CONSTRUCTION OF STABLE PERIODIC ORBITS FOR THE SPIN–ORBIT PROBLEM OF CELESTIAL MECHANICS

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ABSTRACT. Birkhoff periodic orbits associated to spin–orbit resonances in Celestial Mechanics and in particular to the Moon–Earth and Mercury–Sun systems are considered. A general method (based on a quantitative version of the Implicit Function Theorem) for the construction of such orbits with particular attention to "effective estimates" on the size of the perturbative parameters is presented and tested on the above mentioned systems. Lyapunov stability of the periodic orbits (for small values of the perturbative parameters) is proved by constructing KAM librational invariant surfaces trapping the periodic orbits.

KEYWORDS: Periodic orbits, Spin–orbit resonances, Stability.

§1. INTRODUCTION AND RESULTS

The study of periodic orbits in Celestial Mechanics is strongly motivated by the abundance of "resonant relations" existing in the solar system. In particular, in this paper, we are concerned with commensurabilities between the revolutional and the rotational period, i.e., with the so–called *spin–orbit* resonances (see, e.g., Celletti 1990, 1994, Goldreich and Peale 1966, 1970, Murdock 1978, Peale 1973, Wisdom 1987). As is well known, most of the evolved satellites of the solar system point always the same face toward the host planet (the most familiar example being, of course, that of our Moon). In such a case one speaks of 1:1 or "synchronous" spin–orbit resonance. The only exception to 1:1 spin–orbit resonances is provided by the Mercury–Sun system, which moves in a 3:2 resonance (in fact, the ratio between the revolutional period of Mercury around the Sun and its period of rotation amounts to 3/2 within a very good approximation).

In §2 we introduce a mathematical model describing an approximation of the spin–orbit problem. In particular we reduce such a problem to the study of a Hamiltonian equation of the form

$$\ddot{x} - \varepsilon f_x(x,t) = 0 , \tag{1.1}$$

where x represents the *librational* angle, ε is a positive "perturbative" parameter measuring the equatorial oblateness of the satellite and $f = f(x,t)$ is a smooth (x– and t–)periodic function, which depends also on the eccentricity of the satellite's orbit assumed to be Keplerian. A spin–orbit resonance of order $p : q$ is a Birkhoff periodic orbit with frequency $\omega = \frac{p}{q}$. We present a (general) method (§3), based on a quantitative version of the classical Implicit Function Theorem (applied to a Poincaré map associated to eq. (1.1)), which allows to construct such periodic orbits. In particular we provide explicit approximations to the initial conditions associated to the periodic orbit and we give explicit "effective" estimate on the equatorial oblateness parameter ε ensuring the existence of the periodic orbit. Results for the 1:1, 3:2, 2:1 resonances in the Moon–Earth and the Mercury–Sun systems are discussed in §4. In particular, we are able to prove the existence of a synchronous periodic orbit for the observed parameters of the Moon. Instead we cannot establish an analogous result for the 3:2 and 2:1 resonances: this suggests a greater robustness (and therefore a bigger probability of capture) of the 1:1 resonance compared with other resonances.

The Mercury–Sun case appears to be different: the existence of the three main resonances cannot be proved for "realistic" values of the parameters and a less pronounced discrepancy (compared with the Moon–Earth case) is found between the 1:1 and 3:2 resonances.

A comparison with the observed data on the libration in longitude given in the Astronomical Almanac is also provided.

Finally we consider the stability of the periodic orbits constructed in §4 and show that Lyapunov stability can be obtained by proving the existence of *librational* KAM invariant surfaces trapping the periodic orbits. In particular, in §5, we show that the "Siegel–Moser conditions" (Siegel and Moser 1971) for the existence of librational invariant surfaces are satisfied in our model–problem. Here, however, we do not pay attention about optimal estimates on the parameters: such estimates will be discussed in a future work.

Details on the results of §4 and §5 are provided, respectively, in Appendix A and B.

We close this introduction by mentioning that a further extension of this work might concern the computation of the actual ephemeris of the Moon: in fact one might use our approximate periodic orbit as a starting point to compute the effective lunar motion, using a strategy similar to that adopted by Hill (Hill, 1878).

§2. THE SPIN–ORBIT MODEL

In this section we discuss briefly the so–called "spin–orbit" model in Celestial Mechanics.

Let S be a triaxial ellipsoidal satellite moving around a central planet P. We denote by T_{rev} and T_{rot} the revolutional period of the satellite around P and the rotational period about an internal spin–axis. A $p:q$ *spin–orbit resonance* occurs whenever

$$\frac{T_{rev}}{T_{rot}} = \frac{p}{q} , \qquad \text{for } p,q \in \mathbf{N},\ q \neq 0 .$$

In particular, when $p=q=1$ we speak of 1:1 or *synchronous* spin–orbit resonance; in this case, the satellite always points the same face to the host planet. As is well known, most of the *evolved* satellites or planets of the solar system (like, e.g., the Moon) are trapped in a 1:1 resonance (Astronomical Almanac 1990). The only exception is provided by Mercury which is observed in a nearly 3:2 resonance. We introduce a mathematical model describing the spin–orbit coupling, assuming that
i) the center of mass of the satellite moves on a Keplerian orbit around P with semimajor axis a and eccentricity e (secular perturbations on the orbital parameters are neglected);
ii) the spin–axis is perpendicular to the orbit plane (i.e., we neglect the so–called "obliquity");
iii) the spin–axis coincides with the shortest physical axis (i.e., the axis whose moment of inertia is largest);
iv) dissipative effects as well as perturbations due to other planets or satellites are neglected.

Let $A<B<C$ be the principal moments of inertia of the satellite, let r and f be, respectively, the instantaneous orbital radius and the true anomaly of the Keplerian orbit, finally let x be the angle between the longest axis of the ellipsoid and the periapsis line (see Figure 1). Under assumptions $i)-iv)$, the equation of motion may be derived from the standard Euler's equations for rigid body and (in normalized units) takes the form

$$\ddot{x} + \varepsilon(\frac{1}{r})^3 \sin(2x-2f) = 0 , \tag{2.1}$$

where $\varepsilon \equiv \frac{3}{2}\frac{B-A}{C}$ is proportional to the equatorial oblateness coefficient $\frac{B-A}{C}$ (and the dot denotes time differentiation). The mean motion has been normalized to one, i.e. $2\pi/T_{rev} = 1$. Notice that (2.1) is trivially integrated when $A = B$ or in the case of zero orbital eccentricity (since $e = 0$ implies $r =$ constant, $f = \frac{2\pi}{T_{rev}}t$).

A "$p:q$ periodic orbit" (or "Birkhoff periodic orbit of rotation number p/q") is a solution of (2.1) such that

$$x(t + 2\pi q) \ = \ x(t) + 2\pi p \ ,$$

namely after q orbital revolutions the satellite makes p rotations about the spin–axis.

Due to assumption i), the quantities r and f are known Keplerian functions of the time; therefore we can expand (2.1) in Fourier series as

$$\ddot{x} \ + \ \varepsilon \sum_{m\neq 0, m=-\infty}^{\infty} W(\frac{m}{2}, e) \ \sin(2x - mt) \ = \ 0 \ , \tag{2.2}$$

where the coefficients $W(\frac{m}{2}, e)$ decay as powers of the orbital eccentricity as $W(\frac{m}{2}, e) \propto e^{|m-2|}$ (see Cayley 1859, for explicit expressions).

