Classical Limits of Euclidean Gibbs States for Quantum Lattice Models

Sergio Albeverio
Abteilung für Stochastik,
Institut für Angewandte Mathematik, Universität Bonn,
D 53115 Bonn (Germany);
Forschungszentrum BiBoS, Bielefeld (Germany);
SFB 237 (Essen–Bochum–Düsseldorf) (Germany);
CERFIM and USI, Locarno (Switzerland)
e-mail albeverio@uni–bonn.de

Yuri Kondratiev
Abteilung für Stochastik,
Institut für Angewandte Mathematik, Universität Bonn,
D 53115 Bonn (Germany);
Forschungszentrum BiBoS, D 33615 Bielefeld (Germany);
Institute of Mathematics, Kiev (Ukraine)
e-mail kondratiev@uni-bonn.de

Yuri Kozitsky
Institute of Mathematics, Maria Curie-Sklodowska University
PL 20-031 Lublin (Poland);
Institute for Condensed Matter Physics, Lviv (Ukraine)
e-mail jkozi@golem.umcs.lublin.pl

Abstract

Models of quantum and classical particles on the d–dimensional lattice $\mathbb{Z}^d$ with pair and many–particle interactions of a general type are considered. The classical model is obtained from the corresponding quantum one when the reduced physical mass of the particle $m = \mu/\hbar^2$ tends to infinity. For these models, it is proved that in the case of pair interactions every net of quantum conditional Gibbs measures weakly converges to a classical conditional Gibbs measure when m tends to infinity. In the case of general type interactions, it is proved that every conditional Gibbs measure of the classical model may be obtained as a weak limit of corresponding conditional Gibbs measures of the quantum model. The convergence of the probability kernels and periodic Gibbs measures for such models has also been proven. The latter convergence is then used to show the convergence of the order parameters which describe the phase transitions in the translation invariant models with pair interactions of the type cosidered.

1 Formulation of the Results

1.1 Introduction

We consider a system of interacting particles performing one– dimensional oscillations around their equilibrium positions which form a d– dimensional lattice $\mathbb{Z}^d$. This system serves as a base for two models. The first one is a quantum model described by the following formal Hamiltonian

$$H = \sum_k H_k^{(0)} + \sum_k U_k(x_k) + \frac{1}{2}\sum_{j,k} J_{jk}x_j x_k, \tag{1.1}$$

where all sums are taken over the lattice $\mathbb{Z}^d$. Here $H_k^{(0)}$ is the Hamiltonian of a harmonic oscillator defined in the complex Hilbert space $L^2(\mathbb{R}, dx_k)$ (see e.g. [11])

$$H_k^{(0)} = -\frac{1}{2m}\Delta + \frac{1}{2}x_k^2, \tag{1.2}$$

Δ stands for the Laplacian and $m > 0$ is the reduced physical mass of the particle, which one obtains dividing the physical mass by $\hbar^2$, where $\hbar$ is the Planck's constant. The second term in (1.1) contributes to the potential energy of the particles making it to be anharmonic. The third term describes the interaction between the particles. Below we also consider a more general choice of the interaction potentials. For the matrix J_{jk}, we assume that there exists $r > 0$ such that $J_{jk} = 0$ whenever the Euclidean distance $|j-k|$ between j and k exceeds this r. We also suppose that

$$c \stackrel{\mathrm{def}}{=} \sup_j \sum_k |J_{jk}| < \infty. \tag{1.3}$$

For the functions $U_k : \mathbb{R} \to \mathbb{R}$, we assume that all of them are continuous and lower semi–bounded such that the following estimate holds, for all $k \in \mathbb{Z}^d$ and all $x \in \mathbb{R}$,

$$U_k(x) \geq cx^2 + b. \tag{1.4}$$

The second model is a classical version of the model described above. Its formal Hamiltonian has the form

$$H^{\rm cl} = \sum_k \left(\frac{1}{2} x_k^2 + U_k(x_k) \right) + \frac{1}{2} \sum_{j,k} J_{jk} x_j x_k, \tag{1.5}$$

which means that in this case only the potential energy is taken into account (see [1]). Formally the latter Hamiltonian may be obtained from the quantum one (1.1) by passing to the limit $m \to +\infty$.

The aim of this work is to study the possible convergence of the Gibbs states of the quantum model to the Gibbs states of the classical model. Let us remark that the convergence of certain Green functions, describing the Gibbs states, in similar models with a special choice of the interaction potentials has already been proved in the paper [1].

For the quantum lattice models, the Gibbs states are constructed as positive functionals on the algebras of observables (see e.g. [12], [16]), in contrast to the case of classical models where they are built by means of conditional probability distributions (see [13], [14]) as measures which solve the equilibrium equations. But for the quantum models with unbounded operators, which we consider here, the algebraic approach does not allow to construct such states of infinite systems. In 1975, S. Albeverio and R. Høegh-Krohn [1] have initiated an approach to the construction of Gibbs states of quantum lattice models which employs the infinite dimensional integration in path spaces (see also [2], [8]–[10], [15], [16], and [21]). Here the state of an infinite system at temperature β^{-1} is defined by a measure ν_β on certain space Ω_β. As in the case of classical systems, this measure solves the equilibrium equation and the Gibbs state as a positive functional may be uniquely reconstructed, analogously to the Euclidean quantum field theory, by means of moments of the of ν_β, which here are the temperature Green functions. That is why the measure ν_β is known as *the Euclidean Gibbs state* of the quantum model. The additional advantage of such *Euclidean* approach is that the Gibbs states in quantum and in classical models may be considered in one setting, that we use in this paper. The main features of our technique may be found in [3] – [7], [15].

1.2 Setup and Main Results

Let $\mathcal{L}$, $\mathcal{L}_{\text{fin}}$ denote the set of all, respectively of all finite, subsets of $\mathbb{Z}^d$. For certain value of inverse temperature $\beta > 0$, consider the space of continuous functions (*temperature loops*) taking equal values at the endpoints of the interval $[0, \beta]$

$$C_\beta^{\text{eq}} \stackrel{\text{def}}{=} \{\omega \in C([0,\beta] \to \mathbb{R}) \mid \omega(0) = \omega(\beta)\},$$

equipped with the norm

$$|\omega|_\beta \stackrel{\text{def}}{=} \sup\{|\omega(\tau)| : \ \tau \in [0,\beta]\}, \tag{1.6}$$

and with the usual Banach space structure. For $\Lambda \in \mathcal{L}$, we put

$$\Omega_{\beta,\Lambda} \stackrel{\text{def}}{=} \{\omega_\Lambda = (\omega_k)_{k\in\Lambda} \mid \omega_k \in C_\beta^{\text{eq}}\}, \quad \Omega_\beta \stackrel{\text{def}}{=} \Omega_{\beta,\mathbb{Z}^d}. \tag{1.7}$$

$\Omega_{\beta,\Lambda}$ will be called *the temperature loop spaces* (TLS), their elements are *the configurations of the temperature loops* (at inverse temperature β and for the domain Λ). The TLS $\Omega_{\beta,\Lambda}$ may be equipped with the product topology and with the σ–algebra $\mathcal{B}_{\beta,\Lambda} = \mathcal{B}(\Omega_{\beta,\Lambda})$ generated by the cylinder subsets of $\Omega_{\beta,\Lambda}$, see e.g. [3], [14], [18], [19]. The product topology is defined by a system of open neighborhoods of a point $\tilde{\omega}_\Lambda \in \Omega_{\beta,\Lambda}$, $\{V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}) \mid \Lambda' \subset \Lambda \ \text{ and } \ \Lambda' \in \mathcal{L}_{\text{fin}}\}$, $\varepsilon_{\Lambda'} = (\varepsilon_j)_{j\in\Lambda'}$ defined as follows

$$\mathcal{B}_{\beta,\Lambda} \supset V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}) \stackrel{\text{def}}{=} \{\omega_\Lambda \in \Omega_{\beta,\Lambda} \mid |\omega_j - \tilde{\omega}_j|_\beta < \varepsilon_j, \ j \in \Lambda'\}, \tag{1.8}$$

where the norm (1.6) has been used.

