Renormalization and Periodic Orbits for Hamiltonian Flows

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Abstract. We consider a renormalization group transformation $\mathcal{R}$ for analytic Hamiltonians in two or more dimensions, and use this transformation to construct invariant tori, as well as sequences of periodic orbits with rotation vectors approaching that of the invariant torus. The construction of periodic and quasiperiodic orbits is limited to near–integrable Hamiltonians. But as a first step toward a non–perturbative analysis, we extend the domain of $\mathcal{R}$ to include any Hamiltonian for which a certain non–resonance condition holds.

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1. Introduction and Results

In this paper we complement and extend the results given in [11], by using a renormalization group (RG) transformation to construct sequences of periodic orbits for near–integrable Hamiltonians, and by extending the domain of this transformation to a larger set of Hamiltonians. The construction of periodic orbits that approximate quasiperiodic motion is a canonical application of RG ideas; see for example [23,6–10]. It relates observed universal accumulation rates to eigenvalues of the linearized RG transformation. This part of our analysis is restricted to near–integrable Hamiltonians, since RG fixed points relevant to critical cases have not yet been obtained rigorously. But the first part, which includes the definition of the RG transformation, does not require near–integrability.

The work presented here is essentially self–contained, but for a motivation of some of our choices, and other background material, the reader is referred to [11].

We start with some definitions. On $\mathbb{C}^d$, consider the two norms $|v| = \sum_j |v_j|$ and $\|v\| = \max_j |v_j|$. Let V and W be two fixed but arbitrary $d \times d$ matrices over $\mathbb{C}$, satisfying $V^T W = \mathrm{I}$, where V^T denotes the transposed of V. Define $\mathcal{D}_{\rho,1} = \{q \in \mathbb{C}^d : |V \mathrm{Im} q| < \rho\}$ and $\mathcal{D}_{\rho,2} = \{p \in \mathbb{C}^d : \|Wp\| < \rho\}$, for every $\rho > 0$. Unless stated otherwise, we will identify $\mathcal{D}_{\rho,1}$ with a complexified d–torus. In particular, a function on $\mathcal{D}_\rho = \mathcal{D}_{\rho,1} \times \mathcal{D}_{\rho,2}$ is assumed to be 2π–periodic in each component of its first argument. If analytic, such a function H may be written as a Fourier–Taylor series

$$H(q,p) = \sum_{(\nu,\alpha)\in I} H_{\nu,\alpha} \, (Wp)^\alpha e^{iq\cdot\nu} , \qquad (q,p) \in \mathcal{D}_\rho , \tag{1.1}$$

where $I = \mathbb{Z}^d \times \mathbb{N}^d$. Here, and in what follows, $x \cdot y$ denotes the standard dot product of two vectors in $\mathbb{C}^d$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$.

Definition 1.1. *Given any $\rho > 0$, define $\mathcal{A}_\rho$ to be the Banach space of all analytic Hamiltonians H on $\mathcal{D}_\rho$ of the form (1.1), for which the norm*

$$|H|_\rho = \sum_{(\nu,\alpha)\in I} |H_{\nu,\alpha}| \rho^{|\alpha|} e^{\rho \|W\nu\|} \tag{1.2}$$

is finite. The identity operator on $\mathcal{A}_\rho$ will be denoted by $\mathbb{I}$. On the product space $\mathcal{A}_\rho^d$ we define the two norms $|f|_\rho = \sum_j |f_j|_\rho$ and $\|f\|_\rho = \max_j |f_j|_\rho$. Let $\mathcal{A}'_\rho$ be the space of all functions in $\mathcal{A}_\rho$, whose first partial derivatives belong to $\mathcal{A}_\rho$. On this space, we consider the following two seminorms: Denoting by $\nabla_j H$ the partial gradient of H with respect to the j–th argument, $j = 1, 2$, we define $|H|'_\rho$ and $\|H\|'_\rho$ to be the sum and maximum, respectively, of the numbers $\|W\nabla_1 H\|_\rho$ and $|V\nabla_2 H|_\rho$.

Our first result is concerned with the possibility of finding a canonical change of variables that eliminates the component of a Hamiltonian in the direction of some predefined subspace of $\mathcal{A}_\rho$. The coordinate changes are restricted to those canonical transformations $U : (q,p) \mapsto (q+Q, p+P)$ for which the one–form $P \cdot dq + p \cdot dQ$ is not just closed, but the differential of a function S. The function $\phi = p \cdot Q - S$, expressed in terms of q and $p + P$ (if possible), satisfies the equation

$$\big(Q(q,p), P(q,p)\big) = (\mathbb{J}\nabla\phi)\big(q, p + P(q,p)\big) , \tag{1.3}$$

where $\mathbb{J}(q,p)=(p,-q)$. Conversely, given $\phi\in\mathcal{A}_\rho$ sufficiently small, we can use (1.3) to define a canonical transformation $U=U_\phi$. Denote by $\widehat{H}$ the Hamiltonian vector field associated with a Hamiltonian H, that is, $\widehat{H}f=(\mathbb{J}\nabla H)\cdot\nabla f$.

Theorem 1.2. *Let $\rho,c>0$, and let $\mathcal{H}$ be some non–empty open set of Hamiltonians $H\in\mathcal{A}_\rho$ satisfying $|H|_\rho<c$. Let $\mathbb{I}^-$ be a projection operator on $\mathcal{A}_{\rho'}$, where $0<\rho'<\rho$, and assume that $\mathbb{I}^-$ satisfies the following non–resonance condition:*
There are constants $a,b>0$, such that for all values of r in the interval $[\rho',(\rho'+8\rho)/9]$,

(a) if $f\in\mathcal{A}_r$, then $\mathbb{I}^-f\in\mathcal{A}_r$ and $|\mathbb{I}^-f|_r\le a|f|_r$;

(b) $\bigl|\mathbb{I}^-H\bigr|_\rho<ab^2c(1-\rho'/\rho)^2/1231$, for all $H\in\mathcal{H}$;

(c) for every $H\in\mathcal{H}$, $\mathbb{I}^-\widehat{H}$ maps $\mathbb{I}^-\mathcal{A}_r'$ onto $\mathbb{I}^-\mathcal{A}_r$, and if $\phi\in\mathbb{I}^-\mathcal{A}_r'$ is nonzero, then $\bigl|\mathbb{I}^-\widehat{H}\phi\bigr|_r>abc\rho^{-1}\|\phi\|_r'$.

Then there exists a map $\mathcal{U}$ that assigns to each $H\in\mathcal{H}$ a canonical transformation $\mathcal{U}_H$ from $\mathcal{D}_{\rho'}$ to $\mathcal{D}_{(\rho'+\rho)/2}$, such that

$$H\circ\mathcal{U}_H\in\mathcal{A}_{\rho'}\,,\qquad \mathbb{I}^-\bigl(H\circ\mathcal{U}_H\bigr)=0\,. \tag{1.4}$$

The map $\mathcal{N}:H\mapsto H\circ\mathcal{U}_H$ is analytic from $\mathcal{H}$ to $\mathcal{A}_{\rho'}$. Furthermore, if $H\in\mathcal{H}$ satisfies $\mathbb{I}^-H=0$, then $\mathcal{U}_H=\mathrm{I}$ and $D\mathcal{N}(H)=\mathbb{I}^+-\mathbb{I}^+\widehat{H}\bigl(\mathbb{I}^-\widehat{H}\mathbb{I}^-\bigr)^{-1}\mathbb{I}^-$, where $\mathbb{I}^+=\mathbb{I}-\mathbb{I}^-$.