We simplify further the model as follows. According to iv), we neglected dissipative forces and gravitational attractions beside that of the central planet; one of the most important contribution comes from the non–rigidity of the satellite, which provokes a tidal torque due to the internal friction. Following Goldreich and Peale 1966, we can write the tidal torque as

$$\mathcal{T} \ = \ -\frac{3}{2}k_2 \ \frac{GM^2R^5}{a^6} \ \sin(2\delta) \ ,$$

where G is the gravitational constant, M is the mass of P, R is the satellite's mean radius, a its semimajor axis and k_2, δ are the so–called *Love number* and *lag angle* of high tide, which depends on the internal structure of the satellite. Since the magnitude of the dissipative effects is small compared to the gravitational term, we simplify further eq. (2.2) retaining only those terms whose magnitude is of the same order or bigger than the average effect of the tidal torque $\mathcal{T}$. Therefore we are led to an equation of the form

$$\ddot{x} \ + \ \varepsilon \sum_{m\neq 0, m=N_1}^{N_2} \tilde{W}(\frac{m}{2}, e) \ \sin(2x - mt) \ = \ 0 \ ,$$

where N_1 and N_2 are suitable integers and $\tilde{W}(\frac{m}{2}, e)$ are truncations of the coefficients $W(\frac{m}{2}, e)$, which are power series in the eccentricity. For example, in the case of the Moon–Earth system we obtain the following equation of motion:

$$\begin{aligned}
&\ddot{x} \ + \ \varepsilon \ [(-\frac{e}{2} + \frac{e^3}{16}) \ \sin(2x - t) \ + \\
&\quad + (1 - \frac{5}{2}e^2 + \frac{13}{16}e^4) \ \sin(2x - 2t) \ + \ (\frac{7}{2}e - \frac{123}{16}e^3) \ \sin(2x - 3t) \ + \\
&\quad + (\frac{17}{2}e^2 - \frac{115}{6}e^4) \ \sin(2x - 4t) \ + \ (\frac{845}{48}e^3 - \frac{32525}{768}e^5) \ \sin(2x - 5t) \ + \\
&\quad + \frac{533}{16}e^4 \sin(2x - 6t) \ + \ \frac{228347}{3840}e^5 \ \sin(2x - 7t) \ ] \ = \ 0 \ ,
\end{aligned} \tag{2.3}$$

having taken $N_1 = 1$ and $N_2 = 7$ in (2.2). In the Mercury–Sun case the above criterion leads to the values $N_1 = -17$ and $N_2 = 6$: however we shall make one more simplification taking again $N_1 = 1$ and $N_2 = 7$.

§3. CONSTRUCTION OF BIRKHOFF PERIODIC ORBITS

Motivated by the model described in the previous section, here we show how to *construct* certain periodic solutions of the second order equations

$$\ddot{x} = \varepsilon f_x(x,t) \ , \tag{3.1}$$

where f is a smooth (say C^2) periodic function of x and t (with period 2π) and ε is a scalar "perturbative parameter".

Equation (3.1) is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon f_x(x,t) \ , \end{aligned} \tag{3.2}$$

which forms the Hamilton equations associated to the time–dependent Hamiltonian $H = \frac{1}{2}y^2 + f(x,t)$. Here y and x are standard symplectic variables; the cylinder $\mathbf{R} \times \mathbf{T}$ is the phase space ($\mathbf{T}$ being the circle $\mathbf{R}/(2\pi\mathbf{Z})$), while $\mathbf{R} \times \mathbf{T}^2$ is the so–called generalized phase space.

We are interested in "continuing" (and constructing) non–degenerate (and, in particular, elliptic) equilibria of (3.1), *for as large as possible values of the parameter* ε, *so as to obtain "Birkhoff periodic orbits"* $t \to x(t)$ *with rotation number (or "frequency")* $\omega = p/q$ (for given positive integers p and q).
This means that $x(t)$ is a periodic solution of (3.1) with period $T = 2\pi q$ which "winds around" the cylinder $\mathbf{R} \times \mathbf{T}$ p times:

$$x(t + 2\pi q) = x(t) + 2\pi p \qquad \Big(y(t + 2\pi q) = y(t) \Big) \ .$$

Since $t \to f(x,t)$ is 2π periodic, by uniqueness of the solution for the Cauchy problem for (3.2), one has that $t \to (x(t), y(t))$ is a Birkhoff periodic orbit with rotation number $\omega = p/q$ of (3.2) if and only if $x(t) \equiv x(t;x,y)$ *and* $y(t) \equiv y(t;x,y)$ *form a solution of* (3.2) *with* $x(0;x,y) = x$, $y(0;x,y) = y$ *and*

$$\begin{aligned} &\int_0^{2\pi q} y(s)ds - 2\pi p = 0 \\ &\int_0^{2\pi q} f_x(x(s),s)ds = 0 \ . \end{aligned} \tag{3.3}$$

Our plan is therefore to solve problem (3.3) with the aid of a quantitative form of the Implicit Function Theorem, which we proceed to formulate.

Implicit Function Theorem *Let $\rho > 0$, $0 < \theta < 1$, $z_0 \in \mathbf{R}^n$ and let A be a compact set of $\mathbf{R}^p$. Let $F : (z, \alpha) \in \overline{B}_\rho(z_0) \times A \to F(z, \alpha) \in \mathbf{R}^n$ ($\overline{B}_\rho(z_0)$ denoting the closed ball of radius ρ and center z_0) be a continuous function with continuous and invertible Jacobian matrix $\frac{\partial F}{\partial z}(z_0, \alpha)$, for any $\alpha \in A$. Denote by $M(\alpha) \equiv \left(\frac{\partial F}{\partial z}(z_0, \alpha)\right)^{-1}$ and by m an upper bound on $\sup_A \|M\|$ ($\|\cdot\|$ denoting the standard "operator norm" on matrices). If*

$$\text{(i)} \quad \sup_{\overline{B}_\rho(z_0) \times A} \left\| I - M\frac{\partial F}{\partial z} \right\| \le \theta \ , \qquad \text{(ii)} \quad \sup_A |F(z_0, \alpha)| \le (1-\theta)\frac{\rho}{m} \ ,$$

then there exists a unique continuous function $\alpha \in A \to z(\alpha) \in \overline{B}_\rho(z_0)$ such that $F(z(\alpha), \alpha) \equiv 0$ for any $\alpha \in A$.

The proof is standard: conditions (i) and (ii) are immediately seen to guarantee that the map $u \in X \equiv C(A, \overline{B}_\rho(z_0)) \to \Phi u$ defined by

$$(\Phi u)(\alpha) \equiv u(\alpha) - M(\alpha)F(u(\alpha), \alpha)$$

is a contraction from X (equipped with the supremum–norm) into itself (with contraction constant θ). Thus, by the contraction mapping theorem, there is a unique fixed point in X which corresponds to $z(\alpha)$ in the thesis of the Implicit Function Theorem.

Remarks. (i) If F is C^2 in z then, choosing $\theta = 1/2$ and

$$\rho \equiv 2m \sup_A |F(z_0, \alpha)| \ ,$$

(and using the mean value theorem) one sees immediately that conditions (i) and (ii) above are enforced by the requirement

$$4m^2 \sup_A |F(z_0, \alpha)| \sup_{\overline{B}_\rho(z_0) \times A} \left\| \frac{\partial^2 F}{\partial z^2} \right\| \le 1 \ . \tag{3.4}$$

This is the form we shall use below. The choice of $\theta = 1/2$ is the one that "optimizes" condition (3.4).
(ii) The above formulation is meaningful also in the case A is a singleton, $A \equiv \{\alpha_0\}$.
(iii) Notice that it is not required to have an initial solution of the equation $F = 0$ but it is enough to have *an approximate solution* z_0 (uniformly in the parameter α).

To *construct* an initial approximation for (3.3) we proceed as follows. Considering ε as a small parameter we write explicitely the *first ε–order* of the general solution of (3.2) with initial data (x, y): let

$$\begin{cases} x_1(t) \equiv x_1(t; y, x) = \displaystyle\int_0^t y_1(s) ds \\ y_1(t) \equiv y_1(t; y, x) = \displaystyle\int_0^t f_x(x + ys, s) ds \ , \end{cases}$$

and let $\xi(t)$ and $\eta(t)$ be the solution of

$$\begin{cases} \dot{\xi} = \eta \ , & \xi(0) = 0 \ , \\ \dot{\eta} = \frac{1}{\varepsilon}[f_x(x + yt + \varepsilon x_1(t) + \varepsilon^2 \xi(t), t) - f_x(x + yt, t)] \ , & \eta(0) = 0 \ . \end{cases} \tag{3.5}$$

Then, as one readily verifies,

$$\begin{aligned} x(t) &\equiv x + yt + \varepsilon x_1(t) + \varepsilon^2 \xi(t) \\ y(t) &\equiv y + \varepsilon y_1(t) + \varepsilon^2 \eta(t) \end{aligned} \tag{3.6}$$

solve (3.2) with initial data $x(0) = x$ and $y(0) = y$. Notice that ξ and η are well defined and bounded also in ε at $\varepsilon = 0$ (interprete the right–hand–side of the second equation in (3.5) as $f_{xx}(x + yt, t)$).

