Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent

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Abstract

In this paper we introduce a method that allows one to prove uniform local results for one-dimensional discrete Schrödinger operators with Sturmian potentials. We apply this method to the transfer matrices in order to study the Lyapunov exponent and the growth rate of eigenfunctions. This gives uniform vanishing of the Lyapunov exponent on the spectrum for all irrational rotation numbers. For irrational rotation numbers with bounded continued fraction expansion, it gives uniform existence of the Lyapunov exponent on the whole complex plane. Moreover, it yields uniform polynomial upper bounds on the growth rate of transfer matrices for irrational rotation numbers with bounded density. In particular, all our results apply to the Fibonacci case.

1 Introduction

In this paper we continue our investigation, started in [6], of one-dimensional quasicrystals. That is, we will be concerned with the family of operators $H_{\lambda,\alpha,\theta}$ on $l^2(\mathbb{Z})$, acting on $u \in l^2(\mathbb{Z})$ by

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + \lambda v_{\alpha,\theta}(n)u(n). \tag{1}$$

Here, λ belongs to $\mathbb{R} \setminus \{0\}$, $\alpha \in (0,1)$ is irrational, θ belongs to $[0,1)$, and $v_{\alpha,\theta}(n)$ is given by

$$v_{\alpha,\theta}(n) = \chi_{[1-\alpha,1)}(n\alpha + \theta \mod 1).$$

These operators arise in standard models of one-dimensional quasicrystals. They have attracted much attention (cf. [1, 2, 5, 11] and references therein) in recent years and they exhibit remarkable spectral properties.

These spectral properties are most conveniently studied within the framework of random operators. To do so, fix λ and α and consider the family (in θ) of operators $(H_{\lambda,\alpha,\theta})_{\theta\in[0,1)}$. This is an ergodic family of discrete random operators. Therefore, the spectral properties of the $H_{\lambda,\alpha,\theta}$ are independent of θ for Lebesgue almost all θ [4]. The general theory of random operators does not yield uniform spectral properties, that is, properties holding everywhere. In fact, one cannot expect uniformity in general, as many counterexamples, such as the almost Mathieu operator or Anderson-type models, show. However, if the family is minimal in the sense of strong convergence, it is not hard to show that the spectrum itself is constant. Moreover, in the one-dimensional minimal case, by a recent result of Last and Simon [15], the absolutely continuous spectrum is constant everywhere. These two results apply in particular to the family $(H_{\lambda,\alpha,\theta})_{\theta\in[0,1)}$. Therefore, there exists a set $\Sigma_{\lambda,\alpha} \subset \mathbb{R}$ such that the spectrum of $H_{\lambda,\alpha,\theta}$ equals $\Sigma_{\lambda,\alpha}$ for all θ, and the absolutely continuous spectrum is independent of θ and empty by [14]. In [6], we exhibited hierarchical structures in the sequences $v_{\alpha,\theta}$ that allow us to prove a further uniform result, namely uniform absence of eigenvalues. More precisely, these structures can be used to show that for arbitrary α and λ for all $E \in \Sigma_{\lambda,\alpha}$ and all θ the difference equation

$$u(n+1) + u(n-1) + \lambda v_{\alpha,\theta}(n)u(n) = Eu(n) \tag{2}$$

has no l^2-solution (cf. [6] and [8] for further details and proofs). Together with the already mentioned results of [14, 15] this gives uniform singular continuous spectrum.

In this paper we introduce a method that allows one to show uniform local results for the family $(H_{\lambda,\alpha,\theta})_{\theta\in[0,1)}$. This method is based on local hierarchies. We will use it to study upper bounds on the growth of solutions of (2) and to study the Lyapunov exponent. The Lyapunov exponent is an important quantity in the study of (2). Upper bounds on the growth of solutions of (2) give information on the so-called resistance [11, 12]. Moreover, uniform upper bounds may be useful in proofs of uniform α-continuity (cf. the general method in [5]). Our investigation is based on a careful analysis of the transfer matrices associated to (2). A key step in our approach is to introduce the set

$$\mathcal{W}(\alpha) \equiv \cup_{\theta\in[0,1)} \mathrm{Sub}(v_{\alpha,\theta}). \tag{3}$$

