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Trotter-Kato product formula and operator-norm convergence

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Abstract

We find necessary and sufficient conditions for the *operator-norm* convergence of the Trotter-Kato product formula. Using them we prove that this convergence takes place: *(i)* if the resolvent of one of the involved operators is compact, either *(ii)* if one operator is relatively compact with respect to another one, or *(iii)* if the product of resolvents of the involved operators is compact.

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Contents

1 Introduction

Let A and B be two non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$, i.e.

$$A \geq 0 \qquad \text{and} \qquad B \geq 0. \tag{1.1}$$

By $\mathcal{H}_0$ we denote the subspace

$$\mathcal{H}_0 := \overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})}. \tag{1.2}$$

It may happen that $\mathcal{H}_0 = \{0\}$. The form sum $H = A \dot{+} B$ is well-defined on the subspace $\mathcal{H}_0$. Kato has shown in [14] that under these conditions the Trotter product formula [20] strongly converges to e^{-tH} uniformly in $t \in [0,T]$, $0 < T < +\infty$, i.e., one has

$$s - \lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-tH}P_0, \qquad t \geq 0, \tag{1.3}$$

where P_0 denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$. Moreover, it was found by Kato in [13, 14] that the Trotter product formula is true not only for the exponential function e^{-x}, $x \geq 0$, but for a whole class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$ which are defined on $\mathbf{R}^1_+ := [0, \infty)$ and characterized by

$$0 \leq f(x) \leq 1, \qquad f(0) = 1, \qquad f'(+0) = -1 \tag{1.4}$$

$$0 \leq g(x) \leq 1, \qquad g(0) = 1, \qquad g'(+0) = -1. \tag{1.5}$$

and which are called *Kato-functions* in the following. In [14] it was proven that for Kato-functions $f(\cdot)$ and $g(\cdot)$ besides (1.3) one always has

$$s - \lim_{n\to\infty} (f(tA/n)g(tB/n))^n = e^{-tH}P_0, \qquad t \geq 0, \tag{1.6}$$

uniformly in $t \in [0,T]$, $0 < T < +\infty$. In the following we call product formulae of this type *Trotter-Kato product formulae.*

Naturally, the problem arises to find conditions such that the strong convergence in (1.3) and (1.6) can be replaced by the operator-norm convergence. A partial solution of this problem was found in [15]. Defining

$$0 \leq \varphi_0(x) := \inf_{0<s\leq x} s^{-1}\left(\frac{1}{f(s)} - 1\right), \tag{1.7}$$

where $0^{-1} := +\infty$, and setting

$$f_0(x) := \begin{cases} 1 & x = 0 \\ (1 + x\varphi_0(x))^{-1} & x > 0, \end{cases} \tag{1.8}$$

which again defines a Kato-function, it was shown in [15] that under condition

$$f_0(tA) \in \mathcal{L}_p(\mathcal{H}), \qquad t > 0, \qquad 1 \leq p < +\infty, \tag{1.9}$$

one gets for Kato-functions $f(\cdot)$ and $g(\cdot)$

$$\lim_{n\to\infty} \left\|(f(tA/n)g(tB/n))^n - e^{-tH}P_0\right\|_1 = 0 \tag{1.10}$$

uniformly in $t \in [a,b]$ for any bounded interval $[a,b] \subseteq (0,\infty)$. Here $\mathcal{L}_p(\mathcal{H})$ and $\|\cdot\|_p$, $1 \leq p < +\infty$, denote the so-called von Neumann-Schatten ideals of compact operators and their norms ($p \geq 1$), in particular, $\mathcal{L}_\infty(\mathcal{H})$ denotes the set of compact operators. Since trace-norm convergence (1.10) implies operator-norm convergence, the condition (1.9) implies the desired convergence. In particular, if $f(x) = e^{-x}$, $x \geq 0$, then $f_0(x) = (1+x)^{-1}$ which yields that condition

$$(I + tA)^{-1} \in \mathcal{L}_p(\mathcal{H}), \qquad t > 0, \quad 1 \leq p < +\infty, \tag{1.11}$$

implies the operator-norm convergence.

However, by the recent papers [10], [16], [17] and [18] the assumption (1.9) is not necessary for the operator-norm convergence. For instance, in [16] it is proven that under the conditions $A \geq I$, $B \geq I$,

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \tag{1.12}$$

and

$$\|Bf\| \leq a\|Af\|, \qquad 0 < a < 1, \quad f \in \mathrm{dom}(A), \tag{1.13}$$

for a certain subclass of Kato-functions one has

$$\|(f(tA/n)g(tB/n))^n - e^{-tH}\| = O(\ln(n)/n) \tag{1.14}$$

uniformly in $t \geq 0$. Moreover, replacing conditions (1.12) and (1.13) by fractional-power conditions

$$\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B^\alpha), \qquad \frac{1}{2} < \alpha < 1, \tag{1.15}$$

and

$$\|B^\alpha f\| \leq a\|A^\alpha f\|, \qquad 0 < a < 1, \quad f \in \mathrm{dom}(A^\alpha), \tag{1.16}$$

for a certain subclass of Kato-functions it was in particular found in [17] that

$$\|(f(tA/n)g(tB/n))^n - e^{-tH}\| = O(\ln(n)/n^{2\alpha-1}) \tag{1.17}$$

uniformly in $t \in [\epsilon, \infty)$, where $\epsilon > 0$. Hence very different conditions can imply operator-norm convergence of the Trotter-Kato product formula.

So, our first aim (see Section 2) will be to look for *necessary* and *sufficient* conditions which guarantee the *operator-norm* convergence of the Trotter-Kato product formula for given non-negative self-adjoint operators A and B and Kato-functions $f(\cdot)$ and $g(\cdot)$. However, in contrast to the papers [10], [11], [16], [17] and [18] we are *not* interested in error bounds.

Using this approach we prove in Section 3 that under the condition

$$f_0(t_0 A) \in \mathcal{L}_\infty(\mathcal{H}) \tag{1.18}$$

for some $t_0 > 0$ one gets operator-norm convergence of the Trotter-Kato product formula for any non-negative self-adjoint operator B and any Kato-function $g(\cdot)$. Notice, that condition (1.18) is weaker than (1.9). Furthermore, it turns out that the simpler condition (cf. (1.11)):

$$(I + t_0 A)^{-1} \in \mathcal{L}_\infty(\mathcal{H}), \qquad t_0 > 0, \tag{1.19}$$

implies the operator-norm convergence for *any* $B \geq 0$, *any regular* Kato-function $f(\cdot)$ (for definitions see Section 3) and *any* Kato-function $g(\cdot)$. Moreover, the converse holds: If A is a non-negative self-adjoint operator such that the operator-norm convergence of the Trotter-Kato product formula takes place for *any* $B \geq 0$, *any regular* Kato-function $f(\cdot)$ and *any* Kato-function $g(\cdot)$, then condition (1.19) has to be satisfied for some $t = t_0 > 0$.

Notice that methods used in [16] and [17] do not cover the case $\alpha = 1/2$ (cf. (1.15)):

$$\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2}). \tag{1.20}$$

Using the results of Section 2 we prove in Section 4 that operator-norm convergence takes place under the condition (1.20) provided that *in addition* one has

$$B^{1/2}A^{-1/2} \in \mathcal{L}_\infty(\mathcal{H}), \qquad A \geq I. \tag{1.21}$$

We note that under condition (1.21) the operator $B^{1/2}$ is relatively $A^{1/2}$-bounded with *relative bound* $a = 0$ which clarifies the difference between conditions (1.21) and (1.16) for $\alpha = 1/2$.

Finally, neither conditions (1.14) or (1.16), nor conditions (1.18) or (1.21) are applicable to the operator-norm convergence of the Trotter product formula for Schrödinger semigroups studied in [4], [8], [9]. It turns out that this situation can be treated on the abstract level and for the Trotter-Kato product case assuming

$$(I + A)^{-1}(I + B)^{-1} \in \mathcal{L}_\infty(\mathcal{H}). \tag{1.22}$$

Using results of Section 2 we show in Section 5 that condition (1.22) guarantees the operator-norm convergence of the Trotter-Kato product formula, however, without error bounds.

2 Operator-norm convergence

We set

$$\begin{aligned}
T(t) &:= f(tA)^{1/2}g(tB)f(tA)^{1/2}, \qquad t \geq 0, && (2.1)\\
R(t) &:= \frac{1}{t}(I - T(t)), \qquad t > 0, && (2.2)\\
F(t) &:= g(tB)^{1/2}f(tA)g(tB)^{1/2}, \qquad t \geq 0, && (2.3)\\
S(t) &:= \frac{1}{t}(I - F(t)), \qquad t > 0,. && (2.4)
\end{aligned}$$

Further, we set $\mathcal{H}(t) := \overline{\mathrm{ran}(f(tA))} \subseteq \mathcal{H}$, $t \geq 0$. By $Q(t)$, $t \geq 0$, we denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}(t)$. By (1.4) one obviously has that

$$s - \lim_{t\to+0} Q(t) = I. \tag{2.5}$$

We note that for each $t \geq 0$ the projection $Q(t)$ commutes with A. We set $\tilde{f}(tA) := f(tA)|\mathcal{H}(t) : \mathcal{H}(t) \longrightarrow \mathcal{H}(t)$, $t \geq 0$. By definition we have $\ker(\tilde{f}(tA)) = \{0\}$ and

$$0 \leq \tilde{f}(tA) \leq I(t), \qquad t \geq 0, \tag{2.6}$$

where $I(t) := I|\mathcal{H}(t)$. Therefore, it makes sense to introduce the family of operators $\{M(t)\}_{t>0}$,

$$M(t) := \frac{1}{t}[\tilde{f}(tA)^{-1} - I(t)] + \frac{1}{t}Q(t)[I - g(tB)]Q(t) : \mathcal{H}(t) \longrightarrow \mathcal{H}(t), \tag{2.7}$$

with domain

$$\mathrm{dom}(M(t)) = \mathrm{ran}(f(tA)) \subseteq \mathcal{H}(t). \tag{2.8}$$

Finally, we put $I_0 := I|\mathcal{H}_0$, see (1.2). Note that in case $f(x) > 0$, $x \in \mathbf{R}^1$, one readily gets

$$\mathcal{H}(t) = \mathcal{H} \qquad \text{and} \qquad Q(t) = I. \tag{2.9}$$

Theorem 2.1 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be Kato-functions. Then the following conditions are equivalent:*

(i) $\lim_{r\to+\infty}\sup_{t\in[a,b]}\|T(t/r)^r - P_0e^{-tH}P_0\| = 0, \qquad [a,b] \subseteq (0,\infty),$

(ii) $\lim_{t\to+0}\|(\lambda I + R(t))^{-1} - P_0(\lambda I_0 + H)^{-1}P_0\| = 0, \qquad \lambda > 0,$

(iii) $\lim_{t\to+0}\|Q(t)(\lambda I(t) + M(t))^{-1}Q(t) - P_0(\lambda I_0 + H)^{-1}P_0\| = 0, \qquad \lambda > 0,$

(iv) $\lim_{t\to+0}\|(\lambda I + S(t))^{-1} - P_0(\lambda I_0 + H)^{-1}P_0\| = 0, \qquad \lambda > 0,$

(v) $\lim_{r\to+\infty}\sup_{t\in[a,b]}\|F(t/r)^r - P_0e^{-tH}P_0\| = 0, \qquad [a,b] \subseteq (0,\infty).$

Proof: We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Since

$$\sup_{x\in[0,1]}|x^r - e^{-r(1-x)}| \le \frac{1}{r}, \qquad r \ge 1, \tag{2.10}$$

by the spectral integral one gets that

$$\|T(t/r)^r - e^{-tR(t/r)}\| \le \frac{1}{r}, \qquad r \ge 1, \quad t \ge 0. \tag{2.11}$$