In order to have the collection of all TLS $\{\Omega_{\beta,\Lambda}, \Lambda \in \mathcal{L}\}$ ordered by inclusion, one may introduce the following embedding mappings. For $\Lambda \subset \Lambda'$, we put $\omega_\Lambda \mapsto \omega_\Lambda \times 0_{\Lambda'\setminus\Lambda} \in \Omega_{\beta,\Lambda'}$. Here 0_Λ is zero configuration in $\Omega_{\beta,\Lambda}$, and

$$\omega_\Lambda \times \xi_{\Lambda'\setminus\Lambda} \stackrel{\text{def}}{=} \zeta_{\Lambda'}, \ \Lambda \subset \Lambda',$$

means a configuration in $\Omega_{\beta,\Lambda'}$ such that $\zeta_k = \omega_k$ for $k \in \Lambda$, and $\zeta_k = \xi_k$ for $k \in \Lambda' \setminus \Lambda$. Having in mind such embeddings, we shall consider every configuration ω_Λ as an element of all TLS $\Omega_{\beta,\Lambda'}$, with $\Lambda \subset \Lambda'$. Along with these embeddings, we define the projections

$$\omega_\Lambda \mapsto (\omega_\Lambda)_{\Lambda'} \stackrel{\text{def}}{=} (\omega_k)_{k\in\Lambda\cap\Lambda'},$$

with $(\omega_\Lambda)_{\Lambda'}$ being zero configuration if $\Lambda \cap \Lambda' = \emptyset$.

For every choice of $\Lambda \in \mathcal{L}_{\rm fin}$, $\Lambda' \in \mathcal{L}$, and $\zeta \in \Omega_\beta$, the configuration

$$(\omega_\Lambda \times \zeta_{\Lambda^c})_{\Lambda'} = (\omega_\Lambda \times \zeta_{\Lambda^c})_{\Lambda'}(\tau), \quad \Lambda^c = \mathbb{Z}^d \setminus \Lambda$$

is a continuous function of $\tau \in [0, \beta]$ in the product topology. Thus for $\Lambda \in \mathcal{L}_{\rm fin}$, the expression

$$\begin{aligned} E_{\beta,\Lambda}(\omega_\Lambda|\zeta) \quad &\stackrel{\rm def}{=} \quad \sum_{j\in\Lambda} \int_{[0,\beta]} U_j(\omega_j(\tau))d\tau + \frac{1}{2}\sum_{j,k\in\Lambda} \int_{[0,\beta]} J_{jk}\omega_j(\tau)\omega_k(\tau)d\tau \\ &+ \quad \sum_{j\in\Lambda, k\in\Lambda^c} \int_{[0,\beta]} J_{jk}\omega_j(\tau)\zeta_k(\tau))d\tau, \end{aligned} \tag{1.9}$$

defines a continuous function $E_{\beta,\Lambda}(.\ |\zeta) : \ \Omega_{\beta,\Lambda} \longrightarrow \mathbb{R}$.

The space of temperature loops $C_\beta^{\rm eq}$ may naturally be embedded into the real Hilbert space $\mathcal{H}_\beta \stackrel{\rm def}{=} L^2([0, \beta])$ with the scalar product

$$(\omega, \xi)_\beta = \int_{[0,\beta]} \omega(\tau)\xi(\tau)d\tau. \tag{1.10}$$

Let $\mathcal{H}_{\beta,j}$, $j \in \mathbb{Z}^d$ be the copy of $\mathcal{H}_\beta$. For $\Lambda \in \mathcal{L}_{\rm fin}$, we put

$$\mathcal{H}_{\beta,\Lambda} \quad \stackrel{\rm def}{=} \quad \bigoplus_{j\in\Lambda} \mathcal{H}_{\beta,j} = \{\omega_\Lambda = (\omega_j)_{j\in\Lambda} \mid \omega_j \in \mathcal{H}_{\beta,j}\}. \tag{1.11}$$

For short, we omit in the sequel the subscript of the type of j when this does not cause any ambiguities.

The scalar product in the Hilbert space $\mathcal{H}_{\beta,\Lambda}$ is

$$(\omega_\Lambda, \xi_\Lambda)_{\beta,\Lambda} = \sum_{j\in\Lambda} (\omega_j, \xi_j)_\beta, \tag{1.12}$$

and for all $\Lambda \in \mathcal{L}_{\rm fin}$, one has $\Omega_{\beta,\Lambda} \subset \mathcal{H}_{\beta,\Lambda}$.

Let us consider the following strictly positive trace class operator on $\mathcal{H}_\beta$

$$S_\beta(m) = (-m\Delta_\beta + 1)^{-1}, \tag{1.13}$$

where Δ_β stands for the Laplace operator on the interval $[0, \beta]$ with periodic boundary conditions, and m is the reduced physical mass of the particle. Then one can define the Gaussian measure γ_β on $\mathcal{H}_\beta$ which has zero mean and $S_\beta(m)$ as a covariance operator. The measure γ_β is uniquely determined by its Fourier transform

$$\int_{\mathcal{H}_\beta} \exp(i(\varphi,\omega)_\beta)\gamma_\beta(d\omega) = \exp(-\frac{1}{2}(\varphi, S_\beta\varphi)_\beta), \quad \varphi \in \mathcal{H}_\beta. \tag{1.14}$$

Actually, the set of continuous loops is a set of full measure (i.e. $\gamma_\beta(\Omega_\beta) = 1$), and the measure γ_β corresponds to the oscillator bridge process of length β [21]. Then the product measure

$$\gamma_{\beta,\Lambda}(d\omega_\Lambda) \stackrel{\rm def}{=} \prod_{j\in\Lambda} \gamma_\beta(d\omega_j), \quad \Lambda \in \mathcal{L}_{\rm fin}, \tag{1.15}$$

is a measure defined on $\mathcal{H}_{\beta,\Lambda}$ and supported on $\Omega_{\beta,\Lambda}$.

Now let the subset $\Lambda \in \mathcal{L}_{\rm fin}$ be fixed. For the model considered, the Gibbs measure in Λ, subject to a chosen configuration $\zeta \in \Omega_\beta$, is

$$\nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \stackrel{\rm def}{=} \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.16}$$

defined, as $\gamma_{\beta,\Lambda}$, on $\mathcal{H}_{\beta,\Lambda}$ and supported on the space $\Omega_{\beta,\Lambda}$. Here

$$Z_{\beta,\Lambda}(\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.17}$$

is the finite volume partition function subject to the external boundary condition ζ_{Λ^c}. The conditions (1.3) and (1.4) imposed on the potentials J_{jk} and U_k provide that the function $\exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}$ is $\gamma_{\beta,\Lambda}$–integrable, thus the objects introduced in (1.16), (1.17) are well–defined.

