PLANCK'S FORMULA IN CLASSICAL MECHANICS

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ABSTRACT

We consider the model studied by Poincaré in connection with Planck's law, when he proved the necessity of quantization, namely a system of N independent identical oscillators, each of which interacts through smooth collisions with a gas particle (mimicing a heat reservoir), according to the laws of classical mechanics. We prove that the expected energy distribution of the oscillators obeys Planck's formula, i.e. Planck's law with an action characteristic of the system in place of Planck's constant. This is obtained by combining two ingredients, namely: the conception of Jeans who, following a perspective introduced by Boltzmann, was thinking of Planck's formula as describing a situation of quasi equilibrium very far from equilibrium, and: Einstein's conception of the thermodynamic role of the energy fluctuations, which is at the basis of nonequilibrium thermodynamics. In turn, the energy fluctuations are estimated by the most advanced mathematical results presently available for the energy exchanges in elementary collision processes.

0. As is well known, quantum mechanics started almost exactly a century ago with Planck's law, which was introduced, at first just as a good interpolation formula, [1,2] in place of the law of energy equipartition predicted by classical mechanics. The quantization of energy then entered physics. [3,4] However, at the beginning it was not clear whether quantization was just an useful device (in Einstein's words, " only a temporary way out";

see [5], page 51), or rather a true necessity, and many attempts were made at undertanding Planck's law in a classical framework, i.e. by considering processes involving continuous variations of energy. A particular effort in this direction was made by Jeans [6,7] who, in a perspective advanced by Boltzmann [8,9] (and also by Lord Rayleigh [10]), was trying to interpret Planck's law as describing a situation of quasi equilibrium very far from equilibrium, in the way familiar today in connection with the "aging" phenomena of glasses (see for example [11]) and spin glasses. Here we show that the point of view of Boltzmann and Jeans can actually be implemented, making use of Einstein's conception of the role of fluctuations, and of the most sophisticated mathematical results presently available for the energy exchanges in atomic collisions. We prove that,for a system of harmonic oscillators suffering smooth collisions with atoms (mimicing a heat reservoir), the expected energy distribution is apparently frozen about the initial one, with the addition of a "thermal part" which has a universal character and coincides with Planck's formula, i.e. Planck's law with an action characteristic of the system in place of Planck's constant.

Let us recall preliminarly some more facts. The core of Jeans' argument was the fact that, according to the laws of classical dynamics, the quantities of energy exchanged by oscillators in elementary events, typically in atomic collisions, decrease exponentially fast as frequency increases. For example, it can occur (see [12,13]) that there exists a frequency ω^* in the range of interest for atomic physics for which the relaxation time to equilibrium is of the order of one second, while it is of the order of 10^5 years for $\omega = 2\omega^*$. Thus for long finite times the high frequency oscillators practically behave as it they were frozen, and their energy distribution had a quasi equilibrium character. But Jeans was unable to extract from the presence of such exponentials a distribution law presenting a universal character, and his proposal was strongly criticized by Poincaré [14].

Apparently the debate was closed by Poincaré himself [15] and by Ehrenfest [16] (for a compact exposition, see also [17], sec. 6.7), who showed that the analytical form of Planck's law implies quantization: namely, if one pretends to deduce Planck's law by the standard methods of equilibrium statistical mechanics, then the energy of a harmonic oscillator of angular frequency ω has necessarily to be be restricted to the "energy levels" $E_n = n\epsilon$, $n = 0, 1, 2, \cdots$, with $\epsilon = \hbar\omega$ and $\hbar$ the (reduced) Planck's constant. The acceptance of this state of affairs by the scientific community is well witnessed by the vivid report of the Physics Meeting of the British Association of the year 1913 published in Nature [18] (see also [19]), where it is stated that *"Mr. Jeans regarded the work of Poincaré as conclusive"*; this was Jeans' "conversion", according to [20]. On the other hand it is well known that Einstein never proved convinced of the necessity of quantization. In fact he stressed this even in connection with Planck's law, when he afforded for it his remarkable interpretation in terms of thermodynamic fluctuations [21](see especially [22]).