Here, the $v_{\alpha,\theta}$ are considered to be words over $\{0,1\}$ and $\mathrm{Sub}(w)$ denotes the set of all finite non-empty subwords of w. It is in fact well known that $\mathrm{Sub}(v_{\alpha,\theta})$ does not depend on θ (cf. Proposition 2.3 below). The set $\mathcal{W}(\alpha)$ is by its very definition particularly appropriate to serve as a basis for the study of uniform local properties of the family of operators. In fact, the transfer matrices associated to (2) can be defined simultaneously in θ by setting

$$M(\lambda, E, w) \equiv T(\lambda, E, w_n) \times \cdots \times T(\lambda, E, w_1) \tag{4}$$

for $w = w_1 ... w_n \in \mathcal{W}(\alpha)$, where, for $a \in \mathbb{R}$ and $E \in \mathbb{C}$, the matrix $T(\lambda, E, a)$ is defined by

$$T(\lambda, E, a) \equiv \begin{pmatrix} E - \lambda a & -1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

If we define for $n \in \mathbb{N}$ the words $v_{\alpha,\theta}^n \equiv v_{\alpha,\theta}(1) \dots v_{\alpha,\theta}(n)$, then the matrices $M(\lambda, E, v_{\alpha,\theta}^n)$ are just the usual transfer matrices as defined, for example, in [4]. Having formulated uniform local properties in terms of functions on $\mathcal{W}(\alpha)$, the crucial step in our method is to partition the words of $\mathcal{W}(\alpha)$ into parts on which the functions can easily be analyzed. This will be done using the local analog of the series of (global) partitions introduced in [6]. This hierarchical structure is similar to the series of (de-) compositions of self-similar tilings [9, 10] and it is naturally connected to the coefficients a_n in the continued fraction expansion of α (for general information on continued fractions see, e.g., [13]),

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, ...]. \tag{6}$$

Recall that $\alpha = [a_1, a_2, ...]$ is said to be a bounded density number if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j < \infty.$$

Our main results can be phrased as follows.

Theorem 1 *For all λ, α, θ, and all $E \in \Sigma_{\lambda,\alpha}$, the limit*

$$\gamma_\theta(E) \equiv \lim_{n\to\infty} \frac{1}{n} \ln \|M(\lambda, E, v^n_{\alpha,\theta})\|$$

exists and equals zero.

Theorem 2 *Suppose α is such that the sequence (a_n) is bounded. Then, for all λ, θ, and all $E \in \mathbb{C}$, the limit*

$$\gamma_\theta(E) \equiv \lim_{n\to\infty} \frac{1}{n} \ln \|M(\lambda, E, v^n_{\alpha,\theta})\|$$

exists and is independent of θ.

Theorem 3 *Let α be a bounded density number and let λ be arbitrary. Then there exist $C > 0$ and $\mu > 0$ such that for all $E \in \Sigma_{\lambda,\alpha}$ and all $n \in \mathbb{N}$, the following inequality holds,*

$$\|M(\lambda, E, v^n_{\alpha,\theta})\| \leq C n^\mu.$$

In fact, we will show the corresponding results not only for words of the form $v^n_{\alpha,\theta}$ but rather uniformly in the length for all words in $\mathcal{W}(\alpha)$. In particular, our results hold as well for the larger family of operators $(H_{\lambda,\omega})_{\omega\in\Omega}$ given by

$$H_{\lambda,\omega}(u) = u(n+1) + u(n-1) + \lambda\omega(n)u(n),$$

where ω belongs to the closure of $\{v_{\alpha,\theta} : \theta\}$ in $\{0,1\}^{\mathbb{Z}}$ with respect to product topology.

Uniform existence of the Lyapunov exponent has so far only been known for those of the above operators that can also be generated by a substitution process [9]. Uniform polynomial upper bounds on the solutions have not been known at all.

To summarize, our method consists of the following steps.

- Introduce the set $\mathcal{W}(\alpha)$ containing uniform local information.
- Formulate local properties of the family of operators in terms of functions on $\mathcal{W}(\alpha)$.
- Partition the words in $\mathcal{W}(\alpha)$ into parts on which the functions can easily be studied.

