n) \times \sigma_m^{-1}(\mathcal{N}_m \cap \mathcal{Z}_m^*)$, there holds (in view of the distribution character of the boundary values of the $\mathcal{A}_{n\ m}$ on X_d^{n+m}):

$$|A_{n\,m}(\sigma_n(z),\ \sigma_m(z'))| \le K(\sum_j |\mathrm{Im}\, z_j|^{-r} + \sum_j |\mathrm{Im}\, z'_j|^{-r}), \tag{104}$$

where $K > 0$ and $r \ge 0$ may be taken independent of $n,\ m \in [1,\ M]$.

By composing σ_n with $z_j = \mathrm{th}\,(\zeta_j/2)$, we obtain a self-conjugate holomorphic diffeomorphism τ_n of the tube

$$\{\zeta = \xi + i\eta \in \mathbf{C}^{nd}\ :\ |\eta_j| < \pi/2,\ 1 \le j \le nd\} \tag{105}$$

onto a complex neighborhood of a_n in $X_d^{(c)\,n}$ such that $\tau_n(0) = a_n$ and the image of the tube

$$\Theta_n = \{\zeta = \xi + i\eta \in \mathbf{C}^{nd}\ :\ 0 < \eta_j < \pi/2,\ 1 \le j \le nd\}. \tag{106}$$

is contained in $\mathcal{Z}_n$. Let

$$B_{n\,m}(\zeta,\ \zeta') = A_{n\,m}(\tau_n(\zeta),\ \tau_m(\zeta')). \tag{107}$$

The functions $B_{n\,m}$ are holomorphic in $\Theta_n \times \Theta_m^*$. Since for $\zeta = \xi + i\eta \in \mathbf{C}$,

$$\mathrm{th}\,(\zeta/2) = \frac{\mathrm{sh}\,\xi + i\sin\eta}{2|\mathrm{ch}\,(\zeta/2)|^2}, \tag{108}$$

the $B_{n\,m}$ satisfy

$$|B_{n\,m}(\zeta,\ \zeta')| \le K' \sum_j \left(\frac{\mathrm{e}^{|\xi_j|}}{|\sin\eta_j|}\right)^r + K' \sum_j \left(\frac{\mathrm{e}^{|\xi'_j|}}{|\sin\eta'_j|}\right)^r, \quad \forall\zeta = \xi + i\eta \in \Theta_n,\ \zeta' = \xi' + i\eta' \in \Theta_m^*. \tag{109}$$

They have boundary values $B_{n\,m}^{(v)}$ in the sense of generalized functions over test-functions of faster than exponential decrease. These boundary values satisfy, for each finite sequence $\{f_n\}$, $f_0 \in \mathbf{C}$, $f_n \in \mathcal{D}(\mathbf{R}^{nd})$ for $n \ge 1$,

$$\sum_{n,\ m} \int B_{n\,m}^{(v)}(\xi,\ \xi') f_n(\xi)\, \overline{f_m(\xi')}\, d\xi\, d\xi' \ge 0. \tag{110}$$

Let now

$$\rho_{n,\ \varepsilon}(\xi) = C(\varepsilon)\ \exp\left(-\sum_{j=1}^{nd} (\xi_j^2/\varepsilon)\right), \tag{111}$$

where $C(\varepsilon)$ is chosen so that $\int \rho_{n,\ \varepsilon}(\xi)\, d\xi = 1$. For each $\mu \in \mathbf{C}^{nd}$, the function $\xi \mapsto \rho_{n,\ \varepsilon}(\xi + \mu)$ is of gaussian decrease, and depends holomorphically on μ. In particular if $\mu_n \in \Theta_n$, $\mu'_m \in \Theta_m^*$,

$$\begin{aligned}&\int_{\mathbf{R}^{nd}\times\mathbf{R}^{md}} B_{n\,m}^{(v)}(t,\ t')\, f_n(\xi)\, \rho_{n,\ \varepsilon}(t - \mu_n - \xi)\, \overline{f_m(\xi')\rho_{m,\ \varepsilon}(t' - \bar{\mu}'_m - \xi')}\, dt\, dt'\, d\xi\, d\xi' \\ &= \int_{\mathbf{R}^{nd}\times\mathbf{R}^{md}} B_{n\,m}(t + \mu_n,\ t' + \mu'_m)\, f_n(\xi)\, \rho_{n,\ \varepsilon}(t - \xi)\, \overline{f_m(\xi')\rho_{m,\ \varepsilon}(t' - \xi')}\, dt\, dt'\, d\xi\, d\xi',\end{aligned} \tag{112}$$

since both sides define analytic functions in $\Theta_n \times \Theta_m^*$ whose boundary values for real μ_n, μ'_m coincide. The lhs satisfies the positivity conditions, by virtue of Eq. (110), if we chose $\mu'_n = \bar{\mu}_n$ for all n. It follows, by letting ε tend to 0 in the rhs, that the functions $B_{n\,m}$ have the property (G.0) of Glaser's theorem 1 and therefore all the properties (G.0)-(G.4) in the sequence of domains $\{\Theta_n\}$. Coming back to the original variables, Glaser's theorem 1 now shows that the same properties, in particular (G.2), extend to the entire tuboid $\{\mathcal{Z}_n\}$. We have thus proved the following