The initial data (x, y) has now to be fixed so as to meet (3.3) (and will, of course, depend on ε). We shall take

$$x = x_0 + \varepsilon x_1 \ , \qquad y = y_0 + \varepsilon y_1 \ , \tag{3.7}$$

with x_i and y_i *independent* of ε. The choice of x_i and y_i will be made in the natural fashion: the problem (3.3) may be formally solved expanding in power series of ε and equating coefficients and x_0, x_1, y_0, y_1 will be taken as the first orders of such formal series. Keeping this in mind, one finds that

$$y_0 = \frac{p}{q} \ , \tag{3.8}$$

and that x_0 has to be a *nondegenerate critical point of the (periodic) function*

$$\beta \to \int_0^{2\pi q} f_x(\beta + y_0 s, s) ds \ ,$$

i.e., x_0 is such that

$$\int_0^{2\pi q} f_x(x_0 + y_0 s, s) ds = 0 \ , \qquad \tau \equiv \int_0^{2\pi q} f_{xx}(x_0 + y_0 s, s) ds \neq 0 \ . \tag{3.9}$$

In fact we shall see in Appendix B that the sign of τ determines the type of the solution: $\tau > 0$ corresponds to *hyperbolic* periodic solutions while $\tau < 0$ to *elliptic* ones. The next "orders" x_1 and y_1 are given by

$$y_1 = -\frac{1}{2\pi q} \int_0^{2\pi q} \int_0^t f_x(x_0 + y_0 s, s) ds dt \ , \tag{3.10}$$

$$x_1 = -\frac{1}{\int_0^{2\pi q} f_{xx}^0(t) dt} \ [y_1 \int_0^{2\pi q} t f_{xx}^0(t) dt + \int_0^{2\pi q} f_{xx}^0(t) x_1(t; y_0, x_0) dt] \ , \tag{3.11}$$

where $f^0_{xx}(t) \equiv f_{xx}(x_0 + y_0 t, t)$.

Having fixed such *approximate* initial data, the function $\xi(t)$ and $\eta(t)$ in (3.5) are determined and hence the whole solution $x(t)$ and $y(t)$ in (3.2) is also uniquely determined and one can proceed to apply the Implicit Function Theorem performing the necessary "a–priori estimates" on the solution $x(t)$, $y(t)$ (see Appendix A for detailed estimates).

We remark that the word "approximate" refers to the equation (3.3) and not to equations (3.2) of which $x(t)$ and $y(t)$ give an *exact* solution with initial data (3.7). We also notice that having choosen the first two nontrivial orders in ε is of course rather arbitrary (for example one could take *higher ε–order approximations*).

§4. PERIODIC SOLUTIONS FOR THE SPIN–ORBIT PROBLEM

Here we apply the theory described in the previous section to the spin–orbit model discussed in §2.

We consider the two most significative examples of spin–orbit coupling, namely the Moon–Earth and Mercury–Sun systems. As everybody looking up in the sky knows, the Moon–Earth system lies in a 1:1 spin–orbit resonance. The Mercury–Sun system lies instead in a 3:2 resonance. Here, besides the 1:1 and 3:2 resonances, we shall consider also the 2:1 resonance since numerical computations show that the 2:1 is surrounded, in phase space, by a "librational region" which appears to be larger than the ones associated to the remaining resonances.

In applying the theory of §3 *we shall fix the value of the eccentricity equal to the astronomically observed one*. For the astronomical observed values of the parameters for the Moon–Earth and Mercury–Sun systems see Table 1.

The mathematical results are listed in Table 2, which shows the maximum value of the perturbing parameter ε for which we can establish the existence of a periodic orbit with frequency $\frac{p}{q}$, associated to a $p : q$ spin–orbit resonance. We stress that the results are obtained for the *true* values of the eccentricities, i.e. $e = 0.0549$ for the Moon and $e = 0.2056$ for Mercury.

Table 2 shows that the existence of a synchronous periodic orbit close to the actual motion of the Moon can be proved for values of the perturbing parameter bigger than the corresponding physical value. Instead a stable 3:2 or 2:1 periodic orbit for the Moon *cannot* be established for values of the parameter consistent with the observations. This remark suggests that the most likely ending state for the Moon is toward the synchronous resonance, validating previous results (Goldreich and Peale 1966, Henrard 1985) on probablities of capture into a resonance. Less evident is the situation for Mercury, for which the discrepancy between the theoretical results on resonance's stability is less pronounced.

In Figure 2 we compare our theoretical results with the actual motion of the Moon as it can be found in the Astronomical Almanac, 1990. More precisely, we plot the value of the x–coordinate over a year every lunar month ($\sim 27^d$). These values are obtained by integrating (with a leap–frog method) the equation of motion (2.3) with initial data $\hat{x} = x_0 + \varepsilon x_1$, $\hat{y} = y_0 + \varepsilon y_1$, where x_0, x_1, y_0, y_1 can be explicitely computed through eq.s (3.8)–(3.11) of §3. Since $(\hat{x}, \hat{y})$ provide a good approximation of the theoretical location of the 1:1 periodic orbit, the successive x–values are almost the same every lunar month. We compare these results with the *libration in longitude* provided by the 1990 Astronomical Almanac. The small oscillations of the observed data are due to the physical librations of the Moon. We remark that the difference between our theoretical periodic orbit and the astronomical variation amounts to some thousandths of degree.

In Figure 3 we compare the values obtained plotting over one lunar month the theoretical solution

$$x(t) \ = \ x_* + y_* t + \varepsilon \ x_1(t) \ ,$$

where $x_1(t) = \sum_{j=1}^{7} c_j \ (\frac{\sin((2y_*-j)t+2x_*)-\sin(2x_*)}{(2y_*-j)^2} - \frac{\cos(2x_*)t}{(2y_*-j)})$ (here x_* and y_* are the values x and y given in (3.7)). More precisely, the value of the angle $x(t)$ was decreased by the revolutional mean motion and compared to the average over one solar year of the librational values provided by the 1990 Astronomical Almanac.

As is well known, periodic orbits are often used as the starting points for computing the effective motion of solar system objects. For example, concerning the Moon, Hill (Hill 1878) found an exact special orbit and computed neighboring trajectories. More precisely, the idea is to solve the variational equations around a suitable periodic orbit and to recover the actual ephemeris within a good precision. We suggest that a similar method might be implemented, using as a starting point the orbits constructed above, to derive a semi–analytical theory of the Moon's physical librations.

§5. KAM STABILITY FOR THE ELLIPTIC CASE

Lyapunov stability of the periodic orbits constructed in §3 can be obtained by proving the existence of surrounding *librational* invariant surfaces. More precisely, according to (Siegel and Moser 1971), we consider the 2–dimensional area–preserving Poincaré map associated to (2.3), having the origin as a fixed point. We reduce ourselves to study the stability of the origin, which implies the stability of the periodic orbit for the system (2.3). As an example, we study the synchronous periodic orbit having period $T = 2\pi$.
We outline the sketch of the proof, referring to the Appendix B for further details. Following (Siegel and Moser 1971), we reduce the Poincaré map for (3.2) to the form

$$\begin{aligned} \eta_1' &= a\eta_1 + b\eta_2 + p(\eta_1, \eta_2) \\ \eta_2' &= c\eta_1 + d\eta_2 + q(\eta_1, \eta_2) \ , \end{aligned} \tag{5.1}$$

where $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix associated to the linear part and $p(\eta_1, \eta_2)$, $q(\eta_1, \eta_2)$

are higher order polynomials (with degree greater or equal than 3) in η_1, η_2. Since, in the elliptic case, the matrix S has complex conjugated eigenvalues $(\lambda, \overline{\lambda})$, we proceed by reducing (5.1) to a diagonal form through a symplectic coordinate change $(\eta_1, \eta_2) \to (\tilde{x}, \tilde{y})$

