Weakly Gibbsian Measures and Quasilocality : a long range pair-interaction example.

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Abstract

We exhibit an example of a measure on a discrete and finite spin system whose conditional probabilities are given in terms of an almost everywhere absolutely summable potential but are discontinuous almost everywhere.

Keywords: Gibbs measures, quasilocality, almost Gibbs, weakly Gibbs

1 Introduction.

This note is intended to provide a simple example of the non-equivalence of different notions introduced recently in order to generalize the standard Gibbs theory. We want to consider the difference between *weakly* Gibbsian measures and *almost* Gibbsian measures.

We shall work on a spin system whose configuration space is given by $\Omega = \{0, +1\}^{\mathbf{Z}^d}$. We denote by s an element of Ω and by s_A an element of $\Omega_A = \{0, +1\}^A$, for $A \subset \mathbf{Z}^d$. Consider potentials (interactions) $\Phi = (\Phi_X)$ which are families of functions

$$\Phi_X : \Omega_X \to \mathbf{R}, \tag{1}$$

indexed by $X \subset \mathcal{L}$, $|X| < \infty$. We say that a potential is $\overline{\Omega}$-pointwise absolutely summable, with $\overline{\Omega} \subset \Omega$, if

$$\sum_{X \ni x} |\Phi_X(s_X)| < \infty \quad \forall x \in \mathbf{Z}^d, \forall s \in \overline{\Omega}. \tag{2}$$

Then, if $\overline{\Omega}$ is in the tail-field, one may define a Hamiltonian in a finite volume V with boundary conditions $\bar{s} \in \overline{\Omega}$ by the usual formula :

$$H(s_V|\bar{s}_{V^c}) = \sum_{X \cap V \neq \emptyset} \Phi_X(s_{X\cap V} \vee \bar{s}_{X \cap V^c}) \tag{3}$$

where $s_V \in \Omega_V$, $\bar{s}_{V^c}$ is the restriction to V^c of $\bar{s} \in \overline{\Omega}$ and, for $X \cap Y = \emptyset$, $s_X \vee s_Y$ denotes the obvious configuration in $\Omega_{X\cup Y}$. We say that a measure on Ω is *weakly* Gibbsian if there exist a translation-invariant set $\overline{\Omega}$, and a $\overline{\Omega}$-pointwise absolutely summable interaction Φ such that $\mu(\overline{\Omega}) = 1$ and for μ there exists a version of the conditional probabilities that satisfy $\forall V \subset \mathbf{Z}^d$, $|V|$ finite, $\forall s_V \in \Omega_V$

$$\mu(s_V|\bar{s}_{V^c}) \;=\; \begin{cases} Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V|\bar{s}_{V^c})) & \text{for } \bar{s} \in \overline{\Omega} \\ 0 & \text{for } \bar{s} \notin \overline{\Omega} \end{cases} \tag{4}$$

A function f on Ω is said to be quasilocal at a certain s, if s is a point of continuity of f in the product topology on Ω, i.e.

$$\forall \epsilon > 0,\, \exists V(s) \; such\, that\, if\; s'_{V(s)} = s_{V(s)},\; we\, have,\; |f(s) - f(s')| < \epsilon. \tag{5}$$

A function f is *essentially* discontinuous at s if

$$\exists \delta > 0, \forall V, \exists s' \, s.t. \, s'_V = s_V \, and \, \exists V', \, V' \supset V \, s.t. \forall s'' \, s.t. \, s''_{V'} = s'_{V'}, \, |f(s'') - f(s)| > \delta.. \tag{6}$$

The difference between discontinuity and essential discontinuity is important in our context. Indeed, the points of essential discontinuity of a system of conditional probabilities can not be tranformed into points of continuity by modifying the conditional probabilities on a set of measure zero.

A measure μ on Ω is said to be *almost* Gibbsian [4] if there exists a version of its conditional probabilities that is quasilocal (as a function of the conditioning) μ-almost everywhere.

The quasilocality *everywhere* of the conditional probabilities is an important characterization of the measure due to a theorem by Kozlov and Sullivan [3, 7]. This theorem states that if a measure has a system of conditional probabilities that is quasilocal everywhere then, modulo a positivity condition on the conditional probabilities, the measure is a Gibbs measure in the standard sense and the converse is also true.

The relationship between the two definitions introduced above (i.e. weakly and almost Gibbsian) is not clear a priori. A generalization of the result of Kozlov and Sullivan was obtained in [4] for almost Gibbsian measures. Almost Gibbsian measures are also weakly Gibbsian. But, for example, the conditional probabilities of some measures obtained after a Renormalization Group Transformation on the Ising model at low temperature have a non-empty set of points of essential discontinuity [2]. While the size measure of this set is unknown, it is shown, as in the Schonman example [6, 5], that those measures are weakly Gibbsian [1].