Our paper is organized as follows. In Section 2 we introduce necessary notation and review some basic facts. Section 3 is concerned with partitions which will be our main tool in the sequel. In Section 4 we study the Lyapunov exponent and prove Theorem 1. Theorem 2 follows from the more general considerations given in [16]. Here we give a more direct proof adapted to the situation at hand. Finally, in Section 5 we prove Theorem 3.

2 Preliminaries

Fix an irrational α and consider the family (in θ) of all sequences of the form $(v_{\alpha,\theta}(k))_{k\in\mathbb{Z}}$. Consider such a sequence as a two-sided infinite word over the alphabet $A \equiv \{0,1\}$ (cf. [17, 18] for the general theory of combinatorics on words). We shall now recall some basic facts about $v_{\alpha,\theta}$ (cf. [1, 3]). Define the words s_n over the alphabet A by

$$s_{-1} \equiv 1, \quad s_0 \equiv 0, \quad s_1 \equiv s_0^{a_1-1}s_{-1}, \quad s_n \equiv s_{n-1}^{a_n}s_{n-2}, \; n \geq 2, \tag{7}$$

where the a_n are the coefficients in the continued fraction expansion of α.

Proposition 2.1 [3] *There exist palindromes π_n, $n \geq 2$, such that $s_{2k} = \pi_{2k}10$ and $s_{2k+1} = \pi_{2k+1}01$ for all $k \in \mathbb{N}$.*

By definition, for $n \geq 2$, s_{n-1} is a prefix of s_n. Therefore, the following ("right"-) limit exists in an obvious sense,

$$c_\alpha \equiv \lim_{n\to\infty} s_n. \tag{8}$$

The relation between c_α and the $v_{\alpha,\theta}$ is given in the following proposition.

Proposition 2.2 [1] *$v_{\alpha,0}$ restricted to $\{1,2,3,\dots\}$ coincides with c_α.*

Some convenient and well-known descriptions of $\mathcal{W}(\alpha)$ are given in the next proposition.

Proposition 2.3 *$\mathcal{W}(\alpha) = \mathrm{Sub}(c_\alpha) = \mathrm{Sub}(v_{\alpha,\theta})$ for all θ.*

Proof. By the irrationality of α, the set $\{\alpha n \bmod 1 : n \in \mathbb{N}\}$ and the sets $\{\alpha n + \theta \bmod 1 : n \in \mathbb{Z}\}$ are dense in $[0,1)$. This and a standard right-continuity argument now yield the assertion. $\square$

The following proposition will turn out to be rather useful.

Proposition 2.4 [6, 18] *For all $n \geq 2$, the word s_n is a prefix of the word $s_{n-1}s_n$.*

Let us end this section by introducing the length $|w|$ of a word w. Here, $|w|$ is defined by $|w| = n$ if $w = w_1 \dots w_n$, $w_i \in A$, by $|w| = 0$ if w is the empty word, and by $|w| = \infty$ in all other cases.

3 Partitions

In this section we introduce the notion of an n-partion of a word and study some of its properties.

Definition 1 (i) *Let w be a finite word over A. Let $n \in \mathbb{N}_0$ be given. An (n,α)-partition of w consists of words a, b, z_j, $j = 1,\dots,l$, with $l \in \mathbb{N}_0$, where the word a is a proper prefix of s_n or s_{n-1}, and the word b is a proper suffix of s_n and $z_j \in \{s_{n-1}, s_n\}$ for $j = 1,\dots,l$, such that the equation*

$$w = a z_1 \dots z_l b$$

holds.
(ii) *An (n,α)-partition of a one-sided infinite word w consists of words a and z_j, $j \in \mathbb{N}$, where the word a is a proper suffix of s_n or s_{n-1}, and $z_j \in \{s_{n-1}, s_n\}$ for $j \in \mathbb{N}$, such that the following equation holds,*

$$w = a z_1 z_2 \dots .$$

In (i) *as well as in* (ii) *the z_j are referred to as blocks in the (n,α)-partition or, more specifically, as blocks of the form s_n if $z_j = s_n$, and as blocks of the form s_{n-1} if $z_j = s_{n-1}$. The position of the block z_j is the interval*

$$I_j = \left\{ |a| + \sum_{r=1}^{j-1} |z_r|, \dots, |a| + \sum_{r=1}^{j} |z_r| \right\}.$$

If it is clear from the context to which α we refer, we will suppress the dependence on α and just write n-partition instead of (n,α)-partition.