This theorem was proved in [11], in the special case where $\mathcal{H}$ is a small neighborhood of a linear Hamiltonian $(q,p)\mapsto\omega\cdot p$, and $\mathbb{I}^-$ is a specific projection adapted to the choice of the vector ω. A similar projection will be considered again here. Its usefulness for renormalization derives from a combination of Theorem 1.2, and Lemma 1.3 below.

Given a nonzero vector $\omega\in\mathbb{R}^d$, and two positive constants σ and κ, let

$$I^-=\bigl\{(\nu,\alpha)\in I:|\omega\cdot\nu|>\sigma\|W\nu\|+\kappa|\alpha|\bigr\}\,,\qquad I^+=I-I^-\,, \tag{1.5}$$

and define $\mathbb{I}^\pm H$ by restricting the sum in (1.1) to the corresponding index sets $I^\pm$,

$$\bigl(\mathbb{I}^\pm H\bigr)(q,p)=\sum_{(\nu,\alpha)\in I^\pm}H_{\nu,\alpha}(Wp)^\alpha e^{iq\cdot\nu}\,. \tag{1.6}$$

The functions $\mathbb{I}^-H$ and $\mathbb{I}^+H$ will be referred to as the non–resonant and resonant part, respectively, of H. Clearly, the projection $\mathbb{I}^-$ satisfies the condition (a) of Theorem 1.2, with $a=1$. Notice that if $H(q,p)$ depends on p only, then $\mathbb{I}^-H=0$.

For the purpose of renormalization, we now focus on vectors $\omega=(1,\omega_2,\ldots,\omega_d)$ whose components span a real algebraic number field of degree d. In particular, $\omega\cdot\nu\neq0$ for every nonzero vector ν in $\mathbb{Z}^d$. Another consequence of this assumption [11] is that there exists an integral $d\times d$ matrix T such that

(T1) T has a simple real eigenvalue $\vartheta > 1$, and $T\omega = \vartheta\omega$.
(T2) All other eigenvalues of T are simple, and of modulus less than 1.
(T3) $\det(T) = \pm 1$.

Such a matrix T provides a way of approximating ω by vectors with rational components: If $w \in \mathbb{Q}^d$ is nonzero, then the vector $T^n w$, when rescaled such that its first component is one, approaches ω as $n \to \infty$. The same approximating sequences can also be found in some Hamiltonian systems [1–10], in the form of periodic orbits that accumulate at an invariant ω–torus. In order to investigate this Hamiltonian "representation" of the arithmetic related to ω, we "lift" the inverse of T, viewed as a map on frequency vectors, to a transformation acting on a space of Hamiltonians. A transformation that has some of the required properties is $H \mapsto \mu^{-1} H \circ T_\mu$, where

$$T_\mu(q,p) = \bigl(Tq, \mu(T^*)^{-1}p\bigr)\,, \qquad \mu \neq 0\,. \tag{1.7}$$

It combines a canonical change of variables (case $\mu = 1$) with a scaling in p, and is part of most RG schemes for Hamiltonians [10–17]. By itself, this transformation is not a dynamical system on any of the spaces $\mathcal{A}_\rho$, since the domain $\mathcal{D}_\rho$ is not left invariant by T_μ. But we can combine it with Theorem 1.2: The following lemma shows that a canonical change of variables, that eliminates the non–resonant part of a Hamiltonian, can "transfer analyticity" from the variable p to the variable q.

After fixing an integral $d \times d$ matrix T satisfying (T1–T3), we adapt the matrices W and V to our choice of T, by assuming that the row vectors $W_1, W_2, \ldots, W_d$ of W are an ordered basis of eigenvectors for T,

$$TW_j = \vartheta_j W_j\,, \qquad |\vartheta_j| \le |\vartheta_i|\,, \qquad 1 \le i \le j \le d\,, \tag{1.8}$$

with $W_1 = \omega$. In addition we fix the parameters $\sigma, \kappa > 0$, with σ restricted by the condition $|\vartheta_2| + \sigma(\vartheta - |\vartheta_2|) < 1$.

Lemma 1.3. *Let $0 < \rho' < \rho$ and $\mu \in \mathbb{C}$ be given such that*

$$|\vartheta_2| + \sigma\bigl(\vartheta - |\vartheta_2|\bigr) < \frac{\rho'}{\rho}\,, \qquad 0 < \left|\frac{\mu}{\vartheta_d}\right| e^{\rho\kappa(\vartheta - |\vartheta_2|)} < \frac{\rho'}{\rho}\,. \tag{1.9}$$

Then every function $H \in \mathbb{I}^+\mathcal{A}_{\rho'}$ extends analytically to $T_\mu\mathcal{D}_\rho$, and $H \mapsto H \circ T_\mu$ is a compact linear map from $\mathbb{I}^+\mathcal{A}_{\rho'}$ to $\mathcal{A}_\rho$, whose operator norm is ≤ 1.

Let now $\mathcal{H}$ be some subset of $\mathcal{A}_\rho$ for which the given projection $\mathbb{I}^-$ satisfies a non–resonance condition, as described in Theorem 1.2.

Definition 1.4. *Given a nonzero complex number μ of modulus less than $|\vartheta_d|$, define*

$$\mathcal{R}_\mu(H) = \frac{\eta}{\mu} H \circ \mathcal{U}_H \circ T_\mu\,, \qquad H \in \mathcal{H}\,, \tag{1.10}$$

where $\eta = \eta(H)$ is determined such that the coefficient $\widetilde{H}_{0,(1,0,\ldots,0)}$ of the renormalized Hamiltonian $\widetilde{H} = \mathcal{R}_\mu(H)$ is equal to one, if this is possible.

The action of this transformation is particularly simple when restricted to functions of the form

$$h_w(q,p) = w \cdot p\,, \qquad w \in \mathbb{C}^d\,. \tag{1.11}$$

In particular, it is independent of the choice of μ, since h_w is invariant under the scaling $\mathcal{S}_z : H \mapsto z^{-1}H(.,z.)$, for any $z \neq 0$. More explicitly, if $w = \beta_1\omega + \beta_2 W_2 + \ldots + \beta_d W_d$, with β_1 nonzero, then $\mathcal{R}_\mu(h_w) = h_{w'}$, where $w' = \eta T^{-1}w$ and $\eta = \vartheta/\beta_1$. This shows e.g. that h_ω is a (trivial) fixed point of $\mathcal{R}_\mu$. For particular choices of the scaling parameter μ, there are other trivial fixed points as well. Such fixed points can be found easily by restricting $\mathcal{R}_\mu$ to the space of Hamiltonians $H \in \mathcal{A}_\rho$ for which $\nabla_1 H = 0$, and thus $\mathcal{U}_H = \mathrm{I}$.

