On a phase transition in a one-dimensional non-homogeneous model

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Abstract

We prove the existence of an infinite order phase transition for a semi-infinite, one-dimensional, non-homogeneous system of classical continuous spins, with long range interactions and general *a priori* distribution.

1 Introduction

In a recent paper [1], a phase transition of infinite order in temperature has been proved for a non-homogeneous, one-dimensional Ising model, with long range interactions. Numerical simulations and analytical considerations [2, 3] suggest that the results of [1] might be valid for more general models.

In this paper we extend the results of [1] to a one-dimensional system of (classical) continuous spins, with general *a priori* distribution of the spins.

More precisely, we consider a system of spins $x_i \in \mathbb{R}$, $i = 1, 2, ...n$, in the presence of a constant, external magnetic field $h \in \mathbb{R}$ described by the Hamiltonian

$$H_{n,h}(x_1, x_2, ...x_n) = -\sum_{j=2}^{n} \frac{1}{j} \sum_{i=1}^{j-1} x_i x_j - \sum_{i=1}^{n} h x_i, \quad H_{1,h} = -h x_1. \tag{1}$$

Obviously, the sequence $\{H_n\}_{n=1,2,...}$ verifies the recurrence relation

$$H_{n,h}(x_1, ...x_n) = H_{n-1,h+x_n/n}(x_1, ...x_{n-1}) - h x_n \qquad n \geq 2. \tag{2}$$

We assume that the *a priori* probability distribution ρ of the spins, in the absence of the interactions, has compact support and is an even 'GHS measure' (see e.g. [4]), i.e. its generating function

$$F(x) = \ln\left(\int e^{xt} \rho(dt)\right) \tag{3}$$

has strictly concave derivative on $[0, \infty)$.

The partition function of the model, for the inverse temperature $\beta \geq 0$, is

$$Z_n(h, \beta) = \int_{\mathbb{R}^n} \exp\left[-\beta H_{n,h}(x_1, ...x_n)\right] \prod_{i=1}^{n} \rho(dx_i). \tag{4}$$

Denoting

$$g_n(h, \beta) = \frac{1}{n\beta} \ln Z_n(h, \beta) \tag{5}$$

the free energy per spin is $-g_n(h, \beta)$, while the magnetization per spin is $g'_n(h, \beta)$.

In this paper we prove the existence of the thermodynamic limit $-\lim_{n\to\infty} g_n(h, \beta)$ and show that the corresponding thermodynamic magnetization per spin is the solution of a certain differential equation. Moreover, the system exhibits a phase transition in β (there exists spontaneous magnetization).

The differential equation satisfied by the magnetization can be guessed, starting from the recurrence of the partition functions resulting from (2),

$$Z_n(h, \beta) = \int_{\mathbb{R}} e^{\beta h t} Z_{n-1}(h + t/n, \beta) \rho(dt) \qquad n \geq 2 \tag{6}$$

or

$$\exp(\beta g_n(h)) = \int \exp[\beta h t + (n-1)\beta(g_{n-1}(h + t/n) - g_n(h))] \rho(dt) \tag{7}$$

(for the sake of simplicity we have omitted the dependence on β in the notation of g_n).

Indeed, assuming that $g_n \to g$ and $(n-1)(g_n - g_{n-1}) \to 0$, with the sequence of derivatives verifying $g_n' \to g'$, as $n \to \infty$, then the thermodynamic limit g should satisfy the equation

$$\exp(\beta g(h)) = \int \exp[\beta h t + \beta g'(h) t] \rho(dt) \tag{8}$$

or

$$\beta g(h) = F[\beta h + \beta g'(h)] \tag{9}$$

where F is given by (3).

In terms of the magnetization per spin $m = g'$, equation (9) becomes

$$m'(h) = \frac{m(h)}{F'(\beta m(h) + \beta h)} - 1. \tag{10}$$

Obviously the solutions of (9) and (10) are related by the following formula

$$g(h, \beta) = \beta^{-1} F(\beta m(0)) + \int_0^h m(t) dt. \tag{11}$$

Further, from simple physical considerations, equation (10) should be supplemented with the following condition

$$\lim_{h \to \infty} m(h) = M > 0 \tag{12}$$

where $[-M, M]$ is the smallest interval containing the compact support of ρ.

We can state our main results as follows.

