PERCOLATION AND GIBBS STATES MULTIPLICITY FOR FERROMAGNETIC ASHKIN–TELLER MODELS ON $\mathbb{Z}^2$

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Abstract. For a region of the nearest neighbor ferromagnetic Ashkin–Teller spin systems on $\mathbb{Z}^2$, we characterize the existence of multiple Gibbs states via percolation. In particular, there are multiple Gibbs states if and only if there exists percolation of any of the spin types. (I.e., the magnetized states are characterized by percolation of the dominant species.) This result was previously known only for the Potts models on $\mathbb{Z}^2$.

Key words and phrases. Percolation, Potts Models, Ashkin–Teller Models, Random Cluster Models, FKG inequalities, Gibbs states.

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INTRODUCTION

An issue that sometimes arises in statistical mechanics concerns the connection between percolation and phase transitions. For the Potts models on $\mathbb{Z}^2$ there are characterization theorems relating the uniqueness of the Gibbs states and the absence of spin-system percolation [$\mathrm{CNPR}_{1\&2}$], [C_1]. Explicitly, for the Ising magnet, the region of non-uniqueness is *characterized* by percolation of + spins in the + state. The analogous result holds for the Potts model and a number of similar results, for various systems, were established in [GLM]. In this paper, we will establish such a result for a region of the Ashkin–Teller models. Specifically, there are multiple limiting Gibbs states precisely at those temperatures which foster percolation of one of the spin types.

We begin with a description of a general Ashkin-Teller model on an arbitrary graph with spins at each vertex. There are four possible spin types, labelled: blue, red+, yellow, and red−. The spins may be regarded as lying equidistant on the unit circle, occurring clockwise in the order just named, with blue at 12 o'clock. There is complete symmetry around the circle, so that interactions receive energy assignments based solely on the relative positions of the spin colors on the circle. Here the model is "completely" ferromagnetic: colors opposite to each other receive the highest energy assignments; the like-like interactions the lowest, and the adjacent colors receive an intermediate energy. Without loss of generality, we may set this intermediate energy level $= 0$. For positive $\mathcal{K}_{\langle i,j\rangle}$, $k_{\langle i,j\rangle}$, we set the like-like interaction between sites i and j along the edge $\langle i,j\rangle = k_{\langle i,j\rangle} - \mathcal{K}_{\langle i,j\rangle}$, and the interaction for spin pairs with colors opposite to each other to $k_{\langle i,j\rangle}$. Although the $\mathbb{Z}^2$ Ashkin-Teller model in our theorem has uniform couplings (and at most one edge between any two sites), some of our proofs will use the flexibility of multiple edges between sites and non-uniform coupling constants. In this paper, we will confine attention to the parameter region $k_{\langle i,j\rangle} \leq \mathcal{K}_{\langle i,j\rangle}/2$ for all $\langle i,j\rangle$.

Let $\vec{s}_i$ denote the Cartesian coordinates of the site i's color on the unit circle. Then the explicit energy value $\mathcal{E}(\vec{s}_i, \vec{s}_j)$ between sites i and j is:

$$\mathcal{E}(\vec{s}_i, \vec{s}_j) = \Gamma \vec{s}_i \cdot \vec{s}_j + \gamma(\vec{s}_i \cdot \vec{s}_j)^2 \tag{1}$$

where $-\Gamma = \frac{\mathcal{K}_{\langle i,j\rangle}}{2}$ and $-\gamma = \frac{\mathcal{K}_{\langle i,j\rangle} - 2k_{\langle i,j\rangle}}{2}$. For any finite graph $\mathcal{G}$ the Hamiltonian is given by $H = \sum_{\langle i,j\rangle \in \mathcal{G}} \mathcal{E}(\vec{s}_i, \vec{s}_j)$, and the Boltzmann weight of any spin configuration is $e^{-\beta H}$ where $1/\beta \propto$ temperature.

The phase diagram of the Ashkin-Teller model is depicted in figure 1; this is the upper right hand quadrant of Baxter's diagram 12.12 [B], slightly tilted. In the notation of Baxter's book, the change of variables (for the uniform case) is as follows: $\epsilon_3 = k$, $\epsilon_0 = k - K$, and $\epsilon_1 = \epsilon_2 = 0$. The present work focuses on the region $0 < k < K/2$, where the Ashkin-Teller model interpolates between the 4-state Potts and $\mathbb{Z}_4$ models. (For more on $\mathbb{Z}_n$ models and their relation to Ashkin-Teller models, see [Ca].) If $0 < k < K/2$, then there is a unique ordering transition [P]. Here we prove that the phase boundary of this unique ferromagnetic transition coincides precisely with the critical percolation boundary for the Ashkin-Teller spin system.

We will start with the forward direction of the characterization theorem: The existence of multiple Gibbs states implies that there is percolation of individual spin types. For the opposite direction we need some additional ammunition, in the form of the result of [GKR] and two lemmas that we will present. Both directions rely on a Wolff/FK-type representation [W], [FK] of the Ashkin-Teller model, similar to that used in [A], [C_2] and [CM_2] for the XY models.