By

$$e^{-tR(t/r)} - P_0e^{-tH}P_0 = T(t/r)^r - P_0e^{-tH}P_0 + e^{-tR(t/r)} - T(t/r)^r \tag{2.12}$$

we find the estimate

$$\|e^{-tR(t/r)} - P_0e^{-tH}P_0\| \le \|T(t/r)^r - P_0e^{-tH}P_0\| + \|e^{-tR(t/r)} - T(t/r)^r\|. \tag{2.13}$$

Applying (i) and (2.11) we immediately find

$$\lim_{r\to+\infty}\|e^{-tR(t/r)} - P_0e^{-tH}P_0\| = 0 \tag{2.14}$$

for each $t \ge 0$. Let $\mathcal{A}$ denote the algebra of all real-valued continuous functions (which are defined on $\mathbf{R}^1_+$ and which tend to zero as $x \longrightarrow +\infty$) endowed with the supremum norm. Then the subalgebra generated by the function e^{-x}, $x \in \mathbf{R}^1_+$, is denoted by $\mathcal{A}_e$. The Stone-Weierstrass theorem implies that $\mathcal{A}_e$ is dense in $\mathcal{A}$. In particular, this yields that for each $\epsilon > 0$ there is a polynomial $\mathcal{P}(\cdot)$ such that

$$\sup_{x\ge 0}|(\lambda + x)^{-1} - \mathcal{P}(e^{-x})| \le \epsilon \tag{2.15}$$

and some fixed $\lambda > 0$. In particular, we have

$$\sup_{x\ge 0}|(\lambda + tx)^{-1} - \mathcal{P}(e^{-tx})| < \epsilon \tag{2.16}$$

uniformly for $t \geq 0$. Hence

$$\|(\lambda I + tR(t/r))^{-1} - \mathcal{P}(e^{-tR(t/r)})\| < \epsilon \tag{2.17}$$

and

$$\|(\lambda I_0 + tH)^{-1} - \mathcal{P}(e^{-tH})\| < \epsilon \tag{2.18}$$

uniformly for $t \geq 0$. By

$$\begin{aligned}(\lambda I + tR(t/r))^{-1} - P_0(\lambda I_0 + tH)^{-1}P_0 = (\lambda I + tR(t/r))^{-1} - \mathcal{P}(e^{-tR(t/r)}) + \\ \mathcal{P}(e^{-tR(t/r)}) - P_0\mathcal{P}(e^{-tH})P_0 + P_0(\mathcal{P}(e^{-tH}) - (\lambda I_0 + tH)^{-1})P_0\end{aligned} \tag{2.19}$$

we get the estimate

$$\begin{aligned}\|(\lambda I + tR(t/r))^{-1} - P_0(\lambda I_0 + tH)^{-1}P_0\| \leq \|(\lambda I + tR(t/r))^{-1} - \mathcal{P}(e^{-tR(t/r)})\| + \\ \|\mathcal{P}(e^{-tR(t/r)}) - P_0\mathcal{P}(e^{-tH})P_0\| + \|P_0(\mathcal{P}(e^{-tH}) - (\lambda I_0 + tH)^{-1})P_0\|.\end{aligned} \tag{2.20}$$

Taking into account (2.17) and (2.18) we obtain

$$\|(\lambda I + tR(t/r))^{-1} - P_0(\lambda I_0 + tH)^{-1}P_0\| \leq 2\epsilon + \|\mathcal{P}(e^{-tR(t/r)}) - P_0\mathcal{P}(e^{-tH})P_0\|. \tag{2.21}$$

Since $\|e^{-tR(t/r)}\| \leq 1$ and $\|e^{-tH}\| \leq 1$ we find a constant $C_{\mathcal{P}} > 0$, which depends on the polynomial $\mathcal{P}(\cdot)$, such that

$$\|\mathcal{P}(e^{-tR(t/r)}) - P_0\mathcal{P}(e^{-tH})P_0\| \leq C_{\mathcal{P}}\|e^{-tR(t/r)} - P_0e^{-tH}P_0\|. \tag{2.22}$$

for $t \geq 0$. Hence

$$\|(\lambda I + tR(t/r))^{-1} - P_0(\lambda I_0 + tH)^{-1}P_0\| \leq 2\epsilon + C_{\mathcal{P}}\|e^{-tR(t/r)} - P_0e^{-tH}P_0\| \tag{2.23}$$

for $t \geq 0$. Thus, using (2.14) we get

$$\lim_{r\to+\infty} \|(\lambda I + tR(t/r))^{-1} - P_0(\lambda I_0 + tH)^{-1}P_0\| = 0. \tag{2.24}$$

for $t > 0$. Setting $t = 1$ and replacing $1/r$ by t we obtain (ii).

$(ii) \Rightarrow (iii)$. Since

$$R(t) = \frac{1}{t}\{I - f(tA) + f(tA)^{1/2}[I - g(tB)]f(tA)^{1/2}\}, \tag{2.25}$$

we find

$$R(t) = \frac{1}{t}(I - Q(t)) + f(tA)^{1/2}M(t)f(tA)^{1/2}, \tag{2.26}$$

which yields

$$\begin{aligned}&(I + R(t))^{-1} = \\ &\frac{t}{1+t}(I - Q(t)) + Q(t)\tilde{f}(tA)^{-1/2}\{\tilde{f}(tA)^{-1} + M(t)\}^{-1}\tilde{f}(tA)^{-1/2}Q(t).\end{aligned} \tag{2.27}$$

Hence

$$f(tA)^{1/2}(I + R(t))^{-1}f(tA)^{1/2} = Q(t)\{\tilde{f}(tA)^{-1} + M(t)\}^{-1}Q(t). \tag{2.28}$$

Since
$$\tilde{f}(tA)^{-1} + M(t) = (1+t)\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\} \geq I(t), \tag{2.29}$$
we finally get
$$\begin{aligned} f(tA)^{1/2}(I+R(t))^{-1}f(tA)^{1/2} = \\ \frac{1}{1+t}Q(t)\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\}^{-1}Q(t). \end{aligned} \tag{2.30}$$
Now we prove that
$$\lim_{t\to+0} \|f(tA)^{1/2}(I+R(t))^{-1}f(tA)^{1/2} - P_0(I_0+H)^{-1}P_0\| = 0. \tag{2.31}$$
To this end we use the representation
$$\begin{aligned} f(tA)^{1/2}(I+R(t))^{-1}f(tA)^{1/2} - P_0(I_0+H)^{-1}P_0 = \\ f(tA)^{1/2}\{(I+R(t))^{-1} - P_0(I_0+H)^{-1}P_0\}f(tA)^{1/2} + \\ [f(tA)^{1/2} - I]P_0(I_0+H)^{-1}P_0 f(tA)^{1/2} + P_0(I_0+H)^{-1}P_0[f(tA)^{1/2} - I], \end{aligned} \tag{2.32}$$
which gives the estimate
$$\begin{aligned} \|f(tA)^{1/2}(I+R(t))^{-1}f(tA)^{1/2} - P_0(I_0+H)^{-1}P_0\| \leq \\ \|(I+R(t))^{-1} - P_0(I_0+H)^{-1}P_0\| + 2\|[I - f(tA)^{1/2}]P_0(I_0+H)^{-1}P_0\|. \end{aligned} \tag{2.33}$$
Since by (1.2) $\mathrm{dom}(H^{1/2}) \subseteq \mathrm{dom}(A^{1/2})$, we have the representation
$$\begin{aligned} [I - f(tA)^{1/2}](I_0+H)^{-1}h = \\ t^{1/2}[I - f(tA)^{1/2}](tA)^{-1/2}A^{1/2}(I_0+H)^{-1/2}(I_0+H)^{1/2}(I_0+H)^{-1}h, \qquad h \in \mathcal{H}_0, \end{aligned} \tag{2.34}$$
which yields the estimate
$$\|[I - f(tA)^{1/2}](I_0+H)^{-1}h\| \leq t^{1/2}C_{1/2}\|h\|, \qquad h \in \mathcal{H}, \tag{2.35}$$
where we have used that the Kato-function $f(\cdot)$ satisfies (cf.(1.4)):
$$C_{1/2} := \sup_{x>0} \frac{1 - f(x)^{1/2}}{x^{1/2}} < +\infty. \tag{2.36}$$
Hence
$$\|[I - f(tA)^{1/2}](I_0+H)^{-1}\| \leq C_{1/2}t^{1/2}. \tag{2.37}$$
Then by (ii), (2.33) and (2.37) we obtain (2.31). Taking into account (2.30) and (2.31) one immediately finds that
$$\lim_{t\to+0} \left\| Q(t)\left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1} Q(t) - P_0(I_0+H)^{-1}P_0 \right\| = 0. \tag{2.38}$$
To show that
$$\lim_{t\to+0} \left\| \left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1} - \{M(t)+I(t)\}^{-1} \right\| = 0 \tag{2.39}$$

we use the identity

$$\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\}^{-1} - \{M(t) + I(t)\}^{-1} = \tag{2.40}$$
$$\{M(t) + I(t)\}^{-1}\left\{I(t) - \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}\left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1}$$

to find the representation

$$\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\}^{-1} - \{M(t) + I(t)\}^{-1} = \tag{2.41}$$
$$\frac{t}{1+t}\{M(t) + I(t)\}^{-1}\left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1} +$$
$$\frac{1}{1+t}\{M(t) + I(t)\}^{-1}Q(t)[I - g(tB)]Q(t)\left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1}.$$

Therefore, the limit (2.39) takes place if

$$\lim_{t\to+0}\left\|[I - g(tB)]Q(t)\left\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right\}^{-1}\right\| = 0. \tag{2.42}$$

Since by (1.2) $\mathrm{dom}(H^{1/2}) \subseteq \mathrm{dom}(B^{1/2})$, one gets the representation

$$[I - g(tB)]Q(t)(M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t))^{-1} = \tag{2.43}$$
$$[I - g(tB)]\left\{Q(t)\left(M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right)^{-1}Q(t) - P_0(I_0 + H)^{-1}P_0\right\} +$$
$$t^{1/2}[I + g(tB)^{1/2}][I - g(tB)^{1/2}](tB)^{-1/2}B^{1/2}(I_0 + H)^{-1/2}(I_0 + H)^{-1/2}P_0$$

and the estimate

$$\|[I - g(tB)^{1/2}]P_0(I_0 + H)^{-1}P_0\| \le S_{1/2}t^{1/2}, \tag{2.44}$$

because for the Kato-function $g(\cdot)$ we have (cf.(1.5)):

$$S_{1/2} := \sup_{x>0}\frac{1 - g(x)^{1/2}}{x^{1/2}} < +\infty. \tag{2.45}$$

Therefore, from (2.43) and (2.44) we obtain the estimate

$$\|[I - g(tB)]Q(t)\{M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\}^{-1}\| \le \tag{2.46}$$
$$\left\|Q(t)\left(M(t) + \frac{1}{1+t}Q(t)g(tB)Q(t)\right)^{-1}Q(t) - P_0(I_0 + H)^{-1}P_0\right\| + 2S_{1/2}t^{1/2}.$$

Applying (2.38) to (2.46) we obtain (2.42) which yields (2.39). Then from (2.38) and (2.39) we obtain (iii) for $\lambda = 1$ and obviously for any $\lambda > 0$.