Proposition 1. *a) The thermodynamic limit* $-\lim_{n\to\infty} g_n(h, \beta)$ *of the free energy per spin exists and is finite.*

The corresponding magnetization (per spin) is $sgn(h) m_\beta(|h|)$ *where* $m_\beta(h)$ *is the unique solution of equation* (10) *with condition* (12). *Moreover,* $\lim_{n\to\infty} g_n(h, \beta) = g(|h|, \beta)$, *where* $g(h, \beta)$ *is determined by* $m_\beta(h)$, *according to* (11) *;*

b) $m_\beta(h) > 0$ *for* $h > 0$, *and* $m_\beta(h)$ *can be extended, as solution of* (10), (12), *to the region* $h + m_\beta(h) \geq 0$, *where* $m_\beta'(h) > 0$*;*

c) The maps $(h, \beta) \to g(h, \beta)$ *and* $(h, \beta) \to m_\beta(h)$ *are real analytic on* $(0, \infty) \times (0, \infty)$;

d) The maps $\beta \to g(0,\beta)$ and $\beta \to m_\beta(0)$ are real analytic on $(0,\infty) \setminus \{\beta_c\}$. In addition,

d_1) $m_\beta(0) = 0$ for $0 \le \beta \le \beta_c = 1/(4F''(0))$, while $m_\beta(0) > 0$ for $\beta > \beta_c$;

d_2) $m'_\beta(0) = 2(1-\sqrt{1-\beta/\beta_c})\beta_c/\beta - 1$ for $0 \le \beta \le \beta_c$ and $m'_{\beta_c}(0) = 1$, while $m'_\beta(0) = m_\beta(0)/F'(\beta m_\beta(0)) - 1$ for $\beta > \beta_c$ and $m'_\beta(0) \to 3$, as $\beta \searrow \beta_c$;

e)For each $\beta > 0$, the map $h \to m_\beta(h)$ is strictly concave on $(0,\infty)$.

This result can be completed with the following proposition, which states that the phase transition is of infinite order.

Proposition 2. *For $\beta - \beta_c > 0$, sufficiently small, there exists some constant $K > 0$ such that*

$$m_\beta(0) \le K \exp\left[-\frac{\pi}{2}/\sqrt{F''(0)(\beta-\beta_c)}\right]. \tag{13}$$

The maps $\beta \to g(0,\beta)$ and $\beta \to m_\beta(0)$ are indefinitely differentiable at β_c.

Proposition 1 can be easily related to the results of [1] and [2].

Indeed, in the Ising case, when $\rho(t) = (\delta(t-1)+\delta(t+1))/2$, we have $F(x) = \ln chx$, and obtain the result of [1], namely $\beta_c = 1/4$.

Let us consider the extremely anisotropic d-vector model, consisting of a system of unitary classical spins $x_i \in \mathbb{R}^d$, associated with a one-dimensional lattice. The interactions of the spins are given by $x_{i1}x_{j1}/\max(i,j)$, $i,j \ge 1$. The *a priori* distribution $\rho_d(dt_1, dt_2, ...dt_d)$ is the uniform distribution on the unit sphere in $\mathbb{R}^d$. Suppose the system in an external magnetic field, directed along the first axis of $\mathbb{R}^d$. This is equivalent to our model with the *a priori* distribution

$$\begin{aligned}\rho(dt) &= \int \rho_d(dt, dt_2, ...dt_d)\\ &= \frac{\Gamma(d/2)}{\Gamma(1/2)\Gamma((d-1)/2)}(1-t^2)^{\frac{d-3}{2}}dt,\ t\in[-1,1] \qquad d\ge 2.\end{aligned}$$

We obtain $F(x) = x^2/2d + ...$ and $\beta_c(d) = d/4$, $d \ge 2$, in agreement with the numerical simulations, for $d = 2, 3$, on the extremely anisotropic d-vector model [2].

The next sections are devoted to the proofs of Proposition 1 and 2.

2 The phase transition

The proof of Proposition 1 is based on the study of equation (10) and the associated two-dimensional, autonomous dynamical system. First, for h sufficiently large, one shows the existence and uniqueness of a solution $m_\beta(h)$ to equation (10), verifying condition (12)(Lemma 1). Then $m_\beta(h)$ is extended to $h \geq 0$ (Lemma 2). The critical properties of the magnetization stated in Proposition 1, are obtained by using the linearized approximation of the dynamical system around its fixed point.