Main Section

We start by rewriting our spins' circle positions: $\vec{s}_i = (a_i\tau_i, b_i\sigma_i)$, where $a_i, b_i \in \{0,1\}$, $a_i = 1 - b_i$, and $\tau_i, \sigma_i \in \{-1,+1\}$. For example, if site $i =$ blue, then $\vec{s}_i = (0,1)$ and correspondingly $a_i = 0$, $b_i = 1$, $\tau_i = \pm 1$, and $\sigma_i = +1$. Note there is some ambiguity with this description: In this example, for instance, τ_i does not have a set value because $a_i = 0$. This is of no consequence; the effect of summing over all configurations of the b, σ, τ variables amounts to an unimportant extra factor of 2 for each site, and does not alter the resultant probability measure.

Expanding the dot products in equation (1) with our new variables, and using the identity $\sigma_i\sigma_j = 2\delta_{\sigma_i\sigma_j} - 1$, we may rewrite our Hamiltonian as:

$$H = H^{\mathbb{K}}_{\underline{b},\underline{\sigma}} + H^{\mathbb{K}}_{\underline{a},\underline{\tau}} \tag{2}$$

$$-H^{\mathbb{K}}_{\underline{b},\underline{\sigma}} = \sum_{\langle i,j\rangle\in\mathcal{G}} [\mathcal{K}_{\langle i,j\rangle} b_i b_j (\delta_{\sigma_i\sigma_j} - 1) + (\mathcal{K}_{\langle i,j\rangle} - k_{\langle i,j\rangle}) b_i b_j] \tag{3}$$

$$-H^{\mathbb{K}}_{\underline{a},\underline{\tau}} = \sum_{\langle i,j\rangle\in\mathcal{G}} [\mathcal{K}_{\langle i,j\rangle} a_i a_j (\delta_{\tau_i\tau_j} - 1) + (\mathcal{K}_{\langle i,j\rangle} - k_{\langle i,j\rangle}) a_i a_j] \tag{4}$$

where $\underline{a}$, $\underline{b}$, $\underline{\sigma}$, $\underline{\tau}$ are configurations of a, b, σ, τ, respectively, and $\mathbb{K}$ denotes a configuration of $\mathcal{K}_{\langle i,j\rangle}$, $k_{\langle i,j\rangle}$ over $\Theta_{\mathcal{G}}$, the set of all edges of $\mathcal{G}$. Notice that $H^{\mathbb{K}}_{\underline{b},\underline{\sigma}}$,

$H^{\mathbb{K}}_{\underline{a},\underline{\tau}}$ each take the form of an Ising Hamiltonian (with couplings determined by parameters $\underline{b}$ and $\underline{a}$), plus an additional term independent of the σ_i's and τ_i's. We denote the Ising terms as $I^{\mathbb{K}}_{\underline{b},\underline{\sigma}}$ and $I^{\mathbb{K}}_{\underline{a},\underline{\tau}}$, and the additional terms as $\psi^{\mathbb{K}}(\underline{b})$ and $\psi^{\mathbb{K}}(\underline{a})$. For example:

$$\psi^{\mathbb{K}}(\underline{b}) = \sum_{\langle i,j\rangle \in \mathcal{G}} (\mathcal{K}_{\langle i,j\rangle} - k_{\langle i,j\rangle}) b_i b_j. \tag{5}$$

The Boltzmann weight can now be written as: $e^{-\beta I^{\mathbb{K}}_{\underline{b},\underline{\sigma}}} e^{-\beta I^{\mathbb{K}}_{\underline{a},\underline{\tau}}} e^{\beta\psi^{\mathbb{K}}(\underline{b})} e^{\beta\psi^{\mathbb{K}}(\underline{a})}$.

It is now convenient to introduce related measures needed for upcoming proofs. Tracing over $\underline{\tau}$, and letting $\mathcal{Z}^{I,\mathbb{K}}_{\underline{a}}$ denote the Ising partition function for $I^{\mathbb{K}}_{\underline{a},\underline{\tau}}$, we arrive at a new measure $\nu_{\mathcal{G}}(\underline{b},\underline{\sigma})$:

$$\nu_{\mathcal{G}}(\underline{b},\underline{\sigma}) \propto \mathcal{Z}^{I,\mathbb{K}}_{\underline{a}} e^{-\beta I^{\mathbb{K}}_{\underline{b},\underline{\sigma}}} e^{\beta\psi^{\mathbb{K}}(\underline{b})} e^{\beta\psi^{\mathbb{K}}(\underline{a})}. \tag{6}$$