$(iii) \Rightarrow (iv)$. The identity

$$g(tB)^{1/2}Q(t)(\lambda I(t) + M(t))^{-1}Q(t)g(tB)^{1/2} = (\lambda I + S(t))^{-1}F(t) + \tag{2.47}$$
$$\lambda(\lambda I + S(t))^{-1}g(tB)^{1/2}Q(t)[I - f(tA)]Q(t)(\lambda I(t) + M(t))^{-1}Q(t)g(tB)^{1/2}$$

yields the representation

$$(\lambda I + S(t))^{-1} = tS(t)(\lambda I + S(t))^{-1} + \tag{2.48}$$
$$g(tB)^{1/2} Q(t)(\lambda I(t) + M(t))^{-1} Q(t) g(tB)^{1/2} +$$
$$\lambda(\lambda I + S(t))^{-1} g(tB)^{1/2}[f(tA) - I] Q(t)(\lambda I(t) + M(t))^{-1} Q(t) g(tB)^{1/2}.$$

Hence

$$(\lambda I + S(t))^{-1} - P_0(\lambda I_0 + H)^{-1} P_0 = tS(t)(\lambda I + S(t))^{-1} + \tag{2.49}$$
$$g(tB)^{1/2} \left\{ Q(t)(\lambda I(t) + M(t))^{-1} Q(t) - P_0(\lambda I_0 + H)^{-1} P_0 \right\} g(tB)^{1/2} +$$
$$[g(tB)^{1/2} - I] P_0(\lambda I_0 + H)^{-1} P_0 g(tB)^{1/2} + P_0(\lambda I_0 + H)^{-1} P_0 [g(tB)^{1/2} - I] +$$
$$\lambda(\lambda I + S(t))^{-1} g(tB)^{1/2} [f(tA) - I] \times$$
$$\left\{ Q(t)(\lambda I(t) + M(t))^{-1} Q(t) - P_0(\lambda I_0 + H)^{-1} P_0 \right\} g(tB)^{1/2} +$$
$$\lambda(\lambda I + S(t))^{-1} g(tB)^{1/2} [f(tA) - I] P_0 (\lambda I_0 + H)^{-1} P_0 g(tB)^{1/2}.$$

Therefore, we get the estimate

$$\|(\lambda I + S(t))^{-1} - P_0(\lambda I_0 + H)^{-1} P_0\| \leq \tag{2.50}$$
$$\frac{t}{\lambda} + 2 \left\| Q(t)(\lambda I(t) + M(t))^{-1} Q(t) - P_0(\lambda I_0 + H)^{-1} P_0 \right\| +$$
$$2\|[I - g(tB)^{1/2}] P_0 (\lambda I_0 + H)^{-1} P_0\| + \|[I - f(tA)] P_0 (\lambda I_0 + H)^{-1} P_0\|.$$

Taking into account (iii), (2.37) and (2.44) we obtain (iv).

$(iv) \Rightarrow (v)$. By $\mathcal{A}_r$ we denote the subalgebra of $\mathcal{A}$ generated by the function $(\mu + x)^{-1}$, $x \in \mathbf{R}^1_+$, for a fixed $\mu > 0$. Again by the Stone-Weierstrass theorem the algebra $\mathcal{A}_r$ is dense in $\mathcal{A}$ in the supremum norm. Hence for any $\epsilon_0 > 0$ there is a polynomial $\mathcal{P}(\cdot)$ such that

$$\sup_{x \geq 0} |e^{-x} - \mathcal{P}((\mu + x)^{-1})| \leq \epsilon_0. \tag{2.51}$$

Hence we find

$$\sup_{t \in [a,b]} \sup_{x \geq 0} |e^{-tx} - \mathcal{P}((\mu + tx)^{-1}| \leq \epsilon_0 \tag{2.52}$$

for each interval $[a, b] \subseteq (0, \infty)$. Therefore,

$$\sup_{t \in [a,b]} \|e^{-tS(t/r)} - \mathcal{P}((\mu I + tS(t/r))^{-1}\| \leq \epsilon_0, \quad r \geq 1, \tag{2.53}$$

and

$$\sup_{t \in [a,b]} \|e^{-tH} - \mathcal{P}((\mu I_0 + tH)^{-1}\| \leq \epsilon_0 \tag{2.54}$$

for the interval $[a, b] \subseteq (0, \infty)$. Since the representation

$$e^{-tS(t/r)} - P_0 e^{-tH} P_0 = e^{-tS(t/r)} - \mathcal{P}((\mu I + tS(t/r))^{-1}) + \tag{2.55}$$
$$\mathcal{P}((\mu I + tS(t/r))^{-1}) - P_0 \mathcal{P}((\mu I_0 + tH)^{-1}) P_0 + P_0 \{\mathcal{P}((\mu I_0 + tH)^{-1}) - e^{-tH}\} P_0$$

yields the estimate

$$\|e^{-tS(t/r)} - P_0 e^{-tH} P_0\| \le \|e^{-tS(t/r)} - \mathcal{P}((\mu I + tS(t/r))^{-1})\| + \\ \|\mathcal{P}((\mu I + tS(t/r))^{-1}) - P_0 \mathcal{P}((\mu I_0 + tH)^{-1}) P_0\| + \|P_0\{\mathcal{P}((\mu I_0 + tH)^{-1}) - e^{-tH}\} P_0\|, \tag{2.56}$$

by (2.53) and (2.54) we find

$$\|e^{-tS(t/r)} - P_0 e^{-tH} P_0\| \le \\ 2\epsilon_0 + \sup_{t\in[a,b]} \|\mathcal{P}((\mu I + tS(t/r))^{-1}) - P_0 \mathcal{P}((\mu I_0 + tH)^{-1}) P_0\|. \tag{2.57}$$

Since $\|(\mu I + tS(t/r))^{-1}\| \le \frac{1}{\mu}$ and $\|(\mu I_0 + tH)^{-1}\| \le \frac{1}{\mu}$, $t \ge 0$, there is a constant $C_{\mathcal{P}} > 0$, which depends only on $\mathcal{P}(\cdot)$ and μ, such that

$$\sup_{t\in[a,b]} \|e^{-tS(t/r)} - P_0 e^{-tH} P_0\| \le \\ 2\epsilon_0 + C_{\mathcal{P}} \sup_{t\in[a,b]} \|(\mu I + tS(t/r))^{-1}) - P_0 (\mu I_0 + tH)^{-1} P_0\| \tag{2.58}$$

for each interval $[a,b] \subseteq (0,\infty)$. Using resolvent series

$$(\mu I + tS(t/r))^{-1} = \frac{1}{t}\sum_{k=0}^{\infty} (\lambda - \frac{\mu}{t})^k (\lambda I + S(t/r))^{-(k+1)} \tag{2.59}$$

and

$$(\mu I_0 + tH)^{-1} = \frac{1}{t}\sum_{k=0}^{\infty} (\lambda - \frac{\mu}{t})^k (\lambda I_0 + H)^{-(k+1)} \tag{2.60}$$

for $0 < \mu \le 2\lambda a$, one gets

$$(\mu I + tS(t/r))^{-1} - P_0 (\mu I_0 + tH)^{-1} P_0 = \\ \frac{1}{t}\sum_{k=0}^{\infty} \left(\lambda - \frac{\mu}{t}\right)^k \left\{(\lambda I + S(t/r))^{-(k+1)} - P_0 (\lambda I_0 + H)^{-(k+1)} P_0\right\}. \tag{2.61}$$

Therefore,

$$\|(\mu I + tS(t/r))^{-1} - P_0 (\mu I_0 + tH)^{-1} P_0\| \le \\ \frac{1}{t}\sum_{k=0}^{\infty} (k+1) \left|1 - \frac{\mu}{t\lambda}\right|^k \|(\lambda I + S(t/r))^{-1} - P_0 (\lambda I_0 + H)^{-1} P_0\|. \tag{2.62}$$

If $0 < \mu < 2\lambda a$, then

$$\eta := \sup_{t\in[a,b]} |1 - \frac{\mu}{t\lambda}| < 1. \tag{2.63}$$

Since $\sum_{k=0}^{\infty} (k+1)\eta^k < +\infty$, one gets from (2.62) the estimate

$$\|(\mu I + tS(t/r))^{-1} - P_0 (\mu I_0 + tH)^{-1} P_0\| \le \\ \frac{1}{a}\|(\lambda I + S(t/r))^{-1} - P_0 (\lambda I_0 + H)^{-1} P_0\| \sum_{k=0}^{\infty} (k+1)\eta^k. \tag{2.64}$$

By (iv) for any $\epsilon_1 > 0$ there exists a $\delta > 0$ such that for $0 < t/r \leq \delta$ one has

$$\|(\lambda I + S(t/r))^{-1} - P_0(\lambda I_0 + H)^{-1}P_0\| \leq \epsilon_1. \tag{2.65}$$

Therefore,

$$\sup_{t\in[a,b]} \|(\lambda I + S(t/r))^{-1} - P_0(\lambda I_0 + H)^{-1}P_0\| \leq \epsilon_1 \tag{2.66}$$

for $b/\delta \leq r$. Hence, by (2.64) we get

$$\|(\mu I + tS(t/r))^{-1} - P_0(\mu I_0 + tH)^{-1}P_0\| \leq \frac{\epsilon_1}{a}\sum_{k=0}^{\infty}(k+1)\eta^k \tag{2.67}$$

for sufficiently large r. The estimates (2.58) and (2.67) give

$$\sup_{t\in[a,b]} \|e^{-tS(t/r)} - P_0 e^{-tH}P_0\| \leq 2\epsilon_0 + \epsilon_1\frac{C_{\mathcal{P}}}{a}\sum_{k=0}^{\infty}(k+1)\eta^k \tag{2.68}$$

for sufficiently large r.

To prove (v) one has only to remark that the estimate similar to (2.10) is valid for $F(t/r)^r$ and $\exp(-tS(t/r))$.

$(v) \Rightarrow (i)$. We start (see (2.1), (2.3)) with the identity

$$F(t)g(tB)^{1/2}f(tA)^{1/2} = g(tB)^{1/2}f(tA)^{1/2}T(t) \tag{2.69}$$

for $t \geq 0$. Hence

$$F(t)^r g(tB)^{1/2}f(tA)^{1/2} = g(tB)^{1/2}f(tA)^{1/2}T(t)^r \tag{2.70}$$

for $t \geq 0$ and $r \geq 1$, or

$$f(tA/r)^{1/2}g(tB/r)^{1/2}F(t/r)^r g(tB/r)^{1/2}f(tA/r)^{1/2} = T(t)^{r+1}. \tag{2.71}$$

Then one gets

$$\begin{aligned} T(t/r)^r = {} & T(t/r)^r(I - T(t/r)) + \\ & f(tA/r)^{1/2}g(tB/r)^{1/2}F(t/r)^r g(tB/r)^{1/2}f(tA/r)^{1/2}, \end{aligned} \tag{2.72}$$

which leads to the representation

$$\begin{aligned} T(t/r)^r - P_0e^{-tH}P_0 = {} & T(t/r)^r(I - T(t/r)) + \\ & f(tA/r)^{1/2}g(tB/r)^{1/2}\{F(t/r)^r - P_0e^{-tH}P_0\}g(tB/r)^{1/2}f(tA/r)^{1/2} + \\ & f(tA/r)^{1/2}[g(tB/r)^{1/2} - I]P_0e^{-tH}P_0 g(tB/r)^{1/2}f(tA/r)^{1/2} + \\ & [f(tA/r)^{1/2} - I]P_0e^{-tH}P_0 g(tB/r)^{1/2}f(tA/r)^{1/2} + \\ & P_0e^{-tH}P_0[g(tB/r)^{1/2} - I]f(tA/r)^{1/2} + P_0e^{-tH}P_0[f(tA/r)^{1/2} - I]. \end{aligned} \tag{2.73}$$