In any case, the paradox concerning equipartition pointed out by Jeans remains: if one takes the point of view of considering long finite times, then for sufficiently low temperatures or high frequencies the oscillators will have a distribution apparently frozen about the initial one (and thus in general differing from equipartition), which moreover seems to be in some cases in agreement with the observations (for recent results in plasma physics, see [23,24]); on the other hand there seems to be no universal law that one could possibly extract from the exponentials described above, inasmuch as they depend on the particular interaction potentials involved.

In the present paper we propose a solution to this problem, making essential use of the role played by the dynamical fluctuations in the energy exchanges, through Einstein's fluctuation formula which is at the basis of nonequilibrium statistical mechanics. We consider the problem of the energy distribution for a system of harmonic oscillators suffering collisions with point particles, mimicing a heat reservoir, and show that, up to verly long times, the energy distribution is just the initial one, with the addition of a "thermal part" having a universal form; moreover, such a universal form is exactly Planck's formula, namely Planck's law with $\hbar$ replaced by an action a_* characteristic of the system. To this end we make use of the most advanced estimates presently available for the energy echanges in elementary collisions, which confirm and extend Jeans' results, and were obtained along a very popular line of research in the mathematical theory of dynamical systems (see for example [25,26], and the reviews [27,28]), and independently in the field of plasma physics (see for example [29,30], and the review [31]). But the key point, which constitutes the core of the present paper, consists in showing how the dynamical fluctuations in the single energy exchanges produce a thermodynamic law of a universal type (which turns out to be exactly Planck's formula) through Einstein's fluctuation formula. So we combine, and in a rather simple way, Einstein's conception of the thermodynamic role of fluctuations with Jeans' conception that Planck's law might actually describe a situation of quasi equilibrium very far from equilibrium.

The proof is given here only for initial data with small oscillator energies; moreover, the role of the critical action a_* is not yet completely clear, and several problems of interpretation seem to remain open. But the result is obtained in such a natural and simple way, as to let us hope it will not prove to be fortuitous.

1. By *Planck's law* we mean as usual the relation

$$U(T) = N\,\frac{\epsilon}{e^{\epsilon/kT}-1}\quad , \tag{1}$$

with $\epsilon = \hbar\omega$, giving the expected energy U of a system of N oscillators of angular frequency ω in thermodynamic equilibrium at absolute temperature T (k being Boltzmann's constant). By *Planck's formula*, on which we concentrate in the present paper, we mean relation (1) with $\epsilon = a_*\omega$, where a_* is an action not necessarily coinciding with $\hbar$. The *zero–point energy* $N\epsilon/2$ was added to the the right hand side of (1) in the year 1911 by Planck himself,[32] thus leading to the formula

$$U(T) = N\,\big(\frac{\epsilon}{e^{\epsilon/kT}-1}+\frac{\epsilon}{2}\big)\ , \tag{2}$$

which we call *Planck's formula with zero–point energy.* This new version satisfies the asymptotic condition $U/(NkT)\to 1$ as $T\to\infty$ or $\epsilon\to 0$, and corresponds to energy levels $E_n = (n+1/2)\epsilon$.

A central role will be played by the remark that the original Planck's formula (1) can be regarded as a solution to a first order differential equation for the function $U = U(T)$, namely

$$kT^2\,\frac{\mathrm{d}U}{\mathrm{d}T} = \epsilon\,U + \frac{U^2}{N}\ . \tag{3}$$

This essentially constitutes the core of Planck's first memoir(*) of October 1900: by integration one has $U = N\epsilon/(C\exp(\epsilon/kT)-1)$, with an integration constant C which is set equal to 1 by the asymptotic condition $U \to \infty$ as T$\to \infty$.

The interplay of the two terms U^2/N and $\epsilon\, U$ in (3) will be essential in the following discussion, as was also originally for Planck. He remarked that, in the integration procedure recalled above, the term U^2/N alone would lead to $U(T) = N\,kT$, i.e. to the "classical" Rayleigh–Jeans equipartition law, which is well verified for high temperatures (or low frequencies), while the term $\epsilon\, U$ alone would lead to Wien's law $U(T) = NC\,\exp\left(-\epsilon/kT\right)$ (C being an integration constant), which is well verified for low temperatures (or high frequencies). The combination $\epsilon\, U + U^2/N$ occurred to Planck ay first just as a good interpolation, apparently fitting well the data in the whole available domain. In any case, from Planck's procedure one can get the idea, which will be used below, that any functional form for the right hand side of (3) produces a corresponding thermodynamic function $U = U(T)$.