Proposition 5 *For any integer $M \ge 1$, there exist a sequence $\{F_{\nu,\ 0} \in \mathbf{C}\}_{\nu\in\mathbf{N}}$ and, for each integer $n \in [1,\ M]$, a sequence $\{F_{\nu,\ n}\}_{\nu\in\mathbf{N}}$ of functions holomorphic in $\mathcal{Z}_n$, such that, for every n and m in $[1,\ M]$, $z \in \mathcal{Z}_n$, $w \in \mathcal{Z}_m^*$,*

$$\mathrm{W}_{m+n}(w_{\leftarrow},\ z) = \sum_{\nu\in\mathbf{N}} \overline{F_{\nu,\ m}(\bar{w})}\, F_{\nu,\ n}(z), \tag{113}$$

where the convergence is uniform on every compact subset of $\mathcal{Z}_m^* \times \mathcal{Z}_n$.

In particular

$$\mathrm{W}_{2n}(\bar{z}_{\leftarrow},\ z) = \sum_{\nu \in \mathbf{N}} |F_{\nu,\ n}(z)|^2, \tag{114}$$

(so that if the temperedness condition holds, each $F_{\nu,\ n}$ has polynomial behavior at infinity and near the reals).

Let now $\{\varphi_m\}_{1\leq m\leq M}$ be a sequence of test-functions, $\varphi_m \in \mathcal{D}(X_d^m)$, $\varphi_0 \in \mathbf{C}$. We continue to denote φ_m a $\mathcal{C}^\infty$ extension of φ_m with compact support over $X_d^{(c)\,m}$. Let $\mathcal{C}(m,\ \varepsilon)$ be, for each $m \in [1,\ M]$ and $\varepsilon \geq 0$, an (md)-cycle, contained in $\mathcal{Z}_m$ for $\varepsilon > 0$, equal to X_d^m for $\varepsilon = 0$, and continuously depending on ε. Using proposition 5 and Schwarz's inequality, we find, for any $z \in \mathcal{Z}_n$ (n being fixed and M chosen arbitrarily such that $n \leq M$),

$$\begin{aligned}
&\left|\sum_{0\leq m\leq M} \int_{\mathcal{C}(m,\ \varepsilon)} \overline{\varphi_m(w)}\, \mathrm{W}_{m+n}(\bar{w}_{\leftarrow},\ z)\, d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_m\right|^2 \\
&= \left|\textstyle\sum_{\nu\in\mathbf{N}} [\sum_{0\leq m\leq M} \int_{\mathcal{C}(m,\ \varepsilon)} \overline{\varphi_m(w)}\, \overline{F_{\nu,\ m}(w)} d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_m] F_{\nu,\ n}(z)\right|^2 \\
&\leq \sum_{\nu\in\mathbf{N}} |F_{\nu,\ n}(z)|^2 \times \\
&\sum_{0\leq m,\ k\leq M} \int_{\mathcal{C}(m,\ \varepsilon)\times\mathcal{C}(k,\ \varepsilon)} \overline{\varphi_m(w)}\, \varphi_k(w')\, \mathrm{W}_{m+k}(\bar{w}_{\leftarrow},\ w')\, d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_m \wedge dw'_1 \wedge \ldots \wedge dw'_k.
\end{aligned} \tag{115}$$

Taking Eq. (114) into account and letting ε tend to 0 then yield:

$$\begin{aligned}
&\left|\sum_{0\leq m\leq M} \int_{X_d^m} \overline{\varphi_m(w)}\, \mathrm{W}_{m+n}(w_{\leftarrow},\ z)\, dw_1 \ldots dw_m\right|^2 \\
&\qquad\qquad \leq \mathrm{W}_{2n}(\bar{z}_{\leftarrow},\ z) \left\|\sum_{0\leq m\leq M} \int_{X_d} \Phi_m^{(b)}(w)\, \varphi_m(w)\, dw\right\|^2 .
\end{aligned} \tag{116}$$

Since the latter holds for any (arbitrarily large) value of M, namely for a dense set of vectors $\mathbf{\Phi}(\varphi)\ \Omega$ in $\mathcal{H}$, this shows that for every n there is a vector $\Phi_n(z) \in \mathcal{H}$ such that

$$\left(\int_{X_d} \Phi_m^{(b)}(w)\, \varphi_m(w)\, dw,\ \Phi_n(z)\right) = \int_{X_d} \mathrm{W}_{m+n}(w_{\leftarrow},\ z)\overline{\varphi_m(w)}\, dw. \tag{117}$$

Integrating similarly in z over a cycle such as $\mathcal{C}(n,\ \varepsilon)$, and letting ε tend to 0 show that Φ_n admits $\Phi_n^{(b)}$ as its boundary value in the sense of distributions and theorem 4 follows.