The thermodynamic limit should follow estimating

$$[\exp \beta n g_n(h,\beta)] \,/\, [\exp \beta n g(|h|\,,\beta)]$$

where g is as in Proposition 1.

Apparently the only difficulty comes from the fact that $g(|h|\,,\beta)$ is not differentiable at $h = 0$. To avoid this technical point we actually estimate

$$\lambda_n(h,\beta) = \frac{\exp \beta n g_n(h,\beta)}{\exp \beta n g_n^*(h,\beta)} \qquad h \in \mathbb{R}. \tag{14}$$

where $g_n^*(h,\beta)$ is a suitable sequence of differentiable functions approximating $g(|h|\,,\beta)$ for large n. From (7), we find the corresponding recurrence for λ_n. By using the properties of $m_\beta(h)$, one obtains suitable bounds for λ_n such that $(\ln \lambda_n(h,\beta))/n \to 0$, as $n \to \infty$, implying the existence of the thermodynamic limit.

We need the following properties of F.

$$\begin{array}{rlll} i) & F'(x) > 0 & for\;\; x > 0 \;\; and & \lim_{x\to\infty} F'(x) = M \\ ii) & F''(x) > 0 & for\;\; x \geq 0 \;\; and & \lim_{x\to\infty} F''(x) = 0 \\ iii) & F'''(x) < 0 & for\;\; x > 0 \;\; and & \lim_{x\to\infty} F'''(x) = 0 \end{array} \tag{15}$$

M being defined in the previous Section.

In (i) and (ii), one recognizes the Griffiths inequalities, while (iii) is the strict GHS inequality. The values of the limits are simple consequences of the formula $\lim_{x\to\infty} \int f(t)e^{xt}\rho(dt)/\int e^{xt}\rho(dt) = f(M)$ applied to real continuous functions.

We first investigate equation (10). This is defined for $h + m \neq 0$, since $F'(x) = 0$ if $x = 0$. But $F(x) = F(-x)$, $x \in \mathbb{R}$, so that it is sufficient to study the equation only on the domain $h + m > 0$. On this domain, the right hand side of (10) verifies the Lipschitz condition, with respect to m, implying the uniqueness of the solutions (with respect to the initial data). In particular, through every point (m_0, h_0) of the domain $h + m > 0$, there passes a unique maximal solution.

Equation (10), with condition (12), is equivalent to the integral equation

$$m(h) = M - \int_h^\infty e^{-\frac{s-h}{M}} m(s) H(m(s) + s) ds \tag{16}$$

where

$$H(x) = \frac{1}{F'(\beta x)} - \frac{1}{M} \qquad x > 0.$$

By (15),

$$H(x) > 0 \qquad H'(x) < 0 \qquad \lim_{x\to\infty} H(x) = \lim_{x\to\infty} H'(x) = 0. \tag{17}$$

We have

Lemma 1. *For h sufficiently large, equation* (10)*, with condition* (12)*, has a unique bounded solution, which extends to a maximal solution $m_\beta(h) \in [0, M]$.*

Proof. Set $\mathcal{M}_a = \{m \in C([a, \infty); \mathbb{R}) : m(x) \in [0, M]\}$.

First we show that there exists h_0 such that for each $a \geq h_0$, equation (16) has a unique solution in $\mathcal{M}_a$. Remark that by (17), there is $h_0 > 0$ such that, for instance,

$$\sup_{s\geq h_0} H(s) \leq \sup_{s\geq h_0} (H(s) + M\,|H'(s)|) \leq \frac{1}{2M}. \tag{18}$$

Let T be the operator defined by the right hand side of equation (16),

$$(Tm)(h) = M - \int_h^\infty e^{-\frac{s-h}{M}} m(s) H(m(s) + s) ds.$$

Using (18) we find that if $m \in [0.M]$, $s \geq a \geq h_0$, then $mH(m + s)$ verifies the Lipschitz condition, with respect to m, with Lipschitz bound $\leq 1/(2M)$. It follows that $T\mathcal{M}_a \subset \mathcal{M}_a$ and T is a strict contraction on $\mathcal{M}_a$. Then by the Banach fixed point theorem, equation (16) has a unique solution in $\mathcal{M}_a$.

On the other side, using the properties of H, it can be easily checked that for a sufficiently large, a function $m \in \mathcal{M}_a$ verifies equation (10) with condition (12), if and only if m satisfies (16).

To conclude the lemma, it is sufficient to remark that each solution in $\mathcal{M}_a$ of equation (10) extends to a unique maximal solution.