This representation readily implies the estimate

$$\begin{aligned} \|T(t/r)^r - P_0e^{-tH}P_0\| \leq {} & \|T(t/r)^r(I - T(t/r))\| + \\ & \|F(t/r)^r - P_0e^{-tH}P_0\| + 2\|[I - g(tB/r)^{1/2}]P_0e^{-tH}P_0\| + 2\|[I - f(tA/r)^{1/2}]P_0e^{-tH}P_0\|. \end{aligned} \tag{2.74}$$

We note that by Lemma 2.1 of [16] one has

$$\|T(t/r)^r(I - T(t/r))\| \leq \frac{1}{1+r}, \qquad r \geq 1. \tag{2.75}$$

Using the representation

$$\begin{aligned}&[I - f(tA/r)^{1/2}]e^{-tH}h = \\ &\qquad r^{-1/2}[I - f(tA/r)^{1/2}](tA/r)^{-1/2}(tA)^{1/2}(I_0 + tH)^{-1/2}(I_0 + tH)^{1/2}e^{-tH}h\end{aligned} \tag{2.76}$$

for $h \in \mathcal{H}_0$, by (2.36) one gets (see Lemma 4.2 of [17]) that

$$\|[I - f(tA/r)^{1/2}]e^{-tH}h\| \leq r^{-1/2}C_{1/2}\|(I_0 + tH)^{1/2}e^{-tH}\| \, \|h\| \leq r^{-1/2}C_{1/2}\|h\|e \tag{2.77}$$

for $h \in \mathcal{H}_0$. Hence

$$\|[I - f(tA/r)^{1/2}]P_0e^{-tH}P_0\| \leq r^{-1/2}C_{1/2}e \tag{2.78}$$

for $t \geq 0$ and $r \geq 1$. Similarly (cf.(2.45)), we prove that

$$\|[I - g(tB/r)^{1/2}]P_0e^{-tH}P_0\| \leq r^{-1/2}S_{1/2}e \tag{2.79}$$

for $t \geq 0$ and $r \geq 1$. Therefore, (2.74), (2.75) and (2.78), (2.79) imply

$$\begin{aligned}&\sup_{t\in[a,b]} \|T(t/r)^r - P_0e^{-tH}P_0\| \leq \\ &\qquad \sup_{t\in[a,b]} \|F(t/r)^r - P_0e^{-tH}P_0\| + \frac{1}{1+r} + \frac{2e}{r^{1/2}}(C_{1/2} + S_{1/2}).\end{aligned} \tag{2.80}$$

Now by (v) we obtain (i). □

Corollary 2.2 *Under assumptions of Theorem 2.1 the conditions $(i) - (v)$ are equivalent to*

(vi) $\lim_{n\to\infty} \sup_{t\in[a,b]} \|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| = 0, \quad [a,b] \subseteq (0,\infty)$,

(vii) $\lim_{n\to\infty} \sup_{t\in[a,b]} \|(g(tB/n)f(tA/n))^n - P_0e^{-tH}P_0\| = 0, \quad [a,b] \subseteq (0,\infty)$,

Proof: By substitutions $A \leftrightarrow B$ and $f(\cdot) \leftrightarrow g(\cdot)$ it is enough to prove only $(i) \Leftrightarrow (vi)$.
$(i) \Rightarrow (vi)$. Since

$$(f(t/n)g(tB/n))^n = f(tA/n)^{1/2}T(t/n)^{n-1}f(tA/n)^{1/2}g(tB/n), \quad n = 1, 2 \ldots, \tag{2.81}$$

we get the representation

$$\begin{aligned}&(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0 = \\ &\quad f(tA/n)^{1/2}T(t/n)^{n-1}(I - T(t/n))f(tA/n)^{1/2}g(tB/n) + \\ &\quad f(tA/n)^{1/2}(T(t/n)^n - P_0e^{-tH}P_0)f(tA/n)^{1/2}g(tB/n) + \\ &\quad [f(tA/n)^{1/2} - I]P_0e^{-tH}P_0f(tA/n)^{1/2}g(tB/n) + \\ &\quad P_0e^{-tH}P_0[f(tA/n)^{1/2} - I]g(tB/n) + P_0e^{-tH}P_0[g(tB/n)^{1/2} - I][g(tB/n)^{1/2} + I],\end{aligned} \tag{2.82}$$

which gives the estimate

$$\begin{aligned}\|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| \le& \\ &\|T(t/n)^{n-1}(I - T(t/n))\| + \|T(t/n)^n - P_0e^{-tH}P_0\| + \\ &2\|[I - f(tA/n)^{1/2}]P_0e^{-tH}P_0\| + 2\|[I - g(tB/n)^{1/2}]P_0e^{-tH}P_0\|.\end{aligned} \tag{2.83}$$

Using (2.75), (2.78) and (2.79) we get

$$\begin{aligned}\|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| \le& \\ &\|T(t/n)^n - P_0e^{-tH}P_0\| + \frac{1}{n} + \frac{2e}{n^{1/2}}(C_{1/2} + S_{1/2})\end{aligned} \tag{2.84}$$

which yields

$$\begin{aligned}\sup_{t\in[a,b]} \|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| \le& \\ &\sup_{t\in[a,b]} \|T(t/n)^n - P_0e^{-tH}P_0\| + \frac{1}{n} + \frac{2e}{n^{1/2}}(C_{1/2} + S_{1/2}).\end{aligned} \tag{2.85}$$

Therefore, (i) implies (vi).

$(vi) \Rightarrow (i)$. From

$$T(t/n)^{n+1} = f(tA/n)^{1/2}g(tB/n)(f(tA/n)g(tB/n))^n f(tA/n)^{1/2}, \quad n = 1, 2, \ldots, \tag{2.86}$$

we obtain the representation

$$\begin{aligned}T(t/n)^n - P_0e^{-tH}P_0 =& T(t/n)^n(I - T(t/n)) + \\ &f(tA/n)^{1/2}g(tB/n)\left((f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\right)f(tA/n)^{1/2} + \\ &f(tA/n)^{1/2}[g(tB/n)^{1/2} + I][g(tB/n)^{1/2} - I]P_0e^{-tH}P_0f(tA/n)^{1/2} + \\ &[f(tA)^{1/2} - I]P_0e^{-tH}P_0f(tA/n)^{1/2} + P_0e^{-tH}P_0[f(tA/n)^{1/2} - I].\end{aligned} \tag{2.87}$$

Hence

$$\begin{aligned}\|T(t/n)^n - P_0e^{-tH}P_0\| \le& \\ &\|T(t/n)^n(I - T(t/n))\| + \|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| + \\ &2\|[I - g(tB/n)^{1/2}]P_0e^{-tH}P_0\| + 2\|[I - f(tA)^{1/2}]P_0e^{-tH}P_0\|.\end{aligned} \tag{2.88}$$

¿From (2.75), (2.78) and (2.79) we obtain

$$\begin{aligned}\sup_{t\in[a/2,b]} \|T(t/n)^n - P_0e^{-tH}P_0\| \le& \\ &\sup_{t\in[a/2,b]} \|(f(tA/n)g(tB/n))^n - P_0e^{-tH}P_0\| + \frac{1}{n+1} + \frac{2e}{n^{1/2}}(C_{1/2} + S_{1/2})\end{aligned} \tag{2.89}$$

for *any* interval $[a/2, b] \subseteq (0, \infty)$. Hence, by (vi) for each $\epsilon > 0$ and interval $[a/2, b] \subseteq (0, \infty)$ there is a $n_0 \ge 1$ such that

$$\sup_{t\in[a/2,b]} \|T(t/n)^n - P_0e^{-tH}P_0\| \le \epsilon \tag{2.90}$$

for $n \geq n_0 \geq 1$. To extend (2.90) to (i) for $r \in \mathbf{R}^1_+$ we proceed as follows. Let $[r] := \sup\{n \in \mathbf{N} : n \leq r\}$, $\mathbf{N} := \{1, 2, \ldots\}$. If $t \in [a, b] \subseteq (0, \infty)$, then $[r]t/r \in [a/2, b]$, $r \geq 1$. Hence, by (2.90) one gets

$$\sup_{t\in[a,b]} \|T([r]t/r[r])^{[r]} - P_0 e^{-[r]tH/r} P_0\| \leq \epsilon \tag{2.91}$$

for $n \geq n_0 \geq 1$ and $t \in [a, b]$, which yields

$$\sup_{t\in[a,b]} \|T(t/r)^{[r]} - P_0 e^{-[r]tH/r} P_0\| \leq \epsilon \tag{2.92}$$

for $n \geq n_0 \geq 1$. Since

$$e^{-[r]tH/r}h - e^{-tH}h = \int_{[r]t/r}^{t} He^{-sH}h ds, \quad h \in \mathrm{dom}(H), \tag{2.93}$$

by Lemma 4.2 of [17] we get the estimate

$$\|e^{-[r]tH/r} - e^{-tH}\| \leq \int_{[r]t/r}^{t} \frac{1}{s} ds = \ln(r/[r]), \qquad t > 0, \tag{2.94}$$

which implies

$$\lim_{r\to+\infty} \sup_{t\in[a,b]} \|e^{-[r]tH/r} - e^{-tH}\| = 0. \tag{2.95}$$

Let X be a non-negative self-adjoint operator obeying $0 \leq X \leq I$. By its spectral representation one gets

$$\|X^n(I - X^\alpha)\| \leq \frac{1}{n}, \qquad n = 1, 2, \ldots, \quad 0 \leq \alpha < 1. \tag{2.96}$$

Using that

$$T(t/r)^{[r]} - T(t/r)^r = T(t/r)^{[r]}(I - T(t/r)^{r-[r]}) \tag{2.97}$$

and applying (2.96) we get the estimate

$$\|T(t/r)^{[r]} - T(t/r)^r\| \leq \frac{1}{[r]}, \qquad r \geq 1, \quad t \geq 0. \tag{2.98}$$

Since

$$T(t/r)^r - P_0 e^{-tH} P_0 = T(t/r)^r - T(t/r)^{[r]} + T(t/r)^{[r]} - P_0 e^{-tH} P_0, \tag{2.99}$$

due to (2.98) we have the estimate

$$\sup_{t\in[a,b]} \|T(t/r)^r - P_0 e^{-tH} P_0\| \leq \sup_{t\in[a,b]} \|T(t/r)^{[r]} - P_0 e^{-tH} P_0\| + \frac{1}{[r]}. \tag{2.100}$$

Taking into account (2.92) and (2.95) we prove (i). $\square$

3 Resolvent Compactness

In this section we prove the operator-norm convergence of the Trotter-Kato product formula under resolvent compactness assumptions. Below we say that the Trotter-Kato product formula *converges locally uniformly away from zero in the operator-norm for the family* $\{F(t)\}_{t\geq 0}$ if the limit (i)

(see Theorem 2.1) exists for any interval $[a,b] \subseteq (0,\infty)$. By Theorem 2.1 and Corollary 2.2 this is equivalent to the convergence in the for other families generated by $f(\cdot)$ and $g(\cdot)$, see (v) and $(vi)-(vii)$. For short we simply say that the Trotter-Kato product formula *converges locally uniformly away from zero in the operator-norm* if one of the limits $(i)-(vii)$ is established.