For $B \in \mathcal{B}_\beta \stackrel{\rm def}{=} \mathcal{B}(\Omega_\beta)$, let $1\!\!1_B(\omega)$ be the indicator function of B. Introduce a family of probability kernels $\{\pi_{\beta,\Lambda} \mid \Lambda \in \mathcal{L}\}$

$$\pi_{\beta,\Lambda}(B|\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} 1\!\!1_B(\omega_\Lambda \times \zeta_{\Lambda^c})\nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta), \tag{1.18}$$

which satisfy the consistency conditions.

$$\begin{aligned} \pi_{\beta,\Lambda}\pi_{\beta,\Lambda'}(B|\zeta) &\stackrel{\rm def}{=} \int_{\Omega_\beta} \pi_{\beta,\Lambda}(d\omega|\zeta)\pi_{\beta,\Lambda'}(B|\omega) \\ &= \pi_{\beta,\Lambda}(B|\zeta). \end{aligned} \tag{1.19}$$

Definition 1.1 *A probability measure ν_β on $(\Omega_\beta, \mathcal{B}_\beta)$ is said to be an Euclidean Gibbs state of the lattice model (1.1), (1.2), at temperature β^{-1} if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equilibrium equation:*

$$\nu_\beta \pi_{\beta,\Lambda} = \nu_\beta,$$

that is

$$\int_{\Omega_\beta} \nu_\beta(d\omega)\pi_{\beta,\Lambda}(B|\omega) = \nu_\beta(B), \tag{1.20}$$

for all $\Lambda \in \mathcal{L}$ and $B \in \mathcal{B}_\beta$.

In order to exclude the states with no physical relevance we impose some *a priori* restrictions. To this end we introduce

$$\mathcal{S}'(\mathbb{Z}^d) \stackrel{\text{def}}{=} \operatorname{ind} \lim_{p \in \mathbb{N}} l^2_{-p}(\mathbb{Z}^d),$$

where

$$l^2_{-p} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-p} x_k^2 < \infty \right\}. \tag{1.21}$$

Definition 1.2 *The class of tempered Gibbs measures $\mathcal{G}_\beta$ for the model considered consists of the Euclidean Gibbs states defined above which are such that their moment sequences obey the condition*

$$(< \|\omega_k\|_\beta >_{\mu_\beta})_{k \in \mathbb{Z}^d} \in \mathcal{S}'(\mathbb{Z}^d), \tag{1.22}$$

where $\| \,.\, \|_\beta$ is the norm in the Hilbert space $\mathcal{H}_\beta$.

Given $\Lambda \in \mathcal{L}$, let us consider a subset of $\Omega_{\beta,\Lambda}$ consisting of constant trajectories, that is

$$\Omega^{\text{cl}}_{\beta,\Lambda} \stackrel{\text{def}}{=} \{\omega_\Lambda \in \Omega_{\beta,\Lambda} \mid (\forall k \in \Lambda)\ (\forall \tau \in [0, \beta])\ \ \omega_k(\tau) = x_k \in \mathbb{R}\}, \tag{1.23}$$

which is isomorphic to $\mathbb{R}^\Lambda$. We also set

$$\Omega_\beta \supset \Omega^{\text{cl}}_{\beta,\mathbb{Z}^d} \stackrel{\text{def}}{=} \Omega^{\text{cl}}_\beta \cong \mathbb{R}^{\mathbb{Z}^d}.$$

Now we construct the Gibbs states also for the classical model described by the Hamiltonian (1.5). To this end we use the following Gaussian measures

$$\chi_{\beta,\Lambda}(dx_\Lambda) \stackrel{\text{def}}{=} \prod_{j \in \Lambda} \chi_\beta(dx_j), \quad x_\Lambda \in \mathbb{R}^\Lambda, \tag{1.24}$$

$$\chi_\beta(dx_j) \stackrel{\text{def}}{=} \sqrt{\frac{\beta}{2\pi}} \exp\{-\frac{\beta}{2} x_j^2\} dx_j, \quad x_j \in \mathbb{R}. \tag{1.25}$$

Given $\Lambda \in \mathcal{L}$, let $\mathcal{B}^{\rm cl}_{\beta,\Lambda}$ be the σ–algebra generated by the cylinder subsets of $\Omega^{\rm cl}_{\beta,\Lambda}$, which is isomorphic to the corresponding algebra generated by the subsets of $I\!\!R^{\Lambda}$ but, on the other hand, is a subalgebra of $\mathcal{B}_{\beta,\Lambda}$. For every $B \in \mathcal{B}_{\beta,\Lambda}$, let also

$$B^{\rm cl} \stackrel{\rm def}{=} \bigcup_{A \in \mathcal{B}^{\rm cl}_{\beta,\Lambda}} B \cap A. \tag{1.26}$$

Obviously each such $B^{\rm cl}$ is an element of $\mathcal{B}^{\rm cl}_{\beta,\Lambda}$. Then for $\Lambda \in \mathcal{L}_{\rm fin}$, let $\gamma^{\rm cl}_{\beta,\Lambda}$ be the Gaussian measure on $\Omega_{\beta,\Lambda}$ such that for every $B \in \mathcal{B}_{\beta,\Lambda}$

$$\gamma^{\rm cl}_{\beta,\Lambda}(B) = \chi_{\beta,\Lambda}(B^{\rm cl}), \tag{1.27}$$

which means that $\gamma^{\rm cl}_{\beta,\Lambda}$ is supported on $\mathcal{B}^{\rm cl}_{\beta,\Lambda}$.

The family of conditional Gibbs measures $\{\mu_{\beta,\Lambda}(.|\xi) | \Lambda \in \mathcal{L}_{\rm fin}\}$ for the classical model described by the Hamiltonian (1.5) is intruduced as follows. For every $\xi \in \Omega^{\rm cl}_{\beta} \subset \Omega_\beta$, we set

$$\mu_{\beta,\Lambda}(d\omega_\Lambda|\xi) = \frac{1}{Y_{\beta,\Lambda}(\xi)} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\xi)\}\gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.28}$$

which is a measure on the TLS $\Omega_{\beta,\Lambda}$ absolutely continuous with respect to the Gaussian measure $\gamma^{\rm cl}_{\beta,\Lambda}$, and

$$Y_{\beta,\Lambda}(\xi) = \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\xi)\gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda). \tag{1.29}$$

Recall that the function $E_{\beta,\Lambda}(.|\zeta)$ is defined by (1.9).