Lemma 3.1 *For every $n \in \mathbb{N}_0$, there exists a unique n-partition of c_α. It is of the form $c_\alpha = z_1 z_2 \dots$.*

Proof. Existence: This is clear from (7) and the definition of c_α in (8).
Uniqueness: This follows by induction: As $s_0 = 0$ and $s_{-1} = 1$, uniqueness is clear for $n = 0$. By (7), every $(n+1)$-partition gives rise to an n-partition and the positions of the s_{n+1} in the $(n+1)$-partition are determined by the positions of s_{n-1} in the n-partition. Thus, uniqueness of the n-partition implies uniqueness of the $(n+1)$-partition. □

For arbitrary $w \in \mathcal{W}(\alpha)$, n-partitions do exist, as can be seen from the next definition. For a further study of these local n-partition and their uniqueness properties, we refer the reader to [16].

Definition 2 *Let $w \in \mathcal{W}(\alpha)$ be given. Then, by Proposition 2.3, there exists a smallest $j \in \mathbb{N}$ with*

$$w = (c_\alpha)_j \dots (c_\alpha)_{j+|w|-1}.$$

Let $n \in \mathbb{N}_0$ be given such that w is not a subword of s_n. Then the restriction of the n-partition of c_α to $(c_\alpha)_j \dots (c_\alpha)_{j+|w|-1}$ induces an n-partition of w. This n-partition will be called the standard n-partition of w.

Lemma 3.2 *Let $w \in \mathcal{W}(\alpha)$ be given. Then there exist $t \in \mathbb{N}$, a suffix x of s_t or s_{t-1}, and a prefix y of s_{t+1} such that $w = xy$.*

Proof. We will consider two cases.
Case 1. $w \in \mathrm{Sub}(s_1)$: By $s_1 = 0^{a_1-1}1$ we see that either $w = 0^k$ or $w = 0^k1$ with $0 \le k \le a_1 - 1$. Thus, w is either a prefix of s_1 or a suffix of s_1.
Case 2. $w \notin \mathrm{Sub}(s_1)$: Define $M \equiv \{n \in \mathbb{N} : w \notin \mathrm{Sub}(s_n)\}$. By assumption M is not empty. By Proposition 2.3 and (8), the set M is bounded. Moreover, it is clear that $n \in M$ implies $n-1 \in M$. Thus, M is of the form $\{1, \dots, t\}$. This means that w is not a subword of s_t, but it is a subword of s_{t+1}. As $s_{t+1} = s_t^{a_{t+1}} s_{t-1}$ by (7), this implies that

$$k \equiv \min\{n \in \mathbb{N}_0 \ : w \in \mathrm{Sub}(s_t^n s_{t-1})\}$$

satisfies $1 \le k \le a_{t+1}$. Then the word $s_t^k s_{t-1}$ admits a $(t+1)$-partition of the form

$$s_t^k s_{t-1} = s_t s_t^{k-1} s_{t-1} = ab,$$

where $a \equiv s_t$ is a suffix of s_t and $b \equiv s_t^{k-1} s_{t-1}$ is considered to be a prefix of s_{t+1}. By minimality of k, this $(t+1)$-partition of $s_t^k s_{t-1}$ induces a $(t+1)$-partition of w with the required properties. □