For $h \geq 0$, fixed, let $\underline{m}$ be the solution of the equation

$$F'(\beta(\underline{m}+h)) - \underline{m} = 0. \tag{19}$$

The map $h \to \underline{m}(h)$ is strictly increasing, with $\underline{m}(0) = 0$, and $\underline{m}(h) \to M$, as $h \to \infty$.

Lemma 2 *Let $m_\beta(h)$ be as in Lemma 1. The $m_\beta(h)$ is defined for $h + m_\beta(h) \geq 0$ and has the following properties:*
i) $m_\beta(h) \in (\underline{m}(h), M)$, $h > 0$ and $m'_\beta(h) > 0$ for all $h > 0$;
ii) if $0 < \beta \leq \beta_c = (4F''(0))^{-1}$, then $m_\beta(h) < h$, $m_\beta(0) = 0$ and $m'_\beta(0) = (1 - 2\beta F''(0) - \sqrt{1 - 4\beta F''(0)})/(2\beta F''(0))$;
iii) if $\beta > \beta_c$, then $m_\beta(0) > 0$ and $m'_\beta(0) = m_\beta(0)/F'(\beta(m_\beta(0)) - 1$;
iv) the map $(h, \beta) \to m_\beta(h)$ is real-analytic on $(0, \infty) \times (0, \infty)$.

This result follows from standard arguments on dynamical systems, so that we only sketch the proof. Consider the two-dimensional, autonomous dynamical system

$$\begin{aligned} \dot{h} &= -F'(\beta(m+h)) \\ \dot{m} &= F'(\beta(m+h)) - m \end{aligned} \tag{20}$$

associated to equation (10), defined for all $(h, m) \in \mathbb{R}^2$. Taking into account the properties of F (see (15)), for each point $(h_0, m_0) \in \mathbb{R}^2 \backslash (0, 0)$ there corresponds a trajectory of (20), defined for all $t \in \mathbb{R}$. Obviously, there exists a unique fixed point $(h, m) = (0, 0)$. Moreover it can be easily checked that $W(h, m) = \frac{1}{2}m^2 + F(\beta(m + h))/\beta$ is a (global) Lyapunov function. Thus, standard arguments [5, 6] imply that $(h, m) = (0, 0)$ is a stable node for $0 < \beta \leq \beta_c = 1/(4F''(0))$ and a stable spiral point for $\beta > \beta_c$.

By the above properties, Lemma 1, and the phase portrait of (20), there exists a unique trajectory $(h(t), m_\beta(t))$ with $h(-\infty) = \infty$ and $m_\beta(-\infty) = M$ with

$$\begin{aligned} \dot{h} &= -F'(\beta(m_\beta(t) + h(t))) < 0 \\ \dot{m}_\beta &= F'(\beta(m_\beta(t) + h(t))) - m_\beta(t) < 0 \end{aligned} \qquad t > t_c \tag{21}$$

where $t_c = \sup\{t : F'(\beta(m_\beta(t) + h(t))) = 0\}$.

Note that $m_\beta(h)$ results by eliminating t between $h(t)$ and $m_\beta(t)$, for $t > t_c$. By virtue of these remarks, the first assertion and property i) of Lemma 2 follow from the phase portrait of equation (20).

The inequality $m_\beta(h) < h$ in ii) is also obtained by investigating the phase portrait of (20).

The rest of the properties in ii) and iii) are immediate from the behaviour of the linearized part of equation (20) around $(0,0)$, which is a node for $\beta \in (0, \beta_c]$, and a spiral point for $\beta > \beta_c$ (the correct value of $m'_\beta(0)$ in ii) is obtained by taking into account that $m_\beta(h) < h$ for $h > 0$).

Finally, since F is analytic, one can apply a standard result to obtain iv).

Proof of Proposition 1. The existence of m_β, with properties a-d) as in Proposition 1, results directly by applying Lemmas 1 and 2, respectively.