It should be pointed out that for given $\Lambda \in \mathcal{L}_{\rm fin}$ and β, each such a measure $\mu_{\beta,\Lambda}(.|\xi)$ is uniquely determined by the external boundary condition ξ_{Λ^c}. Like in the quantum case (see (1.18)), the family $\{\mu_{\beta,\Lambda}(.|\xi)|\Lambda \in \mathcal{L}_{\rm fin}\}$ may be used to introduce the probability kernels

$$\rho_{\beta,\Lambda}(B|\xi) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} 1\!\!1_B(\omega_\Lambda \times \xi_{\Lambda^c})\mu_{\beta,\Lambda}(d\omega_\Lambda|\xi), \tag{1.30}$$

which satisfy the consistency condition analogous to (1.19). Then the Gibbs state of the classical model at given temperature β^{-1}, as well as its tempered Gibbs measure, are defined by Definition 1.1 and Definition 1.2 respectively. Like $\gamma^{\rm cl}_{\beta,\Lambda}$, they are measures μ_β on the space Ω_β, supported on its subspace

$\Omega^{\rm cl}_\beta \cong I\!\!R^{\mathbb{Z}^d}$, which satisfies the growth restrictions (1.22) and the equilibrium equation

$$\int_{\Omega_\beta} \mu_\beta(d\omega)\rho_{\beta,\Lambda}(B|\omega) = \mu_\beta(B), \tag{1.31}$$

for all $\Lambda \in \mathcal{L}$ and $B \in \mathcal{B}(\Omega_\beta)$.

In the sequel we shall use some notions (see e.g. [20]) concerning the weak convergence of measures on metric spaces. Consider a measure space $(X, \mathcal{B}(X))$, where X is a real separable metric space and $\mathcal{B}(X)$ is the Borel σ–algebra of its subsets. Let $\mathcal{M}(X)$ be the space of all probability measures defined on $\mathcal{B}(X)$. Let also $C_{\rm b}(X)$ stand for the space of all bounded real valued continuous functions on X. The topology on the space $\mathcal{M}(X)$ is defined by a system of open neighborhoods of a point $\mu \in \mathcal{M}(X)$, given as follows

$$\begin{aligned} & V_\mu(f_1, \ldots, f_n; \varepsilon_1, \ldots, \varepsilon_n) = \\ & \Big\{ \nu \in \mathcal{M}(X) | \, | \int f_i d\nu - \int f_i d\mu | < \varepsilon_i, \ i = 1, \ldots, n \Big\} \end{aligned} \tag{1.32}$$

with arbitrarily chosen $n \in I\!\!N$, $\varepsilon_1, \ldots, \varepsilon_n$ in $(0, +\infty)$, and $f_1, \ldots, f_n$ in $C_{\rm b}(X)$. Such a topology is said to be *the weak topology* on $\mathcal{M}(X)$. If a net of measures $\{\mu_\alpha\}$ converges to a measure $\mu \in \mathcal{M}(X)$ in this topology, we write $\mu_\alpha \Rightarrow \mu$. It should be pointed out that this convergence holds if and only if

$$\int f d\mu_\alpha \to \int f d\mu, \quad \forall f \in C_{\rm b}(X).$$

For every $B \in \mathcal{B}(X)$, the boundary

$$\partial B \stackrel{\rm def}{=} \bar{B} \setminus {\rm int} B,$$

is, as usually, the difference between its closure and interior.

Let $\Lambda \in \mathcal{L}_{\rm fin}$, $\beta > 0$, and $\zeta \in \Omega_\beta$ be fixed. Then the measures γ_β, $\gamma_{\beta,\Lambda}$, and $\nu_{\beta,\Lambda}(.|\zeta)$, defined by (1.14), (1.15), and (1.16) respectively, depend only on the value of the reduced physical mass m through the operator $S_{\beta,\Lambda}(m)$. We consider them as nets labeled by $m \in (0, +\infty)$ (which we indicate in the sequel by writing $\gamma_{\beta,\Lambda} \stackrel{\rm def}{=} \gamma^{(m)}_{\beta,\Lambda}$, $\nu_{\beta,\Lambda}(.|\zeta) \stackrel{\rm def}{=} \nu^{(m)}_{\beta,\Lambda}(.|\zeta)$) and study their weak convergence when $m \to +\infty$. Having such a convergence we also establish the convergence of the probability kernels $\pi_{\beta,\Lambda} \stackrel{\rm def}{=} \pi^{(m)}_{\beta,\Lambda}$, defined in (1.18), to the classical kernels (1.30). For given β, $\Lambda \in \mathcal{L}_{\rm fin}$, and $\xi \in \Omega_\beta^{\rm cl} \cong I\!R^{Z\!\!Z^d}$, let $\Upsilon_{\beta,\Lambda}(\xi)$ be the subset of Ω_β consisting of such ζ, that for every $k \in \Lambda$,

$$\xi_k = \int_{[0,\beta]} \zeta_k(\tau) d\tau. \tag{1.33}$$

Theorem 1.1 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$, and $\xi \in \Omega_\beta^{\rm cl} \cong I\!R^{Z\!\!Z^d}$ be chosen, Then for every $\zeta \in \Upsilon_{\beta,\Lambda}(\xi)$, the net $\{\nu_{\beta,\Lambda}^{(m)}(.|\zeta)\}$ weakly converges to the classical conditional Gibbs measure $\mu_{\beta,\Lambda}(.|\xi)$ given by (1.28).*

For given β, Λ, we set

$$\mathcal{B}_{\beta,\Lambda}^{(0)} \stackrel{\rm def}{=} \{B \in \mathcal{B}_{\beta,\Lambda} \mid \gamma_{\beta,\Lambda}^{\rm cl}(\partial B) = 0 \ \}. \tag{1.34}$$

Theorem 1.2 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$, and $\xi \in \Omega_\beta^{\rm cl} \cong I\!R^{Z\!\!Z^d}$ be chosen, Then for every $B \in \mathcal{B}_{\beta,\Lambda}^{(0)}$ and all $\zeta \in \Upsilon_{\beta,\Lambda}(\xi)$, the net $\{\pi_{\beta,\Lambda}^{(m)}(B|\zeta)\}$ converges to the value of the classical kernel $\rho_{\beta,\Lambda}(B|\xi)$. In particular such a convergence holds for $B = V_{\Lambda'}$ given by (1.8).*

Now let us consider more general Hamiltonians instead of (1.1) and (1.5), namely:

$$\tilde{H} = \sum_{k \in Z\!\!Z^d} H_k^{(0)} + \sum_{\Lambda \in \mathcal{L}_{\rm fin}} \Phi_\Lambda(x_\Lambda), \tag{1.35}$$

and

$$\tilde{H}^{\rm cl} = \frac{1}{2} \sum_{k \in Z\!\!Z^d} x_k^2 + \sum_{\Lambda \in \mathcal{L}_{\rm fin}} \Phi_\Lambda(x_\Lambda), \tag{1.36}$$

respectively. To describe the properties of the potentials Φ_Λ we define

$$\mathcal{L}(\Lambda) \stackrel{\rm def}{=} \{\Lambda' \in \mathcal{L} \mid \Lambda \cap \Lambda' \neq \emptyset\}, \ \Lambda \in \mathcal{L}_{\rm fin}. \tag{1.37}$$

For one–point subsets $\Lambda = \{k\}$, we simply write $\mathcal{L}(\{k\}) = \mathcal{L}_k$, that is the collection of all subsets containing given $k \in Z\!\!Z^d$. Each Φ_Λ, $\Lambda \in \mathcal{L}\}$ is defined on the set of vectors $x_\Lambda = (x_j)_\Lambda \in I\!R^\Lambda$, and is assumed to possess the following properties