2. Einstein's contribution,[21,22] which will be at the core of the present dynamical approach, consisted, so to say, in a physical substantiation of Planck's formal procedure. This he obtained by interpreting the left hand side of (3) in terms of energy fluctuations, through the relation

$$kT^2 \frac{\mathrm{d}U}{\mathrm{d}T} = DE \ , \tag{4}$$

which we call *Einstein's general (or thermodynamic) fluctuation formula*. Here DE is the variance of the energy E of the system of oscillators, i.e. $DE = \overline{(E-U)^2}$ with $U = \overline{E}$, the bar denoting expectation with respect to a suitable probability distribution.

The thermodynamic relevance of Einstein's fluctuation formula (4) is that, if the variance DE is somehow given as a function of the expected energy U, say

$$DE = f(U) \ , \tag{5}$$

then (4) appears as a differential equation from which the thermodynamic energy $U = U(T)$ can be recovered by integration, as in Planck's procedure recalled above. In fact, with *"a simple calculation"* (i.e. the check of the equivalence of (3) and (1) recalled above) Einstein found that Planck's formula (1) is equivalent to the functional relation

$$DE = \epsilon\, U + U^2/N \ , \tag{6}$$

which we call *Einstein's special fluctuation formula* for a system of oscillators.(*) One can thus say, in Einstein's very words, that the general and the special fluctuation formulæ

(*) Planck was in fact working in terms of the entropy $S = S(U)$, instead of the function $U = U(T)$, using the relation $\frac{\mathrm{d}^2 S}{\mathrm{d}U^2} = -\frac{k}{U(\epsilon+U)}$; this is equivalent to (3) by $T = \frac{\mathrm{d}U}{\mathrm{d}S}$.

(*) In terms of relative fluctuations, one thus gets $DE/U^2 = \epsilon/U + 1/N$, a relation exhibiting the characteristic singularity ϵ/U (in Einstein's words: *"an unevenness in the distribution of the radiation energy, which is the more significant the smaller is the quantity of the energy involved"* (see [22], page 419). This is just due to the presence of the "nonclassical"term $\epsilon\, U$ in (3). As is well known, the relation $DE/U^2 = \epsilon/U$, holding in Wien's approximation, was interpreted by Einstein as corresponding to the existence of the photon.

(4) and (6) *"exaust the thermodynamic content of Planck's formula"*, and that in such a sense *" a mechanics compatible with the energy fluctuation (6) must then necessarily lead to Planck's formula".*[22] We will show below that the laws of classical mechanics lead, for the energy of a system of material oscillators colliding with the atoms of a gas, to a fluctuation formula of the analytic form (6) and thus lead for the expected energy, in Einstein's sense, to Planck's formula.

We add now a few comments concerning the general fluctuation formula (4), which plays for us a fundamental role and is quite subtle. First of all, such a formula appears as a trivial identity in the canonical formalism of equilibrium statistical mechanics, and as such was discovered by Einstein himself in his paper [33]. But for our purposes it is essential to consider the extension to nonequilibrium situations, which was given by Einstein himself in the papers [21,22], and is now taken, in several forms, as the basis of nonequilibrium statistical mechanics (see for example [34]). Einstein's well known procedure consists in introducing an expansion of entropy up to second order near a maximum (i.e. near an equilibrium state), which leads to a gaussian distribution for the energy; then he makes use of Boltzmann's relation $S = k \log W$ between entropy and probability, which provides a relation between variance and mean, thus leading to (4). We are ourselves working on a more direct proof of Einstein's formula in a nonequilibrium context, but we refrain from saying more here, and just take the formula for granted. We only mention that Einstein's formula should hold in the spirit of the central limit theorem, and this requires taking into consideration a system of a large number N of oscillators.