By (10),

$$m''_\beta(h) = -\frac{(1 + m'_\beta(h))^3 \gamma_\beta(h)}{m_\beta(h)} \tag{22}$$

where

$$\gamma_\beta(h) = \beta F''(\beta(m_\beta(h) + h))) - \frac{m'_\beta(h)}{(1 + m'_\beta(h))^2}. \tag{23}$$

Suppose that property e) is not true, i.e. there is some $h_1 > 0$ such that $m''_\beta(h_1) \geq 0$. According to Proposition 1b), $m'_\beta(h)$ is finite and strictly positive on $(0, \infty)$. But (15) i) and (12) imply that $m'_\beta(h) \to 0$, as $h \to \infty$. Then, there exists $h_2 \in (h_1, \infty)$ such that $m''_\beta(h_2) < 0$. By continuity, one can find $h_* \in [h_1, h_2)$ such that $m''_\beta(h_*) = 0$. Let $\widetilde{h} = sup\left\{h \in (h_1, h_2) : m''_\beta(h) = 0\right\}$. We have $\widetilde{h} < h_2$ and $m''_\beta(h) < 0$ on $(\widetilde{h}, h_2)$. On the other side, remark that $\gamma'_\beta(\widetilde{h}) < 0$, since

$$\gamma'_\beta(h) = \frac{m'_\beta(h) - 1}{\left(1 + m'_\beta(h)\right)^3} m''_\beta(h) + \beta^2(1 + m'_\beta(h))F'''(\beta(m_\beta(h) + h))$$

and $F''' < 0$ (property iii) in (15)). By continuity there is $h_{**} \in (\widetilde{h}, h_2)$ such that $\gamma'_\beta(h_{**}) < 0$. Then, by virtue of (22), we get $m''_\beta(h_{**}) > 0$, in contradiction with the properties of $\widetilde{h}$, so that the argument of e) is complete.

It remains to prove the existence of the thermodynamic limit.

As in the beginning of this section, we should define g_n^* conveniently and estimate the ratio λ_n introduced in (14).

Let m_β and $g^*(h) = g(|h|, \beta)$ as in Proposition 1 (for simplicity, we omit the dependence of β in notations in the irrelevant cases). From the properties of m_β, it results that if $h > 0$, then

$$0 \leq g'(0) \leq g'(h) \leq M \qquad g''(h) > 0 \qquad \sup_{h \geq 0} g''(h) < \infty \tag{24}$$

verified by $g^*(h)$, too.

Since the functions $g^*(h)$ and $F(x)$ are even, equation (9) can be replaced by

$$\beta g^*(h) = F(\beta h + \beta g^{*\prime}(h))\,, \quad h \in \mathbb{R} \tag{25}$$

which has been extended to $h = 0$, by setting $g^{*\prime}(0) = \pm g'(0))$.

Define g_n^* by

$$g_n^*(h) = g^*(h) + a_n(h) = g^*(h) + (n)^{-1/2} \exp(-\sqrt{n} g'(0)\,|h|). \tag{26}$$

Obviously, g_n^* is even, convex and has continuous second derivative. ¿From (24), if $h \geq 0$, then

$$0 \leq g_n^{*\prime}(h) \leq g^{*\prime}(h) \leq M \qquad |g_n^{*\prime\prime}(h)| \leq C_0 \cdot \sqrt{n} \tag{27}$$

for some constant $C_0 > 0$.

By using (14), we find

$$\lambda_1(h) = \exp[\beta g_1(h) - \beta g^*(h) - \beta a_1(h)].$$

Applying the Lagrange theorem,

$$0 \geq F(\beta h) - F(\beta h + \beta g^{*\prime}(h)) = -\beta g^{*\prime}(h) F'(\beta h + \theta \beta g^{*\prime}(h)) \geq -\beta M^2.$$

Further, by (24)

$$\exp\left[-\beta(M^2 + 1)\right] \leq \lambda_1(h) \leq 1. \tag{28}$$

Observe that (7) provides a recurrence for λ_n

$$\lambda_n(h) = e^{\delta_{n,1}(h)} \int e^{\delta_{n,2}(h,t)} \lambda_{n-1}(h + t/n) \rho(dt) \tag{29}$$

where

$$\delta_{n,1}(h) = -\beta g^*(h) + \beta[(n-1)a_{n-1}(h) - n a_n(h)]$$

and

$$\delta_{n,2}(h,t) = \beta h t + \beta(n-1)[g_{n-1}^*(h + t/n) - g_{n-1}^*(h)].$$

¿From the definition (26), $|a_{n-1}(h) - a_n(h)| \leq 1/2(n-1)\sqrt{n}$, so that

$$|\delta_{n,1} + \beta g^*(h)| \leq C_1 \cdot \frac{1}{\sqrt{n}} \tag{30}$$

for some constant C_1.