$$\begin{aligned} \tilde{x}' &= \lambda \tilde{x} + \tilde{p}(\tilde{x}, \tilde{y}) \\ \tilde{y}' &= \overline{\lambda} \tilde{y} + \tilde{q}(\tilde{x}, \tilde{y}) \ , \end{aligned} \tag{5.2}$$

for some complex polynomials $\tilde{p}(\tilde{x}, \tilde{y})$, $\tilde{q}(\tilde{x}, \tilde{y})$. Next we perform a symplectic transformation which conjugates (5.2) to the form

$$\begin{aligned} \xi' &= \xi \ e^{i(\gamma_0 + \gamma_1 \xi\eta + \gamma_2 (\xi\eta)^2 + \ldots)} \\ \eta' &= \eta \ e^{-i(\gamma_0 + \gamma_1 \xi\eta + \gamma_2 (\xi\eta)^2 + \ldots)} \ , \end{aligned}$$

where the coefficients γ_j depend on ε and e. According to Siegel and Moser the existence of an invariant curve around the origin is guaranteed by the condition that at least one of the coefficients γ_j, for $j \geq 1$, is non–zero. In particular, an explicit computation shows that the leading order in ε is given by

$$\gamma_1 \ = \ \frac{(3 + \varepsilon T^2) T}{2(3 - 4\varepsilon T^2 + \varepsilon^2 T^4)} \ \sqrt{2\varepsilon T^2 - \frac{2}{3} \varepsilon^2 T^4} \ ,$$

which implies that the origin of the Poincaré map is a stable fixed point, providing the stability of the above periodic orbits of the differential system (2.3).

APPENDIX A:

In this appendix we perform the main estimates in order to prove the results of §4, using the Implicit Function Theorem of §3. We denote by $x_* = x \equiv x_0 + \varepsilon x_1$, $y_* = y \equiv y_0 + \varepsilon y_1$ as in (3.7) and $z_0 \equiv (x_*, y_*)$. In particular we provide explicit estimates to check condition (3.4), i.e.

$$4\ \|M\|^2\ \|\frac{\partial^2 F}{\partial(x,y)^2}\|_\rho\ \|F(x_*, y_*)\|\ \leq\ 1\ ,$$

with

$$\rho\ =\ 2\ \|M\|\ \|F(x_*, y_*)\|\ ;$$

here the parameter vector α is replaced by ε varying in the interval $A \equiv [0, \varepsilon_0]$. For simplicity we have denoted $\|\cdot\| \equiv \sup_A |\cdot|$, $\|\cdot\|_\rho \equiv \sup_{\overline{B}_\rho(z_0)\times A} \|\cdot\|$. From (3.3), (3.6) the function $F(x,y) = (F_1(x,y), F_2(x,y))$ is explicitely given by

$$F_1(x,y) \equiv yT + \varepsilon_0 \int_0^T \int_0^t f_x(x+ys,s)dsdt + \varepsilon_0^2 \int_0^T \eta(t;y,x)dt - 2\pi p$$

$$F_2(x,y) \equiv \int_0^T f_x(x+yt+\varepsilon_0 x_1(t;y,x) + \varepsilon^2 \xi(t;y,x),t)dt\ .$$

Let $f_x(x,t)$ be as in (2.3) – (3.2); the norm of $F(x_*, y_*)$ is obtained as

$$\|F(x_*, y_*; \varepsilon)\| \equiv \sup\{\|F_1(x_*, y_*; \varepsilon)\|, \|F_2(x_*, y_*; \varepsilon)\|\}$$

$$\|F_1(x_*, y_*; \varepsilon)\| \leq \varepsilon^2 \Big[\|f_{xx}\|\ \Big[|x_1| + |y_1|\frac{T}{3}\Big]\ \frac{T^2}{2} + \|\eta\|T\Big]$$

$$\|F_2(x_*, y_*; \varepsilon)\| \leq \varepsilon_0^2 \Big[\|f_{xx}\|T\|\xi\| + \frac{1}{2}\|f_{xxx}\|T\|\xi\|^2\varepsilon_0^2\Big]\ ,$$

(having choosen in $\mathbf{R}^2$ the sup–norm) where

$$\|x_1(t; x_*, y_*)\| \leq \sum_{j=1}^{7} |c_j|\ |\frac{\sin((2y_* - j)T + 2x_*) - \sin(2x_*)}{(2y_* - j)^2} - \frac{\cos(2x_*)T}{(2y_* - j)}| \qquad (A.1)$$

and

$$\|\xi\| \leq \frac{T^2}{2}\ \frac{\|f_{xx}\|\|x_1(t; x_*, y_*)\|}{1 - \varepsilon_0 \frac{T^2}{2}\|f_{xx}\|}$$
$$\|\eta\| \leq T\|f_{xx}\|\ \Big(\|x_1(t; x_*, y_*)\| + \varepsilon_0\|\xi\|\Big)\ . \qquad (A.2)$$

- Estimate of the norm of M: let

$$M \equiv \Big(\frac{\partial F(x_*, y_*; \varepsilon)}{\partial(x,y)}\Big)^{-1}\ .$$

Let us explicit the derivatives as follows. If we denote by

$$G_1(x,y) \equiv Ty - 2\pi p + \varepsilon \int_0^T y_1(t;y,x)dt$$
$$G_2(x,y) \equiv \int_0^T f_x(x+yt+\varepsilon x_1(t;y,x),t)dt \ ,$$

then, omitting the arguments of the functions, we obtain:

$$\begin{aligned}
\frac{\partial F_1}{\partial x} &= \frac{\partial G_1}{\partial x} + \varepsilon^2 \int_0^T \eta_x \ dt \\
\frac{\partial F_1}{\partial y} &= \frac{\partial G_1}{\partial y} + \varepsilon^2 \int_0^T \eta_y \ dt \\
\frac{\partial F_2}{\partial x} &= \frac{\partial G_2}{\partial x} + \varepsilon^2 \int_0^T f_{xx} \ \xi_x \ dt \\
\frac{\partial F_2}{\partial y} &= \frac{\partial G_2}{\partial y} + \varepsilon^2 \int_0^T f_{xx} \ \xi_y \ dt \ .
\end{aligned}$$

Denoting by

$$H_G \equiv \Big(\frac{\partial G(x_*,y_*)}{\partial(x,y)}\Big) \ ,$$

we can write $M^{-1} \equiv H_G + \varepsilon^2 \tilde H$, where

$$\tilde H \ \equiv \ \begin{pmatrix} \int_0^T \ \eta_x \ dt & \int_0^T \ \eta_y \ dt \\ \int_0^T \ f_{xx} \ \xi_x \ dt & \int_0^T \ f_{xx} \ \xi_y \ dt \end{pmatrix} \ .$$

Therefore an estimate on M can be obtained through the above quantities as

$$\|M\| \ \leq \ \frac{\|H_G^{-1}\|}{1-\varepsilon_0^2\|H_G^{-1}\| \ \|\tilde H\|} \ .$$

Now we provide estimates on H_G^{-1} and $\tilde H$. Let

$$H_G \equiv \Big(\frac{\partial G(x_*,y_*)}{\partial(x,y)}\Big) \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ ;$$

then

$$\|H_G^{-1}\| = \|\frac{1}{\alpha\delta-\beta\gamma}\| \ \cdot \ \sup\{\|\beta\|+\|\delta\|, \ \|\alpha\|+\|\gamma\|\} \ .$$

Denoting by $a = 2x_*$ and by $b_j = 2y_* - j$, one has

$$
\begin{aligned}
&\alpha = \varepsilon\|f_{xx}\|\frac{T^2}{2}\\
&\beta = T + \varepsilon\sum_{i=1}^{7} c_j\ \{-\frac{T}{b_j^2}\cos(a+b_jT) + \frac{2}{b_j^3}\sin(a+b_jT) - \frac{T}{b_j^2}\cos(a) - \frac{2}{b_j^3}\sin(a)\}\\
&\gamma = \|f_{xx}\|T\ (1+\varepsilon\|f_{xx}\|\frac{T^2}{2})\\
&\delta = \|f_{xx}\|T\ (\frac{T}{2} + \varepsilon\|f_{xx}\|\frac{T^3}{24})\ .
\end{aligned}
$$