Now we expand the $e^{-\beta I^{\mathbb{K}}_{\underline{b},\underline{\sigma}}}$ term into Random Cluster [FK] Ising ($q = 2$) weights to obtain the measure

$$\phi(\underline{b},\underline{\sigma},\omega) \propto \mathcal{Z}^{I,\mathbb{K}}_{\underline{a}} e^{\beta\psi^{\mathbb{K}}(\underline{b})} e^{\beta\psi^{\mathbb{K}}(\underline{a})} B^{\mathbb{K}}_{p}(\omega) \prod_{\langle i,j\rangle\in\omega} \delta_{\sigma_i\sigma_j} \tag{7}$$

where $\omega \subset \Theta_{\mathcal{G}}$ is an Ising FK bond configuration, and $B^{\mathbb{K}}_{p}(\omega)$ is the Bernoulli weight for ω with probability $p_{\langle i,j\rangle} = 1 - e^{-\beta\mathcal{K}_{\langle i,j\rangle} b_i b_j}$ of the bond $\langle i,j\rangle$ being occupied. Specifically,

$$B^{\mathbb{K}}_{p}(\omega) = \prod_{\langle i,j\rangle\in\omega} p_{\langle i,j\rangle} \prod_{\langle i,j\rangle\notin\omega} (1 - p_{\langle i,j\rangle}).$$

Let $C(\omega)$ = the number of connected components of the configuration ω (where sites not touching bonds are considered to be individual components.) Summing over $\underline{\sigma}$ and $\underline{b}$, we arrive at the marginal distribution:

$$\mu_{\mathcal{G}}(\omega) \propto \mathcal{Z}_{\underline{a}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{a})} \sum_{\underline{b}} e^{\beta\psi^{\mathbb{K}}(\underline{b})} B_p^{\mathbb{K}}(\omega) 2^{C(\omega)}. \tag{8}$$

Notice that $p_{\langle i,j\rangle}$ is nonzero only if b_i and b_j are one; it is observed that ω bonds represent full spin alignment so that each connected cluster must be monochrome – either of the blue or yellow type.

So far, we have only considered free boundary conditions on the graph $\mathcal{G}$. Also of interest are *blue* boundary conditions. Let $\partial\mathcal{G}$ denote a set of "boundary" sites in $\mathcal{G}$. Consider the analogous developments under the boundary conditions that all sites of $\partial\mathcal{G}$ are fixed at blue; we denote the corresponding measures by $\nu_{\mathcal{G}}^{\mathcal{B}}(-)$, $\phi_{\mathcal{G}}^{\mathcal{B}}(-)$ and $\mu_{\mathcal{G}}^{\mathcal{B}}(-)$ respectively. Of course, $\nu_{\mathcal{G}}^{\mathcal{B}}(-)$ is just the marginal distribution of a Canonical Gibbs measure. In equation (7), the terms $\delta_{\sigma_i\sigma_j}$ must be modified if i and/or j is a boundary site, and in addition the partition function $\mathcal{Z}_{\underline{a}}^{I,\mathbb{K}}$ has to be recomputed. Finally, in the counting of clusters one arrives at $2^{C_{\mathrm{w}}(\omega)-1}$ where $C_{\mathrm{w}}(\omega)$ is the number of components counted as though all sites of $\partial\mathcal{G}$ are identified as a single site. Hence, in the $\mu_{\mathcal{G}}^{\mathcal{B}}(-)$ measure, the connected component of the boundary represents sites that are all blue.

We are now ready for the first direction of our characterization proof.

Theorem 1a. *In the region $0 \le k_{\langle i,j\rangle} \le \mathcal{K}_{\langle i,j\rangle}/2$ of the above described Ashkin-Teller model on $\mathbb{Z}^2$, the presence of multiple Gibbs states implies that there is percolation of blue spins in the "blue" state: the state obtained as the limit of finite volume conditional measures with all boundary spins set to blue.*

Proof. Let $\vec{s}_0$ be the spin at the origin, and let $\hat{e}_y$ be the unit vector in the blue direction. The superscript "$\mathcal{B}$" will denote blue boundary conditions on $\mathcal{G}$. By use of yet another (bi-layer) graphical representation, Theorem III.7 in [CM_1] demonstrates that non-uniqueness of Gibbs states in the region $k_{\langle i,j\rangle} \leq \mathcal{K}_{\langle i,j\rangle}/2$ of the Ashkin–Teller model is equivalent to positive spontaneous magnetization. So for this direction of the argument, it suffices to assume that we have this positive magnetization. Let $\langle - \rangle^{\mathcal{B}}_{\mathcal{G},w}$ be the expectation with respect to a measure w under blue boundary conditions on $\mathcal{G}$.

From positivity of the magnetization, we have:

$$\langle b_0\sigma_0\rangle^{\mathcal{B}}_{\mathcal{G},\nu} = \langle \vec{s}_0 \cdot \hat{e}_y\rangle^{\mathcal{B}}_{\mathcal{G},\nu} \geq \epsilon > 0 \tag{9}$$

for some $\epsilon > 0$, for all finite $\mathcal{G} \subset \mathbb{Z}^2$. Let E be the event that the origin is connected to the boundary of $\mathcal{G}$ through ω bonds. Recalling the measure described in (7), we see that:

$$\begin{aligned}\langle b_0\sigma_0\rangle^{\mathcal{B}}_{\mathcal{G},\nu} &= \langle b_0\sigma_0\rangle^{\mathcal{B}}_{\mathcal{G},\phi}\\ &= \langle b_0\sigma_0|E\rangle^{\mathcal{B}}_{\mathcal{G},\phi}\,\phi^{\mathcal{B}}_{\mathcal{G}}(E) + \langle b_0\sigma_0|E^c\rangle^{\mathcal{B}}_{\mathcal{G},\phi}\,\phi^{\mathcal{B}}_{\mathcal{G}}(E^c).\end{aligned} \tag{10}$$