Theorem 1.2 and Lemma 1.3 can be combined as follows. (Additional symmetries of $\mathcal{R}_\mu$ are mentioned at the end of Section 3.)

Theorem 1.5. *Let $0 < \rho < \sigma/\kappa$. If $\rho' < \rho$ is sufficiently close to ρ, and $\mu \in \mathbb{C}$ satisfies (1.9), then there exists an open neighborhood $\mathcal{H}$ of $\{H \in \mathbb{I}^+\mathcal{A}'_s : H_{0,0} = 0,\ |H - h_\omega|'_s < \kappa\rho\}$ in $\mathcal{A}_\rho$, where $s = (\rho' + 8\rho)/9$, such that the transformation $\mathcal{R}_\mu$ is well defined, analytic, and compact, as a map from $\mathcal{H}$ to $\mathcal{A}_\rho$. The same holds for $\mathcal{R}_{z\mu} = \mathcal{S}_z \circ \mathcal{R}_\mu$, for all z in some open neighborhood $Z \subset \mathbb{C}$ of the unit circle, and $\mathcal{S}_z \circ \mathcal{R}_\mu = \mathcal{R}_\mu \circ \mathcal{S}_z$, for all $z \in Z$.*

This theorem is obtained by verifying conditions (b) and (c) of Theorem 1.2 for the given domain $\mathcal{H}$. For simplicity, we have chosen $b > 0$ close to zero. This may not be sufficient to be used in the construction of a nontrivial RG fixed point, although the approximate fixed point found in [10], for the golden mean case $T = \left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]$, does have a relatively small non–resonant part. For such a task, it is likely that all parameters (including σ, κ, ρ, ρ') will have to be fine–tuned simultaneously, and a sharper version of Theorem 1.2 may be needed as well.

As mentioned earlier, $\mathcal{R}_\mu$ acts trivially on Hamiltonians that only depend on the action variable p. This makes it possible to compute all eigenvalues and eigenvectors of the derivative $D\mathcal{R}_\mu(h_\omega)$ of $\mathcal{R}_\mu$ at the fixed point h_ω. The eigenvectors are precisely the monomials $(q,p) \mapsto (Wp)^\alpha$, and the point 0 in the spectrum of $D\mathcal{R}_\mu(h_\omega)$ corresponds to functions with torus–average zero; see [11] for details.

In what follows, ρ is a fixed positive real number less than $\sigma\kappa$, and the scaling parameter μ is assumed to be real, satisfying

$$0 < \mu < \left|\frac{\vartheta_d^2}{\vartheta_1}\right| . \tag{1.12}$$

In this case, h_ω is an isolated fixed point of $\mathcal{R}_\mu$, and $D\mathcal{R}_\mu(h_\omega)$ has precisely d eigenvalues outside the open unit disk. One of them is $\lambda_0 = \vartheta/\mu$, associated with constant Hamiltonians, and the other $d-1$ eigenvalue–eigenvector pairs are

$$\lambda_j = \vartheta_1/\vartheta_j\,, \qquad h_{W_j}(q,p) = W_j \cdot p\,, \qquad j = 2,\ldots,d. \tag{1.13}$$

The corresponding local unstable manifold $\mathcal{W}^u$ of h_ω is simply the d–dimensional affine subspace of $\mathcal{A}_\rho$ that is tangent to the expanding eigenspace at h_ω.

As usual in the theory of renormalization, the transformation $\mathcal{R}_\mu$ is merely a tool for constructing and analyzing certain objects that are of interest outside this theory. We start with a discussion of the local stable manifold $\mathcal{W}^s$ of $\mathcal{R}_\mu$ at the fixed point h_ω.

By definition, if H lies on $\mathcal{W}^s$, then the sequence of Hamiltonians $H_n = \mathcal{R}_\mu^n(H)$ converges to h_ω, as $n \to \infty$. This fact can be used to define a sequence of canonical transformations

$$\mathcal{V}_n(H) = V_0(H) \circ V_1(H) \circ \ldots \circ V_{n-1}(H)\,, \qquad V_k(H) = T_\mu^k \circ \mathcal{U}_{H_k} \circ T_\mu^{-k}\,. \tag{1.14}$$

Formally, $H \circ \mathcal{V}_n(H)$ approaches a constant multiple of h_ω, as n tends to infinity. But we cannot expect convergence on an open subset of phase space, unless H is integrable. One of the things that can be extracted from the transformations $\mathcal{V}_n(H)$ is the limit of $\mathcal{V}_n(H) \circ \Upsilon$, as $n \to \infty$, where $\Upsilon(q) = (q, 0)$. This limit yields the function Γ_H described below.

Theorem 1.6. *Given two positive real numbers r' and $r > r' + \rho$, there exists an open neighborhood B of h_ω in $\mathcal{A}_r$, and for every $H \in B$ a complex number c_H and an analytic function $\Gamma_H: \mathcal{D}_{r',1} \to \mathcal{D}_r$, such that the following holds. If $H \in \mathcal{W}^s \cap B$ then*

$$\big(\mathbb{J}\nabla H\big) \circ \Gamma_H = c_H\, \omega \cdot \nabla\Gamma_H \qquad \text{on } \mathcal{D}_{r',1}\,. \tag{1.15}$$

For $H = h_\omega$, the values of c_H and Γ_H are 1 and Υ, respectively. Furthermore, $H \mapsto c_H$ is an analytic function on B, and $H \mapsto \Gamma_H - \Upsilon$ is an analytic map from B to some Banach space of analytic 2π-periodic functions on $\mathcal{D}_{r',1}$.

This theorem is essentially identical to [11, Theorem 1.7]. Thus, we will not repeat a proof here. Notice that (1.15) is the equation of an invariant d–torus for H, with rotation vector proportional to ω; or more precisely (if c_H is not real), the equation of an invariant d–torus for $c_H^{-1}H$, with rotation vector ω. By construction, this torus lies on the energy surface $H^{-1}(0)$. Another property of this torus is that it is "centered at $p = 0$", in the sense that the integral

$$K(\gamma) = \frac{1}{2\pi} \oint_\gamma p \cdot dq \tag{1.16}$$

vanishes, if γ is any closed curve on the torus. This follows from the fact that Γ_H is the limit of canonical transformations $\mathcal{V}_n(H)$, that leave $p \cdot dq$ invariant up to a differential of a one–form.