On the other side, expanding $g^*_{n-1}(h+t/n)$ in t/n, up to the second order terms, we get

$$\delta_{n,2} - \beta ht - \beta g^*_{n-1}(h)t = -\beta g^{*\prime}_{n-1}(h)\frac{t}{n} + \beta\frac{n-1}{2}g^{*\prime\prime}_{n-1}(h+\theta t/n)\frac{t^2}{n^2}$$

with $|\theta| \leq 1$. Invoking (27) and the condition $|t| \leq M$, one finally finds

$$\left|\delta_{n,2} - \beta ht - \beta g^*_{n-1}(h)t\right| \leq C_2 \cdot \frac{1}{\sqrt{n}} \tag{31}$$

for some constant C_1.

We evaluate (29) by means of (30) and (31). We obtain

$$\lambda_{n-1}(h+\frac{\theta_n M}{n})\exp[-\frac{C(\beta)}{\sqrt{n}}] \leq \lambda_n(h)\exp[-\varepsilon_n(h)] \leq \lambda_{n-1}(h+\frac{\theta_n M}{n})\exp[\frac{C(\beta)}{\sqrt{n}})] \tag{32}$$

for some positive constant $C(\beta)$ and $|\theta_n| \leq 1$. Here, $\varepsilon_n(h) = -\beta g^*(h) + F(\beta h + \beta g^{*\prime}_{n-1}(h))]$. Since $\lambda_n(h)$ is even, it is sufficient to consider $h \geq 0$. Since g^* satisfies (25),

$$\varepsilon_n(h) = -F(\beta h + \beta g^{*\prime}(h)) + F(\beta h + \beta g^{*\prime}_{n-1}(h)) \tag{33}$$

so that $\lim_{n\to\infty}\varepsilon_n(h) = 0$, because $\lim_{n\to\infty} g^{*\prime}_n(h) = g^{*\prime}(h)$.

Iterating (32) and using (28), we have

$$\exp[-C(\beta)\sum_{i=2}^{n}\frac{1}{\sqrt{i}}. - \beta(M^2+1)] \leq \lambda_n(h)\exp[-\sum_{i=2}^{n}\varepsilon_n)] \leq \exp C(\beta)\sum_{i=2}^{n}\frac{1}{\sqrt{i}}).$$

Then it is immediate that $(\ln\lambda_n(h))/n \to 0$ and, obviously, $g_n(h) \to g^*(h) = g(|h|)$, as $n \to \infty$, concluding the proof.

3 The order of the phase transition

Proposition 2 follows from suitable estimations on the higher derivatives $\partial^k m_\beta(0)/\partial^k\beta$ of the magnetization, using the equations in variations associated to (10).

We know that $m_\beta(0) \equiv 0$ for $\beta \leq \beta_c$ and $\beta \to m_\beta(0)$ is real analytic on (β_c, ∞) (as a simple consequence of the general theory of ordinary differential equations). It appears that the only difficulty comes from the behaviour at β_c.

We first prove (13).

Since $m''_\beta(h) < 0$ on $(0, \infty)$ (Proposition 1 e)), we can choose m'_β as independent variable, instead of h. In estimations, this appears to be more convenient, in order to handle the singularities. Let $h_\beta(u)$ denote the inverse of $(0, \infty) \ni h \to m'_\beta(h) \in (0, m'_\beta(0))$. Set also $m^*_\beta(u) = (m_\beta \circ h_\beta)(u)$.

Let $\Gamma_\beta(u) = (\gamma_\beta \circ h_\beta)(u)$ with γ_β as in (23). Clearly, $\Gamma_\beta(u) > 0$, by Proposition 1 e) and (22). We can write

$$\Gamma_\beta(u) = \widetilde{\Gamma}_\beta(u) - \beta\left[F''(0) - F''(\beta(m^*_\beta(u) + h_\beta(u))\right]$$

where

$$\widetilde{\Gamma}_\beta(u) = (\frac{1}{1+u} - \frac{1}{2})^2 + (\beta - \beta_c)F''(0) > \Gamma_\beta(u), \quad u \in (0, m'_\beta(0)]$$

with $\beta - \beta_c \geq 0$.