Given E, $b_0\sigma_0 = 1$; it is easy to see that the second term vanishes. Thus,

$$\langle b_0\sigma_0\rangle^{\mathcal{B}}_{\mathcal{G},\nu} = \phi^{\mathcal{B}}_{\mathcal{G}}(E) = \mu^{\mathcal{B}}_{\mathcal{G}}(E). \tag{11}$$

Hence, (9) implies $\mu^{\mathcal{B}}_{\mathcal{G}}(E) \geq \epsilon > 0 \quad \forall$ finite $\mathcal{G} \subset \mathbb{Z}^2$. So we have percolation of ω bonds. The blue boundary condition now forces the percolating cluster to, in fact, be blue. In the thermodynamic limit, this gives us percolation of blue spins. $\square$

For the second direction of the argument we will make use of a result by Gandolfi, Keane, and Russo [GKR]. Their result requires a measure on $\mathbb{Z}^2$ that

- is invariant under translations and axis reflections
- is ergodic under vertical and horizontal translations
- satisfies the FKG condition: positive events are positively correlated.

Under these three conditions, if there is percolation, then an infinite cluster is unique with probability one. Furthermore, all other spin-types lie in finite star-connected clusters. (The definition of star-connectedness is as follows: two sites are said to be star-connected if they are nearest neighbors or next nearest neighbors; i.e. if neither their x nor their y coordinates differ, in modulus, by more than one.)

Let $\rho^{\mathcal{B}}(\underline{b}) \stackrel{\text{def}}{=} \lim_{\mathcal{G}\nearrow\mathbb{Z}^2} \rho^{\mathcal{B}}_{\mathcal{G}}(\underline{b})$, where $\rho^{\mathcal{B}}_{\mathcal{G}}(\underline{b})$ is the $b-$marginal distribution of $\nu^{\mathcal{B}}_{\mathcal{G}}(\underline{b},\underline{\sigma})$. We will demonstrate in the Appendix that this measure satisfies the above conditions.

Theorem 1b. *In the region $k \leq K/2$ of the Ashkin–Teller model on $\mathbb{Z}^2$, percolation of blues implies the existence of multiple Gibbs states.*

Proof. We remind the reader that $b = 1$ for the blue and yellow spins, whereas $b = 0$ for the red spins. The FKG property of $\rho^{\mathcal{B}}(\underline{b})$ (see Appendix), then, actually establishes the FKG property for the ordering blue, yellow $\geq$ reds. Suppose that we have percolation of blues. By Theorem 1a, if there were no percolation of blues in the blue state, then we would not see percolation in any purported state, all states being equivalent. Thus, blues are percolating in the blue state. Then certainly the blue-yellow spin combination percolates under these conditions. Since our blue measure satisfies the conditions of the GKR theorem, the blue-yellow infinite cluster is unique WP1, and all red clusters lie in finite star-connected clusters. Now we may produce at least 2 distinct Gibbs states: one corresponding to the blue-yellow percolation – a "green" state, and one for red percolation – a "red" state. We have just learned that these are mutually exclusive situations: Consider the event that the origin is part of an infinite cluster, given that the origin is blue or yellow; this event has positive probability in the green state, but has *zero* probability in the red state. Hence, these states are distinct, and we have non-uniqueness of Gibbs states. $\square$

Together, Theorems 1a & 1b give us our desired characterization.

APPENDIX

To facilitate the oncoming handling of boundary conditions, we will consider $\mathcal{G}$ to be a general finite graph (not necessarily a subset of $\mathbb{Z}^2$) and we will use the more general Hamiltonian $H = H^{\mathbb{L}}_{\underline{a},\underline{\tau}} + H^{\mathbb{K}}_{\underline{b},\underline{\sigma}}$ where $\mathbb{L}$, analogous to $\mathbb{K}$, represents a configuration of $\mathcal{L}_{\langle i,j\rangle}$, $l_{\langle i,j\rangle}$ on $\Theta_{\mathcal{G}}$ for the a & τ variables.

At this point, we take interest in the marginal distribution $\rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}) = \sum_{\underline{\sigma}} \nu_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b},\underline{\sigma})$, with weights denoted by $\mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b})$:

$$\mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}) = \mathcal{Z}_{\underline{a}}^{I,\mathbb{L}} \mathcal{Z}_{\underline{b}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{b})+\beta\psi^{\mathbb{L}}(\underline{a})} \tag{12}$$

where $\psi^{\mathbb{K}}(\underline{b}) = \sum_{\langle i,j\rangle\in\mathcal{G}}(\mathcal{K}_{\langle i,j\rangle} - k_{\langle i,j\rangle})b_i b_j$, and $\psi^{\mathbb{L}}(\underline{a}) = \sum_{\langle i,j\rangle\in\mathcal{G}}(\mathcal{L}_{\langle i,j\rangle} - l_{\langle i,j\rangle})a_i a_j$.