The assumption $H \in \mathcal{W}^s$, used to prove (1.15), replaces the non–degeneracy assumption in traditional KAM theory. To discuss the connection between these two conditions, we consider families $\beta \mapsto H_\beta$,

$$H_\beta = H \circ R_\beta\,, \qquad R_\beta(q,p) = (q, p + \beta)\,, \tag{1.17}$$

generated from a fixed Hamiltonian H by a translation in the action variable p. To be more specific, let $r > \rho$, and consider a Hamiltonian $h \in \mathcal{A}_r$ of the form

$$h(q,p) = \omega \cdot p + \tfrac{1}{2}\, p \cdot Mp + f(p)\,, \qquad f(p) = \mathcal{O}(|p|^3)\,, \tag{1.18}$$

where M is a real symmetric $d \times d$ matrix, such that the quadratic form $p \mapsto p \cdot Mp$ is non–degenerate, when restricted to the $d - 1$ dimensional contracting subspace of T^*. It is straightforward to check that if h is sufficiently close to h_ω, then the family $\beta \mapsto H_\beta$, for

$H = h$, intersects the stable manifold $\mathcal{W}^s$ transversally. Since this property persists under small perturbations, every Hamiltonian $H \in \mathcal{A}_r$ near h has an invariant $c\omega$–torus on the energy surface $H^{-1}(0)$. The torus is given by $\Gamma' = R_{\beta'} \circ \Gamma_{H_{\beta'}}$, where β' is the value of the parameter β for which H_β belongs to $\mathcal{W}^s$.

Under suitable assumptions on H, the renormalization transformation $\mathcal{R}_\mu$ can also be used to construct periodic orbits for H, whose rotation vectors are "rational approximants" for $c\omega$. Recall that a curve $\gamma : \mathbb{R} \to \mathcal{D}_\rho$ is a (lifted) orbit for H if it satisfies the first order differential equation $\gamma' = (\mathbb{J}\nabla H) \circ \gamma$. A periodic orbit that closes (modulo $2\pi\mathbb{Z}^d$ in the variable q) after a time $2\pi\tau > 0$, but not earlier, can be parametrized as follows:

$$\gamma(t) = \bigl(tw + Q_0 + Q(t/\tau)\,,\, P_0 + P(t/\tau)\bigr)\,, \tag{1.19}$$

where $w = (\gamma(2\pi\tau) - \gamma(0))/(2\pi\tau)$, and where Q, P are periodic functions with fundamental period 2π and average zero. The rotation vector w belongs to $\mathbb{R}\mathbb{Z}^d$, that is, w is a real scalar multiple of some vector in $\mathbb{Z}^d$. In what follows, given a nonzero vector $w \in \mathbb{R}\mathbb{Z}^d$, we denote by $\tau(w)$, or τ for short, the value of the smallest positive real number t such that tw belongs to $\mathbb{Z}^d$.

In order to construct periodic orbits that depend continuously on H, near the Hamiltonian (1.18) that is invariant under translations $q \mapsto q + u$, we will need to limit the number of ways this symmetry can be broken by a perturbation. We shall do this by restricting to Hamiltonians $H(q, p)$ that are even functions of q — a property that is preserved under the transformation $\mathcal{R}_\mu$. The corresponding "even" subspace of $\mathcal{A}_r$, $r > 0$, will be denoted by $\mathcal{B}_r$. We note that the approximate RG fixed point of [10] lies in such a space $\mathcal{B}_r$.

Let ρ' and ε be fixed positive real numbers (to be chosen below).

Definition 1.7. *For every nonzero vector $w \in \mathbb{R}\mathbb{Z}^d$, we define $\mathcal{H}(w)$ to be the set of all Hamiltonians $H \in \mathcal{B}_\rho$, such that a constant multiple of H has a periodic orbit γ with rotation vector w, on the surface of constant energy zero, with $K(\gamma) = 0$. The functions Q and P in (1.19) are assumed to be 2π–periodic, and to have average zero. In addition, we require P to be even, Q odd, and $Q_0 = 0$.*
A subset $\Sigma(w)$ of $\mathcal{H}(w)$ is defined by requiring also that $P_0 = kV^*Vw$, with $|k| < \varepsilon^2/\tau$, and that Q, P extend analytically to the strip $|\mathrm{Im}z| < \rho'/\tau$, satisfying the bounds $|VQ(z)| < \varepsilon$ and $\|WP(z)\| < \varepsilon$.

Theorem 1.8. *There exist $\rho', \varepsilon > 0$ such that the following holds. If $w \in \mathbb{R}\mathbb{Z}^d$ is sufficiently close to ω, and h_w lies on $\mathcal{W}^u$, then there exists an open neighborhood $B(w)$ of h_w in $\mathcal{B}_\rho$, such that $\Sigma(w) \cap B(w)$ is an analytic manifold of codimension d that intersects $\mathcal{W}^u$ transversally at h_w .*

Consider now such a rotation vector $w \in \mathbb{R}\mathbb{Z}^d$, and let $\Sigma_n(w) = \mathcal{R}_\mu^{-n}(\Sigma(w) \cap B(w))$, for all $n \ge 0$. By the λ–Lemma [25,26], this defines a sequence of codimension d manifolds that accumulate at $\mathcal{W}^s$ as follows (see also Section 4). Assume that $\beta \mapsto H_\beta$ is an analytic d–parameter family of Hamiltonians in the domain of $\mathcal{R}_\mu$, that intersect $\mathcal{W}^s$ transversally at $\beta = \beta'$. Then there exists an open neighborhood B' of β' in $\mathbb{C}^d$, such that for sufficiently large n, the set $\{\beta \in B' : H_\beta \in \Sigma_n(w)\}$ contains a single point, say β_n , and the ratio $|\beta_n - \beta'|/|\beta_{n+1} - \beta'|$ converges to $|\lambda_2|$, as $n \to \infty$.

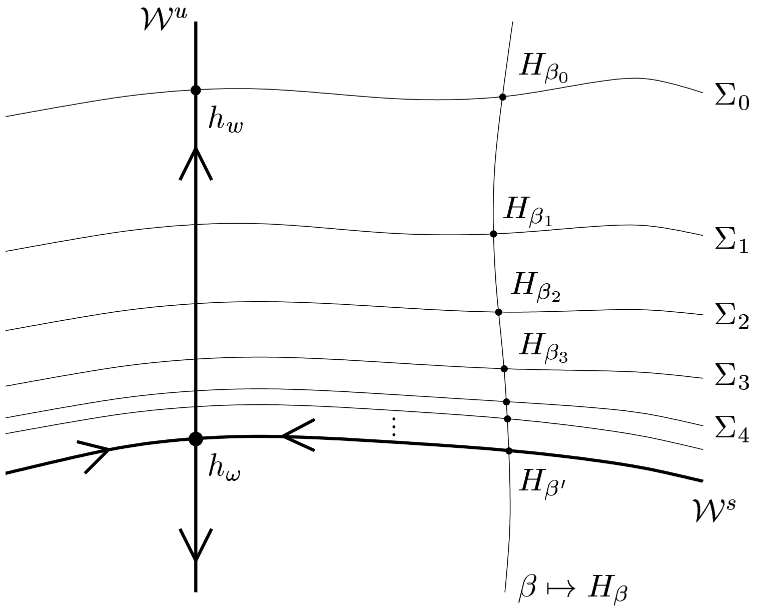

Figure 1. Accumulation of the hypersurfaces Σ_n at the stable manifold $\mathcal{W}^s$