Lemma 3.1 *(cf. [21]) Let* $\{Y(t)\}_{t>0}$ *be a sequence of non-negative self-adjoint operators on* $\mathcal{H}$ *and let* X *be a non-negative compact operator on* $\mathcal{H}$ *such that*

$$0 \leq Y(t) \leq X, \qquad t > 0. \tag{3.1}$$

If

$$w - \lim_{t\to+0} Y(t) = Y, \tag{3.2}$$

then $Y \in \mathcal{L}_\infty(\mathcal{H})$ *and, moreover,*

$$\lim_{t\to+0} \|Y(t) - Y\| = 0. \tag{3.3}$$

Proof: By (3.1) there is a sequence of contractions $\{\Gamma(t)\}_{t>0}$, $\Gamma(t) : \overline{\mathrm{ran}(\sqrt{X})} \longrightarrow \mathcal{H}$, such that

$$\sqrt{Y(t)} = \Gamma(t)\sqrt{X}, \qquad t > 0. \tag{3.4}$$

We set

$$Z(t) := \Gamma(t)^*\Gamma(t) : \overline{\mathrm{ran}(\sqrt{X})} \longrightarrow \overline{\mathrm{ran}(\sqrt{X})}. \tag{3.5}$$

By (3.1) and (3.2) one gets $0 \leq Y \leq X$ which yields the existence of a contraction $\Gamma : \overline{\mathrm{ran}(\sqrt{X})} \longrightarrow \mathcal{H}$ such that

$$\sqrt{Y} = \Gamma\sqrt{X}. \tag{3.6}$$

Hence $Y \in \mathcal{L}_\infty(\mathcal{H})$. Let $Z := \Gamma^*\Gamma : \overline{\mathrm{ran}(\sqrt{X})} \longrightarrow \overline{\mathrm{ran}(\sqrt{X})}$. Then

$$Y(t) - Y = \sqrt{X}(Z(t) - Z)\sqrt{X}, \qquad t > 0, \tag{3.7}$$

which yields

$$w - \lim_{t\to+0}(Z(t) - Z) = 0. \tag{3.8}$$

Since $\sqrt{X}$ is compact, by (3.7) and (3.8) one immediately gets (3.3). □

The next theorem can be regarded as a generalization of Theorem 3.4 of [15] to compact operators, i.e., to $p = \infty$.

Theorem 3.2 *Let* A *and* B *be non-negative self-adjoint operators on* $\mathcal{H}$ *and let* $f(\cdot)$ *and* $g(\cdot)$ *be Kato-functions. If condition* $f_0(t_0A) \in \mathcal{L}_\infty(\mathcal{H})$ *is satisfied for some* $t_0 > 0$*, then the Trotter-Kato formula converges locally uniformly away from zero in the operator-norm. Moreover, one has*

$$(\lambda I_0 + H)^{-1} \in \mathcal{L}_\infty(\mathcal{H}) \tag{3.9}$$

for $\lambda > 0$.

Proof: By [1], [2] and [14] the above conditions on A, B and $f(\cdot)$, $g(\cdot)$ (see (1.1), (1.2) (1.4), (1.5)) imply that

$$s-\lim_{t\to+0}(\lambda I+S(t))^{-1}=P_0(\lambda I_0+H)^{-1}P_0 \tag{3.10}$$

for $\lambda>0$. From the identity (2.47) we obtain

$$\begin{aligned}g(tB)^{1/2}Q(t)(\lambda I(t)+M(t))^{-1}Q(t)g(tB)^{1/2}&=(\lambda+S(t))^{-1}F(t)+\\ \lambda g(tB)^{1/2}Q(t)(\lambda I(t)+M(t))^{-1}&Q(t)[I-f(tA)]g(tB)^{1/2}(\lambda I+S(t))^{-1}\end{aligned}$$

for $t>0$ and $\lambda>0$. Using $s-\lim_{t\to+0}g(tB)^{1/2}=s-\lim_{t\to+0}f(tA)=s-\lim_{t\to+0}F(t)=s-\lim_{t\to+0}Q(t)=I$ and (3.10) we find

$$w-\lim_{t\to+0}Q(t)(\lambda I(t)+M(t))^{-1}Q(t)=P_0(\lambda I_0+H)^{-1}P_0 \tag{3.11}$$

for each $\lambda>0$. By definitions (1.7) and (1.8) one has

$$\varphi_0(x)\geq\varphi_0(y),\qquad 0<x\leq y, \tag{3.12}$$

and

$$0\leq f(x)\leq f_0(x)\leq 1,\qquad x>0. \tag{3.13}$$

Therefore, we get

$$f(tA)\leq f_0(tA),\qquad t\geq 0, \tag{3.14}$$

which yields (cf. (2.5), (2.6))

$$\tilde{f}(tA)\leq f_0(tA)Q(t),\qquad t>0. \tag{3.15}$$

Since $f_0(x)>0$ for $x>0$, we find

$$f_0(tA)^{-1}Q(t)\leq\tilde{f}(tA)^{-1},\qquad t>0, \tag{3.16}$$

which yields

$$\frac{1}{t}[f_0(tA)^{-1}-I]Q(t)\leq\frac{1}{t}[\tilde{f}(t)^{-1}-I(t)],\qquad t>0. \tag{3.17}$$

By (2.7) we finally get

$$\frac{1}{t}[f_0(tA)^{-1}-I]Q(t)\leq M(t),\qquad t>0, \tag{3.18}$$

or, equivalently (cf. (1.8)),

$$A\varphi_0(tA)Q(t)\leq M(t),\qquad t>0. \tag{3.19}$$

Hence

$$\begin{aligned}&Q(t)(\lambda I(t)+M(t))^{-1}Q(t)\leq\\ &\qquad(\lambda I+A\varphi_0(tA))^{-1}Q(t)\leq(\lambda I+A\varphi_0(tA))^{-1},\quad t>0,\end{aligned} \tag{3.20}$$

where we used that $Q(t)$ commutes with A. Since $\varphi_0(x)$ is a non-increasing function (3.12), we have

$$\varphi_0(tA)\geq\varphi_0(t_0A),\qquad 0<t\leq t_0, \tag{3.21}$$

which yields

$$(\lambda I + A\varphi_0(t_0A))^{-1} \geq (\lambda I + A\varphi_0(tA))^{-1}, \qquad 0 < t \leq t_0. \tag{3.22}$$

Hence

$$Q(t)(\lambda I(t) + M(t))^{-1}Q(t) \leq (\lambda I + A\varphi_0(t_0A))^{-1}, \qquad 0 < t \leq t_0. \tag{3.23}$$

Since

$$(\lambda I + A\varphi_0(t_0A))^{-1} = t_0(I + (\lambda t_0 - 1)f_0(t_0A))^{-1}f_0(t_0A), \tag{3.24}$$

we obtain

$$(\lambda I + A\varphi_0(t_0A))^{-1} \in \mathcal{L}_\infty(\mathcal{H}) \tag{3.25}$$

for $\lambda t_0 > 1$. Taking into account (3.23) we obtain that assumption (3.1) is satisfied. Therefore, (3.11) and Lemma 3.1 give the proof of the condition (*iii*) of Theorem 2.1 as well as (3.9). $\square$

Notice that $0 \leq \varphi_0(x) \leq 1$, $x > 0$, implies

$$1 + x\varphi_0(x) \leq 1 + x, \qquad x > 0, \tag{3.26}$$

which together with (1.8) yields

$$\frac{1}{1+x} \leq f_0(x), \qquad x > 0. \tag{3.27}$$

Therefore, $f_0(t_0A) \in \mathcal{L}_\infty(\mathcal{H})$ implies that

$$(I + t_0A)^{-1} \in \mathcal{L}_\infty(\mathcal{H}). \tag{3.28}$$

The converse is certainly *not* true as it is demonstrated by the following example.

Example 3.3 Let $\{\Delta_n\}_{n=0}^\infty$ be a partition of $\mathbf{R}^1_+$ given by

$$\Delta_0 = [0,1), \qquad \Delta_n := [n^n, (n+1)^{n+1}), \qquad n = 1, 2, \ldots. \tag{3.29}$$

We define a Borel function $f(\cdot)$ setting

$$\varphi(x) := \begin{cases} 1 & : \quad x \in \Delta_0 \\ \frac{1}{n^n} & : \quad x \in \Delta_n, \qquad n = 1, 2, \ldots \end{cases} \tag{3.30}$$

and

$$f(x) := (1 + x\varphi(x))^{-1}, \qquad x \in \mathbf{R}^1_+. \tag{3.31}$$

A straightforward computation shows (see (1.7), (1.8)) that $\varphi_0(x) = \varphi(x)$ and $f_0(x) = f(x)$, $x \in \mathbf{R}^1_+$. Let A be a non-negative self-adjoint operator with simple discrete spectrum given by $\{\lambda_n\}_{n=0}^\infty$, $\lambda_n = n^n$. Since $\lim_{n\to\infty}\lambda_n = +\infty$ the resolvent of A is compact. Further, we have

$$f_0(t_0\lambda_n) = f(t_0\lambda_n) = \left(1 + n^n n^{-n}\right)^{-1} = \frac{1}{2}, \qquad n = 1, 2, \ldots, \quad t_0 = 1, \tag{3.32}$$

which yields $\lim_{n\to\infty} f_0(\lambda_n) = 1/2$. Therefore, the essential spectrum of $f_0(A)$ contains the point $\{1/2\}$ which is impossible if $f_0(A)$ is compact. By the way, if the spectrum of A is given by $\lambda_n := (n+1)^{n+1} - 1$, then $\lim_{n\to\infty} f_0(\lambda_n) = 0$. Hence $f_0(A) \in \mathcal{L}_\infty(\mathcal{H})$. $\square$

Let $f(\cdot)$ be a Borel function obeying (1.4). We set

$$b(x) := \sup_{0 \le s \le x} s f(s), \qquad x \in \mathbf{R}^1_+, \tag{3.33}$$

and

$$r(x) := \sup_{s \in [x,\infty)} f(s), \qquad x \in \mathbf{R}^1_+. \tag{3.34}$$

We note that $b(\cdot)$ is non-decreasing and $0 \le b(x) \le x$, $x \in \mathbf{R}^1_+$, while $r(x)$ is non-increasing such that $0 \le r(x) \le 1$, $x \in \mathbf{R}^1_+$.

Lemma 3.4 *Let $f(\cdot)$ be a Kato-function. For any non-negative operator A obeying $(I+A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$ one has $f_0(A) \in \mathcal{L}_\infty(\mathcal{H})$ if and only if*

$$\lim_{x \to +\infty} \frac{b(x)}{x} = 0 \tag{3.35}$$

and

$$0 \le r(x) < 1 \tag{3.36}$$

for $x > 0$.

Proof: If $(I+A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$, then the spectrum of A consists of isolated non-negative eigenvalues of finite multiplicity with the only accumulation point at infinity. Denoting by $\{\lambda_n\}_{n=1}^\infty$ the spectrum of A counting multiplicities we deduce that the operator $f_0(A)$ is compact if and only if a unique accumulation point of $\{f_0(\lambda_n)\}_{n=1}^\infty$ is at zero. This yields $\lim_{n\to\infty} f_0(\lambda_n) = 0$. Hence $(I+A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$ implies $f_0(A) \in \mathcal{L}_\infty(\mathcal{H})$ if and only if for any sequence of non-negative numbers $\{\lambda_n\}_{n=1}^\infty$ tending to infinity as $n \longrightarrow \infty$ one has that $\lim_{n\to\infty} f_0(\lambda_n) = 0$. This yields

$$\lim_{x \to \infty} f_0(x) = 0. \tag{3.37}$$

Using the representation (1.8) one gets that $f_0(t_0 A) \in \mathcal{L}_\infty(\mathcal{H})$ is equivalent to

$$\lim_{x \to \infty} x \varphi_0(x) = +\infty. \tag{3.38}$$

Since $\varphi_0(x)$ is non-increasing one has that $\varphi_0(x_0) = 0$ for some $x_0 \ge 0$ yields $\varphi_0(x) = 0$ for all $x \ge x_0$. Therefore, if $f_0(t_0 A) \in \mathcal{L}_\infty(\mathcal{H})$, then $\varphi_0(x) > 0$ for all $x \in \mathbf{R}^1_+$. By (3.13) one has

$$f(x) \le f_0(x) = \frac{1}{1 + x\varphi_0(x)}, \qquad x \in \mathbf{R}^1_+, \tag{3.39}$$

and consequently

$$x f(x) \le \frac{x}{1 + x\varphi_0(x)} \le \frac{1}{\varphi_0(x)} \tag{3.40}$$

for $x \in \mathbf{R}^1_+$. Hence

$$b(x) = \sup_{s \in [0,x]} s f(s) \le \frac{1}{\varphi_0(x)}, \qquad x \in \mathbf{R}^1_+. \tag{3.41}$$

Therefore,

$$\frac{b(x)}{x} \le \frac{1}{x\varphi_0(x)}, \qquad x \in \mathbf{R}^1_+. \tag{3.42}$$

Applying (3.38) we obtain (3.35). If there is a $x_0 > 0$ such that $r(x_0) = 1$, then there is a non-increasing sequence $\{x_n\}_{n=1}^{\infty}$, $x_0 \le x_n$, tending to x_0 such that $\lim_{n\to\infty} f(x_n) = 1$. Since by (1.7)

$$\varphi_0(x_1) \le \frac{1}{x_n}\left(\frac{1}{f(x_n)} - 1\right), \qquad n = 1, 2, \ldots, \tag{3.43}$$

one gets that $\varphi_0(x_1) = 0$. Thus $\varphi_0(x) = 0$ for all $x \ge x_1$ which contradicts to (3.38). Hence (3.36) is true.