Remarks
1. This proof is valid for some other spaces besides de Sitter space. What is really used is that the space is real-analytic and that the Wightman distributions are boundary values of functions $\mathrm{W}_{m+n}(w_{\leftarrow},\ z)$ holomorphic in products of the form $U_m^* \times U_n$, where the U_n are connected complex tuboids.
2. Neither temperedness nor locality have been used.
3. By using the PCT property, the BW analyticity and the Reeh-Schlieder property it is possible to restate the full Bisognano-Wichmann theorem in the de Sitter case. We do not give here the details.

A Appendix. A lemma of analytic completion

In this appendix we prove a simple lemma of analytic completion by applying the convex tube theorem, according to which any function which is holomorphic in a tube $\mathbf{R}^n + iB$, where B is a domain in $\mathbf{R}^n$, can be analytically continued in the convex hull of this tube. (See [4, 35, 15, 14]). $\mathbf{C}_+$ denotes the upper half-plane.

Lemma 3 *(i) Let*

$$P = \{z \in \mathbf{C}^N \ : \ |z_j| < 1, \ \mathrm{Im}\, z_j > 0, \ \forall j = 1, \dots, \ N\} \tag{118}$$

Let D be a domain in $\mathbf{C}^N$, containing P, and Ω a domain in $\mathbf{C} \times \mathbf{C}^N$ of the form

$$\Omega = \mathcal{N} \cap (\mathbf{C}_+ \times D), \tag{119}$$

where $\mathcal{N}$ is an open neighborhood, in $\mathbf{C}^{1+N}$, of the set

$$\big((\mathbf{R} \setminus \{0\}) \times D\big) \cup \big((\overline{\mathbf{C}_+} \setminus \{0\}) \times \{z \in \mathbf{C}^N \ : \ |z_j| < 1, \ \mathrm{Im}\, z_j = 0, \ \forall j = 1, \dots, \ N\}\big). \tag{120}$$

Then any function holomorphic in Ω has a holomorphic extension in $\mathbf{C}_+ \times D$.
(ii) Let D' be a domain in $\mathbf{C}^N$, containing

$$Q = \{z \in \mathbf{C}^N \ : \ |z_j| < 1, \ \forall j = 1, \dots, \ N\}, \tag{121}$$

and Ω' a domain in $\mathbf{C} \times \mathbf{C}^N$ of the form

$$\Omega' = (\mathbf{C}_+ \times Q) \cup (\mathcal{N}' \cap (\mathbf{C}_+ \times D')), \tag{122}$$

where $\mathcal{N}'$ is an open neighborhood, in $\mathbf{C}^{1+N}$, of $(\mathbf{R} \setminus \{0\}) \times D'$. Then any function holomorphic in Ω' has a holomorphic extension in $\mathbf{C}_+ \times D'$.

Remark 5 *By setting $w = \mathrm{e}^{\pi\sigma}$ the upper half-plane can be replaced by the strip $\{\sigma \ : \ 0 < \mathrm{Im}\, \sigma < 1\}$, and $\mathbf{R} \setminus \{0\}$ by the boundary of that strip.*

1. We start by proving Lemma 3 (i) for the case when $D = P$. This follows from:

Lemma 4 *Let $a \in (0, \ 1)$ and Δ'_a a domain in $\mathbf{C} \times \mathbf{C}^N$ of the form $\mathcal{V} \cap (\mathbf{C}_+ \times P)$, where $\mathcal{V}$ is an open neighborhood in $\mathbf{C}^{1+N}$ of*

$$\begin{aligned} &\{(w, \ z) \in \mathbf{C}^{1+N} \ : \ w \in \mathbf{R} \ : \ a < |w| < 1/a\}, \quad z \in P\} \ \cup \\ &\quad \{(w, \ z) \in \mathbf{C}^{1+N} \ : \ w \in \mathbf{C}_+ \cup (-1/a, \ -a) \cup (a, \ 1/a), \ \ |z_j| < 1, \ \mathrm{Im}\, z_j = 0, \ \forall j = 1, \dots, \ N\}. \end{aligned} \tag{123}$$