(a) (continuity) Φ_Λ is a continuous function on $I\!R^\Lambda$;
(b) (finite range) for every $j \in Z\!\!Z^d$, the collection $\mathcal{L}_j$ contains only finite number of elements Λ, for which, Φ_Λ is not identically zero;
(c) (superstability) there exist $a > 0$ and $b \in I\!R$ such that for all $\Lambda \in \mathcal{L}_{\rm fin}$ and all $x_\Lambda \in I\!R^\Lambda$, the following estimate holds:

$$\sum_{\Lambda' \subset \Lambda} \Phi_{\Lambda'}(x_{\Lambda'}) \geq \sum_{j \in \Lambda} (a x_j^2 + b). \tag{1.38}$$

The question of the existence and a number of properties of such models are discussed e.g in the papers [18], [19]. Obviously, the Hamiltonian (1.1) is a partial case of the latter. The assumed finiteness of the range of the potentials Φ_Λ provides that for every $\tau \in [0, \beta]$, the sum

$$\sum_{\Lambda' \in \mathcal{L}(\Lambda)} \Phi_{\Lambda'}((\omega_\Lambda \times \zeta_{\Lambda^c})_{\Lambda'}(\tau))$$

contains only finite number of terms, therefore, it is a continuous function of τ. Now we define instead of (1.9)

$$E_{\beta,\Lambda}(\omega_\Lambda|\zeta) \stackrel{\text{def}}{=} \sum_{\Lambda' \in \mathcal{L}(\Lambda)} \int_{[0,\beta]} \Phi_{\Lambda'}((\omega_\Lambda \times \zeta_{\Lambda^c}(\tau))_{\Lambda'})d\tau, \tag{1.39}$$

then the conditional Gibbs measures $\nu^{(m)}_{\beta,\Lambda}$ and $\mu_{\beta,\Lambda}$ are defined by (1.16) and (1.28) respectively with this choice of the function $E_{\beta,\Lambda}$. By means of these measures one may define also probability kernels $\pi_{\beta,\Lambda}$ and $\rho_{\beta,\Lambda}$, using for this (1.18), (1.30) respectively. In what follows, one may study the weak convergence of measures and probability kernels as it was performed above. But in this general case, one has a somewhat weaker statement instead of Theorem 1.1.

Theorem 1.3 *For every $\Lambda \in \mathcal{L}_{\text{fin}}$, $\beta > 0$, and $\xi \in \Omega^{\text{cl}}_\beta \cong \mathbb{R}^{\mathbb{Z}^d}$, the conditional Gibbs measure of the classical model $\mu_{\beta,\Lambda}(.|\xi)$ is a weak limit when $m \to +\infty$ of the net of the conditional Gibbs measures of the quantum model $\{\nu^{(m)}_{\beta,\Lambda}(.|\zeta)\}$ with certain boundary conditions $\zeta \in \Omega_\beta$.*

Remark 1.1 *In contrast to the case of models described by the Hamiltonians (1.1), (1.5), the above theorem does not claim that every net of quantum conditional Gibbs measures converges to a classical Gibbs measure. It does not exclude the possibility to obtain, as a limit when $m \to +\infty$, measures which are absent among conditional Gibbs measures of the model described by the Hamiltonian (1.36).*

As a corollary of Theorem 1.3, one may obtain also a weaker form of Theorem 1.2.

Theorem 1.4 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\text{fin}}$, and $\xi \in \Omega^{\text{cl}}_\beta \cong \mathbb{R}^{\mathbb{Z}^d}$ be chosen, Then there exists $\zeta \in \Omega_\beta$ such that for every $B \in \mathcal{B}^{(0)}_{\beta,\Lambda}$, the net $\{\pi^{(m)}_{\beta,\Lambda}(B|\zeta)\}$ converges to the value on the pair (b, ξ) of the classical kernel $\rho_{\beta,\Lambda}(B|\xi)$. In particular such a convergence holds for $B = V_{\Lambda'}$ given by (1.8).*

Remark 1.2 *Above we have restricted ourselves to the case of one–dimensional oscillations of the particles. This is done only in order to avoid further complications of notations and to make our consideration more transparent. The generalization to the vector case may be performed in a standard way with no additional troubles.*

1.3 Periodic Gibbs States and Order Parameters

It is fairly well known that, for $d \geq 2$, the models considered – both quantum and classical – may undergo a phase transition when the inverse temperature β exceeds a certain value β_*. The typical feature of this phenomenon is the nonuniqueness of the tempered Gibbs measures introduced by Definition 1.2. More precisely, the model considered undergoes the phase transition if there exists β_* such that for $\beta > \beta_*$, the class of tempered Gibbs measures $\mathcal{G}_\beta$ consists of more than one element. But in most nontrivial cases there are no possibilities to describe the phase transition on this level. Much more realistic approach is based on the use of the *order parameter*, which becomes being positive for $\beta > \beta_*$. This parameter should describe the symmetry breaking. This means that the symmetry of the formal Hamiltonian, which is inherited by the unique tempered Gibbs measure for $\beta < \beta_*$, is no longer proper for each such a measure existing in the number more than one for $\beta > \beta_*$.

Having the weak convergence of the Euclidean Gibbs measures of the quantum model to the Gibbs measure of the classical model we may study the possible connections between the order parameters in these models. For this purpose, the most convenient objects, of the type of those ones considered above, are the models possessing the translation invariance. Below we deal with the translation invariant models with the pair interaction, that is described by the Hamiltonians of the type of (1.1) and (1.5). For these models, the translation invariance may be obtained if one puts all U_k to be independent of k and realizes J_{jk} as a suitable function $J(.)$ of the Euclidean distance $|j-k|$ between j and k. In order to have local Gibbs measures invariant one may impose the periodic boundary conditions instead of those ones established by means of configurations out of Λ. Consider this construction in more details.

Let a system of particles be described by the following formal translation invariant Hamiltonians

$$H_{\rm tr} = \sum_k H_k^{(0)} + \sum_k U(x_k) + \frac{1}{2}\sum_{j,k} J(|j-k|)x_j x_k, \tag{1.40}$$

and

$$H_{\rm tr}^{\rm cl} = \sum_k \left(\frac{1}{2}x_k^2 + U(x_k)\right) + \frac{1}{2}\sum_{j,k} J(|j-k|)x_j x_k, \tag{1.41}$$

which correspond to the Hamiltonians (1.1) and (1.5) respectively. Here the Hamiltonian $H_k^{(0)}$ is given by (1.2) and all sums, as before, are taken over the whole lattice $\mathbb{Z}^d$. We also suppose that the function J vanishes when its argument exceeds some $r > 0$, and in addition U and J are assumed to obey the conditions (1.3), (1.4). Consider now a box $\Lambda \in \mathcal{L}_{\rm fin}$

$$\Lambda = \{k = (k_1, \dots, k_d) \in \mathbb{Z}^d \ \ | \ \ k_l^{(0)} \le k_l \le k_l^{(1)}, \ \ l = 1, \dots, d; \\ k_l^{(0)} < k_l^{(1)}; \ \ k_l^{(0)}, k_l^{(1)} \in \mathbb{Z}^d \ \}.$$