3. In order to show how classical mechanics leads to a fluctuation formula of the analytic form (6), we will concentrate on a model which is particularly important for the present discussion, because is the one considered by Poincaré in his celebrated paper[15] where he proved the necessity of quantization. Poincaré's model consists of a system of N independent identical subsystems, each involving a harmonic oscillator (also called a hard spring) of angular frequency ω, suffering smooth collisions with a point particle (also called atom, or gas particle) on a line. This model is a prototype of a class of models concerned with the approach to equipartition between "internal" degrees of freedom (high frequency oscillators, but one can also think of rotators) and low frequency oscillators (as a limit case, zero frequency oscillators, i.e. free particles, or centers of mass). In fact, such a model is essentially equivalent to that of a system of N diatomic molecules on a line, as discussed for example in the works [12,25]; but one can also think of a system of diatomic molecules suffering independent smooth collisions with a wall, analogous to the system of rotators colliding with a wall, studied in [13].

Since the times of Jeans and, through the work of Landau and Teller,[35,36] up to some recent works[25] along the lines of Nekhoroshev's theorem, it is well known that, under very general conditions, at each single collision the energy exchange δe between oscillator and atom is bounded by $(\delta e)^2 \leq \mathrm{const}\, \eta$, with

$$\eta = \mathcal{E}\, \exp(-\omega\tau) \ , \tag{7}$$

τ and $\mathcal{E}$ depending on the particular form of the interaction potential; τ also depends on the incoming velocity v of the atom, typically as $\tau \simeq v^{-a}$ with $a > 0$, so that for sufficiently low gas temperatures (i.e. for small v) or high frequencies the oscillator is actually frozen, i.e. essentially doesn't exchange energy at all. The condition to be satisfied is that the frequency be sufficiently large, more precisely one should have $\omega > (t_{\rm coll})^{-1}$, where $t_{\rm coll}$ is the "collision time", i.e. essentially the time required for the atom to cross the interaction potential (this is the point where the smoothness of the potential plays a role). Analogous results were obtained by a group around O'Neil (see [29,30], and the review [31]) in the field of plasma physics, apparently in agreement with the observations. [23,24]

The existence of dynamical fluctuations in the energy exchanges was obviously known since a long time, first of all to Jeans, and even to Planck himself, who for example described them in an impressive way in his book [37] (but only after the third edition, not available in english; see section 143, page 152, of the fifth edition). But their role was not appreciated, and their connection to thermodynamics through Einstein's fluctuation formula was not taken into account at all. From a purely dynamical point of view, the existence of relevant fluctuations in the energy exchanges in atomic collisions was pointed out in recent times in [13], and even before in plasma physics [29]. For example, in [13], studying numerically a model of rotators impinging on a wall, it was found that, for small initial energies of the rotator, the energy exchange δe in any single collision actually decomposes into a drift and a fluctuating term, being given by what we are accostumed to call Benettin's formula, namely

$$\delta e = \eta + \sqrt{b\eta}\,\cos\varphi \ ; \tag{8}$$

here η is exactly the exponentially small terms (7), which depends on the details of the molecular interaction potentials, while φ is an angle related to the initial phase of the rotator, and b is a quantity independent of ω, with the dimensions of an energy. So for small η, i.e. for low temperatures or high frequencies, the fluctuating term in general dominates over the drift term η, and an exponentially large number n_* of collisions, say

$$n_* \simeq \sqrt{b/\mathcal{E}}\ \exp(\omega\tau/2) \ , \tag{9}$$

is required for the two contributions to become comparable, as pointed out in [13]. For any shorter time, i.e. for any smaller number of collisions, the rotator energy can be conceived to perform a kind of random walk about the initial energy e_0.