4 The Lyapunov exponent

Our study of uniform properties of the Lyapunov exponent will be based on the study of the functions

$$L(\lambda, E) : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}, \;\; L(\lambda, E)(w) = \ln \|M(\lambda, E, w)\|,$$

for arbitrary $E \in \mathbb{C}$ and λ. Here, $\|A\|$ denotes the norm of the linear operator on $\mathbb{C}^2$ with matrix A with respect to the standard orthonormal basis $\{(0,1), (1,0)\}$ of $\mathbb{C}^2$. By the submultiplicativity of $\|\cdot\|$, this function is subadditive, where a function $F : \mathcal{W}(\alpha) \longrightarrow \mathbb{R}$ is called subadditive if it satisfies

$$F(ab) \le F(a) + F(b) \tag{9}$$

for all words $a, b \in \mathcal{W}(\alpha)$ with $ab \in \mathcal{W}(\alpha)$. As the matrices T and hence the matrices M have determinant 1, we see that the function $L(\lambda, E)$ is in fact nonnegative. In the sequel we will sometimes suppress the dependence on E and λ and just write L instead of $L(\lambda, E)$ if it is clear from the context to which E and λ we refer.

Lemma 4.1 *Let λ and α be given. Fix $E \in \Sigma_{\lambda,\alpha}$. Then, the following equation holds,*

$$\lim_{|w|\to\infty, w\in\mathcal{W}(\alpha)} \frac{1}{|w|} L(\lambda, E)(w) = 0.$$

Remark. Let G be a function on $\mathcal{W}(\alpha)$. Then, the equation $\lim_{|w|\to\infty, w\in\mathcal{W}(\alpha)} G(w) = a$ means that for all $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that $|G(w) - a| \leq \epsilon$ for all $w \in \mathcal{W}(\alpha)$ with $|w| \geq n_\epsilon$. In the sequel we will suppress the "$w \in \mathcal{W}(\alpha)$" under the limit if it is clear to which $\mathcal{W}(\alpha)$ we refer.

Proof. Choose an arbitrary $\epsilon > 0$. By Lemma 5 of [1], we have $\lim_{n\to\infty} \frac{1}{|s_n|} L(s_n) = 0$. Therefore, there exists $n_1 \in \mathbb{N}$ with

$$\frac{1}{|s_n|} L(s_n) \leq \frac{\epsilon}{2} \tag{10}$$

for all $n \in \mathbb{N}$ with $n \geq n_1$. Set $C \equiv \max\{L(0), L(1)\}$. As $|s_n|$ tends to infinity for $n \to \infty$, we can find $n_2 \in \mathbb{N}$ such that all $n \geq n_2$ satisfy

$$\frac{2C|s_{n_1+1}|}{|s_n|} \leq \frac{\epsilon}{2}. \tag{11}$$

Define $n_\epsilon \equiv \max\{|s_{n_1+2}|, |s_{n_2+1}|\}$. Choose an arbitrary $w \in W$ with $|w| \geq n_\epsilon$ and let $w = az_1 \ldots z_l b$ be the standard $(n_1 + 1)$-partition of w. Define p and q by $p \equiv \#\{j \in \{1, \ldots, l\} : z_j = s_{n_1+1}\}$ and $q \equiv \#\{j \in \{1, \ldots, l\} : z_j = s_{n_1}\}$. Using the subadditivity of L (9), we can calculate

$$\begin{aligned}
\frac{1}{|w|} L(w) &\leq \frac{1}{|w|} \left(L(a) + \sum_{j=1}^{l} L(z_j) + L(b) \right) \\
&\leq \frac{|a|C}{|w|} + \frac{pL(s_{n_1+1})}{|w|} + \frac{qL(s_{n_1})}{|w|} + \frac{|b|C}{|w|} \\
(10) \quad &\leq \frac{2C|s_{n_1+1}|}{|w|} + \frac{p|s_{n_1+1}|\frac{\epsilon}{2}}{|w|} + \frac{q|s_{n_1}|\frac{\epsilon}{2}}{|w|} \\
(11) \quad &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \left(\frac{p|s_{n_1+1}| + q|s_{n_1}|}{|w|} \right) \\
&\leq \epsilon.
\end{aligned}$$

As L is nonnegative, this proves the lemma. □

Proof of Theorem 1. The theorem follows immediately from Proposition 2.3 and Lemma 4.1. □

In [16], results on the existence of certain limits of subadditive functions F on $\mathcal{W}(\alpha)$ are proven. These results imply in particular that the limit $\lim_{|w|\to\infty} \frac{F(w)}{|w|}$ exists for arbitrary subadditive functions on $\mathcal{W}(\alpha)$ if α has bounded continued fraction expansion. Here, we give a direct proof of this latter result for nonnegative functions F. This will apply in particular to the functions L introduced in the last lemma. Thus, it will be sufficient for a proof of Theorem 2. We start with a simple proposition similar to Proposition 3.5 of [16].