Given Λ and $l = 1, \dots, d$, set

$$|j_l - k_l|_\Lambda \stackrel{\rm def}{=} \min\{|j_l - k_l|; \ k_l^{(1)} - k_l^{(0)} + 1 - |j_l - k_l| \ \}, \tag{1.42}$$

and

$$|j-k|_\Lambda^2 \stackrel{\rm def}{=} \sum_{l=1}^d |j_l - k_l|_\Lambda^2, \tag{1.43}$$

which defines a periodic metric in Λ. For this metric, one can observe

$$|j-k|_\Lambda \le |j-k|. \tag{1.44}$$

Chosen Λ and some β, we introduce the following continuous real valued function on $\Omega_{\beta,\Lambda}$

$$E_{\beta,\Lambda}^{\rm tr}(\omega_\Lambda) \stackrel{\rm def}{=} \sum_{j\in\Lambda} \int_{[0,\beta]} U(\omega_j(\tau)d\tau + \frac{1}{2}\sum_{j,k\in\Lambda} \int_{[0,\beta]} J(|j-k|_\Lambda)\omega_j(\tau)\omega_k(\tau)d\tau, \tag{1.45}$$

which will be used to construct the periodic local Gibbs measures instead of the function (1.9). Thereby we define

$$\nu_{\beta,\Lambda}^{\rm tr}(d\omega_\Lambda) \stackrel{\rm def}{=} \frac{1}{Z_{\beta,\Lambda}^{\rm tr}} \exp\{-E_{\beta,\Lambda}^{\rm tr}(\omega_\Lambda)\}\gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda), \tag{1.46}$$

Here

$$Z^{\rm tr}_{\beta,\Lambda} \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-E^{\rm tr}_{\beta,\Lambda}(\omega_\Lambda)\}\gamma^{(m)}_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.47}$$

and the Gaussian measure $\gamma^{(m)}_{\beta,\Lambda}$ is the same as in (1.16). Furthermore,

$$\mu^{\rm tr}_{\beta,\Lambda}(d\omega_\Lambda) = \frac{1}{Y^{\rm tr}_{\beta,\Lambda}} \exp\{-E^{\rm tr}_{\beta,\Lambda}(\omega_\Lambda)\}\gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.48}$$

where

$$Y^{\rm tr}_{\beta,\Lambda} = \int_{\Omega_{\beta,\Lambda}} \exp\{-E^{\rm tr}_{\beta,\Lambda}(\omega_\Lambda)\gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda), \tag{1.49}$$

and the Gaussian measure $\gamma^{\rm cl}_{\beta,\Lambda}$ is the same as in (1.28). Then an analog of Theorem 1.1 for the translation periodic Gibbs measures reads as follows.

Theorem 1.5 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$ be chosen, Then the net of periodic Gibbs measures (1.46) $\{\nu^{\rm tr}_{\beta,\Lambda}\}$ weakly converges to the classical periodic Gibbs measure $\mu^{\rm tr}_{\beta,\Lambda}$ given by (1.48).*

Now we introduce the order parameters which become being positive for $\beta > \beta_*$, manifesting the appearance of the long range order. From now on, in addition to the previous assumptions, we assume that the anharmonic potential U is an even function, which means that the symmetry being broken is Z_2. Then the corresponding order parameters are defined by means of the following moments of the periodic Gibbs measures

$$P_\Lambda(m) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \left(\frac{1}{|\Lambda|} \int_{[0,\beta]} \sum_{j\in\Lambda} \omega_j(\tau)d\tau \right)^2 \nu^{\rm tr}_{\beta,\Lambda}(d\omega), \tag{1.50}$$

$$Q_\Lambda \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \left(\frac{1}{|\Lambda|} \int_{[0,\beta]} \sum_{j\in\Lambda} \omega_j(\tau)d\tau \right)^2 \mu^{\rm tr}_{\beta,\Lambda}(d\omega). \tag{1.51}$$

Thus one puts (see e.g. [17])

$$P(m) \stackrel{\rm def}{=} \lim_{\Lambda\nearrow\mathbb{Z}^d} P_\Lambda(m) \tag{1.52}$$

$$Q \stackrel{\rm def}{=} \lim_{\Lambda\nearrow\mathbb{Z}^d} Q_\Lambda, \tag{1.53}$$

which are the order parameters for the translation invariant quantum and classical models respectively. Here $|\Lambda|$ stands for the cardinality of Λ. A simple corollary of Theorem 1.5 yields

$$\lim_{m\to+\infty} P_\Lambda(m) = Q_\Lambda. \tag{1.54}$$

Another relation between the moments $P_\Lambda(m)$ nd Q_Λ may be established for a special choice of the function U and under some additional condition imposed on the function J.

Proposition 1.1 *For the considered quantum model, let J be a nonnegative and monotone decreasing function and U has the following form*

$$U_j(x_j) = ax_j^2 + x_j^4, \quad a \in \mathbb{R}. \tag{1.55}$$

Then for every $\Lambda \in \mathcal{L}_{\text{fin}}$, the moment $P_\Lambda(m)$ is a monotone function of $m \in (0, +\infty)$. That is, for arbitrary $m' > m$,

$$P_\Lambda(m') \geq P_\Lambda(m).$$

The proof of this assertion is based on the properties of J, U and on the inequality (1.44). Combining these relations one concludes that in this case, the convergence of the sequence $\{P_\Lambda(m)\}$, when $\Lambda \nearrow \mathbb{Z}^d$, to a positive limit implies the appearance of the long range order for all $m' > m$, as well as in the classical model.

2 The Proofs

The proof of all our theorems is based on the following lemma, which is proven in the final part of this section.

Lemma 2.1 *For every $\Lambda \in \mathcal{L}_{\text{fin}}$, $\beta > 0$, the net of measures $\{\gamma_{\beta,\Lambda}^{(m)}\}$ converges weakly to the measure $\gamma_{\beta,\Lambda}^{\text{cl}}$.*

First we prove the weaker theorem.

Proof of Theorem 1.3. The assumptions concerning the properties of the potentials Φ_Λ yield that the density

$$F_{\beta,\Lambda}(.|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\{-E_{\beta,\Lambda}(.|\zeta)\} \tag{2.1}$$

is a bounded continuous function on the Banach space $\Omega_{\beta,\Lambda}$. Then, for every $G \in C_{\text{b}}(\Omega_{\beta,\Lambda})$, the product FG also belongs to $C_{\text{b}}(\Omega_{\beta,\Lambda})$. Thus by Lemma 2.1 one has

$$\int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\nu_{\beta,\Lambda}^{(m)}(d\omega_\Lambda|\xi) \longrightarrow \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\mu_{\beta,\Lambda}(d\omega_\Lambda|\xi),$$

provided the net of densities (2.1) $\{F_{\beta,\Lambda}(.|\zeta)|m \in (0,+\infty)\}$ converges to the density

$$F^{\rm cl}_{\beta,\Lambda}(.|\xi) \stackrel{\rm def}{=} \frac{1}{Y_{\beta,\Lambda}(\xi)} \exp\{-E_{\beta,\Lambda}(.|\xi)\} \tag{2.2}$$

in the topology of the space $C_{\rm b}(\Omega_{\beta,\Lambda})$. We choose $\zeta = \xi$, then the densities $F_{\beta,\Lambda}(.|\xi)$ and $F^{\rm cl}_{\beta,\Lambda}(.|\xi)$ differ only in the normalizing factors. For these factors, one has