The reason for considering the manifolds $\Sigma_n(w)$ is the fact that $\Sigma_n(w) \subset \mathcal{H}(T^n w)$. Thus, by considering families of the type (1.17), we can construct infinite sequences of periodic orbits for a single Hamiltonian H. As part of the proof of the theorem below, and under its assumptions, we will show that

$$-\frac{1}{n}\ln\bigl|\beta_n - \beta'\bigr| = \ln|\lambda_2| + \mathcal{O}\bigl(\tfrac{1}{n}\bigr)\,. \tag{1.20}$$

Theorem 1.9. *Let $r > \rho$ and $w \in \mathbb{R}\mathbb{Z}^d$ be given, $w \neq 0$. Let $h \in \mathcal{B}_r$ be a Hamiltonian of the form (1.18), with M as described after (1.18). If h is sufficiently close to h_ω in $\mathcal{B}_r$, then there exists an open neighborhood B of h in $\mathcal{B}_r$, and a positive integer N, such that for every Hamiltonian $H \in B$, and every $n \geq N$, some constant multiple of H has a periodic orbit γ_n with frequency vector $w_n = (\vartheta^{-1}T)^n w$, lying on the energy surface $H^{-1}(0)$, and satisfying*

$$-\frac{1}{n}\ln\bigl|\gamma_n(0) - \Gamma'(0)\bigr| = \ln|\lambda_2| + \mathcal{O}\bigl(\tfrac{1}{n}\bigr)\,, \tag{1.21}$$

where Γ' is the invariant torus described after (1.18).

Our reason for using $K(\gamma) = 0$ as one of the conditions in the definition of $\Sigma(w)$, is the resulting identity

$$\frac{1}{2\pi\tau(w_n)}\oint_{\gamma_n} p\cdot dq = w_n\cdot\beta_n\,, \tag{1.22}$$

valid for large n, if $w \in \mathbb{R}\mathbb{Z}^d$ is chosen sufficiently close to ω. The normalized integral in this equation can be regarded as the coordinate of γ_n in the direction of w_n, since it changes by an amount $w_n \cdot v$ under a translation $p \mapsto p+v$. It appears that in this direction, the orbits γ_n accumulate faster than in some other directions: A straightforward calculation shows that if $\nabla_1 H = 0$, then the difference $w_n \cdot \beta_n - \omega \cdot \beta'$ is of the order $|\lambda_2|^{-2n}$. The same might be true more generally, as K is the functional that appears in the variational equation for orbits on fixed energy surfaces.

2. Eliminating Non–Resonant Modes

Our goal in this section is to prove Theorem 1.2, concerning the existence of a canonical change of coordinates that eliminates the component of a Hamiltonian in the direction of a given "non–resonant" subspace of $\mathcal{A}_\rho$. We start by giving some basic estimates involving the evaluation, multiplication, differentiation, and composition of functions in the spaces $\mathcal{A}_\rho$.

Proposition 2.1. *Let $\rho, \delta > 0$. Consider $f, g \in \mathcal{A}_\rho$, and $P, Q \in \mathcal{A}_\rho^d$, and $h \in \mathcal{A}_{\rho+\delta}$. Define $U(q,p) = \bigl(q + Q(q,p), P(q,p)\bigr)$, for all (q,p) in $\mathcal{D}_\rho$. Then*

(a) $|f(q,p)| \le |f|_\rho$ *for all (q,p) in $\mathcal{D}_\rho$.*

(b) $fg \in \mathcal{A}_\rho$ *and* $|fg|_\rho \le |f|_\rho |g|_\rho$.

(c) $|h|_\rho + \delta |h|'_\rho \le |h|_{\rho+\delta}$.

(d) $h \circ U \in \mathcal{A}_\rho$ *and* $|h \circ U|_\rho \le |h|_{\rho+\delta}$, *if* $|VQ|_\rho \le \delta$ *and* $\|WP\|_\rho \le \rho + \delta$.

The proof of these estimates is straightforward and will be omitted.

Define $\{H, \phi\} = \nabla_1 H \cdot \nabla_2 \phi - \nabla_2 H \cdot \nabla_1 \phi$.

Proposition 2.2. *Let $r, \delta > 0$ and $0 < \varepsilon < \frac{1}{2}$. Denote by B' the set of all functions $\phi \in \mathcal{A}'_r$ that satisfy $\|\phi\|'_{r+2\delta} < \varepsilon\delta$. Then for every function $\phi \in B'$, the equation (1.3) has a unique solution $Q, P \in \mathcal{A}_r^d$ satisfying $\|WP\|_r \le \delta$. The corresponding canonical transformation $U_\phi \colon (q,p) \mapsto \bigl(q + Q(q,p), p + P(q,p)\bigr)$ is analytic from $\mathcal{D}_r$ to $\mathcal{D}_{r+2\delta}$. If H is any function in $\mathcal{A}_{r+2\delta}$, then $H \circ U_\phi$ belongs to $\mathcal{A}_r$, and*

$$
\begin{aligned}
\bigl|H \circ U_\phi\bigr|_r &\le |H|_{r+2\delta}\,,\\
\bigl|H \circ U_\phi - H\bigr|_r &\le \frac{2}{3}\varepsilon |H|_{r+2\delta}\,,\\
\bigl|H \circ U_\phi - H - \{H,\phi\}\bigr|_r &\le \frac{1}{3}\varepsilon^2 |H|_{r+2\delta}\,.
\end{aligned}
\tag{2.1}
$$

Furthermore, the maps $\phi \mapsto (Q,P)$ and $\phi \mapsto H \circ U_\phi$ are analytic on B'.

Proof. Denote by B the set of all $P \in \mathcal{A}_r^d$ satisfying $\|WP\|_r \le \delta$. Let $\phi \in B'$, and define a map $F : B \to \mathcal{A}_r^d$ by setting $F(P) = -(\nabla_1 \phi) \circ G$, where $G(q,p) = (q, p + P(q,p))$. If

$P \in B$ then, by using Proposition 2.1, we obtain

$$\begin{aligned}
\|WDF(P)h\|_r &= \max_i \bigl| h \cdot \bigl(\nabla_2 (W\nabla_1\phi)_i\bigr) \circ G \bigr|_r \\
&\le \max_i \|Wh\|_r \bigl| V\bigl(\nabla_2(W\nabla_1\phi)_i\bigr) \circ G \bigr|_r \\
&\le \max_i \|Wh\|_r \bigl| V\nabla_2(W\nabla_1\phi)_i \bigr|_{r+\delta} \\
&\le \max_i \|Wh\|_r \delta^{-1} \bigl| (W\nabla_1\phi)_i \bigr|_{r+2\delta} \\
&= \|Wh\|_r \delta_2^{-1} \|W\nabla_1\phi\|_{r+2\delta} \le \|Wh\|_r \,,
\end{aligned} \tag{2.2}$$

for all $h \in \mathcal{A}_r^d$. This, together with the bound $\|WF(0)\|_r \le \delta/2$, shows that F is a contraction on B, for the norm $\|W.\|_r$. Thus, equation (1.3) has a unique solution $(Q, P) \in \mathcal{A}_r^d \times B$. By Proposition 2.1, this solution satisfies

$$\begin{aligned}
|VQ|_r &= \bigl| (V\nabla_2\phi) \circ G \bigr|_r \le |V\nabla_2\phi|_{r+\delta} < \varepsilon\delta \,, \\
\|WP\|_r &= \|(W\nabla_1\phi) \circ G\|_r \le \|W\nabla_1\phi\|_{r+\delta} < \varepsilon\delta \,.
\end{aligned} \tag{2.3}$$