We provide here an example of measure μ on Ω that is weakly Gibbsian but whose conditional probabilities are essentially discontinuous on a set of μ-measure 1. Our example is similar to the one given in [4] but different in the sense that we are able to prove that the set of points of discontinuity is of measure 1 with respect to the measure under study, the form of our interaction being extremely simple (pair-interactions). Besides our model is defined in any dimension d (but is not translation invariant, as in [4]).

2 Results

Consider the measure on Ω

$$\mu(ds) = \frac{\exp -H(s)}{Z}\mu_0(ds) \tag{7}$$

with μ_0 the product measure, $H(s)$ is formally given by

$$H(s) = \sum_{i,j\in\mathbf{Z}^d}^{\infty} 2^{|i|+|j|}s_i s_j \tag{8}$$

and obviously $Z = \sum_s \exp -H(s)\mu_0(ds)$. Z is well-defined because $\exp -H(s)$ is a limit of measurable functions and and it is uniformly bounded on Ω, besides it is obviously non-zero.

Remark. Our argument in this note can be easily generalized to a Hamiltonian defined like in (8) but with $2^{|i|+|j|}$ replaced by ϕ_{ij} with ϕ_{ij} such that $\phi_{ij} \geq 0$ and $\phi_{ii} \to \infty$ as $|i| \to \infty$.

It is easy to see that $H(s)$ is finite on the set

$$\Omega_g = \{s \in \Omega | \exists \overline{V}(s)\, a\, cube\, ,\, \forall n \notin \overline{V}(s)\; s_n = 0\}. \tag{9}$$

Proposition 1

$$\mu(\Omega_g) = 1 \tag{10}$$

Proof. Let us prove that $\mu(\Omega_g^c) = 0$. One has, $\Omega_g^c = \cap_{N=0}^{\infty} \Omega_N$ with,

$$\Omega_N = \{s \in \Omega | \exists n\, s.\, t.\, |n| \geq N\; s_n = 1\} \tag{11}$$

and also,

$$\Omega_N \subset \cup_{n:|n|\geq N} \overline{\Omega}_n \tag{12}$$

with,

$$\overline{\Omega}_n = \{s \in \Omega | s_n = 1\}. \tag{13}$$

Now, we write,

$$\mu(\overline{\Omega}_n) = \mu(s_n = 1) = \sum_{s:s_n=1} \frac{\exp -H(s)}{Z}, \tag{14}$$

and using the Hamiltonian (8), we see that this expression is bounded from above by

$$e^{-2^{2|n|}} \sum_{s:s_n=0} \frac{\exp -H(s)}{Z} = e^{-2^{2|n|}} \mu(s_n = 0) \leq e^{-2^{2|n|}}, \tag{15}$$

because we have $H(s) - H(\overline{s}) \geq 2^{2|n|}$ if s is such that $s_n = 1$ and $\overline{s}$ such that $\overline{s}_n = 0$, $\overline{s}_{\mathbf{Z}^d \setminus \{n\}} = s_{\mathbf{Z}^d \setminus \{n\}}$.

From the bound (15) it is easy to conclude that $\mu(\Omega_N) \leq cN^{d-1} e^{-2^{2N}}$ and thus that $\mu(\Omega_g^c) = 0$, which concludes the proof.

We can then easily compute the conditional distribution of the spin at the origin and get

$$\mu(s_0|\overline{s}_{\{0\}^c}) = \frac{1}{1+\exp(-(1-2s_0)\sum_{k\in\mathbf{Z}^d} 2^{|k|}\overline{s}_k)} \tag{16}$$

Obviously, this conditional probability is expressed in terms of long-range two-body interactions and we may define the relative energy function of the spin at the origin.

$$h_0(s) = (1-2s_0)\sum_{k\in\mathbf{Z}^d} 2^{|k|}s_k. \tag{17}$$

Obviously, as $H(s)$, it is (absolutely) summable on Ω_g. To conclude, we need to show that this function is not only discontinuous but essentially discontinuous on Ω_g.

Proposition 2 *h_0 is essentially discontinuous on Ω_g in the product topology on Ω.*

Proof. From the definition of h_0 in (17) and Ω_g in (9), it is easy to see that if $s \in \Omega_g$ and $s' \in \Omega_g$ are such that $s_{\overline{V}(s)} = s'_{\overline{V}(s)}$ and $s_k \neq s'_k$, then $|h_0(s) - h_0(s')| \geq 2^{|k|}$ ($k \in \overline{V}^c(s)$). It is then easy to see that h_0 is essentially discontinuous on Ω_g; for any $s \in \Omega_g$, in (6) take $\delta = 1$ and $\forall V$, choose V' and s' as follows, take V' a finite (connected) set such that $V' \supset (V \cup \overline{V}(s))$ and s' a configuration such that $s'_{V\cup\overline{V}(s)} = s_{V\cup\overline{V}(s)}$ and $s'_{V'\setminus(V\cup\overline{V}(s))} \neq s_{V'\setminus(V\cup\overline{V}(s))}$. Then for any configuration s'' such that $s''_{V'} = s'_{V'}$, it is clear that one has $|h_0(s'') - h_0(s)| > 1$.

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