Then any function f holomorphic in Δ'_a has a holomorphic extension in the domain

$$\Delta_a = \bigcup_{0<\theta<\pi} W_a(\theta) \times Z(\theta), \tag{124}$$

where:

$$Z(\theta) = \{z \in \mathbf{C}^N \ : \ \forall j = 1, \dots N, \ \ \mathrm{Im}\, z_j > 0, \ 2\,\mathrm{Im}\, \log\left(\frac{1+z_j}{1-z_j}\right) < \theta \ \}, \tag{125}$$

$$W_a(\theta) = \{w \in \mathbf{C} \ : \ 0 < \mathrm{Im}\, w, \ 0 < \mathrm{Im}\, \Phi(w, \ a) < \pi - \theta \ \}, \tag{126}$$

$$\Phi(w, \ a) = i\pi - \log\left(\frac{w - a^{-1}}{w - a}\right) \ - \ \log\left(\frac{w + a}{w + a^{-1}}\right), \quad (\mathrm{Im}\, w \neq 0). \tag{127}$$

Remark 6 The function $w \mapsto \mathrm{Im}\, \Phi(w, \ a)$ is the bounded harmonic function in the upper half-plane with boundary values equal to 0 on the real segments $(-a^{-1}, \ -a)$ and $(a, \ a^{-1})$, and to π on the other real points. $\pi - \mathrm{Im}\, \Phi(w, \ a)$ is the sum of the angles under which these two segments are seen from the point w.

Proof. We shall make use (at several places and in several complex variables) of the following conformal map. For $A > 0$ and $B > 0$, we denote $L(A,\ B)$ the open lunule in the W-plane bounded by the real segment $[-A,\ A]$ and the circular arc going through the points $-A$, iB, and A. This domain is conformally mapped onto the strip $\{\lambda \in \mathbf{C}\ :\ 0 < \mathrm{Im}\,\lambda < 2\mathrm{Arctg}\,(B/A)\}$ by the map

$$W \mapsto \lambda = \log\left(\frac{A+W}{A-W}\right), \tag{128}$$

whose inverse is

$$\lambda \mapsto W = A\,\mathrm{th}\,(\lambda/2). \tag{129}$$

Both the hypotheses of Lemma 4 and the function Φ are left invariant by the transformation $w \mapsto -1/w$. In fact, denoting $b = a^{-1}$ and

$$\mu(w) = w - 1/w \tag{130}$$

we have

$$\frac{(w-b)(w+a)}{(w-a)(w+b)} \quad = \quad \frac{\mu-(b-a)}{\mu+(b-a)} \quad \equiv -1/\varphi(\mu), \tag{131}$$

$$\Phi(w,\ a) = \log\,\varphi(\mu). \tag{132}$$

A function f holomorphic in Δ'_a can be rewritten in the form:

$$f(w,\ z) = f_s(w,\ z)\ +\ (w+w^{-1})f_a(w,\ z), \tag{133}$$

where

$$f_s(w,\ z) = \frac{1}{2}(f(w,\ z)+f(-w^{-1},\ z)), \quad f_a(w,\ z) = \frac{1}{2(w+w^{-1})}(f(w,\ z)-f(-w^{-1},\ z)). \tag{134}$$

Both f_s and f_a are easily seen to have the properties postulated for f itself, and they are moreover invariant under the transformation $w \mapsto -w^{-1}$. They can therefore be written as holomorphic functions $F_{s,a}(\mu,\ z)$ of $\mu = w - w^{-1}$ and z. We now perform the change of coordinates

$$\mu \mapsto \omega = \log\varphi(\mu), \quad z_j \mapsto \zeta_j = 2\,\log\,((1+z_j)/(1-z_j)), \tag{135}$$

i.e. define

$$G_{s,a}(\omega,\ \zeta) = F_{s,a}(\mu,\ z) \ \ \text{with} \tag{136}$$

$$\mu = (b-a)\mathrm{th}\,(\omega/2), \tag{137}$$

$$z_j = \mathrm{th}\,(\zeta_j/4). \tag{138}$$

The functions $G_{s,a}$ are holomorphic in a domain of the following form, which is the image of the domain Δ'_a into the space of variables (ω, ζ) (after taking the successive maps given in Eqs. (130),(131) and (135) into account):

$$U_1 = \mathcal{V}_1 \cap \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ : 0 < \mathrm{Im}\,\omega < \pi, \ \ 0 < \mathrm{Im}\,\zeta_j < \pi\}, \tag{139}$$

where $\mathcal{V}_1$ is an open neighborhood, in $\mathbf{C}^{1+N}$, of the set

$$\begin{aligned} S_1 &= \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in \mathbf{R},\ \ 0 < \mathrm{Im}\,\zeta_j < \pi\} \cup \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ 0 \le \mathrm{Im}\,\omega < \pi,\ \ \mathrm{Im}\,\zeta_j = 0\} \\ &= \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in \mathbf{R},\ \ 0 \le \mathrm{Im}\,\zeta_j < \pi\} \cup \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ 0 \le \mathrm{Im}\,\omega < \pi,\ \ \mathrm{Im}\,\zeta_j = 0\}. \end{aligned} \tag{140}$$