By (22),

$$\frac{1}{m^*_\beta(u)} \frac{dm^*_\beta(u)}{du} = -\frac{u}{(1+u)^3 \Gamma_\beta(u)} \tag{34}$$

and a straightforward computation implies

$$\begin{aligned} \ln(\frac{m^*_\beta(u_1)}{m^*_\beta(u_2)}) &= \frac{1}{2}\left[\frac{1}{\sqrt{F''(0)(\beta-\beta_c)}} \arctan \frac{1-u}{2(1+u)\sqrt{F''(0)(\beta-\beta_c)}}\right]_{u_2}^{u_1} \\ &+ \frac{1}{2}\ln\frac{\widetilde{\Gamma}_\beta(u_2)}{\widetilde{\Gamma}_\beta(u_1)} + \int_{u_1}^{u_2} \frac{u\Delta_\beta(u)}{(1+u)^3} du \end{aligned} \tag{35}$$

for $0 < u_1 < u_2 \leq m'_\beta(0)$. Here,

$$\Delta_\beta(u) = \frac{1}{\Gamma_\beta(u)} - \frac{1}{\widetilde{\Gamma}_\beta(u)} \geq 0 \tag{36}$$

for $\beta \geq \beta_c$, $u \in (0, m'_\beta(0)]$.

In (35), let $u_2 = m'_\beta(0)$ and $u_1 < 1$ (obviously, $m_\beta(0) = m^*_\beta(m'_\beta(0))$). From Proposition 1d$_2$), there is some $\beta_0 > \beta_c$ such that $u_2 = m'_\beta(0) > 1$, provided that $\beta_0 \geq \beta \geq \beta_c$. Now, inequality (13) follows easily by estimating the right hand side of (35), where only the first term is relevant, since the second term is bounded and the last one is positive, by (36).

In order to complete the proof of Proposition 2, it would be sufficient to show that $\partial^k z_\beta(0)/\partial^k\beta \to 0$ as $\beta \searrow \beta_c$, where $z_\beta(h) = \beta g(h,\beta)$, with g as in Proposition 1.

The starting point is the family of equations associated to (9) for $h \geq 0$, namely

$$\frac{d}{dh}\frac{\partial^k z_\beta(h)}{\partial\beta^k} = \frac{1}{F'(\beta m_\beta(h)+\beta h)}\frac{\partial^k z_\beta(h)}{\partial\beta^k} + Z_{\beta,k}(h)\ ,\ k=1,2,..., \tag{37}$$

where

$$Z_{\beta,1}(h) = -h$$

and

$$Z_{\beta,k}(h) = -\frac{F''(z'_\beta+\beta h)}{F'^3(z'_\beta+\beta h)}\frac{\partial^{k-1} z_\beta(h)}{\partial\beta^{k-1}}\frac{\partial z_\beta(h)}{\partial\beta} + \frac{\partial}{\partial\beta}Z_{\beta,k-1}(h) \quad k\geq 2. \tag{38}$$

We integrate (37) on some interval $[h,h_0]$ and find

$$\frac{\partial^k z_\beta(h)}{\partial\beta^k} = \frac{\partial^k z_\beta(h_0)}{\partial\beta^k}E_\beta(h,h_0) - \int_h^{h_0} E_\beta(h,h')Z_{\beta,k}(h')dh'\ ,h\geq 0 \tag{39}$$

with

$$E_\beta(h_1,h_2) = \exp(-\int_{h_1}^{h_2}\frac{1}{F'(\beta m_\beta(h)+\beta h)}dh). \tag{40}$$

We obtain a convenient inequality for the right hand side of (39), implying the desired behaviour of $\partial^k z_\beta(0)/\partial^k\beta$.

To this end, we first show that if h_0 and $\beta_0-\beta_c>0$ are sufficiently small, then

$$E_\beta(h_1,h_2) \leq C\cdot(m_\beta(h_1)/m_\beta(h_2))^2 \tag{41}$$

for $0\leq h_1\leq h_2\leq h_0$ and $\beta\in(\beta_c,\beta_0]$ (in the rest of the paper, C denotes various constants with respect to $h\in[0,h_0]$ and $\beta\in(\beta_c,\beta_0]$).