Proposition 4.2 *Fix some irrational $\alpha \in (0, 1)$.*

(i) *Let $w \in \mathcal{W}(\alpha)$ be given. If $|w| < |s_n|$ for some $n \in \mathbb{N}$, with $n \geq 2$, then w is a subword of s_{n+2}.*

(ii) *Let $w \in \mathcal{W}(\alpha)$ be a subword of s_n for some $n \in \mathbb{N}$. Then there exist words $x, y \in \mathcal{W}(\alpha)$ with $xwy = s_{n+3}$ and $|x|, |y| \geq |s_{n+1}|$.*

Proof. (i) If w is a subword of s_n, we are done. Otherwise, there is a standard n-partition of w. By the condition on the length of w, this partition shows that w is a subword of either $s_n s_n$ or of $s_n s_{n-1} s_n$. Using (7) twice, we see that there is a word a such that

$$s_{n+2} = a s_n s_{n-1} s_n \tag{12}$$

As $s_{n-1} s_n = s_n c$ for a suitable word c by Proposition 2.4, we see that both $s_n s_n$ and $s_n s_{n-1} s_n$ are subwords of s_{n+2} and the proof of (i) is finished.
(ii) This follows immediately from (12). □

Now, we can prove the following lemma.

Lemma 4.3 *Let $\alpha = [a_1, a_2, \ldots]$ be given such that there exists $C \in \mathbb{R}$ with $|a_n| \leq C$ for all $n \in \mathbb{N}$. Then, the limit $\lim_{|w|\to\infty} \frac{F(w)}{|w|}$ exists for every nonnegative, subadditive function F on $\mathcal{W}(\alpha)$.*

Proof. Define $F^{(n)} \equiv \max\left\{\frac{F(s_n)}{|s_n|}, \frac{F(s_{n-1})}{|s_{n-1}|}\right\}$. We will show the inequalities $(*)$ and $(**)$ given by

$$(*) \quad \limsup_{|w|\to\infty} \frac{F(w)}{|w|} \leq \inf_{n\in\mathbb{N}} F^{(n)}, \qquad (**) \quad \liminf_{|w|\to\infty} \frac{F(w)}{|w|} \geq \inf_{n\in\mathbb{N}} F^{(n)}.$$

Ad $(*)$: For fixed $n \in \mathbb{N}$, the same reasoning as in the proof of Lemma 4.1 gives $\limsup_{|w|\to\infty} \frac{F(w)}{|w|} \leq F^{(n)}$. Here we used the nonnegativity of F. As $n \in \mathbb{N}$ was arbitrary, this proves $(*)$.
Ad $(**)$: Suppose the contrary. Then there exists a $\delta > 0$ and a sequence $(w_k)_{k\in\mathbb{N}}$ of words in $\mathcal{W}(\alpha)$ with $|w_k| \to \infty$ for $k \to \infty$ such that

$$\frac{F(w_k)}{|w_k|} \leq \inf_{n\in\mathbb{N}} F^{(n)} - \delta \tag{13}$$

for all $k \in \mathbb{N}$. Assume w.l.o.g. that $|w_k| \geq |s_1|$ for all $k \in \mathbb{N}$. Define for $k \in \mathbb{N}$ the number $n(k)$ by the inequality $|s_{n(k)-1}| \leq |w_k| < |s_{n(k)}|$. Then $n(k)$ is not smaller than 2 for all k and by Proposition 4.2, there exist words x_k and y_k satisfying the following.

(P1) $x_k w_k y_k = s_{n(k)+5}$,

(P2) $|x_k|, |y_k| \geq |s_{n(k)+3}| \geq |w_k|$.