Lemma 1.

The measure $\rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b})$ *is strong FKG.*

Proof. This is almost identical to a result found in [C$_2$], with slightly different measures, so here it is: In this case the strong FKG property (which does imply the usual "weaker" FKG condition) is equivalent to the lattice condition. For this measure, the lattice condition states that for any two configurations η_1 and η_2 of b-variables on our graph, $\rho(\eta_1 \wedge \eta_2)\rho(\eta_1 \vee \eta_2) \geq \rho(\eta_1)\rho(\eta_2)$. (Here all subscripts and superscripts have been dropped.) The object $\eta_1 \vee \eta_2$ is a new configuration for which each site i chooses the higher value between b_i from η_1 and b_i from η_2. Similarly, $\eta_1 \wedge \eta_2$ chooses the lower value at each site.

Since b_i can only take the values 0 and 1, it is necessary and sufficient to check that:

$$\mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}^*)|_{b_u,b_v=0}\mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}^*)|_{b_u,b_v=1} \geq \mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}^*)|_{b_u=0,b_v=1}\mathcal{R}_{\mathcal{G}}^{\mathbb{L},\mathbb{K}}(\underline{b}^*)|_{b_u=1,b_v=0} \tag{13}$$

for arbitrary sites u, v, and for $\underline{b}^* =$ a *fixed* configuration of spins on all sites of $\mathcal{G}$, excluding u & v. Since $a_i = 1 - b_i$, and because $H_{\underline{a},\underline{\tau}}^{\mathbb{L}}$ and $H_{\underline{b},\underline{\sigma}}^{\mathbb{K}}$ are identical in form, it is sufficient to check this lattice condition for $H = H_{\underline{b},\underline{\sigma}}^{\mathbb{K}}$. Our desired inequality is as follows:

$$\begin{aligned}\left(\mathcal{Z}_{\underline{b}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{b})}\right)\Big|_{b_u,b_v=0} &\left(\mathcal{Z}_{\underline{b}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{b})}\right)\Big|_{b_u,b_v=1} \\ &\geq \left(\mathcal{Z}_{\underline{b}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{b})}\right)\Big|_{b_u=0,b_v=1} \left(\mathcal{Z}_{\underline{b}}^{I,\mathbb{K}} e^{\beta\psi^{\mathbb{K}}(\underline{b})}\right)\Big|_{b_u=1,b_v=0}.\end{aligned} \tag{14}$$

For ease of notation, we define:

$$-H_{\emptyset} \stackrel{\text{def}}{=} \sum_{\langle i,j\rangle : i,j\notin\{u,v\}} \mathcal{K}_{\langle i,j\rangle} b_i b_j (\delta_{\sigma_i\sigma_j} - 1)$$

$$-H_u \stackrel{\text{def}}{=} \sum_{\langle i,u\rangle : i\neq v} \mathcal{K}_{\langle i,u\rangle} b_i(\delta_{\sigma_i\sigma_j} - 1)$$

and $-H_v$ is defined accordingly. With this notation, after canceling the factor $e^{2\beta\sum_{\langle i,j\rangle\neq\langle u,v\rangle}(\mathcal{K}_{\langle i,j\rangle}-k_{\langle i,j\rangle})b_ib_j}_{b_u,b_v=0}$ from both sides, equation (14) reduces to:

$$e^{\beta(\mathcal{K}_{\langle u,v\rangle}-k_{\langle u,v\rangle})}\mathrm{Tr}[e^{-\beta(H_\emptyset+H_u+H_v)}e^{\beta\mathcal{K}_{\langle u,v\rangle}(\delta_{\sigma_u\sigma_v}-1)}]\mathrm{Tr}[e^{-\beta H_\emptyset}]$$
$$\geq \mathrm{Tr}[e^{-\beta(H_\emptyset+H_u)}]\mathrm{Tr}[e^{-\beta(H_\emptyset+H_v)}] \quad (15)$$

where the trace is understood to be taken over $\underline{\sigma}$.

Now we will simplify things further by proving the following inequality:

$$e^{\beta(\mathcal{K}_{\langle u,v\rangle}-k_{\langle u,v\rangle})}\mathrm{Tr}[e^{-\beta(H_\emptyset+H_u+H_v)}e^{\beta\mathcal{K}_{\langle u,v\rangle}(\delta_{\sigma_u\sigma_v}-1)}] \geq \mathrm{Tr}[e^{-\beta(H_\emptyset+H_u+H_v)}]. \quad (16)$$

Dividing by the RHS, and letting $\mathbb{E}_H(-)$ be the Ising expectation with respect to the Hamiltonian H, this inequality is equivalent to

$$e^{\beta(\mathcal{K}_{\langle u,v\rangle}-k_{\langle u,v\rangle})}\mathbb{E}_{H_\emptyset+H_u+H_v}[(1-e^{-\beta\mathcal{K}_{\langle u,v\rangle}})\delta_{\sigma_u\sigma_v} + e^{-\beta\mathcal{K}_{\langle u,v\rangle}}] \geq 1. \quad (17)$$

Using the fact that $\mathbb{E}_{H_\emptyset+H_u+H_v}(\delta_{\sigma_u\sigma_v}) \geq 1/2$, the left hand side of the equation is bounded below by $e^{\beta(\frac{\mathcal{K}_{\langle u,v\rangle}}{2}-k_{\langle u,v\rangle})}\cosh(\frac{\beta\mathcal{K}_{\langle u,v\rangle}}{2})$, which is always ≥ 1, since we are in the region where $\frac{\mathcal{K}_{\langle i,j\rangle}}{2} \geq k_{\langle i,j\rangle}$.