The domain U_1 of Eq. (139) is not a tube. We can however inscribe in it increasing unions of topological products of lunules which are isomorphic to tubes. In fact, for every $A > \frac{1}{\pi}$, there exists an $\varepsilon > 0$ such that U_1 contains

$$\begin{aligned} V_A = &\{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in L(A,\ \varepsilon),\ \ \zeta_j \in L(A,\ \pi - 1/A)\} \\ &\cup \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in L(A,\ \pi - 1/A),\ \ \zeta_j \in L(A,\ \varepsilon)\}. \end{aligned} \tag{141}$$

Using the conformal map (128) in all variables, we can map V_A into a tube whose holomorphy envelope is its convex hull. Returning to the variables $(\omega,\ \zeta)$, and taking the limit $A \to \infty$ shows that the functions $G_{s,a}$ are holomorphic in the interior of the convex hull of S_1, namely

$$\bigcup_{0<\theta<\pi} \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ 0 < \mathrm{Im}\,\omega < \pi - \theta,\ \ 0 < \mathrm{Im}\,\zeta_j < \theta,\ \forall j\}. \tag{142}$$

This set is the image of the domain Δ_a introduced in Eq. (124) under the mapping $w \mapsto \mu = w - w^{-1} \mapsto \omega$, $z \mapsto \zeta$ defined in Eq. (135), and therefore the assertion of Lemma 4 follows.

Lemma 3 (i) in the special case $D = P$ follows from the latter by letting a tend to 0.
2. We now prove Lemma 3 (ii) in the case when $D' = \rho\, Q$ for some real $\rho > 1$. The proof of this is the same as that of Lemma 4, except that the change of coordinates (138) is replaced by

$$z_j = \exp(i\zeta_j), \quad (1 \le j \le N). \tag{143}$$

This again allows the use of the tube theorem.
3. Lemma 3 (ii) follows from this by using chains of polydisks, and (i) follows in the same way from the special case $D = P$ and (ii).

B Appendix. A lemma of Hall and Wightman

In [23], Hall and Wightman prove the following lemma

Lemma 5 *Let $M \in L_+(\mathbf{C})$ be such that $\mathrm{T}_+ \cap M^{-1}\,\mathrm{T}_+ \neq \emptyset$. There exists a continuous path $t \mapsto M(t)$ from the interval $[0,\ 1]$ into $L_+(\mathbf{C})$ such that $M(0) = 1$, $M(1) = M$ and that, for every $z \in \mathrm{T}_+ \cap M^{-1}\,\mathrm{T}_+ \subset \mathbf{C}^{d+1}$, $M(t)z \in \mathrm{T}_+$ holds for all $t \in [0,\ 1]$.*

This lemma is proved in [23] for the case $d+1 \le 4$ (a very clear exposition also appears in [32]). It is extended to all dimensions in [24]. We give another proof based on holomorphic continuation. As noted in the above references, if $M \in L_+(\mathbf{C})$ is such that the statement in Lemma 5 holds, then it holds for $\Lambda_1 M \Lambda_2$ for any $\Lambda_1,\ \Lambda_2 \in L_+^\uparrow$, as well as for M^{-1}. It is therefore sufficient to consider the case when M is one of the normal forms classified by Jost in [24]. M can then be written in the form:

$$M = \begin{pmatrix} M_1(i) & 0 \\ 0 & M_2(i) \end{pmatrix} \tag{144}$$

where $t \mapsto M_1(t)$ is a one-parameter subgroup of the $p \times p$ Lorentz group, real for real t, with $p \le 3$, and $t \mapsto M_2(t)$ is a one-parameter subgroup of the $(d+1-p) \times (d+1-p)$ orthogonal group, real for real t. In the generic case $p \le 2$, $M_1(t) = 1$ if $p = 1$ and, if $p = 2$, $M_1(t) = [\exp at]$ for some real a with $|a| \le \pi$. We focus on this case first. Replacing M by M^{-1} if necessary, we may assume $0 < a \le \pi$. For any $z \in \mathrm{T}_+$ the set $\Delta(z,\ M) = \{t \in \mathbf{C}\ :\ M(t)z \in \mathrm{T}_+\}$ is invariant under real translations, i.e. is a union of open strips parallel to the real axis. Let

$$E(M) = \{\mathrm{T}_+ \cap M^{-1}\mathrm{T}_+\} = \{z \in \mathbf{C}^{d+1}\ :\ \mathbf{R} \cup (i + \mathbf{R}) \subset \Delta(z,\ M)\}.$$