Concerning the estimate on $\tilde{H}$ we have:

$$
\|\tilde{H}\|\ \equiv\ \sup\{\|H_{11}\| + \|H_{12}\|,\ \|H_{21}\| + \|H_{22}\|\}\ ,
$$

and

$$
\begin{aligned}
&\|H_{11}\| \leq T\|\eta_x\|\\
&\|H_{12}\| \leq T\|\eta_y\|\\
&\|H_{21}\| \leq \|f_{xx}\|T\|\xi_x\|\\
&\|H_{22}\| \leq \|f_{xx}\|T\|\xi_y\|\ ,
\end{aligned}
$$

where $\|\eta_x\|$, $\|\eta_y\|$, $\|\xi_x\|$, $\|\xi_y\|$ may be estimated as follows:

$$
\begin{aligned}
&\|\eta_x\| \leq T\|f_{xx}\|\ \Big(\|\frac{\partial x_1}{\partial x}\| + \varepsilon_0\|\xi_x\|\Big) + T\|f_{xxx}\|\Big(\|x_1\| + \varepsilon_0\|\xi\|\Big)\Big(1 + \varepsilon_0\|\frac{\partial x_1}{\partial x}\| + \varepsilon_0^2\|\xi_x\|\Big)\\
&\|\eta_y\| \leq T\|f_{xx}\|\ \Big(\|\frac{\partial x_1}{\partial y}\| + \varepsilon_0\|\xi_y\|\Big) + T\|f_{xxx}\|\Big(\|x_1\| + \varepsilon_0\|\xi\|\Big)\Big(\frac{T}{2} + \varepsilon_0\|\frac{\partial x_1}{\partial y}\| + \varepsilon_0^2\|\xi_y\|\Big)\\
&\|\xi_x\| \leq \frac{T^2\|f_{xx}\|\|\frac{\partial x_1}{\partial x}\| + T^2\|f_{xxx}\|\ \Big(\|x_1\| + \varepsilon_0\|\xi\|\Big)\Big(1 + \varepsilon_0\|\frac{\partial x_1}{\partial x}\|\Big)}{2\Big(1 - \frac{T^2}{2}\|f_{xx}\|\varepsilon_0 - \frac{T^2}{2}\|f_{xxx}\|\varepsilon_0^2\big(\|x_1\| + \varepsilon_0\|\xi\|\big)\Big)}\\
&\|\xi_y\| \leq \frac{T^2\|f_{xx}\|\|\frac{\partial x_1}{\partial y}\| + T^2\|f_{xxx}\|\Big(\|x_1\| + \varepsilon_0\|\xi\|\Big)\Big(\frac{T}{2} + \varepsilon_0\|\frac{\partial x_1}{\partial y}\|\Big)}{2\Big(1 - \frac{T^2}{2}\|f_{xx}\|\varepsilon_0 - \frac{T^2}{2}\|f_{xxx}\|\varepsilon_0^2\big(\|x_1\| + \varepsilon_0\|\xi\|\big)\Big)}\ .
\end{aligned}
\tag{A.3}
$$

The estimates for the derivatives of $\|x_1(t;x,y)\|$ (computed at the point (x_*, y_*)) are derived similarly as in $(A.1)$.

• Estimate on $\|\frac{\partial^2 F}{\partial(x,y)}\|$:

In order to give the estimate on $\|\frac{\partial^2 F}{\partial(x,y)}\|$ we need to provide the norms of the second derivatives of the functions ξ, η (the estimates on ξ, η and their first derivatives were already given in $(A.2)$, $(A.3)$). We find:

$$
\begin{aligned}
\|\xi_{xx}\| \leq & \Big[1-\frac{T^2}{2}\|f_{xx}\|\varepsilon_0-\frac{T^2}{2}\|f_{xxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big)\varepsilon_0^2\Big]^{-1} \;\; \frac{1}{2}\Big[T^2\|f_{xx}\|\|\frac{\partial^2 x_1}{\partial x^2}\| \\
& + 2T^2\Big(\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0\|\xi_x\|\Big)\ \|f_{xxx}\|\ \Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big) \\
& + T^2\|f_{xxxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big)\Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)^2 \\
& + \varepsilon_0 T^2\|\frac{\partial^2 x_1}{\partial x^2}\|\|\|f_{xxx}\|\Big(\|x_1\|+\|\varepsilon_0\|\xi\|\Big)\Big] \\
\|\eta_{xx}\| \leq & T\|f_{xx}\|\Big(\|\frac{\partial^2 x_1}{\partial x^2}\|+\varepsilon_0\|\xi_{xx}\|\Big)+2T\Big(\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0\|\xi_x\|\Big)\|f_{xxx}\| \\
& \Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)+T\|f_{xxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\
& \Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)^2+\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\
& T\|f_{xxx}\|\varepsilon_0\Big(\|\frac{\partial^2 x_1}{\partial x^2}\|+\varepsilon_0\|\xi_{xx}\|\Big) \\
\|\xi_{yy}\| \leq & \frac{T}{2}\Big[1-\frac{T^2}{2}\|f_{xx}\|\varepsilon_0-\frac{T^2}{2}\|f_{xxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big)\varepsilon_0^2\Big]^{-1} \;\; \Big[T\|f_{xx}\|\|\frac{\partial^2 x_1}{\partial y^2}\| \\
& + 2T\Big(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|\Big)\ \|f_{xxx}\|\ \Big(\frac{T}{3}+\varepsilon_0\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0^2\|\xi_y\|\Big) \\
& + \|f_{xxxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\
& \Big(\frac{T^3}{6}+T\varepsilon_0^2(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|)^2+\varepsilon_0\frac{T^2}{3}(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|)\Big) \\
& + \varepsilon_0 T\|\frac{\partial^2 x_1}{\partial y^2}\|\|\|f_{xxx}\|\Big(\|x_1\|+\|\varepsilon_0\|\xi\|\Big)\Big] \\
\|\eta_{yy}\| \leq & T\|f_{xx}\|\Big[\|\frac{\partial^2 x_1}{\partial y^2}\|+\varepsilon_0\|\xi_{yy}\|\Big]+2T\Big[\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|\Big]\|f_{xxx}\| \\
& \Big(\frac{T}{2}+\varepsilon_0\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0^2\|\xi_y\|\Big)+\|f_{xxxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big)\Big(\frac{T^3}{3}+T\varepsilon_0^2(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|)^2 \\
& + \varepsilon_0 T^2(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|)\Big)+\varepsilon_0 T\|f_{xxx}\|(\|x_1\|+\varepsilon_0\|\xi\|)\Big(\|\frac{\partial^2 x_1}{\partial y^2}\|+\varepsilon_0\|\xi_{yy}\|\Big) \\
\|\xi_{yx}\| \leq & \frac{1}{2}\Big[1-\frac{T^2}{2}\|f_{xx}\|\varepsilon_0-\frac{T^2}{2}\|f_{xxx}\|(\|x_1\|+\varepsilon_0\|\xi\|)\varepsilon_0^2\Big]^{-1} \;\; \Big[T^2\|f_{xx}\|\|\frac{\partial^2 x_1}{\partial y\partial x}\| \\
& + T^2\Big(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|\Big)\|f_{xxx}\|\Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)+\|f_{xxxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\
& \Big(1+\varepsilon_0(\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)^2 T^2\Big(\frac{T}{3}+\varepsilon_0\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0^2\|\xi_y\|\Big) \\
& + \varepsilon_0 T^2\|\frac{\partial^2 x_1}{\partial y\partial x}\|\|\|f_{xxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\
& + \|f_{xxx}\|\Big(\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0\|\xi_x\|\Big)T^2\Big(\frac{T}{3}+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_y\|\Big)\Big]
\end{aligned}
$$

$$\begin{aligned}\|\eta_{yx}\| \leq{}& T\|f_{xx}\|\Big(\|\frac{\partial^2 x_1}{\partial y\partial x}\|+\varepsilon_0\|\xi_{yx}\|\Big)+T\Big(\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0\|\xi_y\|\Big)\|f_{xxx}\| \\ & \Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|\Big)+\|f_{xxxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big)\Big(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\| \\ & +\varepsilon_0^2\|\xi_x\|\Big)\ T\Big(\frac{T}{2}+\varepsilon_0\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0^2\|\xi_y\|\Big)+\varepsilon_0 T\|f_{xxx}\|\Big(\|x_1\|+\varepsilon_0\|\xi\|\Big) \\ & \Big(\|\frac{\partial^2 x_1}{\partial y\partial x}\|+\varepsilon_0\|\xi_{yx}\|\Big)+\|f_{xxx}\|\Big(\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0\|\xi_x\|\Big) \\ & T\Big(\frac{T}{2}+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_y\|\Big)\ .\end{aligned}$$