Conversely, by (1.4) there is a $\delta > 0$ such that for $x \in [0, \delta]$ we have

$$\frac{1}{x}\left(\frac{1}{f(x)} - 1\right) \ge \frac{1}{2}. \tag{3.44}$$

Since by (3.33)

$$\frac{b(x)}{x} \ge f(x), \qquad x \in \mathbf{R}^1_+, \tag{3.45}$$

we get

$$\frac{1}{f(x)} \ge \frac{x}{b(x)}, \qquad x \in \mathbf{R}^1_+. \tag{3.46}$$

Then, using $1 - f(x) \ge 1 - r(\delta)$ for $x \ge \delta$, we obtain

$$\frac{1 - f(x)}{x f(x)} \ge \frac{1 - r(\delta)}{b(x)}, \qquad x \ge \delta. \tag{3.47}$$

Therefore, by (3.44) and (3.47) one gets

$$\varphi_0(x) \ge \min\left\{\frac{1}{2}, \frac{1 - r(\delta)}{b(x)}\right\}, \qquad x \in \mathbf{R}^1_+, \tag{3.48}$$

where we have used that $1/b(x)$ is non-increasing. If $\lim_{x\to\infty} b(x) = b_0 < \infty$, then

$$\varphi_0(x) \ge c_0 := \min\left\{\frac{1}{2}, \frac{1 - r(\delta)}{b_0}\right\}, \qquad x \in \mathbf{R}^1_+. \tag{3.49}$$

Hence (3.38) is proven. If $\lim_{x\to+\infty} b(x) = +\infty$, then

$$\varphi_0(x) \ge \frac{1 - r(\delta)}{b(x)} \tag{3.50}$$

for sufficiently large x. Hence

$$x\varphi_0(x) \ge (1 - r(\delta))\frac{x}{b(x)} \tag{3.51}$$

for sufficiently large x. Then applying (3.35) and (3.36) to (3.51) we again get (3.38). □

Remark 3.5 In the following a Kato-function is called *regular* if it satisfies (3.35) and (3.36). For instance, $f(x) = \frac{1}{1+x}$ and $f(x) = e^{-x}$ are regular Kato-functions while $f(x) = 1 - |\sin(x)|$ is *not* a regular one.

An obvious consequence of Theorem 3.2 and Lemma 3.4 is that the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm for *any* regular Kato-function $f(\cdot)$, *any* Kato-function $g(\cdot)$ and *any* non-negative self-adjoint operator B if the resolvent of operator A is *compact*. It turns out that these assumptions are not only *sufficient* but also *necessary*. To prove this we need the following lemma.

Lemma 3.6 *Let $f(x) = \frac{1}{1+x}$ and let $g(\cdot)$ be another Kato-function. Further, let D be a non-negative self-adjoint operator such that $\|D\| < 1$. If the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm for a couple of non-negative self-adjoint operators A and B, then the same holds for the couple of operators $A + D$ and B.*

Proof: We are going to verify the condition (iii) of Theorem 2.1. Since the Trotter-Kato product formula converges in the operator-norm for self-adjoint operators A and B, by Theorem 2.1(iii) for the Kato-function $f(x) = (1+x)^{-1} > 0$ on $\mathbf{R}^1_+$, i.e. for $Q(t) = I$ (see (2.9)), one has

$$\lim_{t\to+0} \|(\lambda I + M(t))^{-1} - P_0(\lambda I_0 + H)^{-1}P_0\| = 0, \qquad \lambda > 0, \tag{3.52}$$

where by (2.7) and (2.9) we have

$$M(t) := A + \frac{1}{t}[I - g(tB)], \qquad t > 0. \tag{3.53}$$

The lemma will be proven if we show that (3.52) implies

$$\lim_{t\to+0} \|(\lambda I + M(t) + D)^{-1} - P_0(\lambda I_0 + H + D_0)^{-1}P_0\| = 0 \tag{3.54}$$

for $\lambda > 0$, where $D_0 := P_0 D P_0$. Let $\lambda \geq 1$. Since $\|D\| < 1$, one has the representations

$$(\lambda I + M(t) + D)^{-1} = (\lambda I + M(t))^{-1} \sum_{k=0}^{\infty} (-1)^k (D(\lambda I + M(t))^{-1})^k \tag{3.55}$$

and

$$(\lambda I_0 + H + D_0)^{-1} = (\lambda I_0 + H)^{-1} \sum_{k=0}^{\infty} (-1)^k (D_0(\lambda I_0 + H)^{-1})^k. \tag{3.56}$$

Since $\lambda \geq 1$ and $\|D\| < 1$, for any $\epsilon > 0$ there is a k_0 such that for $k \geq k_0$ one has

$$\|(\lambda I + M(t))^{-1} \sum_{k=k_0}^{\infty} (-1)^k (D(\lambda I + M(t))^{-1})^k\| \leq \epsilon \tag{3.57}$$

and

$$\|(\lambda I_0 + H)^{-1} \sum_{k=k_0}^{\infty} (-1)^k (D_0(\lambda I_0 + H)^{-1})^k\| \leq \epsilon \tag{3.58}$$

uniformly in $t > 0$. Moreover, by (3.52) there is $\delta > 0$ such that one has

$$\|(\lambda I + M(t))^{-1}(D(\lambda I + M(t))^{-1})^k - P_0(\lambda I_0 + H)^{-1}(D_0(\lambda I_0 + H)^{-1})^k P_0\| \leq \frac{\epsilon}{k_0} \tag{3.59}$$

for $k = 0, 1, \ldots, k_0 - 1$ and $t \in (0, \delta]$. Summing up (3.55) - (3.59) we get

$$\|(\lambda I + M(t) + D)^{-1} - P_0(\lambda I_0 + H + D_0)^{-1}P_0\| \leq 3\epsilon \tag{3.60}$$

for $t \in (0, \delta]$. Therefore, the limit (3.54) is proven. □

Theorem 3.7 *Let A be an non-negative self-adjoint operator. The Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm for any regular Kato-function $f(\cdot)$, any Kato-function $g(\cdot)$ as well as for any non-negative self-adjoint operator B if and only if one has $(I + A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$.*

Proof: If $(I + A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$, then by Lemma 3.4 one has $f_0(A) \in \mathcal{L}_\infty(\mathcal{H})$. Applying Theorem 3.2 we verify the sufficient part of the theorem.

Conversely, assume that the Trotter-Kato product formula converges in the operator-norm for *any* regular Kato-function $f(\cdot)$, for *any* Kato-function $g(\cdot)$ and for *any* $B \geq 0$. In particular, this holds for $f(x) = (1 + x)^{-1}$. Therefore, one gets that (3.52) is valid for *each* Kato-function $g(\cdot)$ and *each* $B \geq 0$, where $M(t)$ is given by (3.53). By the von Neumann theorem (see e.g. [12]) for any $\epsilon > 0$ there is a Hilbert-Schmidt operator V with the Hilbert-Schmidt norm $\|V\|_2 \leq \epsilon$ such that the spectrum of $A + V$ is purely pointwise. Let $\epsilon < 1/2$. Setting $D = \frac{1}{2}I + V$ we construct a non-negative self-adjoint operator such that $\|D\| < 1$ and the spectrum of $A + D$ is purely pointwise. Note that $\sigma_{ess}(A + D) = \sigma_{ess}(A + \frac{1}{2}I) = \frac{1}{2} + \sigma_{ess}(A)$. If $(I + A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$ is not satisfied, then $\sigma_{ess}(A) \neq \emptyset$. Hence $\sigma_{ess}(A + D) \neq \emptyset$. Let $\mu \in \sigma_{ess}(A + D)$. Of course $\mu \geq \frac{1}{2}$. There is an orthonormal sequence of eigenfunctions $\{\varphi_n\}_{n=1}^\infty$ and a sequence of eigenvalues $\{\mu_n\}_{n=1}^\infty$ such that

$$(A + D)\varphi_n = \mu_n\varphi_n, \qquad n = 1, 2, \ldots, \tag{3.61}$$

and

$$\lim_{n\to\infty} \mu_n = \mu. \tag{3.62}$$

Notice that the function $g(x) = 1 - |\sin(x)|$ is from the Kato class. By $\mathcal{K}$ we denote the closed linear span of $\{\varphi_n\}_{n=1}^\infty$. Setting

$$B\varphi_n = 2^{n+1}\pi\varphi_n, \qquad n = 1, 2, \ldots, \tag{3.63}$$

and

$$Bf = 0, \qquad f \in \mathcal{H} \ominus \mathcal{K}, \tag{3.64}$$

we define a non-negative self-adjoint operator B. Obviously, we have

$$(I - g(t_nB))\varphi_n = |\sin(t_nB)|\varphi_n = |\sin(t_n2^{n+1}\pi)|\varphi_n = 0 \tag{3.65}$$

for $t_n := 2^{-n}$, $n = 1, 2, \ldots$. Therefore, by (3.53), (3.61) and (3.65) one gets

$$(\lambda I + M(t_n) + D)\varphi_n = (\lambda + \mu_n)\varphi_n, \qquad n = 1, 2, \ldots. \tag{3.66}$$

Since $P_0 = I$, due to (3.61) and (3.63), we have

$$(\lambda I + H + D)\varphi_n = (\lambda + \mu_n + 2^{n+1}\pi)\varphi_n, \qquad n = 1, 2, \ldots. \tag{3.67}$$

Therefore,

$$\begin{aligned} &\|(\lambda I + M(t_n) + D)^{-1}\varphi_n - (\lambda I + H + D)^{-1}\varphi_n\| \leq \\ &\qquad |(\lambda + \mu_n)^{-1} - (\lambda + \mu_n + 2^{n+1}\pi)^{-1}|. \end{aligned} \tag{3.68}$$

By (3.62) we find

$$\lim_{n\to\infty} \|(\lambda I + M(t_n) + D)^{-1}\varphi_n - (\lambda + H + D)^{-1}\varphi_n\| = (\lambda + \mu)^{-1}. \tag{3.69}$$

Now taking into account (3.54) we obtain

$$\lim_{n\to\infty} \|(\lambda I + M(t_n) + D)^{-1}\varphi_n - (\lambda + H + D)^{-1}\varphi_n\| = 0, \tag{3.70}$$

which contradicts (3.69). Therefore, the assumption $\sigma_{ess}(A) \neq \emptyset$ is false which implies immediately that $(I + A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$. □

4 Relative Compactness

In this section we treat the case $\alpha = 1/2$ which was not covered by our paper [17].