Denote $z(s) = (z^{(0)},\ z^{(1)},\ sz^{(2)},\ \ldots,\ sz^{(d)})$. If $z \in E(M)$, then $z(s) \in E(M)$ for all $s \in [0,\ 1]$. The set $\Delta(z(0),\ M)$ contains the segment $i[0,\ 1]$, and hence $i[0,\ 1] \subset \Delta(z',\ M)$ for all z' in a sufficiently small neighborhood $\mathcal{N}$ of $z(0)$. For $n \in \overline{V_+} \setminus \{0\}$ and $b \in \mathbf{C}$, with $\mathrm{Im}\, b \ge 0$, the function $(t,\ z) \mapsto H_{n,\ b}(t,\ z) = (n \cdot M(t)z + b)^{-1}$ is holomorphic in $\{(t,\ z)\ :\ z \in E(M),\ t \in \Delta(z,\ M)\}$. Applying Lemma 3ii), with w replaced by the variable σ of Remark 5, it follows that $H_{n,\ b}$ is holomorphic in $\{t\ :\ 0 < \mathrm{Im}\, t < 1\} \times E(M)$. Let us now assume that for some $z \in E(M)$ and some $t \in i[0,\ 1]$ the corresponding point $\zeta = M(t)z$ belongs to the complement of T_+; then, as explained below, one can determine n and b satisfying the previous conditions and such that $n \cdot \zeta + b = 0$, which therefore contradicts the previously proved analyticity property of the corresponding function $H_{n,\ b}$. In fact, for any complex point $\zeta = \xi + i\eta$ in the complement of T_+ (i.e. $\eta \notin V_+$), one can find $n \in \mathbf{R}^{d+1}$ and $c \in \mathbf{R}$ such that $n \cdot \eta + c = 0$, while $n \cdot r + c > 0$ for all $r \in V_+$. This implies $n \in \overline{V_+} \setminus \{0\}$ and $c \geq 0$. Hence there is a $b \in \mathbf{C}$ with $\mathrm{Im}\, b = c \geq 0$ such that $n \cdot \zeta + b = 0$ (while $\mathrm{Im}\,(n \cdot q + b) > 0$ for all $q \in \mathrm{T}_+$). This proves Lemma 5 for all dimensions.

C Appendix. Sketch of the proof of Glaser's theorem 1

This section closely follows the original [17] with a few unimportant alterations, mainly intended for the cases when U_n might not be simply connected. The notations are those of Glaser's theorem 1, and we also denote $\widetilde{U}_n$ the universal covering space of U_n, ι_n the canonical projection of $\widetilde{U}_n$ onto U_n. If $\mathcal{V}$ is a complex manifold, $\mathcal{A}(\mathcal{V})$ denotes the set of holomorphic functions on $\mathcal{V}$. It is clear that (G.1) $\Rightarrow$ (G'.1). The latter implies (G.0), by inserting $g_n(z_n) = f(z_n - p_n)\delta(\mathrm{Im}\,(z_n - p_n))$ in G'.1. In turn (G.0) implies (G.4) by inserting $f_n(h_n) = \sum_\alpha a_\alpha(n)\partial^\alpha\delta(h_n)$ in (G.0).

1. The first step of the proof is to show that (G.4) $\Rightarrow$ (G.3). Assume that (G.4) holds. Let, for each $n \in [1,\ M]$, $R_n > 0$ be such that the closure of the polydisk $P_n = \{z_n \in \mathbf{C}^{N_n}\ :\ |z_{n,j} - p_{n,j}| < R_n \forall j\}$ is contained in U_n. For any $\{z_n \in P_n\}_{1 \leq n \leq M}$ and every finite sequence $b(n)_\alpha$,

$$Q_z(b,\ b) = Q_p(a,\ a), \tag{145}$$

$$a(n)_\alpha = \alpha! \sum_{\gamma \leq \alpha} \frac{b_\gamma (z_n - p_n)^{\alpha - \gamma}}{\gamma!\,(\alpha - \gamma)!} \tag{146}$$

Although the sequence $\{a(n)_\alpha\}_{\alpha \in \mathbf{N}^{N_n}}$ is infinite, the convergence of the power series for $Q_z(b,\ b)$ and a limiting argument show that $Q_z(b,\ b) \geq 0$. Thus the property (G.4) propagates everywhere, i.e. (G.3) holds.