Consider now $H \in \mathcal{A}_{r+2\delta}$. In order to prove (2.1), let

$$\bigl(f(z)\bigr)(q,p) = H\Bigl(q + z\nabla_2\phi\bigl(q, p + zP(q,p)\bigr)\,,\ p - z\nabla_1\phi\bigl(q, p + zP(q,p)\bigr)\Bigr)\,. \tag{2.4}$$

From the bounds (2.3) and Proposition 2.1, it follows that this equation defines a function f, from an open neighborhood of the disk $|z| \le 2/\varepsilon$ to $\mathcal{A}_r$, satisfying $|f(z)|_r \le |H|_{r+2\delta}$. By using the representation

$$f(s) = f(0) + sf'(0) + \frac{1}{2\pi i} \oint\limits_{|z|=2/\varepsilon} \frac{f(z)}{z-s} \Bigl(\frac{s}{z}\Bigr)^2 dz \,, \tag{2.5}$$

we obtain the bound

$$\begin{aligned}
\bigl| H \circ U_\phi - H - \{H, \phi\} \bigr|_r &= \bigl| f(1) - f(0) - f'(0) \bigr|_r \\
&\le \left| \frac{1}{2\pi i} \oint\limits_{|z|=2/\varepsilon} \frac{f(z)dz}{(z-1)z^2} \right|_r \le \frac{1}{3}\varepsilon^2 |H|_{r+2\delta} \,.
\end{aligned} \tag{2.6}$$

The first two inequalities in (2.1) are proved similarly. The analyticity of the maps that assign to $\phi \in B'$ the functions $P, Q \in \mathcal{A}_r^d$ and $H \circ U_H \in \mathcal{A}_r$, follows by the implicit function theorem and the chain rule.

■

Proof of Theorem 1.2.

We start with an informal description of the proof. Consider $H_0 = H \in \mathcal{H}$. Our goal is to define functions $\phi_0, \phi_1, \phi_2, \ldots$ such that if we set

$$G_n = U_{\phi_0} \circ U_{\phi_1} \circ \ldots \circ U_{\phi_{n-1}} - \mathrm{I}\,, \qquad H_n = H \circ (\mathrm{I} + G_n)\,, \qquad n = 1, 2, \ldots\,, \tag{2.7}$$

then $G_n \to G_\infty$ and $H_n \to H_\infty = H \circ (\mathrm{I} + G_\infty)$ as $n \to \infty$, with $\mathbb{I}^- H_\infty = 0$. Starting with $n = 0$, we define $\phi_n = \mathbb{I}^- \phi_n$ to be the solution of the equation

$$\mathbb{I}^-\{H_n, \phi_n\} = -\mathbb{I}^- H_n\,. \tag{2.8}$$

If $\mathbb{I}^- H_n$ is small, say of the order ε_n , then the same should be true for ϕ_n . By using that $H_{n+1} = H_n \circ U_{\phi_n}$, and thus

$$\mathbb{I}^- H_{n+1} = \mathbb{I}^-\bigl(H_n \circ U_{\phi_n} - H_n - \{H_n, \phi_n\}\bigr)\,, \tag{2.9}$$

we see from equation (2.1) that $\mathbb{I}^- H_{n+1}$ is of the order $\varepsilon_{n+1} \approx \varepsilon_n^2$. Now the process is repeated for $n = 1, 2, \ldots$.

In order to be more precise, assume now that $\mathcal{H}$ and $\mathbb{I}^-$ satisfy the assumptions of Theorem 1.2. Let $t_0 = (1 - \rho'/\rho)/9$ and $\rho_0 = \rho$. Define

$$t_n = (2/3)^n t_0\,, \quad \delta_n = t_n \rho\,, \quad \rho_n = \rho' + 9\delta_n\,, \qquad n = 1, 2, \ldots\,, \tag{2.10}$$

so that $\rho_{n+1} = \rho_n - 3\delta_n$, for all $n \ge 0$. Given $H \in \mathcal{H}$, we intend to verify inductively that for all $m > 0$, the function H_m belongs to $\mathcal{A}_{\rho_m}$ and satisfies the bounds

$$\begin{aligned} |H_m|_{\rho_m} &< c\,, \\ |H_m - H_{m-1}|_{\rho_m} &< \frac{c}{4} t_0 b (2/3)^{2(3/2)^m}\,, \\ |\mathbb{I}^- H_m|_{\rho_m} &< \frac{ac}{3}(t_0 b)^2 (2/3)^{4(3/2)^m}\,. \end{aligned} \tag{2.11}$$

By assumption, these bounds hold for $m = 0$, if we set $H_{-1} = 0$.

Let now $n \ge 0$ be fixed, and assume that (2.11) has been verified for all $m \le n$. We start by showing that equation (2.8) can be solved. By using Proposition 2.1.c we obtain

$$\begin{aligned} |H_n - H|'_{\rho_n - \delta_n} &\le \sum_{m=1}^{n} |H_m - H_{m-1}|'_{\rho_m - \delta_m} \le \sum_{m=1}^{n} \delta_m^{-1} |H_m - H_{m-1}|_{\rho_m} \\ &< \sum_{m=1}^{n} \frac{bc}{4\rho} (3/2)^n (2/3)^{2(3/2)^m} < \frac{bc}{2\rho}\,. \end{aligned} \tag{2.12}$$

Thus, if ϕ belongs to $\mathbb{I}^- \mathcal{A}'_{\rho_n - \delta_n}$, then

$$\bigl|\mathbb{I}^-\{H_n - H, \phi\}\bigr|_{\rho_n - \delta_n} \le a |H_n - H|'_{\rho_n - \delta_n} \|\phi\|'_{\rho_n - \delta_n} \le \frac{abc}{2\rho} \|\phi\|'_{\rho_n - \delta_n}\,. \tag{2.13}$$