Having shown (16), the Lemma is implied by the following "alteration" of (15):

$$\mathrm{Tr}[e^{-\beta(H_\emptyset+H_u+H_v)}]\mathrm{Tr}[e^{-\beta H_\emptyset}] \geq \mathrm{Tr}[e^{-\beta(H_\emptyset+H_u)}]\mathrm{Tr}[e^{-\beta(H_\emptyset+H_v)}], \quad (18a)$$

which is tantamount to: $\mathbb{E}_{H_\emptyset+H_u}(e^{-\beta H_v}) \geq \mathbb{E}_{H_\emptyset}(e^{-\beta H_v})$.

Let $N(v)$ = {nearest neighbors of v, excluding u}, $\{T\}$ = the set of all subsets of $N(v)$, and

$$\Upsilon(T) = \prod_{i\in T}(1-e^{-\beta b_i \mathcal{K}_{\langle i,v\rangle}})\prod_{i\notin T} e^{-\beta b_i \mathcal{K}_{\langle i,v\rangle}}. \tag{19}$$

Now expanding FK style, (18a) continues to transform into:

$$\sum_T \left[\Upsilon(T)\mathbb{E}_{H_\emptyset+H_u}\left(\prod_{i\in T}\delta_{\sigma_i\sigma_v}\right)\right] \geq \sum_T \left[\Upsilon(T)\mathbb{E}_{H_\emptyset}\left(\prod_{i\in T}\delta_{\sigma_i\sigma_v}\right)\right]. \tag{18b}$$

This will certainly be true if for arbitrary T, the individual expectations in (18b) obey the inequality. As in [C$_1$], these expectations can be expressed with the Ising FK representation, in terms of probabilities of cluster events. Let $\mathbb{P}_H^{\text{RC}}(-)$, $\langle - \rangle_H^{\text{RC}}$ be the Random Cluster (RC) probability and expectation corresponding to the Ising Hamiltonian H, and let $|T|$ be the number of sites in T. Given a bond configuration ω from this representation, let $C_T(\omega)$ be the number of connected components containing sites in T. Converting from spin system expectations:

$$\begin{aligned}\mathbb{E}_H\left(\prod_{i\in T}\delta_{\sigma_i\sigma_v}\right) &= \sum_{n=1}^{|T|}(\frac{1}{2})^{n-1}\mathbb{P}_H^{\text{RC}}\{C_T=n\}\\ &= \left\langle \frac{1}{2^{C_T-1}}\right\rangle_H^{\text{RC}}.\end{aligned} \tag{20}$$

So we want:

$$\left\langle \frac{1}{2^{C_T-1}}\right\rangle_{H_\emptyset+H_u}^{\text{RC}} \geq \left\langle \frac{1}{2^{C_T-1}}\right\rangle_{H_\emptyset}^{\text{RC}} \tag{21}$$

which is true because $(\frac{1}{2})^{C_T-1}$ is an increasing function of bond configurations (added bonds $\to$ smaller C_T), and $\mathbb{P}_{H_\emptyset+H_u}^{\text{RC}}(-)$ FKG dominates $\mathbb{P}_{H_\emptyset}^{\text{RC}}(-)$. By definition, one measure FKG dominates a second measure if it assigns higher probabilities to all positive events than does the second measure. In this case, positive events will receive higher probabilities with H_u added to the Hamiltonian $H_\emptyset$. $\square$

Let us now consider $\rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$, which is defined as the marginal distribution obtained from the blue measure $\nu_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$, and which has weights:

$$R_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(\underline{b}) = e^{\beta\psi^{\mathbb{K}}(\underline{b})+\beta\psi^{\mathbb{L}}(\underline{a})}\mathcal{Z}_{\underline{a}}^{I,\mathbb{L}}\sum_{\underline{\sigma}} e^{\beta I_{\underline{b},\underline{\sigma}}^{\mathbb{K}}}\chi_{(\underline{\sigma}_{\partial\mathcal{G}}=1)}\chi_{(\underline{b}_{\partial\mathcal{G}}=1)} \tag{22}$$

where χ is the indicator function. Expanding into ω bonds with the constraint that all Ising spins on the boundary $\partial\mathcal{G}$ are fixed at $\sigma_i = 1$,

$$R_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(\underline{b}) = e^{\beta\psi^{\mathbb{K}}(\underline{b})+\beta\psi^{\mathbb{L}}(\underline{a})}\mathcal{Z}_{\underline{a}}^{I,\mathbb{L}}\sum_{\omega} B_p^{\mathbb{K}}(\omega)2^{C_{\mathrm{w}}(\omega)-1}\chi_{(\underline{b}_{\partial\mathcal{G}}=1)}. \tag{23}$$