Clearly an analogous description is expected to hold for models of the Poincaré type, involving oscillators, with the angle φ denoting the initial phase of the oscillator. This was shown numerically in [38]. An analytical confirmation was given in the work [26], where, for the case of an oscillator, the energy exchange δe in a single collision was represented as a Fourier series in the angle φ, whose n–th term was found to decrease essentially faster than η^n. Moreover, from that work one can also extract a condition that guarantees that the first two terms, namely those appearing in (8), are the dominating ones; the condition is that the initial oscillator action a_0 be sufficiently small, say $a_0 < a_*$ with a suitable threshold or critical action a_*, which can in principle be estimated (this is an important point, on which we plan to come back in the near future). Thus. for example, the occurrence in (8) of the drift term η, entailing a one–directional tendency towards equilibrium, turns out to be just due to the choice of initial data with small oscillator's energy, because the general formula predicts instead a drift of an opposite sign for initial data with high oscillator's energy. Another important point is that for $a_0 < a_*$ the constant b entering formula (8) for a single collision can be shown to be given by $b = 4e_0 = 4a_0\omega$, where e_0 is the oscillator energy before collision, and a_0 the corresponding action.

For simplicity's sake we consider here the simplest possible case where it can be shown how the purely dynamical fluctuations are converted into thermodynamic fluctuations. To this end we limit ourselves to the case of initial data with small oscillator energy, say with $a_0 < a_*$, so that the single energy exchange can be taken as given by formula (8) with $b = 4e_0$, namely

$$\delta e = \eta + 2\sqrt{e_0\eta}\ \cos\varphi \ , \tag{10}$$

where $e_0 = a_0\omega$ is the oscillator energy before collision; we refer to (10) as the "dynamical fluctuation formula". So we study a system of N identical independent harmonic oscillators, in contact with a heat reservoir at a certain temperature; the latter is modelled by gas atoms suffering independent smooth collisions with the single oscillators, the single energy exchanges being given by the dynamical fluctuation formula (10). This is the main limitation of the model discussed here. We also introduce two further simplifications, which however in our opinion are only of a marginal relevance and could rather easily be removed. The first one consists in assuming that all gas atoms have before collision the same velocity v, instead of following a Maxwell–Boltzmann distribution (this is particularly appropriate if the model describes a system of diatomic molecules impinging on a wall, because the particle's velocity is then essentially constant up to very large times); secondly, we assume that the "time of flight" $t_{\rm fl}$ between two successive collisions of an atom with a single oscillator is a constant, so that the current time t is just proportional (through $t_{\rm fl}$) to the number k of collisions suffered by a single oscillator. The discussion of the general case is deferred to future work.

4. We come now to the connection between the dynamical fluctuation formula (10) and Einstein's special fluctuation formula (6). The quantity of interest is the energy E of the global system of N independent identical oscillators, namely $E = \sum_i e^{(i)}$, where $e^{(i)}$ denotes the energy of the i–th oscillator.

Let us consider the i–th oscillator. By (10), its energy $e_k^{(i)}$ after k collisions could be given explicitly in terms of $k+1$ random variables, namely the initial energy $e_0^{(i)}$ and the phases (which are assumed to be identically – uniformly – distributed) $\varphi_1^{(i)}, \cdots \varphi_k^{(i)}$. But, in order to perform the average over the phases (which we denote by an overline), it is more expedient to make reference to the formula

$$e_k^{(i)} = e_0^{(i)} + k\eta + 2\sqrt{\eta}\sum_{j=1}^{k}\sqrt{e_{j-1}^{(i)}}\ \cos\varphi_j^{(i)} \ ,$$

and just remark that the energies $e_l^{(i)}$ are independent of the phases $\varphi_j^{(i)}$ for $l < j$. Thus, for the mean energy $u_k^{(i)} := \overline{e_k^{(i)}}$ after k collisions one gets the expression

$$u_k^{(i)} = e_0^{(i)} + k\eta \ . \tag{11}$$

Analogously, with $\overline{\cos(\varphi_j^{(i)})^2} = 1/2$, one finds $\overline{(e_k^{(i)})^2} = (e_0^{(i)} + k\eta)^2 + 2\eta \sum_{j=1}^k u_{j-1}^{(i)}$, and thus, with (11), the variance $De_k^{(i)} := \overline{(e_k^{(i)})^2} - (u_k^{(i)})^2$ is given by

$$De_k^{(i)} = 2e_0^{(i)} k\eta + k^2\eta^2$$

(this is in fact the simplified expression obtained if $k-1$ is identified with k, as can be done for a sufficiently large k). Now the "time" k can be eliminated using (11), and one gets

$$De_k^{(i)} = 2e_0^{(i)}(u_k^{(i)} - e_0^{(i)}) + (u_k^{(i)} - e_0^{(i)})^2 \ .$$