2. As our next step, we prove that (G.4) implies that (G'.1) holds within the same sequence of polydisks $\{P_n\}_{n \in [1,\ M]}$ just used. We first prove this in the form of condition (G".1), i.e. in case g_n is a finite linear combination of Dirac measures, i.e.

$$g_n(z_n) = \sum_{r=1}^{L} c_{n,r}\delta(z_n - t_{n,r}), \tag{147}$$

with $t_{n,r} \in P_n$. Indeed

$$\sum_{n,\ m=0}^{M} \sum_{r,\ s=1}^{L} c_{n,r}\overline{c_{m,s}} A_{n\ m}(t_{n,r},\ \overline{t_{m,s}}) = Q_p(a,\ a), \tag{148}$$

with

$$a(n)_\alpha = \sum_{r=1}^{L} c_{n,r}(t_{n,r} - p_n)^\alpha. \tag{149}$$

The sequence $\{a(n)_\alpha\}$ is again infinite, but we can still conclude that $Q_p(a,\ a) \geq 0$, i.e. that (G".1), and hence (G'.1) hold in the sequence of domains $\{P_n\}_{n \in [1,\ M]}$.

3. To prove that (G'.1) in the sequence of polydisks $\{P_n\}_{n \in [1,\ M]}$ implies the property (G.2) within the same sequence, we introduce a Hilbert space $E = E_0 \oplus \ldots E_M$ as follows: $E_0 = \mathbf{C}$. For each $n \in [1,\ M]$, $E_n = \mathcal{A}(U_n) \cap L^2(P_n,\ \lambda_n)$. It is well-known (see [B]) that E_n is a closed subspace of $L^2(P_n,\ \lambda_n)$, and that the convergence of a sequence $\{\psi_\nu \in E_n\}_{\nu \in \mathbf{N}}$ in the sense of E_n implies its uniform convergence on every compact subset of P_n The operator A_P defined on E by

$$(g,\ A_P\, f) = \sum_{m,\ n=0}^{M} \int_{P_n \times P_m} \overline{g_n(p_n)}\, A_{n\ m}(p_n,\ \bar{q}_m)\, f_m(q_m)\, d\lambda_n(p_n)\, d\lambda_m(q_m) \tag{150}$$

is Hilbert-Schmidt and positive by virtue of the property (G'.1). The spectral decomposition of this operator therefore yields the existence of a sequence $\{\varphi_\nu = (\varphi_{\nu,\, 0}, \ldots,\, \varphi_{\nu,\, M}) \;:\; \nu \in \mathbf{N}\}$ of eigenvectors of A_P corresponding to non-negative eigenvalues. Hence there is a sequence $\{f_\nu \in E\}$ such that

$$A_{n\, m}(p_n,\; \bar{q}_m) = \sum_{\nu \in \mathbf{N}} f_{\nu,\; n}(p_n)\, \overline{f_{\nu,\; m}(q_m)} \tag{151}$$

holds for all $n,\; m \in [0,\; M]$, uniformly on every compact subset of $P_n \times P_m$.
4. We now show that for each n and ν, $f_{\nu,\; n}$ extends to a function holomorphic on $\widetilde{U}_n$. Let $z_n \in P_n$ and suppose that the closure of a polydisk P'_n with radius R' centered at z_n is contained in U_n. The Taylor coefficients of $A_{n\; n}$ at z_n satisfy

$$\frac{\partial^\alpha\, \bar{\partial}^\alpha}{\alpha!^2} A_{n\; n}(z_n,\; \bar{z}_n) = \sum_\nu \left| \frac{\partial^\alpha f_{\nu,\; n}(z_n)}{\alpha!} \right|^2 \leq C\, R'^{2|\alpha|}. \tag{152}$$

Hence the power series for $f_{\nu,\; n}$ at (z_n) converges in P'_n. Moreover the expansion (151) continues to hold in $P'_n \times P'_m$. To see this it suffices, by Schwartz's inequality, to prove that $\sum_\nu |f_{\nu,\; n}(w_n)|^2$ converges in P'_n. We fix $n \geq 1$ and temporarily denote $g_{\nu,\; \alpha} = \partial^\alpha f_{\nu,\; n}(z_n)/\alpha!$. For any $\kappa \in (0,\; 1)$, by Schwartz's inequality,

$$\left| \sum_\alpha |g_{\nu,\; \alpha}| (\kappa^2 R')^{|\alpha|} \right|^2 \leq (1-\kappa^2)^{-N_n} \;\sum_\alpha |g_{\nu,\; \alpha}|^2 (\kappa^2 R'^2)^{|\alpha|}, \tag{153}$$

and by Eq. (152),

$$\sum_\nu \left| \sum_\alpha |g_{\nu,\; \alpha}| (\kappa^2 R')^{|\alpha|} \right|^2 \leq C(1-\kappa^2)^{-2N_n}. \tag{154}$$