Thus,

$$\|\frac{\partial^2 F}{\partial(x,y)}\| \;=\; \sup\{\|\frac{\partial^2 F_1}{\partial y^2}\|+\|\frac{\partial^2 F_1}{\partial x^2}\|+2\|\frac{\partial^2 F_1}{\partial y\partial x}\|,\ \|\frac{\partial^2 F_2}{\partial y^2}\|+\|\frac{\partial^2 F_2}{\partial x^2}\|+2\|\frac{\partial^2 F_2}{\partial y\partial x}\|\}\ ,$$

and

$$\begin{aligned}\|\frac{\partial^2 F_1}{\partial y^2}\|+\|\frac{\partial^2 F_1}{\partial x^2}\|+2\|\frac{\partial^2 F_1}{\partial y\partial x}\| \leq{}& \varepsilon_0\|f_{xxx}\|\Big[\frac{T^2}{2}+\frac{T^3}{3}+\frac{T^4}{12}\Big] \\ & +\varepsilon_0^2\Big[\|\eta_{xx}\|+\|\eta_{yy}\|+2\|\eta_{xy}\|\Big] \\ \|\frac{\partial^2 F_2}{\partial y^2}\|+\|\frac{\partial^2 F_2}{\partial x^2}\|+2\|\frac{\partial^2 F_2}{\partial y\partial x}\| \leq{}& \|f_{xxx}\|\Big[(\frac{T^3}{3}+(\varepsilon_0\|\frac{\partial x_1}{\partial y}\|+\varepsilon_0^2\|\xi_y\|)T^2+(\varepsilon_0\|\frac{\partial x_1}{\partial y}\| \\ & +\varepsilon_0^2\|\xi_y\|)^2 T)+T(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|)^2+2(\frac{T^2}{2}+(\varepsilon_0\|\frac{\partial x_1}{\partial y}\| \\ & +\varepsilon_0^2\|\xi_y\|)T)(1+\varepsilon_0\|\frac{\partial x_1}{\partial x}\|+\varepsilon_0^2\|\xi_x\|)\Big] \\ & +\|f_{xx}\|T\Big[\varepsilon_0\|\frac{\partial^2 x_1}{\partial y^2}\|+\varepsilon_0^2\|\xi_{yy}\|+\varepsilon_0\|\frac{\partial^2 x_1}{\partial x^2}\|+\varepsilon_0^2\|\xi_{xx}\| \\ & +2\varepsilon_0\|\frac{\partial^2 x_1}{\partial y\partial x}\|+2\varepsilon_0^2\|\xi_{yx}\|\Big]\ .\end{aligned}$$

In the above formulae, the function $x_1(t;x,y)$ and its derivatives are estimated on the domain of radius ρ, rather then being computed on the point (x_*,y_*). Let us remark that the function $x_1(t;x,y)$ involves a term which dominates due to the resonance relation; therefore, we evidentiate this term labelling it with the index k. In particular it is $k=2$ for the 1:1 resonance, $k=3$ for the 3:2, $k=4$ for the 2:1. More precisely, we write $x_1(t)\equiv x_1(t;x,y)$ as

$$\begin{aligned}x_1(t) ={}& \sum_{j=1,j\neq k}^{7} c_j\Big[\frac{\sin((2y-j)t+2x)-\sin(2x)}{(2y-j)^2}-\frac{\cos(2x)t}{(2y-j)}\Big] \\ & + c_k\Big[\frac{\sin((2y-k)t+2x)-\sin(2x)}{(2y-k)^2}-\frac{\cos(2x)t}{(2y-k)}\Big]\ .\end{aligned}$$

Denoting by $M \equiv T(2\rho + |2y_* - k|)$, $b \equiv 2(|x_*| + \rho)$, $N_l \equiv |2\rho - |2y_* - l||$ for $l = 1, ..., 7$, we obtain the following estimates:

$$\|x_1(t)\| \leq \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{2}{N_j^2} + \frac{T}{N_j}\Big] + |c_k| \ \Big[T^2 \frac{\sinh(M) - M}{M^2} + bT^2 \frac{\cosh(M) - 1}{M^2}\Big]$$

$$\|\frac{\partial x_1(t)}{\partial x}\| \leq 2 \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{2}{N_j^2} + \frac{T}{N_j}\Big] + 2|c_k| \ \Big[T^2 \frac{\cosh(M) - 1}{M^2} + bT^2 \frac{\sinh(M) - M}{M^2}\Big]$$

$$\|\frac{\partial^2 x_1(t)}{\partial x^2}\| \leq 4 \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{2}{N_j^2} + \frac{T}{N_j}\Big] + 4|c_k| \ \Big[T^2 \frac{\sinh(M) - M}{M^2} + bT^2 \frac{\cosh(M) - 1}{M^2}\Big]$$

$$\begin{aligned}\|\frac{\partial x_1(t)}{\partial y}\| \leq & \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{8}{N_j^3} + \frac{T}{N_j^2}\Big] + 2|c_k| \ \Big[2T^3 \frac{\sinh(M) - M}{M^3} + T^3 \frac{\cosh(M) - 1}{M^2} \\ & + 2bT^3 \frac{\cosh(M) - 1}{M^3} + bT^3 \frac{\sinh(M)}{M^2}\Big]\end{aligned}$$

$$\begin{aligned}\|\frac{\partial^2 x_1(t)}{\partial y \partial x}\| \leq & \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{16}{N_j^3} + \frac{8T}{N_j^2}\Big] + 4|c_k| \ \Big[2bT^3 \frac{\sinh(M) - M}{M^3} + bT^3 \frac{\cosh(M) - 1}{M^2} \\ & + 2T^3 \frac{\cosh(M) - 1}{M^3} + T^3 \frac{\sinh(M)}{M^2}\Big]\end{aligned}$$

$$\begin{aligned}\|\frac{\partial^2 x_1(t)}{\partial y^2}\| \leq & \ 4 \sum_{j=1, j\neq k}^{7} |c_j| \Big[\frac{12}{N_j^4} + \frac{6T}{N_j^3} + \frac{T^2}{N_j^2}\Big] + 4|c_k| \ \Big[T^4 \Big(6 \frac{\sinh(M) - M}{M^4} \\ & + 4 \frac{\cosh(M) - 1}{M^3} + \frac{\cosh(M)}{M^2}\Big) + bT^4 \Big(6 \frac{\cosh(M) - 1}{M^4} \\ & + 4 \frac{\sinh(M)}{M^3} + \frac{\cosh(M)}{M^2}\Big)\Big] \ .\end{aligned}$$

APPENDIX B:

In this appendix, following (Siegel and Moser, 1971) we provide details about the existence of KAM librational invariant surfaces around the synchronous periodic orbit. We do not claim to obtain optimal estimates on the parameters ensuring the stability of the periodic orbit, but just to prove that there exist invariant surfaces around the periodic orbit for *suitably* small values of the parameters. Therefore we consider the simpler problem with zero eccentricity, i.e.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\varepsilon \sin(2x - 2t) \equiv \varepsilon g(x,t) \end{aligned} \tag{B.1}$$

and perform all computations to first order in ε. Under suitable coordinate transformations, we reduce $(B.1)$ to the form

$$\begin{aligned} \xi' &= \xi \; e^{i(\gamma_0+\gamma_1\xi\eta+\gamma_2(\xi\eta)^2+\ldots)} \\ \eta' &= \eta \; e^{-i(\gamma_0+\gamma_1\xi\eta+\gamma_2(\xi\eta)^2+\ldots)} \end{aligned} \tag{B.2}$$

and we show that the first coefficient $\gamma_1 = \gamma_1|_{e=0}$ of the normal form is different from zero. Since this coefficient depends analytically on ε, e, we can conclude that the condition $\gamma_1(\varepsilon, e) \neq 0$ is satisfied for sufficiently small values of the parameters ε, e. We postpone to a later work the problem of proving the existence of invariant surfaces for realistic values of the parameters.