An analogous relation also holds for the global system of oscillators. Indeed, let us consider the total energy $E_k = \sum_i e_k^{(i)}$ when each oscillator has suffered k collisions. By the central limit theorem the random variable E_k is normally distributed with a mean U_k and a variance DE_k which are obtained by adding up the corresponding quantities for each oscillator, namely

$$U_k = N(e_0 + k\eta)$$
$$DE_k = N(2e_0 k\eta + k^2\eta^2),$$

where e_0 is defined by $Ne_0 := \sum e_0^{(i)}$, i.e. as the system's initial energy *per* oscillator. Now the number k of collisions can be eliminated in the expression for DE_k, just using $Nk\eta = U_k - Ne_0$, and this provides a relation between variance and expected energy for the global system. Namely, denoting by U the system's mean energy at any "time" k, and by DE the corresponding variance at the same "time", one has

$$DE = 2e_0(U - Ne_0) + \frac{1}{N}(U - Ne_0)^2 \ . \tag{12}$$

This is our dynamical analogue of Einstein's special fluctuation formula (6), and is seen to coincide with it if in the latter one substitutes U with $U - Ne_0$ and takes $\epsilon = 2e_0 \equiv 2a_0\omega$, where e_0 and a_0 are the initial energy and action *per* oscillator respectively, i.e. $e_0 = \sum e_0^{(i)}/N$, $a_0 = \sum a_0^{(i)}/N$.

Through Einstein's general fluctuation formula (4), relation (12) then leads, for the mean energy of the system of oscillators as a function of temperature, to the original Planck's formula (1) with $U - Ne_0 = U - N\epsilon/2$ in place of U, and $\epsilon = 2e_0$, namely exactly to Planck's formula with zero–point energy (2) and $\epsilon = 2a_0\omega$, i.e.

$$U = N\Big(\frac{2a_0\omega}{e^{2a_0\omega/kT} - 1} + a_0\omega\Big) \ . \tag{13}$$

This is our analogue of Planck's formula, and constitutes the main result of the present paper. A further expression for the expected energy can be given in the following way. One remarks that formula (13) holds for any choice of the initial data $a_0^{(i)}$ in the interval $(0, a_*)$. Thus, if one assumes that they are uniformly distributed in that interval, one gets for the expected value of $a_0^{(i)}$ exactly $a_*/2$, and so one finally gets

$$U = N\Big(\frac{a_*\omega}{e^{a_*\omega/kT} - 1} + \frac{a_*\omega}{2}\Big), \tag{14}$$

namely Planck's formula with zero–point energy (2) and $\epsilon = a_* \omega$, in which the critical action a_* takes the place of Planck's constant.

5. So we have shown how Planck's formula occurs in classical physics in a quasi equilibrium context, at least for a special set of initial data, namely with oscillator energies smaller than those of the particles mimicing the heat reservoir; we hope to be able to deal with the general case in the future. ¿From the quasi equilibrium point of view considered here, Planck's formula (13) or (14) should be read in the following way: the second term at the right hand side is nothing but the initial energy Ne_0, while the first term gives the additional "thermal energy", that the system acquires from the heat reservoir, and is actually increasing, very slowly, with time; this corresponds to the fact that the temperature T appearing in (13) and (14) is a function of time, increasing at an extremely low rate. Obviously, the fact that one should have a "thermal part" increasing very slowly with time was well known to Jeans, and what we have added here is essentially that such a "thermal energy" is distributed according to a law which has a quite universal character, and even formally coincides with Planck's formula. The situation here seems to be similar to that occurring in the familiar example of the theory of adiabatic invariants, dealing with a pendulum whose length l changes very slowly with time according to a given law $l = l(t)$. In such a case, the pendulum's energy E too turns out to vary with time, but there exists a functional relation $E = \mathcal{E}(l)$ such that one has, within a certain approximation and up to a certain time, $E(t) \simeq \mathcal{E}(l(t))$. In our case, the functional relation between expected energy and temperature is just Planck's formula.