Invoking that the a_n are bounded by C, we can conclude from (7)

$$\frac{|w_k|}{|s_{n(k)+5}|} \geq \frac{|s_{n(k)-1}|}{|s_{n(k)+5}|} \geq \frac{1}{(C+1)^6}. \tag{14}$$

Now, choose $\epsilon > 0$ with $\bar{\delta} \equiv \delta \frac{1}{(C+1)^7} - \epsilon > 0$. As $|w_k|$ tends to infinity, we see from (P2) and the already proven part $(*)$ that for $k \in \mathbb{N}$ large enough, the inequalities

$$\frac{F(x_k)}{|x_k|} \leq \inf_{n\in\mathbb{N}} F^{(n)} + \epsilon \qquad \text{and} \qquad \frac{F(y_k)}{|y_k|} \leq \inf_{n\in\mathbb{N}} F^{(n)} + \epsilon \tag{15}$$

hold. Fix such a k. Then we get from (P1) and the subadditivity of F the following chain of inequalities:

$$\begin{aligned}
\frac{F(s_{n(k)+5})}{|s_{n(k)+5}|} &\leq \frac{F(x_k)}{|x_k|}\frac{|x_k|}{|s_{n(k)+5}|}+\frac{F(w_k)}{|w_k|}\frac{|w_k|}{|s_{n(k)+5}|}+\frac{F(y_k)}{|y_k|}\frac{|y_k|}{|s_{n(k)+5}|} \\
(13),\,(15) \quad &\leq \left(\inf_{n\in\mathbb{N}} F^{(n)}+\epsilon\right)\left(\frac{|x_k|+|y_k|}{|s_{n(k)+5}|}\right)+\left(\inf_{n\in\mathbb{N}} F^{(n)}-\delta\right)\left(\frac{|w_k|}{|s_{n(k)+5}|}\right) \\
(14) \quad &\leq \inf_{n\in\mathbb{N}} F^{(n)}+\epsilon-\delta\frac{1}{(C+1)^6} \\
&\leq \inf_{n\in\mathbb{N}} F^{(n)}-\bar{\delta}.
\end{aligned}$$

Similarly, one can show

$$\frac{F(s_{n(k)+5+1})}{|s_{n(k)+5+1}|}\leq \inf_{n\in\mathbb{N}} F^{(n)}+\epsilon-\delta\frac{1}{(C+1)^7}\leq \inf_{n\in\mathbb{N}} F^{(n)}-\bar{\delta}.$$

Thus, we arrive at the obvious contradiction $F^{(n(k)+5+1)} \leq \inf_{n\in\mathbb{N}} F^{(n)}-\bar{\delta}$. This proves (**) and finishes the proof of the lemma. □

Proof of Theorem 2. This is an immediate application of Lemma 4.3 to the function $L(\lambda, E)$. □

5 Upper bounds on the growth rate of eigenfunctions

In this section we provide polynomial upper bounds on the transfer matrices. The main result of [11] (cf. [12] as well) can be phrased as follows.

Theorem 4 *Let λ be given. Let $\alpha=[a_1,a_2,...]$ be an irrational number with bounded density. Then there exist $C>0$ and $\mu>0$ such that for all $E\in\Sigma_{\lambda,\alpha}$, the equation*

$$\|M(\lambda,E,w)\|\leq C|w|^{\mu}$$

holds for every prefix w of c_α.

We will need one more lemma for the proof of Lemma 5.2 of this section. This lemma seems to be new. It may be useful in other situations as well.

Lemma 5.1 *For $w=w_1...w_n$, $w_i\in A$, define $w^R\equiv w_n...w_1$. Then for all $E\in\mathbb{C}$ and all λ, we have*

$$\|M(\lambda,E,w)\|=\|M(\lambda,E,w^R)\|.$$

Proof. Suppress the dependence on λ. Let $J\equiv\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Clearly, J is an idempotent unitary operator, that is, $J^*=J^{-1}=J$. A short calculation gives

$$JT(a,E)J^*=T(a,E)^{-1}.$$

This equation and the fact that J is a unitary operator yield

$$\|M(w,E)\|=\|T(w_n,E)\dots T(w_1,E)\|=\|JT(w_n,E)J^*\dots JT(w_1,E)J^*\|=\|T(w_n,E)^{-1}\dots T(w_1,E)^{-1}\|.$$

Moreover, it is well known for a 2×2 matrix A with $\det A=1$ that $\|A\|=\|A^{-1}\|$. This yields

$$\|M(w,E)\|=\|T(w_n,E)^{-1}\dots T(w_1,E)^{-1}\|=\|T(w_1,E)\dots T(w_n,E)\|=\|M(w^R,E)\|.$$

The proof of the lemma is finished. □

Now we can prove the main result of this section.