B.1 Linearization of (B.1)

Let $(\overline{x}, \overline{y})$ be initial conditions on the periodic orbit and let $P(\overline{x}, \overline{y}) \equiv (x(T), y(T)) = (x', y')$ be the Poincaré map at time $T = 2\pi q$. By $(B.1)$ we can rewrite the Poincaré map as

$$\begin{aligned} x' &= \overline{x} + yT + \varepsilon \int_0^T \int_0^s g(x(\tau; \overline{x}, \overline{y}), \tau) d\tau ds \\ y' &= \overline{y} + \varepsilon \int_0^T g(x(s; \overline{x}, \overline{y}), s) ds \ , \end{aligned} \tag{B.3}$$

which has $(\overline{x}, \overline{y})$ as a fixed point. We shift the fixed point to the origin by means of the canonical transformation

$$\begin{aligned} \eta_1 &= x - \overline{x} \\ \eta_2 &= y - \overline{y} \ , \end{aligned}$$

so that $(B.3)$ becomes

$$\begin{aligned} \eta_1' &= \eta_1 + \eta_2 T + \overline{y}T + \varepsilon \int_0^T \int_0^s g(x(\tau; \eta_1 + \overline{x}, \eta_2 + \overline{y}), \tau) d\tau ds \\ \eta_2' &= \eta_2 + \varepsilon \int_0^T g(x(s; \eta_1 + \overline{x}, \eta_2 + \overline{y}), s) ds \ . \end{aligned} \tag{B.4}$$

We recall that $(\overline{x},\overline{y})$ are power series in ε:

$$\overline{x} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$$
$$\overline{y} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + ... \ ,$$

where, in particular, $x_0 = 0$, $y_0 = 1$. Since

$$x(\tau;\overline{x},\overline{y}) = \overline{x} + \overline{y}\tau + \varepsilon \int_0^\tau \int_0^s g(x(t;\overline{x},\overline{y}),t) \ dtds \ ,$$

up to first order in ε we have

$$x(\tau;\overline{x},\overline{y}) = x_0 + y_0\tau + O(\varepsilon) \ .$$

Therefore, disregarding $O(\varepsilon^2)$, $(B.4)$ reduces to

$$\begin{aligned} \eta_1' &= \eta_1 + \eta_2 T + y_0 T + \varepsilon y_1 T + \varepsilon \int_0^T \int_0^s g(\eta_1 + x_0 + (\eta_2 + y_0)\tau,\tau) d\tau ds \\ \eta_2' &= \eta_2 + \varepsilon \int_0^T g(\eta_1 + x_0 + (\eta_2 + y_0)s, s) ds \ . \end{aligned} \tag{B.5}$$

The development of the r.h.s. of $(B.5)$ in power series of η_1, η_2 is performed using the periodicity conditions

$$\begin{aligned} y_1 T &= -\int_0^T \int_0^s g(x_0 + y_0\tau,\tau) \ d\tau ds \\ \int_0^T & g(x_0 + y_0 s, s) \ ds = 0 \ . \end{aligned} \tag{B.6}$$

By means of $(B.6)$ we obtain

$$\begin{aligned} y_1 T &+ \int_0^T \int_0^s g(\eta_1 + x_0 + (\eta_2 + y_0)\tau,\tau) d\tau ds \\ &= \int_0^T \int_0^s [g(\eta_1 + x_0 + (\eta_2 + y_0)\tau,\tau) - g(x_0 + y_0\tau,\tau)] d\tau ds \\ &= \int_0^T \int_0^s [g_x(x_0 + y_0\tau,\tau) \ (\eta_1 + \eta_2\tau) + \frac{1}{2} g_{xx}(x_0 + y_0\tau,\tau) \ (\eta_1 + \eta_2\tau)^2 \\ &\quad + \frac{1}{6} g_{xxx}(x_0 + y_0\tau,\tau) \ (\eta_1 + \eta_2\tau)^3] \ d\tau ds \\ &= -T^2\eta_1 - \frac{T^3}{3}\eta_2 + \frac{2}{3}T^2\eta_1^3 + \frac{2}{3}T^3\eta_1^2\eta_2 + \frac{1}{3}T^4\eta_1\eta_2^2 + \frac{1}{15}T^5\eta_2^3 \ . \end{aligned}$$

In a similar way, we obtain

$$\int_0^T g(\eta_1 + x_0 + (\eta_2 + y_0)s, s)ds = \int_0^T [g(\eta_1 + x_0 + (\eta_2 + y_0)s, s) - g(x_0 + y_0 s, s)]ds$$
$$= \int_0^T [g_x(x_0 + y_0 s)\ (\eta_1 + \eta_2\ s) + \frac{1}{2} g_{xx}(x_0 + y_0 s)\ (\eta_1 + \eta_2\ s)^2$$
$$+ \frac{1}{6} g_{xxx}(x_0 + y_0 s)\ (\eta_1 + \eta_2\ s)^3]ds$$
$$= -2T\eta_1 - T^2\eta_2 + \frac{4}{3}T\eta_1^3 + 2T^2\eta_1^2\eta_2 + \frac{4}{3}T^3\eta_1\eta_2^2 + \frac{1}{3}T^4\eta_2^3\ .$$

Therefore neglecting $O(\varepsilon^2)$ and polynomial terms of order higher than 3 in η_1, η_2, we rewrite $(B.5)$ as

$$\begin{aligned} \eta_1' &= a\eta_1 + b\eta_2 + p(\eta_1, \eta_2) \\ \eta_2' &= c\eta_1 + d\eta_2 + q(\eta_1, \eta_2)\ , \end{aligned} \tag{B.7}$$

where

$$a = 1 - \varepsilon T^2 \qquad b = T - \varepsilon\frac{T^3}{3} \qquad c = -2T\varepsilon \qquad d = 1 - \varepsilon T^2$$
$$p(\eta_1, \eta_2) = \varepsilon[\frac{2}{3}T^2\eta_1^3 + \frac{2}{3}T^2\eta_1^2\eta_2 + \frac{1}{3}T^4\eta_1\eta_2^2 + \frac{1}{15}T^5\eta_2^3]$$
$$q(\eta_1, \eta_2) = \varepsilon[\frac{4}{3}T\eta_1^3 + 2T^2\eta_1^2\eta_2 + \frac{4}{3}T^3\eta_1\eta_2^2 + \frac{1}{3}\eta_2^3]\ .$$

Provided $\varepsilon < \frac{3}{T^2}$, the eigenvalues (λ, μ) of the linear part are complex conjugated, $\mu = \overline{\lambda}$, and precisely $\lambda = \lambda_1 + i\lambda_2$, $\mu = \lambda_1 - i\lambda_2$, with

$$\lambda_1 = 1 - \varepsilon T^2\ , \qquad \lambda_2 = \sqrt{2\varepsilon T^2 - \frac{2}{3}\varepsilon^2 T^4}\ .$$

Remark: Notice that the solution is of *elliptic* type if the eigenvalues of the linear part of eq. $(B.7)$ are complex conjugate. Such eigenvalues are determined as the solution of the secular equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0\ .$$