The role of the critical action a_*, which formally takes the place of Planck's constant, remains at the moment not yet understood, and should be investigated more carefully. We just make here the following comment. It is obvious that a_* and $\hbar$ are in principle completely unrelated; indeed for any model the critical action a_* is a priori proportional to a characteristic action entering the model, which in turn can take any value. For example for interatomic potentials of the Lennard–Jones type a characteristic action is given by $\sqrt{mV_0}\,\sigma$, where m is the mass of the particle, V_0 the depth of the potential and σ the distance at which the potential vanishes, so that the characteristic action can take any positive value. However, we would like to mention that for realistic systems one has with a very good approximation (see [39], page 1110, where V_0 is denoted by ϵ) $\sqrt{mV_0}\,\sigma \simeq 2Z\hbar$, where Z is the atomic number, and consequently a_* might very well turn out to be, for realistic systems, of the order of magnitude of Planck's constant $\hbar$. But we don't insist on this point here, as also refrain from commenting on a possible extension of the present result to the black body, where the characteristic action of the classical model is just $e^2/c \simeq (137)^{-1}\hbar$, with e and c the elementary charge and the speed of light respectively (see the last page of Einstein's paper [21]).

Concerning the argument of Poincaré [15] and Ehrenfest, [16] according to which it should be impossible to obtain Planck's formula without introducing quantization, something appears to be wrong, because apparently we have produced here a counterexample. But obvioulsy, in problems of this type which are concerned with the impossibility of doing something, as in the case of the celebrated von Neumann theorem on quantum mechanics, [40] the question is rather to understand the relevance of the hypotheses which are introduced. This is a very interesting problem which we are presently studying, and on which we hope to come back in the near future.

Another comment, of an historical character, is that in Planck's book (section 143, formula 261, of the fifth edition) one can find an expression for the energy fluctuation of a forced oscillator under quite general conditions, which is very near to a formula of the type $DE = 2e_0U$, and would thus lead, with Einstein's general fluctuation formula, to Wien's law with $2e_0$ in place of $\hbar\omega$. It is thus clear that it is just the general fluctuation formula of Einstein the main ingredient which is lacking in Planck's book, and that the essence of the present paper consists in a combination of that general formula, which is at the basis of nonequilibrium thermodynamics, with an explicit estimate for the energy exchanges in elementary collision processes.

A final comment concerns a possible classical conception of the photon, which was suggested by Einstein on the basis of his interpretation of Planck's formula in terms of energy fluctuations. A first relevant quotation is the following one, which is a comment he made in his last days on his original discovery of the photon (see [5], page 51): *"This way of looking at the problem showed in a drastic and direct way that a type of immediate reality has to be ascribed to Planck's quanta, that radiation must, therefore, possess a kind of molecular structure in energy ... This interpretation, which is looked upon as essentially final by almost all contemporary physicists, appears to me only a temporary way out".* And again (page 420): *" When a body absorbs or emits thermal energy by a quasiperiodic mechanism, the statistical properties of the mechanism are such* as they would if *the energy were propagated in whole quanta of megnitude $h\nu$."* In our opinion, this hint of Einstein for a "statistical interpretation of the photon" relies on his special fluctuation formula (6). Indeed, in connection with the corresponding formula for the relative fluctuation in Wien's limit, namely $DE/U^2 = h\nu/U$, he had made the following comment (see [22], page 415): *"If U becomes of the order of magnitude of $h\nu$, the relative energy fluctuation becomes of the order of magnitude of* 1*, i.e. the fluctuation of the energy is alternatively present and not present: it behaves, in essence, as something with limited divisibility. But nevertheless bounded energy quanta of definite magnitude need not necessarily exist."* In other words, the analogue of the photon is obtained, according to Einstein, any time one gets his special fluctuation formula; and this apparently was shown here to follow from a *mechanics* which is nothing but Newton's classical mechanics.

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