This, together with the assumption (c) of Theorem 1.2, implies that $\mathbb{I}^-\widehat{H_n}$ maps $\mathbb{I}^-\mathcal{A}'_{\rho_n-\delta_n}$ onto $\mathbb{I}^-\mathcal{A}_{\rho_n-\delta_n}$, and that

$$\bigl|\mathbb{I}^-\widehat{H_n}\phi\bigr|_{\rho_n-\delta_n} \ge \bigl|\mathbb{I}^-\widehat{H}\phi\bigr|_{\rho_n-\delta_n} - \bigl|\mathbb{I}^-\{H_n - H, \phi\}\bigr|_{\rho_n-\delta_n} \ge \frac{abc}{2\rho}\|\phi\|'_{\rho_n-\delta_n}\,, \tag{2.14}$$

for all $\phi \in \mathbb{I}^-\mathcal{A}'_{\rho_n-\delta_n}$. Consequently, equation (2.8) has a unique solution $\phi_n \in \mathbb{I}^-\mathcal{A}'_{\rho_n-\delta_n}$, and this solution satisfies the bound

$$\|\phi_n\|'_{\rho_n-\delta_n} \le \frac{2\rho}{abc}|\mathbb{I}^-H_n|_{\rho_n-\delta_n} < \frac{2\rho}{3}t_0^2 b(2/3)^{4(3/2)^n} \le \delta_n\varepsilon_n\,, \tag{2.15}$$

where

$$\varepsilon_n = \frac{2}{3}t_0 b(2/3)^{3(3/2)^n}\,. \tag{2.16}$$

In fact, the solution can be obtained by using a convergent Neumann series (for the operator being inverted) that is dominated in norm by a geometric series with ratio 1/2. We note that $t_0 b \le 1$, and thus $\varepsilon_n < 1/2$. This follows from the fact that

$$\begin{aligned} abc\|\phi\|'_{\rho_0-\delta_0} &\le \rho|\mathbb{I}^-\{H,\phi\}|_{\rho_0-\delta_0} \\ &\le \rho a|H|'_{\rho_0-\delta_0}\|\phi\|'_{\rho_0-\delta_0} \le a t_0^{-1} c\|\phi\|'_{\rho_0-\delta_0}\,, \end{aligned} \tag{2.17}$$

for all $\phi \in \mathbb{I}^-\mathcal{A}'_{\rho_0-\delta_0}$. Now we can use Proposition 2.2 to prove the three bounds (2.11) for $m = n+1$. The first of these bounds is straightforward. For the second one we have

$$\begin{aligned} |H_{n+1} - H_n|_{\rho_n-3\delta_n} &\le \frac{2}{3}\varepsilon_n|H_n|_{\rho_n-\delta_n} < (2/3)^2 c t_0 b(2/3)^{3(3/2)^n} \\ &\le (2/3)^{7/2} c t_0 b(2/3)^{(3/2)^{n+1}} < \frac{c}{4}t_0 b(2/3)^{(3/2)^{n+1}}\,, \end{aligned} \tag{2.18}$$

and the bound on $\mathbb{I}^-H_{n+1}$ is obtained by using identity (2.9):

$$\begin{aligned} \bigl|\mathbb{I}^-H_{n+1}\bigr|_{\rho_n-3\delta_n} &\le \frac{a}{3}\varepsilon_n^2|H_n|_{\rho_n-\delta_n} \\ &< \frac{4ac}{27}(t_0 b)^2(2/3)^{6(3/2)^n} < \frac{ac}{4}(t_0 b)^2(2/3)^{4(3/2)^{n+1}}\,. \end{aligned} \tag{2.19}$$

This proves that (2.11) holds for all positive integers m. In particular, we find that the sequence of Hamiltonians H_n converges in $\mathcal{A}_{\rho'}$ to a function H_∞ that satisfies $\mathbb{I}^-H_\infty = 0$.

Next, we estimate the functions G_n defined in equation (2.7). The bounds (2.15) and (2.3) show that $g_n = U_{\phi_n} - \mathrm{I}$ satisfies $\|g_n\|_{\rho_n-3\delta_n} < \varepsilon_n\delta_n$, for any $n \ge 0$, where

$$\bigl\|(Q,P)\bigr\|_r = \max\bigl\{|VQ|_r\,, \|WP\|_r\bigr\}\,, \qquad (Q,P) \in \mathcal{A}_r^d \times \mathcal{A}_r^d\,. \tag{2.20}$$

Thus, by using the identity

$$G_n = \sum_{m=0}^{n-1} g_m \circ U_{\phi_{m+1}} \circ \ldots \circ U_{\phi_{n-1}}\,, \tag{2.21}$$

and Proposition 2.1.d, we obtain the bound

$$\|G_n\|_{\rho_n} \leq \sum_{m=0}^{n-1} \|g_m\|_{\rho_{m+1}} < \frac{1}{2}\sum_{m=0}^{n-1} \delta_m < \frac{3}{2}\delta_0 \,. \tag{2.22}$$

By Proposition 2.2, we also have

$$\begin{aligned} \|G_{n+1} - G_n\|_{\rho'} &= \big\|g_n + (G_n \circ U_{\phi_n} - G_n)\big\|_{\rho'} \\ &\leq \|g_n\|_{\rho_n - 3\delta_n} + \big\|G_n \circ U_{\phi_n} - G_n\big\|_{\rho_n - 3\delta_n} \\ &< \varepsilon_n\delta_n + \frac{2}{3}\varepsilon_n\|G_n\|_{\rho_n-\delta_n} < 2\varepsilon_n\delta_0 \,. \end{aligned} \tag{2.23}$$

This shows that the sequence $\{G_n\}$ converges in $\mathcal{A}^d_{\rho'} \times \mathcal{A}^d_{\rho'}$, and that the limit G_∞ defines an analytic map $\mathcal{U}_H = \mathrm{I} + G_\infty$ from $\mathcal{D}_{\rho'}$ to $\mathcal{D}_{(\rho'+\rho)/2}$. The identity $H_\infty = H \circ (\mathrm{I} + G_\infty)$ follows from the fact that H is analytic on $\mathcal{D}_\rho$. Since $\mathrm{I} + G_n$ is a canonical transformation, for every $n \geq 0$, the limit $\mathcal{U}_H$ is a canonical transformation as well.

In order to verify the remaining claims in Theorem 1.2, let us consider a fixed but arbitrary curve $z \mapsto H_z \in \mathcal{H}$, which is analytic on an open neighborhood Z of zero in $\mathbb{C}$. For each $z \in Z$, our construction of $\mathcal{U}_{H_z}$ defines a sequence of functions $\phi_{z,n} \in \mathcal{A}'_{\rho_n-\delta_n}$ and $H_{z,n} \in \mathcal{A}_{\rho_n}$ and $G_{z,n} \in \mathcal{A}^{2d}_{\rho_n}$. Since we have only used bounds that are uniform on $\mathcal{H}$, all of these functions depend analytically on z. The same applies to the limit $n \to \infty$, since convergence was proved to be uniform on $\mathcal{H}$. Thus, the maps $H \mapsto H_\infty$ and $H \mapsto G_\infty$ are analytic on $\mathcal{H}$.