$$Z_{\beta,\Lambda}(\xi) \to Y_{\beta,\Lambda}(\xi), \quad m \to +\infty,$$

which is implied by the representations (1.17), (1.29) and by Lemma 2.1. ■

Proof of Theorem 1.1. Remind that in the case considered the function $E_{\beta,\Lambda}(\omega_\Lambda|\zeta)$ is given by (1.9). Then the density function (2.1) may be written as

$$F_{\beta,\Lambda}(\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_{[0,\beta]} \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \Psi_{\beta,\Lambda}(\omega_\Lambda),$$

with a certain $\Psi_{\beta,\Lambda} \in C_{\rm b}(\Omega_{\beta,\Lambda})$. Therefore, for arbitrary function $G \in C_{\rm b}(\Omega_{\beta,\Lambda})$, one has

$$\begin{aligned}
\int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\nu^{(m)}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \\
\exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_{[0,\beta]} \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \gamma^{(m)}_{\beta,\Lambda}(d\omega_\Lambda) \\
\longrightarrow \quad \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \\
\exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_{[0,\beta]} \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda) \\
= \quad \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{I\!R^\Lambda} G(x_\Lambda)\Psi_{\beta,\Lambda}(x_\Lambda) \\
\exp\{- \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} x_j \int_{[0,\beta]} \zeta_k(\tau)d\tau\} \chi_{\beta,\Lambda}(dx_\Lambda) \\
= \quad \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \\
\exp\{- \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk}\omega_j(0)\xi_k\} \gamma^{\rm cl}_{\beta,\Lambda}(d\omega_\Lambda) \\
= \quad \int_{I\!R^\Lambda} G(\omega_\Lambda)\mu_{\beta,\Lambda}(d\omega_\Lambda|\xi),
\end{aligned}$$

where ζ satisfies the condition (1.33). ■

The proof of Theorem 1.5 may be performed by repetition of the arguments just used.

To prove Theorem 1.2 and Theorem 1.4 we will need certain assertion concerning the weak convergence, which we borrow from see [20] p. 40 .

Proposition 2.1 *A net $\{\mu_\alpha\}$ in $\mathcal{M}(X)$ weakly converges to a measure $\mu \in \mathcal{M}(X)$ if and only if*

$$\lim_{\alpha} \mu_\alpha(B) = \mu(B),$$

for every $B \in \mathcal{B}(X)$ such that $\mu(\partial B) = 0$.

Proof of Theorem 1.2 and Theorem 1.4. Taking into account the above proposition one immediately gets the convergence of the nets of probability kernels on $B \in \mathcal{B}^{(0)}_{\beta,\Lambda}$, which follows from Theorem 1.1 and Theorem 1.3 respectively. To complete the proof we ought to show that

$$V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}) \in \mathcal{B}^{(0)}_{\beta,\Lambda},$$

for all $\Lambda \in \mathcal{L}$, finite $\Lambda' \subset \Lambda$, and all $\varepsilon_j > 0$, $j \in \Lambda'$. To this end we remark that each $V_{\Lambda'}$ may be written as

$$V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}) \ = \ \times_{j\in\Lambda'} K_{\varepsilon_j}(\tilde{\omega}_j),$$

$$K_{\varepsilon_j}(\tilde{\omega}_j) \stackrel{\mathrm{def}}{=} \{\omega_j \in C^{\mathrm{eq}}_\beta \mid |\omega_j - \tilde{\omega}_j|_\beta < \varepsilon_j\}.$$

Therefore one gets (see (1.26)

$$(V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}))^{\mathrm{cl}} \ \stackrel{\mathrm{def}}{=} \ \bigcup_{A\in\mathcal{B}^{\mathrm{cl}}_{\beta,\Lambda}} (A \cap V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}))$$

and obtains

$$(V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'}))^{\mathrm{cl}} \ = \ \times_{j\in\Lambda'} I_j(\tilde{\omega}_j, \varepsilon_j),$$

where $I_j(\tilde{\omega}_j, \varepsilon_j)$ is an open (possibly empty) interval of the real line $I\!R$. Then taking into account (1.27), (1.24), and (1.25) one concludes that

$$\gamma^{\mathrm{cl}}_{\beta,\Lambda}(\partial V_{\Lambda'}(\tilde{\omega}_\Lambda|\varepsilon_{\Lambda'})) = 0.$$

■

Now it remains to prove Lemma 2.1. To this end we use the following known property of Gaussian measures on a Hilbert space $\mathcal{H}$ (see e.g. pp. 153–155 of book [20]).

Proposition 2.2 *Let a net of Gaussian measures $\{\gamma_\alpha\}$ on a separable real Hilbert space $\mathcal{H}$ be given. Let also each measure γ_α have zero mean and a trace class operator on $\mathcal{H}$, S_α, as a covariance operator. Suppose that the net $\{S_\alpha\}$ converges in the trace norm to an operator S. Then there exists a Gaussian measure γ on the space $\mathcal{H}$ such that the operator S is its covariance operator, and $\gamma_a \Rightarrow \gamma$.*

Proof of Lemma 2.1. First of all we construct explicitly the covariance operators of the Gaussian measures $\gamma_{\beta,\Lambda}$ and $\gamma^{\mathrm{cl}}_{\beta,\Lambda}$ defined by (1.15) and (1.27) respectively. The first one implies

$$S_{\beta,\Lambda}(m) = \sum_{j\in\Lambda} S_{\beta,j}(m) P_{\beta,j}, \tag{2.3}$$

where $P_{\beta,j}$ is the projector from $\mathcal{H}_{\beta,\Lambda}$ onto the space $\mathcal{H}_{\beta,j}$. Here, as before, we omit the subscript j when this does not cause any ambiguities.

In the sequel we will need the base of the Hilbert space $\mathcal{H}_\beta = \mathcal{H}_{\beta,j}$, which we choose as the following orthonormal set the eigenfunctions of Δ_β

$$\begin{aligned} e_q(\tau) &= \sqrt{\frac{2}{\beta}}\cos q\tau, \text{ for } q>0; \qquad q\in\mathcal{Q} \stackrel{\mathrm{def}}{=} \{\frac{2\pi}{\beta}n \mid n\in\mathbb{Z}\}, \\ e_q(\tau) &= \sqrt{\frac{2}{\beta}}\sin q\tau, \text{ for } q<0; \qquad e_0(\tau) = \sqrt{\frac{1}{\beta}}. \end{aligned} \tag{2.4}$$

The operator $S_{\beta,\Lambda}(m)$ acts on $\mathcal{H}_{\beta,\Lambda}$ as a positive compact operator, hence it has the canonical representation:

$$S_{\beta,\Lambda}(m) = \sum_{j\in\Lambda}\left(\sum_{q\in\mathcal{Q}}(mq^2+1)^{-1}W_q\right)P_{\beta,j}. \tag{2.5}$$

where W_q is the projector from $\mathcal{H}_\beta$ onto the direction e_q (see (2.4)). Having this representation we prove Lemma 2.1 by showing that the net of the covariance operators $\{S_{\beta,\Lambda}(m)|m\in(0,+\infty)\}$ converges in the trace norm to the covariance operator of the measure $\gamma^{\mathrm{cl}}_{\beta,\Lambda}$ defined by (1.27), (1.24), (1.25). Let us construct the covariance operator of the latter measure. To this end we write down its Fourier transformation, which should have the form (1.14), valid for all Gaussian measures.