In particular for any $\varepsilon > 0$, it is possible to chose S such that

$$\sum_{\nu \geq S} \left| \sum_\alpha |g_{\nu,\; \alpha}| (\kappa^2 R')^{|\alpha|} \right|^2 \leq \varepsilon \tag{155}$$

so that, for any ζ with $|\zeta| < \kappa^2 R'$,

$$\sum_{\nu \geq S} |f_{\nu,\; n}(z_n + \zeta)|^2 \leq \varepsilon, \tag{156}$$

$$\sum_\nu |f_{\nu,\; n}(z_n + \zeta)|^2 \leq C(1-\kappa^2)^{-2N_n}. \tag{157}$$

Therefore there exists a function $\widetilde{f}_{\nu,\; n}$ holomorphic on $\widetilde{U}_n$ and a component $\widehat{P}_n$ of $\iota_n^{-1}(P_n)$ such that $\widetilde{f}_{\nu,\; n}$ coincides with $f_{\nu,\; n} \circ \iota_n$ on $\widehat{P}_n$. The expansion

$$\widetilde{A}_{n\; m}(\zeta_n,\; \bar{\zeta}_m) \stackrel{def}{=} A_{n\; m}(\iota_n(\zeta_n),\; \overline{\iota_m(\zeta_m)}) = \sum_\nu \widetilde{f}_{\nu,\; n}(\zeta_n)\, \overline{\widetilde{f}_{\nu,\; m}(\zeta_m)} \tag{158}$$

holds uniformly on every compact subset of $\widetilde{U}_n \times \widetilde{U}_m$. Therefore the sequence $\{\widetilde{A}_{n\; m}\}$ posesses the property (G".1) in $\{\widetilde{U}_n\}$. It follows that the $\{A_{n\; m}\}$ possess the property (G".1) in $\{U_n\}$, since the points $t_{n,\; r}$ can be lifted in an arbitrary way to points in $\widetilde{U}_n$.
5. For each $n \in [1,\; M]$, we now define a measure ρ_n on U_n as follows. Let first $d\mu_n(p_n) = \mathrm{e}^{-\varphi_n(p_n)}\, d\lambda_n(p_n)$ where the smooth real function φ_n is chosen such that $\int_{U_n} d\mu_n(p_n) = 1$. We then define $d\rho_n(p_n) = (1 + A_{n\; n}(p_n,\, \bar{p}_n))^{-1}\, d\mu_n(p_n)$. Let $F_0 = \mathbf{C}$, and, for each $n \in [1,\; M]$, let $F_n = \mathcal{A}(U_n) \cap L^2(U_n,\; \rho_n)$. Note, e.g. that for any fixed $q_n \in U_n$, the function $p_n \mapsto A_{n\; n}(p_n,\, \bar{q}_n)$ belongs to F_n. Let $F = F_0 \oplus \ldots \oplus F_M$. The operator $\mathbf{A}$ defined on F by

$$(g,\; \mathbf{A}\, f) = \sum_{m,\; n=0}^{M} \int_{U_n \times U_m} \overline{g_n(p_n)}\, A_{n\; m}(p_n,\; \bar{q}_m)\, f_m(q_m)\, d\rho_n(p_n)\, d\rho_m(q_m) \tag{159}$$

is Hilbert-Schmidt and positive, and we again conclude that there exists a sequence $\{h_\nu \in F \ : \ \nu \in \mathbf{N}\}$ such that

$$A_{n\ m}(p_n,\ \bar{q}_m) = \sum_{\nu \in \mathbf{N}} h_{\nu,\ n}(p_n)\,\overline{h_{\nu,\ m}(q_n)} \tag{160}$$

holds uniformly on every compact subset of $U_n \times U_m$ as well as in the sense of $F_n \otimes F_m$.

This concludes the proof of Glaser's theorem 1.

Remarks

1. The extension to the case when the U_n are complex manifolds which are separable at infinity is straightforward.

2. The requirement that U be simply connected in Glaser's theorem 2 is necessary as the following example shows. Let $U = \mathbf{C} \setminus \{0\}$ and $A(p,\ \bar{p}) = |p|$. This satisfies the assumptions of Glaser's theorem 2 since $A(p,\ \bar{p}) = \sqrt{p}\sqrt{\bar{p}}$. But there cannot be a sequence f_ν of functions holomorphic on U such that $A(p,\ \bar{p}) = \sum_\nu |f_\nu(p)|^2$, since $|f_\nu(p)| \leq \sqrt{|p|}$ implies that f_ν is analytic at 0, hence entire and necessarily 0.

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