Theorem 4.1 *Let $A \geq I$ and $B \geq 0$ be two self-adjoint operators such that*

$$\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2}). \tag{4.1}$$

Further, let $f(\cdot)$ and $g(\cdot)$ be Kato-functions. If

$$C_0 := \sup_{x>0} \frac{xf(x)}{1 - f(x)} < +\infty \tag{4.2}$$

and

$$B^{1/2}A^{-1/2} \in \mathcal{L}_\infty(\mathcal{H}), \tag{4.3}$$

then the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm.

Proof: From (4.2) it follows that

$$0 \leq f(x) \leq \frac{C_0}{C_0 + x}, \qquad x > 0. \tag{4.4}$$

First we show that

$$\lim_{t\to+0} \left\| \left(I + \frac{1}{t}[I - f(tA)]\right)^{-1} - (I + A)^{-1} \right\| = 0. \tag{4.5}$$

Notice that (4.4) implies

$$1 - f(x) \geq 1 - \frac{C_0}{C_0 + x}, \qquad x > 0, \tag{4.6}$$

which yields

$$\frac{1}{t}[1 - f(tx)] \geq \frac{x}{C_0 + tx}, \qquad x > 0, \quad t \geq 0, \tag{4.7}$$

or, due to $C_0 \geq 1$ (see (1.4), (4.2)),

$$\left(1 + \frac{1}{t}[1 - f(tx)]\right)^{-1} \leq \frac{C_0}{C_0 + x} + t, \qquad x > 0, \quad t \geq 0. \tag{4.8}$$

Therefore,

$$\sup_{x\geq N} \left(1 + \frac{1}{t}[1 - f(tx)]\right)^{-1} \leq \frac{C_0}{C_0 + N} + t, \qquad t \geq 0. \tag{4.9}$$

Since

$$\left(1+\frac{1}{t}[1-f(tx)]\right)^{-1}-(1+x)^{-1}= \tag{4.10}$$
$$\left(1+\frac{1}{t}[1-f(tx)]\right)^{-1}\left(1-\frac{1}{xt}[1-f(tx)]\right)\frac{x}{1+x},$$

we get the estimate

$$\left|\left(1+\frac{1}{t}[1-f(tx)]\right)^{-1}-(1+x)^{-1}\right|\le\left|1-\frac{1}{xt}[1-f(tx)]\right|. \tag{4.11}$$

By (1.4) for each $\epsilon>0$ there is a $\delta>0$ such that

$$\left|1-\frac{1}{y}[1-f(y)]\right|\le\epsilon \tag{4.12}$$

for $y\in(0,\delta]$. Hence for $t\in(0,\delta/N]$ and $x\in(0,N]$ one gets

$$\left|1-\frac{1}{xt}[1-f(xt)]\right|\le\epsilon. \tag{4.13}$$

Therefore, for any $\epsilon>0$ and $N>0$, there is a $\delta'>0$ such that

$$\sup_{x\in(0,N]}\left|1-\frac{1}{xt}[1-f(tx)]\right|\le\epsilon \tag{4.14}$$

for $t\in(0,\delta']$. Hence, (4.11) yields

$$\lim_{t\to+0}\sup_{x\in(0,N]}\left|\left(1+\frac{1}{t}[1-f(xt)]\right)^{-1}-\frac{1}{1+x}\right|=0 \tag{4.15}$$

for each N, $0<N<+\infty$. Since by (4.9)

$$\sup_{x>0}\left|\left(1+\frac{1}{t}[1-f(xt)]\right)^{-1}-\frac{1}{1+x}\right|\le \tag{4.16}$$
$$\max\left\{\sup_{x\in(0,N]}\left|\left(1+\frac{1}{t}[1-f(xt)]\right)^{-1}-\frac{1}{1+x}\right|,\ \frac{C_0}{C_0+N}+t,\ \frac{1}{1+N}\right\},$$

one gets that

$$\lim_{t\to+0}\sup_{x>0}\left|\left(1+\frac{1}{t}[1-f(xt)]\right)^{-1}-\frac{1}{1+x}\right|=0 \tag{4.17}$$

which proves (4.5).

Next we verify the condition (ii) of Theorem 2.1. Let $\lambda=1$. ¿From (2.7) and (2.27) we find the representation

$$I+R(t)=\left(I+\frac{1}{t}[I-f(tA)]\right)^{1/2}\{I+X(t)\}\left(I+\frac{1}{t}[I-f(tA)]\right)^{1/2} \tag{4.18}$$

where

$$X(t) := f(tA)^{1/2}A^{1/2}\left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2}A^{-1/2}B^{1/2}\times \qquad (4.19)$$
$$[I-g(tB)](t\tilde{B})^{-1}B^{1/2}A^{-1/2}f(tA)^{1/2}A^{1/2}\left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2},$$

where $\tilde{B} = B|\mathrm{ran}(B) : \overline{\mathrm{ran}(B)} \longrightarrow \overline{\mathrm{ran}(B)}$. Hence one obtains

$$(I+R(t))^{-1} = \left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2}\{I+X(t)\}^{-1}\left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2}. \qquad (4.20)$$

Since by (4.8)

$$\sup_{t>0}\left\|f(tA)^{1/2}A^{1/2}\left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2}\right\| \le C_0^{1/2}, \qquad (4.21)$$

in virtue of (4.5) one easily gets that

$$s-\lim_{t\to+0} f(tA)^{1/2}A^{1/2}\left(I+\frac{1}{t}[I-f(tA)]\right)^{-1/2} = A^{1/2}(I+A)^{-1/2}. \qquad (4.22)$$

Since (1.5) implies

$$s-\lim_{t\to+0}[I-g(tB)](t\tilde{B})^{-1}h = h, \qquad h \in \mathrm{ran}(B), \qquad (4.23)$$

by (4.19), (4.22) and (4.23) we get

$$s-\lim_{t\to+0} X(t) = (I+A)^{-1/2}B(I+A)^{-1/2}. \qquad (4.24)$$

Moreover, by condition (4.3) the operator $B^{1/2}A^{-1/2}$ is compact. Since it is involved into (4.19), by standard arguments (see e.g. [19], Chapter 2) and by (4.22), (4.23) we get, instead of (4.24), the limit

$$\lim_{t\to+0}\|X(t)-(I+A)^{-1/2}B(I+A)^{-1/2}\| = 0. \qquad (4.25)$$

Using (4.25) we find that

$$\lim_{t\to+0}\|(I+X(t))^{-1}-(I+(I+A)^{-1/2}B(I+A)^{-1/2})^{-1}\| = 0. \qquad (4.26)$$

Taking into account (4.5), (4.20) and (4.26) we obtain

$$\lim_{t\to+0}\|(I+R(t))^{-1}-(I+A)^{-1/2}(I+(I+A)^{-1/2}B(I+A)^{-1/2})^{-1}(I+A)^{-1/2}\| = 0 \qquad (4.27)$$

which yields

$$\lim_{t\to+0}\|(I+R(t))^{-1}-(I+H)^{-1}\| = 0. \qquad (4.28)$$

Then a reference to Theorem 2.1 (*ii*) completes the proof. □

5 Resolvent Product Compactness

In this section we consider another kind of compactness condition. Since the product of resolvents is involved in this condition, we call it the *resolvent product compactness*. We start with two technical lemmata.

Lemma 5.1 *Let X be a non-negative self-adjoint operator on the separable Hilbert space $\mathcal{H}$ and let Q be an orthogonal projection on $\mathcal{H}$. Then one has*

$$Q(I_Q + QXQ)^{-1}Q \le (I+X)^{-1}, \tag{5.1}$$

where $I_Q := I|Q\mathcal{H}$.

Proof: Obviously, one has

$$I + X \le I + n(I-Q) + X, \qquad n = 0,1,2,\ldots, \tag{5.2}$$

which implies

$$(I + n(I-Q) + X)^{-1} \le (I+X)^{-1}, \qquad n = 0,1,2,\ldots. \tag{5.3}$$

A straightforward computation shows that

$$s - \lim_{n\to\infty}(I + n(I-Q) + X)^{-1} = Q(I_Q + QXQ)^{-1}Q, \tag{5.4}$$

where $I_Q := I|Q\mathcal{H}$. Hence from (5.3) and (5.4) we obtain (5.1). □

Lemma 5.2 *Let $\{X(t)\}_{t\ge0}$ be a sequence of non-negative self-adjoint operators on the subspace $\{\mathcal{H}(t)\}_{t\ge0}$ of the separable Hilbert space $\mathcal{H}$ and let $\{Q(t)\}_{t\ge0}$ be the corresponding sequence of orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}(t)$. If for each $\lambda > 0$ the weak resolvent convergence takes place, i.e.,*

$$w - \lim_{t\to+0} Q(t)(\lambda I(t) + X(t))^{-1}Q(t) = Q(0)(I(0) + X(0))^{-1}Q(0), \tag{5.5}$$

then for each $\lambda > 0$ the strong resolvent convergence holds, i.e.,

$$s - \lim_{t\to+0} Q(t)(\lambda I(t) + X(t))^{-1}Q(t) = Q(0)(\lambda I(0) + X(0))^{-1}Q(0), \tag{5.6}$$

where $I(t) = I|\mathcal{H}(t)$, $t \ge 0$. In particular, one has

$$s - \lim_{t\to+0} Q(t)(\lambda I(t) + X(t))^{-1/2}Q(t) = Q(0)(\lambda I(0) + X(0))^{-1/2}Q(0), \tag{5.7}$$

for $\lambda > 0$.

Proof: The proof is based on two observations. Firstly, we note that (5.5) implies

$$\lim_{t\to+0} \|Q(t)(\lambda I(t) + X(t))^{-1/2}Q(t)h\|^2 = \|Q(0)(I(0) + X(0))^{-1/2}Q(0)h\|^2 \tag{5.8}$$

for $h \in \mathcal{H}$ and $\lambda > 0$. Secondly, by the representation

$$Q(t)(\lambda I(t) + X(t))^{-1/2}Q(t) = \frac{1}{\pi}\int_0^{+\infty} \frac{dx}{\sqrt{x}}\, Q(t)(I(t) + x(\lambda I(t) + X(t)))^{-1}Q(t), \quad t \ge 0, \tag{5.9}$$

we find from (5.5) that

$$w-\lim_{t\to+0} Q(t)(\lambda I(t)+X(t))^{-1/2}Q(t) = Q(0)(\lambda I(0)+X(0))^{-1/2}Q(0) \tag{5.10}$$

for each $\lambda > 0$. By a standard argument (see e.g. [12]) we obtain that (5.8) and (5.10) imply the strong convergence of the resolvents in (5.9), i.e., that (5.7) is valid. However, the relation (5.7) immediately implies (5.6). □

The following theorem is motivated by the fact that neither *resolvent compactness* or *relative compactness* (see (1.19), (1.21)), nor *smallness conditions* (see (1.13), (1.16)) allow to prove the operator-norm convergence of the Trotter-Kato product formula, for example (see [4], [8], [9]), in the case of increasing potentials of the Schrödinger semigroup generators. See also [3] and [5] - [7]. The following theorem treats this case on an *abstract* level.

Theorem 5.3 *Let A and B be non-negative self-adjoint operators and let $f(\cdot)$ and $g(\cdot)$ be Kato-functions such that conditions (4.2) and*

$$S_0 := \sup_{x>0} \frac{xg(x)}{1-g(x)} < +\infty \tag{5.11}$$

are satisfied. If

$$(I+A)^{-1}(I+B)^{-1} \in \mathcal{L}_\infty(\mathcal{H}), \tag{5.12}$$

then the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm . Moreover, one has $(I_0+H)^{-1} \in \mathcal{L}_\infty(\mathcal{H}_0)$.