Indeed, in the identity

$$\frac{1}{(1+u)^2\Gamma_\beta(u)} = 2\frac{u}{(1+u)^3\Gamma_\beta(u)} - \frac{d\ln\widetilde{\Gamma}_\beta(u)}{du} + \frac{(1-u)\Delta_\beta(u)}{(1+u)^3}$$

we use formula (34) and take into account that

$$\frac{1}{F'(\beta m^*_\beta(u)+\beta h_\beta(u))}h'_\beta(u) = -\frac{1}{(1+u)^2\Gamma_\beta(u)}$$

obtaining

$$\int_{h_1}^{h_2} \frac{1}{F'(\beta m_\beta(h) + \beta h)} dh = 2\ln\frac{m_\beta(h_2)}{m_\beta(h_1)} + \ln\frac{\widetilde{\Gamma}_\beta(m'_\beta(h_2))}{\widetilde{\Gamma}_\beta(m'_\beta(h_1))} - \int_{m'_\beta(h_2)}^{m'_\beta(h_1)} \frac{(1-u)\Delta_\beta(u)}{(1+u)^3} du\,. \tag{42}$$

Suppose that h_0 and $\beta_0 - \beta_c > 0$ are sufficiently small. Then the last two terms in the right hand side of (42) are bounded, provided that $0 \leq h_1 \leq h_2 \leq h_0$ and $\beta \in (\beta_c, \beta_0]$ (to estimate the third term one uses the continuity of Γ_β and the property $\lim_{\beta\searrow\beta_c} \Gamma_\beta(0) > 0$). We get

$$\left| \int_{h_1}^{h_2} \frac{1}{F'(\beta m_\beta(h) + \beta h)} dh - 2\ln\frac{m_\beta(h_2)}{m_\beta(h_1)} \right| \leq C \tag{43}$$

implying (41).

Consider (39) for $k = 1$. We introduce (41) in (39) and find

$$\left| \frac{\partial z_\beta(h)}{\partial\beta} \right| \leq C \cdot m_\beta(h)^2 (1 - \int_h^{h_0} \frac{h'}{m_\beta(h')^2} dh') \tag{44}$$

for $h \in [0, h_0]$ $\beta \in (\beta_c, \beta_0]$.

We have in mind a bound for the integral in (44). Since $m_\beta(0) \to 0$ as $\beta \searrow \beta_c$, we should investigate the behaviour of $h'/m_\beta(h')^2$ for h' and$\beta - \beta_c$ small. However, it is sufficient to observe that $m_\beta(h) > hm'_\beta(h_0)$ and $m_\beta(h) > F'(\beta m_\beta(h) + h)$. Then, clearly,

$$\int_h^{h_0} \frac{h'}{m_\beta(h')^2} dh' \leq C \int_h^{h_0} \frac{1}{F'(\beta m_\beta(h') + \beta h)} dh'.$$

Moreover, by virtue of (43) and (13), if h_0 and $\beta_0 - \beta_c > 0$ are sufficiently small, then

$$\int_h^{h_0} \frac{1}{F'(\beta m_\beta(h') + \beta h')} dh' \leq \int_0^{h_0} \frac{1}{F'(\beta m_\beta(h) + \beta h)} dh \leq C \cdot |\ln(m_\beta(0))| \tag{45}$$

for $h \in [0, h_0]$, $\beta \in (\beta_c, \beta_0]$. Consequently, for $h \in [0, h_0]$, $\beta \in (\beta_c, \beta_0]$,

$$\left| \frac{\partial z_\beta(h)}{\partial\beta} \right| \leq C \cdot |\ln(m_\beta(0)|\, m_\beta(h)^2, \tag{46}$$

verifying $\partial z_\beta(0)/\partial\beta \to 0$ as $\beta \searrow \beta_c$.

Consider (39) for $k \geq 2$. We proceed by induction: first we find that if $h \in [0, h_0]$ and $\beta \in (\beta_c, \beta_0]$, then $|Z_{\beta,k}(h)|$ is bounded by some polynomial expression (with constant coefficients) depending only on $|1/F'|$ and $|\partial^j z_\beta/\partial\beta^j|$ $j = 1, ..., k-1$; further we apply (41) and (45) in a similar way as for $k = 1$. We finally obtain that if h_0 and $\beta_0 - \beta_c > 0$ are sufficiently small, then there is some integer n_k and a constant C_k such that

$$\frac{\partial^k z_\beta(h)}{\partial\beta^k} \leq C_k \cdot |\ln(m_\beta(0)|^{n_k}\, m_\beta^2(h) \tag{47}$$

for all $h \in [0, h_0]$ and $\beta \in (\beta_c, \beta_0]$.

The last inequality implies that $\partial^k z_\beta(0)/\partial\beta^k \to 0$, as $\beta \searrow \beta_c$, so that the proof of Proposition 2 is complete.

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