$$\begin{aligned} \Gamma^{\mathrm{cl}}_{\beta,\Lambda}(\varphi_\Lambda) &\stackrel{\mathrm{def}}{=} \int_{\mathcal{H}_{\beta,\Lambda}} \exp\{i(\varphi_\Lambda,\omega_\Lambda)_{\beta,\Lambda}\}\gamma^{\mathrm{cl}}_{\beta,\Lambda}(d\omega_\Lambda) & (2.6)\\ &= \exp\{-\frac{1}{2}(\varphi_\Lambda, S^{\mathrm{cl}}_{\beta,\Lambda}\varphi_\Lambda)_{\beta,\Lambda}\}, & (2.7)\end{aligned}$$

where the scalar product $(.,.)_{\beta,\Lambda}$ is defined by (1.12). On the other hand, (1.27) implies that the measure $\gamma^{\rm cl}_{\beta,\Lambda}$ is supported on the subset $\Xi_{\beta,\Lambda} \subset \Omega_{\beta,\Lambda} \subset \mathcal{H}_{\beta,\Lambda}$, where it coincides with the measure $\chi_{\beta,\Lambda}$ given by , (1.24), (1.25). This yields in (2.6)

$$\begin{aligned}
\Gamma^{\rm cl}_{\beta,\Lambda}(\varphi_\Lambda) &= \left(\frac{\beta}{2\pi}\right)^{\frac{|\Lambda|}{2}} \int_{I\!R^\Lambda} \exp\left\{ i \sum_{j\in\Lambda} x_j \int_{[0,\beta]} \varphi_j(\tau) d\tau \right\} \\
&\quad \exp\left\{ -\frac{\beta}{2} \sum_{j\in\Lambda} x_j^2 \right\} \prod_{j\in\Lambda} dx_j \\
&= \exp\left\{ -\frac{1}{2\beta} \sum_{j\in\Lambda} \left(\int_{[0,\beta]} \varphi_j(\tau) d\tau \right)^2 \right\} \\
&= \exp\left\{ -\frac{1}{2} \sum_{j\in\Lambda} (e_0, \varphi_j)^2_\beta \right\}. \qquad (2.8)
\end{aligned}$$

Here we have used the eigenfunction e_0 given by (2.4). Comparing the latter form of $\Gamma^{\rm cl}_{\beta,\Lambda}$ with the definition (2.6), one concludes that

$$S^{\rm cl}_{\beta,\Lambda} = \sum_{{\rm j}\in\Lambda} W_0 P_{\beta,j}, \qquad (2.9)$$

where W_0 is a projector in $\mathcal{H}_{\beta,j}$ on the direction e_0. Now we use the canonical representation (2.5) and obtain

$$S_{\beta,\Lambda}(m) - S^{\rm cl}_{\beta,\Lambda} = \sum_{j\in\Lambda} \left(\sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2+1} W_q \right) P_{\beta,j},$$

which yields

$$\begin{aligned}
{\rm trace}(S_{\beta,\Lambda}(m) - S^{\rm cl}_{\beta,\Lambda}) &= |\Lambda| \sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2+1} \leq |\Lambda| \sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2} \\
&= \frac{1}{m} \left(\frac{|\Lambda|\beta^2}{2\pi^2} \sum_{n\in I\!N} \frac{1}{n^2} \right) \longrightarrow 0, \quad m \to +\infty.
\end{aligned}$$

■

References

[1] Albeverio, S., Høegh-Krohn, R.: *Homogeneous Random Fields and Quantum Statistical Mechanics.* J. Funct. Anal. **19**, 242– 272 (1975)

[2] Albeverio, S., Høegh-Krohn, R.: *Mathematical Theory of Feynman Path Integrals.* Lecture Notes in Math. **523**, Springer, Berlin (1976)

[3] Albeverio, S., Kondratiev, Yu. G., Röckner, M., Tsikalenko, T.V.: *Uniqueness of Gibbs States for Quantum Lattice Systems.* Probab. Theory Relat. Fields **108**, 193–218 (1997)

[4] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Absence of Critical Points for a Class of Quantum Hierarchical Models.* Commun. Math. Phys. **187**, 1–18 (1997)

[5] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Suppression of Critical Fluctuations by Strong Quantum Effects in Quantum Lattice Systems.* Commun. Math. Phys. **194**, 493–521 (1998)

[6] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V., Röckner M.: *Uniqueness for Gibbs States of Quantum Lattices in Small Mass Regime.* (submitted to Commun. Math. Phys.)

[7] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Correlation Inequalities for Temperature Loop Measures.* (in preparation)

[8] Barbulyak, V.S. Kondratiev, Yu.G. *Functional Integrals and Quantum Lattice Systems: I. Existence of Gibbs States.* Reports of Nat. Acad. Sci. of Ukraine. No 9, 38–40 (1991)

[9] Barbulyak, V.S. Kondratiev, Yu.G.*Functional Integrals and Quantum Lattice Systems: II. Periodic Gibbs States.* Reports of Nat. Acad. Sci. of Ukraine. No 8, 31–34 (1991)

[10] Barbulyak, V.S. Kondratiev, Yu.G. *Functional Integrals and Quantum Lattice Systems: III. Phase Transitions.* Reports of Nat. Acad. Sci. of Ukraine. No 10, 19–21 (1991)

[11] Berezin, F.A., Shubin, M.A.: *The Schrödinger Equation.* Kluwer Academic Publishers, Dordrecht/Boston/London (1991)

[12] Bratteli O., Robinson D.W.: *Operator Algebras and Quantum Statistical Mechanics. II.* Springer, New York/Hedelberg/Berlin (1981)

[13] Dobrushin, R.L.: *Prescribing a System of Random Variables by Conditional Distributions.* Theory Prob. Appl. **15**, 458–486 (1970)

[14] Georgii, H.O.: *Gibbs Measures and Phase Transitions.* vol.9, Walter de Gruyter, Berlin New York: Springer (1988)

[15] Globa, S.A., Kondratiev, Yu.G.: *The construction of Gibbs states of quantum lattice systems.* Selecta Math. Sov. **9**, 297–307 (1990)

[16] Klein, A., Landau, L.: *Stochastic Processes Associated with KMS States.* J. Funct. Anal. **42**, 368–428 (1981)

[17] Kondratiev, Yu.G.: *Phase Transitions in Quantum Models of Ferroelectrics.* In: Stochastic Processes, Physics, and Geometry II, eds. S.Albeverio at all, 465–475, World Scientific: Singapore, New Jersey (1994)

[18] Park, Y.M., Yoo, H.J.: *A Characterization of Gibbs States of Lattice Boson Systems.* J.Stat.Phys. **75**, 215– 239 (1994)

[19] Park, Y.M., Yoo, H.J.: *Uniqueness and Clustering Properties of Gibbs States for Classical and Quantum Unbounded Spin Systems.* J.Stat.Phys. **80**, 223–271 (1995)

[20] Parthasarathy, K.R.: *Probability Measures on Metric Spaces.* Academic Press: New York and London (1967)

[21] Simon, B.: *Functional Integrals in Quantum Physics.* Academic Press: New York (1986)