Proof: Notice that (2.7) implies

$$(I(t)+M(t))^{-1} \le \left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1}, \qquad t>0. \tag{5.13}$$

Hence, there is a family of contractions $\{\Gamma(t)\}_{t>0}$, $\Gamma(t) : \mathcal{H}(t) \longrightarrow \mathcal{H}(t)$, such that

$$(I(t)+M(t))^{-1/2} = \Gamma(t)\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1/2}, \qquad t>0. \tag{5.14}$$

Again by (2.7) we get

$$(I(t)+M(t))^{-1} \le \left(I(t)+\frac{1}{t}Q(t)[I-g(tB)]Q(t)\right)^{-1}, \qquad t>0. \tag{5.15}$$

Applying Lemma 5.1 we obtain

$$Q(t)\left(I(t)+\frac{1}{t}Q(t)[I-g(tB)]Q(t)\right)^{-1}Q(t) \le \left(I+\frac{1}{t}[I-g(tB)]\right)^{-1}, \qquad t>0. \tag{5.16}$$

Therefore, there exists a family of contractions $\{\Pi(t)\}_{t>0}$, $\Pi(t) : \mathcal{H} \longrightarrow \mathcal{H}(t)$, such that

$$(I(t)+M(t))^{-1/2}Q(t) = \Pi(t)\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2}, \qquad t>0. \tag{5.17}$$

Setting

$$\Gamma h := (I_0 + H)^{-1/2} P_0 (I + A)^{1/2} h, \qquad h \in \mathrm{dom}(A^{1/2}), \tag{5.18}$$

and

$$\Pi h := (I_0 + H)^{-1/2} P_0 (I + B)^{1/2} h, \qquad h \in \mathrm{dom}(B^{1/2}), \tag{5.19}$$

we define two contractions acting from $\mathcal{H}$ into $\mathcal{H}_0$. Therefore, we find the representations

$$(I_0 + H)^{-1/2} P_0 f = \Gamma (I + A)^{-1/2} f, \qquad f \in \mathcal{H}, \tag{5.20}$$

and

$$(I_0 + H)^{-1/2} P_0 f = \Pi (I + B)^{-1/2} f, \qquad f \in \mathcal{H}. \tag{5.21}$$

Hence, one gets the representation

$$P_0 (I_0 + H)^{-1} P_0 = \Gamma (I + A)^{-1/2} (I + B)^{-1/2} \Pi^*. \tag{5.22}$$

By assumption (5.12) we immediately obtain $(I_0 + H)^{-1} \in \mathcal{L}_\infty(\mathcal{H}_0)$. Notice that by definition of $Q(t)$ we have

$$Q(t) \left(I(t) + \frac{1}{t}[\tilde{f}(tA)^{-1} - I(t)] \right)^{-1} Q(t) = \left(I + \frac{1-t}{t}[I - f(tA)] \right)^{-1} f(tA) \tag{5.23}$$

for $t > 0$. Since

$$\begin{aligned} &\left(I + \frac{1}{t}[I - f(tA)] \right)^{-1} - \left(I + \frac{1-t}{t}[I - f(tA)] \right)^{-1} f(tA) = \\ &\quad \frac{1}{1-t} \frac{1-t}{t}[I - f(tA)] \left(I + \frac{1-t}{t}[I - f(tA)] \right)^{-1} [I - f(tA)] \left(I + \frac{1}{t}[I - f(tA)] \right)^{-1}, \end{aligned} \tag{5.24}$$

one gets the representation

$$\begin{aligned} &\left(I + \frac{1}{t}[I - f(tA)] \right)^{-1} - Q(t) \left(I(t) + \frac{1}{t}[\tilde{f}(tA)^{-1} - I(t)] \right)^{-1} Q(t) = \\ &\quad \frac{1}{1-t} \frac{1-t}{t}[I - f(tA)] \left(I + \frac{1-t}{t}[I - f(tA)] \right)^{-1} [I - f(tA)] \times \\ &\quad \left\{ (I + \frac{1}{t}[I - f(tA)])^{-1} - (I + A)^{-1} \right\} + \\ &\quad \frac{t}{1-t} \frac{1-t}{t}[I - f(tA)] \left(I + \frac{1-t}{t}[I - f(tA)] \right)^{-1} [I - f(tA)](tA)^{-1} A (I + A)^{-1} \end{aligned} \tag{5.25}$$

for $t > 0$. Hence, we obtain the estimate

$$\begin{aligned} &\left\| \left(I + \frac{1}{t}[I - f(tA)] \right)^{-1} - \left(I(t) + \frac{1}{t}[\tilde{f}(tA)^{-1} - I(t)] \right)^{-1} Q(t) \right\| \le \\ &\quad \frac{1}{1-t} \left\| \left(I + \frac{1}{t}[I - f(tA)] \right)^{-1} - (I + A)^{-1} \right\| + C_1 \frac{t}{1-t} \end{aligned} \tag{5.26}$$

for $t > 0$, where $C_1 := \sup_{x>0}(1 - f(x))/x < +\infty$. Then taking into account (4.5) we get from (5.26) that

$$\lim_{t \to +0} \left\| \left(I + \frac{1}{t}[I - f(tA)] \right)^{-1} - \left(I(t) + \frac{1}{t}[\tilde{f}(tA)^{-1} - I(t)] \right)^{-1} Q(t) \right\| = 0. \tag{5.27}$$

By (5.27) and again by (4.5) we finally obtain

$$\lim_{t\to+0}\left\|\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1}Q(t)-(I+A)^{-1}\right\|=0. \tag{5.28}$$

Furthermore, applying the same argument as in (4.6) - (4.17), but now for the operator B, by (5.11) we obtain the analogue of (4.5):

$$\lim_{t\to+0}\left\|\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1}-(I+B)^{-1}\right\|=0. \tag{5.29}$$

Both relations (5.28) and (5.29) imply

$$\lim_{t\to+0}\left\|Q(t)\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1/2}Q(t)-(I+A)^{-1/2}\right\|=0. \tag{5.30}$$

and

$$\lim_{t\to+0}\left\|\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2}-(I+B)^{-1/2}\right\|=0. \tag{5.31}$$

Since under conditions of the Theorem the results of [1], [2] and [14] imply (3.11)(see the Proof of Theorem 3.2), applying Lemma 5.2 to (3.11) we find that

$$s-\lim_{t\to+0}Q(t)(I(t)+M(t))^{-1/2}Q(t)=P_0(I_0+H)^{-1/2}P_0. \tag{5.32}$$

Taking into account (5.14), (5.30) and (5.32) we obtain

$$s-\lim_{t\to+0}\Gamma(t)Q(t)(I+A)^{-1/2}=P_0(I_0+H)^{-1/2}P_0. \tag{5.33}$$

Using (5.20) one gets

$$s-\lim_{t\to+0}\Gamma(t)Q(t)(I+A)^{-1/2}=\Gamma(I+A)^{-1/2}, \tag{5.34}$$

which yields

$$s-\lim_{t\to+0}\Gamma(t)Q(t)=\Gamma. \tag{5.35}$$

Similarly, from (5.17), (5.21), (5.31) and (5.32) we find

$$s-\lim_{t\to+0}\Pi(t)=\Pi. \tag{5.36}$$

¿From (5.14) and (5.17) we obtain the representation

$$\begin{aligned}&Q(t)(I(t)+M(t))^{-1}Q(t)= \\ &\Gamma(t)\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1/2}Q(t)\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2}\Pi(t)^*\end{aligned} \tag{5.37}$$

for $t > 0$. Using the identity

$$\begin{aligned} & Q(t)(I(t)+M(t))^{-1}Q(t) - P_0(I_0+H)^{-1}P_0 = \\ & \qquad \Gamma(t)Q(t)\left\{\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1/2} Q(t) - (I+A)^{-1/2}\right\} \times \\ & \qquad \left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2} \Pi(t)^* + \\ & \qquad \Gamma(t)Q(t)(I+A)^{-1/2}\left\{\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2} - (I+B)^{-1/2}\right\}\Pi(t)^* + \\ & \qquad (\Gamma(t)Q(t)-\Gamma)(I+A)^{-1/2}(I+B)^{-1/2}\Pi(t)^* + \\ & \qquad \Gamma(I+A)^{-1/2}(I+B)^{-1/2}(\Pi(t)^*-\Pi^*) \end{aligned} \tag{5.38}$$

one gets the estimate

$$\begin{aligned} & \|Q(t)(I(t)+M(t))^{-1}Q(t) - P_0(I_0+H)^{-1}P_0\| \leq \\ & \quad \left\|\Gamma(t)Q(t)\left(I(t)+\frac{1}{t}[\tilde{f}(tA)^{-1}-I(t)]\right)^{-1/2} Q(t) - (I+A)^{-1/2}\right\| + \\ & \quad \left\|\left(I+\frac{1}{t}[I-g(tB)]\right)^{-1/2} - (I+B)^{-1/2}\right\| + \\ & \quad \|(\Gamma(t)Q(t)-\Gamma)(I+A)^{-1/2}(I+B)^{-1/2}\| + \|(I+A)^{-1/2}(I+B)^{-1/2}(\Pi(t)^*-\Pi^*)\| \end{aligned} \tag{5.39}$$

for $t > 0$. Since $(I+A)^{-1/2}(I+B)^{-1/2} \in \mathcal{L}_\infty(\mathcal{H})$, by (5.35), (5.36) and standard arguments ([19], Chapter 2) one obtains that

$$\lim_{t\to+0} \|(\Gamma(t)Q(t)-\Gamma)(I+A)^{-1/2}(I+B)^{-1/2}\| = 0 \tag{5.40}$$

and

$$\lim_{t\to+0} \|(I+A)^{-1/2}(I+B)^{-1/2}(\Pi(t)^*-\Pi^*)\| = 0. \tag{5.41}$$

Taking into account (5.30) and (5.31) we finish the proof because (5.39)-(5.41) guarantee the condition (*iii*) of Theorem 2.1. □

6 Remarks

We conclude this paper by some remarks.

1. In the present paper we study *only* the *operator-norm convergence* of the Trotter-Kato product formula (for t away from zero and infinity) and *not* the estimate of *error bounds* for the convergence as it is done in [10], [11], [16] - [18].

2. Notice that in fact the necessary and sufficient conditions (*ii*) and (*iv*) of Theorem 2.1 are a *norm-operator* analogous of conditions for the *strong* convergence of the exponential Trotter formula formulated by Chernoff in [1] and [2].

3. The condition of type *(iii)* of Theorem 2.1 was used at the first time by Kato in [13] in order to prove the *strong* convergence of the Trotter-Kato product formula.

4. A necessity of condition $(I + A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$ in the Theorem 3.7 is due to the fact that the Kato-function $g(\cdot)$ is *not regular*. It would be interested to prove the similar result under the assumption that the *both* Kato-functions $f(\cdot)$ and $g(\cdot)$ are *regular*.

5. Theorem 4.1 extends results of [17] to the case $\alpha = 1/2$ assuming in *addition* that $B^{1/2}$ is relatively compact with respect to $A^{1/2}$ (see also remark 3 in Section 7 of [17]), however, without error bounds. It seems that Theorem 4.1 remains true under a weaker assumption that $B^{1/2}$ is relatively $A^{1/2}$-bounded with the *relative zero-bound*.

6. The resolvent product compactness condition of Section 5 has never been used before. It applies very well to the case of increasing (at infinity) potentials for Schrödinger semigroup generators. However, the resolvent product compactness does not give error bounds for convergence of the Trotter product formula similar to [4], [8] and [9]. It would be interested to reformulate this our observation in such a way that besides the fact of the norm-convergence one also obtains the corresponding error bounds.

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