The eigenvalues are complex conjugate, if the discriminant is negative, i.e. $\Delta \equiv (a+d)^2 - 4 < 0$, namely $-2 < a+d < 2$. Using the definition of d in terms of the function $g = g(x,t)$ and integrating by parts, one obtains

$$d = 1 + \varepsilon\int_0^T g_x(x_0 + y_0 s, s)s ds = 2 + \varepsilon T\int_0^T g_x(x_0 + y_0 s, s)ds - a\ ,$$

namely $a + d = 2 + \varepsilon T\int_0^T g_x(x_0 + y_0 s, s)ds$, so that the condition for ellipticity becomes

$$-4\ < \varepsilon T\int_0^T g_x(x_0 + y_0 s, s)ds\ < 0\ .$$

B.2 Reduction to diagonal form

Next step is to reduce $(B.7)$ to the form

$$\begin{aligned} \tilde{x}' &= \lambda\tilde{x} + \tilde{p}(\tilde{x},\tilde{y}) \\ \tilde{y}' &= \overline{\lambda}\tilde{y} + \tilde{q}(\tilde{x},\tilde{y}) \ . \end{aligned} \qquad (B.8)$$

Let $\zeta \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $z \equiv \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Retaining only linear terms, we have that $\zeta' = S\zeta$ and we want to look for a coordinate change $\zeta = Cz$, such that

$$z' = C^{-1}SCz \equiv Tz \qquad \text{with} \qquad T \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \ .$$

Setting $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the change of variables is provided by

$$\begin{aligned} \eta_1 &= \alpha\tilde{x} + \beta\tilde{y} \\ \eta_2 &= \gamma\tilde{x} + \delta\tilde{y} \ , \end{aligned}$$

with inverse transformation

$$\begin{aligned} \tilde{x} &= \delta\eta_1 - \beta\eta_2 \\ \tilde{y} &= -\gamma\eta_1 + \alpha\eta_2 \ , \end{aligned}$$

under the area–preserving requirement

$$\alpha\delta - \beta\gamma \ = \ 1 \ .$$

Therefore we obtain

$$\alpha = \frac{b\gamma}{\lambda - a} \ , \qquad \beta = \frac{b\delta}{\overline{\lambda} - a} \ , \qquad \gamma = \frac{(\lambda - a)(\overline{\lambda} - a)}{\overline{\lambda} - \lambda} \ \frac{1}{b\delta} \ ,$$

while δ is a free parameter. Under such transformation $(B.2)$ is reduced to

$$\begin{aligned} \tilde{x}' &= \lambda\tilde{x} + \tilde{p}(\tilde{x},\tilde{y}) \\ \tilde{y}' &= \overline{\lambda}\tilde{y} + \tilde{q}(\tilde{x},\tilde{y}) \ , \end{aligned}$$

where

$$\begin{aligned} \tilde{p}(\tilde{x},\tilde{y}) &= \delta p(\alpha\tilde{x} + \beta\tilde{y}, \gamma\tilde{x} + \delta\tilde{y}) - \beta q(\alpha\tilde{x} + \beta\tilde{y}, \gamma\tilde{x} + \delta\tilde{y}) \\ \tilde{q}(\tilde{x},\tilde{y}) &= -\gamma p(\alpha\tilde{x} + \beta\tilde{y}, \gamma\tilde{x} + \delta\tilde{y}) + \alpha q(\alpha\tilde{x} + \beta\tilde{y}, \gamma\tilde{x} + \delta\tilde{y}) \ . \end{aligned}$$

Notice that $\tilde{q}(\tilde{x}, \tilde{y}) = \overline{\tilde{p}(\tilde{x}, \tilde{y})}$.

B.3 Normal form and computation of γ_1

We look for a near–to–identity canonical transformation of the form

$$\begin{aligned}\tilde{x} &= \Phi(\xi, \eta) = \xi + \Phi_2(\xi, \eta) + \Phi_3(\xi, \eta) + \ldots \\ \tilde{y} &= \Psi(\xi, \eta) = \eta + \Psi_2(\xi, \eta) + \Psi_3(\xi, \eta) + \ldots ,\end{aligned}$$

where $\Phi_j(\xi, \eta)$ and $\Psi_j(\xi, \eta)$ are polynomial functions in ξ, η of degree j. For *elliptic* normal forms (Siegel and Moser 1971), the functions Φ_j and Ψ_j are aimed to transform $(B.8)$ to $(B.2)$, which we rewrite as

$$\begin{aligned}\xi \equiv u\xi &= e^{iw}\xi \\ \eta \equiv v\eta &= e^{-iw}\eta ,\end{aligned}$$

where $w = \gamma_0 + \gamma_1 \xi\eta + \gamma_2(\xi\eta)^2 + \ldots$ The functional equations for Φ and Ψ are

$$\begin{aligned}\Phi(u\xi, v\eta) &= \tilde{p}(\Phi(\xi, \eta), \Psi(\xi, \eta)) \\ \Psi(u\xi, v\eta) &= \tilde{q}(\Phi(\xi, \eta), \Psi(\xi, \eta)) .\end{aligned}$$

Since $\tilde{p}$ and $\tilde{q}$ are third–degree polynomials, we easily obtain $\Phi_2(\xi, \eta) = \Psi_2(\xi, \eta) = 0$, while $\Phi_3(\xi, \eta)$, $\Psi_3(\xi, \eta)$ must satisfy the relations

$$\begin{aligned}\Phi_3(\lambda\xi, \overline{\lambda}\eta) + i\lambda\gamma_1\xi^2\eta &= \lambda\Phi_3(\xi, \eta) + \tilde{p}(\xi, \eta) \\ \Psi_3(\lambda\xi, \overline{\lambda}\eta) - i\lambda\gamma_1\xi\eta^2 &= \overline{\lambda}\Psi_3(\xi, \eta) + \tilde{q}(\xi, \eta) .\end{aligned} \qquad (B.9)$$

Let

$$\begin{aligned}\tilde{p}(\xi, \eta) &= p_{30}\xi^3 + p_{21}\xi^2\eta + p_{12}\xi\eta^2 + p_{03}\eta^3 \\ \Phi_3(\xi, \eta) &= \Phi_{30}\xi^3 + \Phi_{21}\xi^2\eta + \Phi_{12}\xi\eta^2 + \Phi_{03}\eta^3\end{aligned}$$

and similarly for $\tilde{q}(\xi, \eta)$ and $\Psi_3(\xi, \eta)$. From $(B.9)$ we obtain

$$\begin{aligned}i\lambda\gamma_1 &= p_{21} \\ i\overline{\lambda}\gamma_1 &= q_{12} ,\end{aligned}$$

namely

$$\gamma_1 = \frac{p_{21} + q_{12}}{2i\lambda_1} ,$$

which provides

$$\gamma_1 = \frac{(3 + \varepsilon T^2)T}{2(3 - 4\varepsilon T^2 + \varepsilon^2 T^4)} \sqrt{2\varepsilon T^2 - \frac{2}{3}\varepsilon^2 T^4} .$$

According to Siegel and Moser, since $\gamma_1 \neq 0$ we can conclude that for ε and e sufficiently small there exists an invariant curve around the elliptic fixed point of the Poincaré map associated to (2.3). This result implies the stability of the periodic orbit for the differential system (2.3).

TABLES

	ε	e
Moon–Earth	$3.45 \cdot 10^{-4}$	0.0549
Mercury–Sun	$1.5 \cdot 10^{-4}$	0.2056

Table 1. Astronomically oberved values for oblateness (ε) and eccentricity (e).

	1 : 1	3 : 2	2 : 1
Moon–Earth	$\varepsilon_0 = 7.1 \cdot 10^{-4}$	$\varepsilon_0 = 7.8 \cdot 10^{-6}$	$\varepsilon_0 = 1.1 \cdot 10^{-5}$
Mercury–Sun	$\varepsilon_0 = 8.8 \cdot 10^{-5}$	$\varepsilon_0 = 5.7 \cdot 10^{-6}$	$\varepsilon_0 = 2.8 \cdot 10^{-5}$

Table 2. Theoretical values for the existence of periodic orbits for $\varepsilon \le \varepsilon_0$.

Figure 1: The spin–orbit geometry.

Figure 2: Values of the libration angle x over 1 year, computed every lunar month. Theoretical predictions ($*$) and astronomical observations ($\circ$).

Figure 3: The difference between the theoretical value of the libration angle x and the revolutional mean motion. The total period corresponds to 1 lunar month. Theoretical predictions ($*$) and the average over one solar year of the astronomical observations ($\circ$).

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