Corollary. *$\rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$ is strong FKG.*

Proof. Without loss of generality, we may assume that $u, v \notin \partial\mathcal{G}$ and that $b_i = 1 \quad \forall i \in \partial\mathcal{G}$. If either of these conditions are violated, the lattice condition holds trivially. Let us define $\widehat{\mathcal{G}} = \mathcal{G}\cup\tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}} = \{\left\langle\widetilde{i,j}\right\rangle\}$, a set of new edges connecting each boundary site in $\mathcal{G}$ to every other boundary site in $\mathcal{G}$. For each added edge $\left\langle\widetilde{i,j}\right\rangle$, set the values $\mathcal{L}_{\left\langle\widetilde{i,j}\right\rangle} = l_{\left\langle\widetilde{i,j}\right\rangle} = k_{\left\langle\widetilde{i,j}\right\rangle} = 0$ and $\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle} \gg 1$. Notice that for all $\widehat{\omega}$ on $\widehat{\mathcal{G}}$ such that all $\left\langle\widetilde{i,j}\right\rangle$ bonds are occupied, $C(\widehat{\omega}) = C_{\mathrm{w}}(\omega)$. Considering the limit as all $\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle} \to \infty$, we find that $p_{\left\langle\widetilde{i,j}\right\rangle} = 1 - e^{-\beta\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle}b_ib_j} \to 1$; consequently, for all $\widehat{\omega}$ on $\widehat{\mathcal{G}}$ having a vacant $\left\langle\widetilde{i,j}\right\rangle$ bond, $B_p^{\widehat{\mathbb{K}}}(\widehat{\omega}) \to 0$. It follows that $\sum_{\widehat{\omega}\text{ on }\widehat{\mathcal{G}}} B_p^{\widehat{\mathbb{K}}}(\widehat{\omega})2^{C(\widehat{\omega})} \underset{\{\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle}\}\to\infty}{\longrightarrow} \sum_{\omega\text{ on }\mathcal{G}} B_p^{\mathbb{K}}(\omega)2^{C_{\mathrm{w}}(\omega)}$.

The key is that $\rho^{\mathbb{L},\mathbb{K},\mathcal{B}}(\underline{b}) = \lim_{\{\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle}\}\to\infty}\rho_{\widehat{\mathcal{G}}}^{\widehat{\mathbb{L}},\widehat{\mathbb{K}}}(\underline{b})$. Recalling that $b_ib_j = 1$ for all $\left\langle\widetilde{i,j}\right\rangle \in \tilde{\mathcal{G}}$, we find that the ratio of the respective weights =

$$e^{\beta\sum_{\tilde{\mathcal{G}}}\mathcal{K}_{\left\langle\widetilde{i,j}\right\rangle}(b_ib_j-1)}\frac{\sum_{\widehat{\omega}} B_p^{\widehat{\mathbb{K}}}(\widehat{\omega})2^{C(\widehat{\omega})}}{\sum_{\omega} B_p^{\mathbb{K}}(\omega)2^{C_{\mathrm{w}}(\omega)}} \to \text{const.} \tag{24}$$

And our result is clear. $\square$

Define the partial ordering $\mathbb{K}' \succ \mathbb{K}$ if for each $\langle i,j\rangle \in \mathcal{G}$, $\mathcal{K}'_{\langle i,j\rangle} \geq \mathcal{K}_{\langle i,j\rangle}, k'_{\langle i,j\rangle} = k_{\langle i,j\rangle}$. To establish the unique existence of an infinite volume limiting measure $\rho^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$, we need the following proposition.

Proposition. *For* $\mathbb{K}' \succ \mathbb{K}$,

$$\rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K}',\mathcal{B}}(-) \underset{\mathrm{FKG}}{\geq} \rho_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$$

where the subscript on the inequality denotes FKG domination of measures, as discussed after equation (21).

Proof. As in [C_2], we will express the ratio of pertinent weights as the Ising Random Cluster expectation of an increasing function. It is sufficient to consider the case where $\mathbb{K}'$ and $\mathbb{K}$ differ only in that $\mathcal{K}'_{\langle u,v\rangle} - \mathcal{K}_{\langle u,v\rangle} = \Delta_{uv} \geq 0$. Assume that we have $\underline{b}$ such that $b_i = 1 \quad \forall i \in \partial\mathcal{G}$. If u and v are both in $\partial\mathcal{G}$, the ratio of weights is simply $e^{\beta\Delta_{uv}b_u b_v}$, a decidedly increasing function of $\underline{b}$. For the case where u and v are not both on the boundary, a bit of manipulation gives:

$$\frac{R_{\mathcal{G}}^{\mathbb{L},\mathbb{K}',\mathcal{B}}(\underline{b})}{R_{\mathcal{G}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(\underline{b})} = 1 + (e^{\beta\Delta_{uv}b_u b_v}) \frac{\sum_\omega B_p^{\mathbb{K}'}(\omega) \sum_{\underline{\sigma}} \prod_{\langle i,j\rangle\in\omega} \delta_{\sigma_i\sigma_j} \chi_{(\underline{\sigma}_{\partial\mathcal{G}}=1)} \delta_{\sigma_u\sigma_v}}{\sum_\omega B_p^{\mathbb{K}}(\omega) 2^{C_{\mathrm{w}}(\omega)-1}}. \tag{25}$$