Lemma 5.2 *Let $\alpha = [a_1, a_2, \dots]$ be an irrational number with bounded density. Let λ be given. Then there exist $C > 0$ and $\mu > 0$ with*

$$\|M(\lambda, E, w)\| \leq C|w|^{\mu}$$

for all $w \in \mathcal{W}(\alpha)$ and all $E \in \Sigma_{\lambda,\alpha}$.

Proof. By Theorem 4, there exist $\hat{C} > 0$ and $\hat{\mu} > 0$ such that

$$\|M(\lambda, E, y)\| \leq \hat{C}|y|^{\hat{\mu}} \tag{16}$$

for every prefix y of c_α. Define

$$C \equiv F^2(\hat{C}^2 + 1), \quad \mu \equiv 2\hat{\mu},$$

where

$$F \equiv \max\left\{\sup\{\|T(\lambda, E)\| : E \in \Sigma_{\lambda,\alpha}\}, \sup\{\|T(0, E)\| : E \in \Sigma_{\lambda,\alpha}\}, 1\right\}.$$

Note that $F < \infty$. We will now show

$$\|M(\lambda, E, w)\| \leq C|w|^{\mu} \tag{17}$$

for all $w \in \mathcal{W}(\alpha)$ and $E \in \Sigma_{\lambda,\alpha}$. Fix $w \in \mathcal{W}(\alpha)$. By Lemma 3.2, there exist $t \in \mathbb{N}$, a suffix x of s_t or s_{t-1}, and a prefix y of s_{t+1} with $w = xy$. In the following we will consider the case $|y|, |x| \geq 1$; the other cases can be treated similarly. The submultiplicativity of the norm $\|\cdot\|$ implies

$$\|M(\lambda, E, w)\| \leq \|M(\lambda, E, x)\| \|M(\lambda, E, y)\|. \tag{18}$$

We will give estimates on $\|M(\lambda, x, E)\|$ and $\|M(\lambda, y, E)\|$. As y is a prefix of s_{t+1} and so a fortiori a prefix of c_α, we can use (16) to estimate

$$\|M(\lambda, E, y)\| \leq \hat{C}|y|^{\hat{\mu}} \leq \hat{C}|w|^{\hat{\mu}}. \tag{19}$$

Next, we will give an estimate for $\|M(\lambda, x, E)\|$. This is a bit harder, as x is not a prefix of c_α. If $|x| \in \{1, 2\}$, we can just estimate

$$\|M(\lambda, E, x)\| \leq F^2 \leq F^2|w|^{\hat{\mu}}$$

by the definition of F. Otherwise, we use Proposition 2.1 together with the fact that x is a suffix of s_t to get

$$x^R = bav, \tag{20}$$

where v is a prefix of c_α and a and b belong to $\{0, 1\}$. Using Lemma 5.1, the definition of F, and (20), we get

$$\|M(\lambda, E, x)\| = \|M(\lambda, E, x^R)\| \leq F^2\hat{C}|v|^{\hat{\mu}} \leq F^2\hat{C}|w|^{\hat{\mu}}. \tag{21}$$

Putting together (18), (19), and (21), we finally arrive at

$$\|M(\lambda, E, w)\| \leq F^2(\hat{C}^2 + 1)|w|^{2\hat{\mu}}.$$

This finishes the proof. □

Proof of Theorem 3. This follows immediately from Proposition 2.3 and Lemma 5.2. □

Remark. It is not hard to show that Theorem 3 implies Lemma 5.2. Thus both results are in fact equivalent.

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