Defining T_{uv} to be the event that u and v are connected by ω bonds, and splitting the summation in the numerator according to this criterion, the ratio (25) becomes:

$$1 + (e^{\beta\Delta_{uv}b_u b_v}) \sinh(\frac{1}{2}\beta\Delta_{uv}b_u b_v)[1 + < \text{ḩ}i_{(T_{uv})} >^{\mathrm{RC}}]; \tag{26}$$

indeed, another increasing function of $\underline{b}$. □

Let $\rho_{\Lambda_{\mathrm{k}}}^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$ be the measure on a finite graph Λ_{k}, and let $\{\Lambda_{\mathrm{k}}\}$ be a nested sequence of finite graphs such that $\Lambda_{\mathrm{k}+1} \supset \Lambda_{\mathrm{k}}$, and $\Lambda_{\mathrm{k}} \nearrow \mathbb{Z}^2$ as $\mathrm{k} \to \infty$. By the previous proposition, $\rho^{\mathbb{L},\mathbb{K}',\mathcal{B}}_{\Lambda_{k+1}}(-)\mid_{\Lambda_k} \underset{\text{FKG}}{\geq} \rho^{\mathbb{L},\mathbb{K},\mathcal{B}}_{\Lambda_{k+1}}(-)\mid_{\Lambda_k}$ for $\mathbb{K}' \succ \mathbb{K}$. Let $Q_k = \{\langle i,j\rangle \in \Theta_{\Lambda_{k+1}} \backslash \Theta_{\Lambda_k}\}$. Notice that for the measure $\rho^{\mathbb{L},\mathbb{K}',\mathcal{B}}_{\Lambda_{k+1}}(-)\mid_{\Lambda_k}$, letting $\mathcal{K}_{\langle i,j\rangle} \to \infty \quad \forall \langle i,j\rangle \in Q_k$ essentially wires the boundary of Λ_k to the boundary of Λ_{k+1} with ω bonds. This forces the limiting measure to have blue boundary conditions on Λ_k, so that:

$$\lim_{\substack{\mathcal{K}_{\langle i,j\rangle}\to\infty \\ \forall\langle i,j\rangle\in Q_k}} \rho^{\mathbb{L},\mathbb{K}',\mathcal{B}}_{\Lambda_{k+1}}(-)\mid_{\Lambda_k} = \rho^{\mathbb{L},\mathbb{K},\mathcal{B}}_{\Lambda_k}(-). \tag{27}$$

It follows that

$$\rho^{\mathbb{L},\mathbb{K},\mathcal{B}}_{\Lambda_k}(-) \underset{\text{FKG}}{\geq} \rho^{\mathbb{L},\mathbb{K},\mathcal{B}}_{\Lambda_{k+1}}(-)\mid_{\Lambda_k}. \tag{28}$$

Hence the limiting measure $\rho^{\mathbb{L},\mathbb{K},\mathcal{B}}(-) \stackrel{\text{def}}{=} \lim_{k\to\infty} \rho^{\mathbb{L},\mathbb{K},\mathcal{B}}_{\Lambda_k}(-)$ exists, and by a squeezing argument is independent of the particular nesting sequence: it is unique. Notice that $\rho^{\mathbb{L},\mathbb{K},\mathcal{B}}(-)$ must also satisfy the lattice condition, and thus retains the strong FKG property.

We now restore $\mathbb{L} = \mathbb{K}$, with uniform couplings on $\mathcal{G} \subset \mathbb{Z}^2$.

Lemma 2. $\rho^{\mathcal{B}}(-)$ *is ergodic under translations, and invariant under translations and axis reflections.*

Proof. Without loss of generality, consider positive local cylinder events A and B. Let $x \in \mathbb{Z}^2$, and let T_x be the linear translation operator which moves a spin at the origin to the site x. Translation invariance follows by translating any sequence $\{\Lambda_k\}$ that defines $\rho^{\mathcal{B}}(A)$ and using this to obtain $\rho^{\mathcal{B}}(T_x A)$. Axis reflection invariance may be seen by simply choosing a nested sequence of Λ_k's symmetric about the axis in question. For ergodicity, we use our measure's FKG property and translation invariance to get $\rho^{\mathcal{B}}(AT_xB) \geq \rho^{\mathcal{B}}(A)\rho^{\mathcal{B}}(T_xB) = \rho^{\mathcal{B}}(A)\rho^{\mathcal{B}}(B)\ \forall x$. Conditioning on the event that the supports of A and $T_x B$ are each surrounded by disjoint circuits of blue, the reverse inequality is demonstrated in the limit as $|x| \to \infty$ by applying FKG dominance of measures. $\square$

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