Assume now that $|\mathbb{I}^- H_{z,n}|_{\rho_n} = \mathcal{O}\big(|z|^k\big)$, for some $n \geq 0$ and $k \geq 1$ Then the inequality (2.14) shows that $\|\phi_{z,n}\|'_{\rho_n-\delta_n}$ is also of order $|z|^k$. This, together with the identity (2.9), and the last inequality in (2.1), implies that $|\mathbb{I}^- H_{z,n+1}|_{\rho_{n+1}} = \mathcal{O}\big(|z|^{2k}\big)$. By applying this to a curve of the form $H_z = H + zh$ with $\mathbb{I}^- H = 0$, we see e.g. that $\mathcal{U}_H = \mathrm{I}$, and that

$$\begin{aligned} H_z \circ \mathcal{U}_{H_z} &= H_{z,1} + z^2 R_1(z) = \mathbb{I}^+ H_{z,1} + z^2 R_2(z) \\ &= \mathbb{I}^+\big(H_z + \{H_z, \phi_{z,0}\}\big) + z^2 R_3(z) \\ &= \mathbb{I}^+ H + z\mathbb{I}^+ h - \mathbb{I}^+ \widehat{H}\phi_{z,0} + z^2 R_4(z) \,, \end{aligned} \tag{2.24}$$

with $R_j(z) \in \mathcal{A}_{\rho'}$ for all $z \in Z$. Furthermore, we have

$$\phi_{z,0} = z\big(\mathbb{I}^- \widehat{H_z} \mathbb{I}^-\big)^{-1} \mathbb{I}^- h = z\big(\mathbb{I}^- \widehat{H} \mathbb{I}^-\big)^{-1} \mathbb{I}^- h + z^2 R_5(z) \,, \tag{2.25}$$

with $R_5(z) \in \mathcal{A}'_{\rho-\delta_0}$, for all $z \in Z$. The last two inequalities yield the given formula for the derivative of $H \mapsto H \circ \mathcal{U}_H$. This concludes the proof of Theorem 1.2.

■

We note that Theorem 1.2 could be applied iteratively, using an increasing sequence of projections $\mathbb{I}^-_0 \leq \mathbb{I}^-_1 \leq \mathbb{I}^-_2 \leq \ldots$. The part of H that is resonant at step k but becomes non–resonant (and gets eliminated) at step $k+1$, could be called a resonance of order k. The iteration scheme of classical KAM theory [18–22] is of this type, except e.g. that the canonical transformation U_{ϕ_0} is used in place of $\mathcal{U}_H$. This is sufficient for near–integrable Hamiltonians, since the remaining non–resonant term is very small.

3. The Renormalization Group Transformation

The transformation $\mathcal{R}_\mu$ implements the abovementioned iteration scheme as a dynamical system. Roughly speaking, the step $H \mapsto H \circ \mathcal{U}_H$ eliminates the non–resonant part of H, or resonance of order zero. In the next step $H \mapsto H \circ T_1$, the order of each remaining resonance is lowered by one. Two additional steps are included in the definition of $\mathcal{R}_\mu$, in order to re-normalize the resulting Hamiltonian: A scaling $\mathcal{S}_\mu$ of the action variable, and a scaling $H \mapsto \eta H$ of the energy (or time). The main effect of iterating $\mathcal{R}_\mu$, besides scaling, is to eliminate resonances of higher and higher order, by successively decreasing orders and eliminating the lowest one.

The remaining part of this section contains our proofs of Lemma 1.3 and Theorem 1.5, followed by some remarks on symmetries.

Proof of Lemma 1.3. For every index (ν,α) in I^+ we have

$$\begin{aligned} \|WT^*\nu\| &= \max_j |W_j \cdot T^*\nu| = \max_j |\vartheta_j||W_j\cdot\nu| \\ &\le \bigl(\vartheta - |\vartheta_2|\bigr)|\omega\cdot\nu| + |\vartheta_2|\|W\nu\| \\ &\le \bigl(\vartheta-|\vartheta_2|\bigr)\kappa|\alpha| + \bigl[|\vartheta_2| + \sigma\bigl(\vartheta-|\vartheta_2|\bigr)\bigr]\|W\nu\|\,. \end{aligned} \tag{3.1}$$

Choose $r > \rho$ in such a way that (1.9) remains true if ρ is replaced by r. Then the inequality (3.1) implies that

$$e^{r\|WT^*\nu\|} \le \left|\frac{\rho'\vartheta_d}{r\mu}\right|^{|\alpha|} e^{\rho'\|W\nu\|}\,. \tag{3.2}$$

Let now $H \in \mathbb{I}^+\mathcal{A}_{\rho'}$. Then for $H\circ T_\mu$ we have the representation

$$\begin{aligned} \bigl(H\circ T_\mu\bigr)(q,p) &= H\bigl(Tq, \mu(T^*)^{-1}p\bigr) \\ &= \sum_{(\nu,\alpha)\in I^+} H_{\nu,\alpha}(Wp)^\alpha \left[\prod_{j=1}^d \left(\frac{\mu}{\vartheta_j}\right)^{\alpha_j}\right] e^{iq\cdot(T^*\nu)}\,. \end{aligned} \tag{3.3}$$

Thus, by using the bound (3.2), we obtain

$$\bigl|H\circ T_\mu\bigr|_r = \sum_{(\nu,\alpha)\in I^+} \bigl|H_{\nu,\alpha}\bigr|\bigl(\rho'\bigr)^{|\alpha|}\left[\prod_{j=1}^d \left|\frac{r\mu}{\rho'\vartheta_j}\right|^{\alpha_j}\right] e^{r\|WT^*\nu\|} \le |H|_{\rho'}\,. \tag{3.4}$$

This shows that $H\mapsto H\circ T_\mu$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}$ to $\mathcal{A}_r$. The assertion now follows from the fact that the inclusion map from $\mathcal{A}_r$ into $\mathcal{A}_\rho$ is compact, and that $|H\circ T_\mu|_\rho \le |H\circ T_\mu|_r$.

■

Proof of Theorem 1.5. Assume that $\kappa\rho < \sigma$. Then we can choose $\rho' < \rho$ and $\kappa' > \kappa$ such that $\kappa\rho^2 < \kappa'\rho\rho' < \sigma\rho'$, and such that the first inequality in (1.9) holds, if κ is replaced by κ'. Let $s = (\rho' + 8\rho)/9$, and define $B = \{H \in \mathbb{I}^+\mathcal{A}'_s : H_{0,0} = 0,\ |H - h_\omega|'_s < \kappa\rho\}$. If $\rho' \le r \le s$, then for every function $\phi \in \mathbb{I}^-\mathcal{A}_r$ we have

$$
\begin{aligned}
\bigl|\{h_\omega, \phi\}\bigr|_r = |\omega \cdot \nabla_1 \phi|_r &= \sum_{(\nu,\alpha)\in I^-} |\phi_{\nu,\alpha}||\omega \cdot \nu| r^{|\alpha|} e^{r\|W\nu\|} \\
&\ge \sigma \sum_{(\nu,\alpha)\in I^-} |\phi_{\nu,\alpha}| r^{|\alpha|} \|W\nu\| e^{r\|W\nu\|} + \kappa' \sum_{(\nu,\alpha)\in I^-} |\phi_{\nu,\alpha}||\alpha| r^{|\alpha|} e^{r\|W\nu\|} \\
&\ge \sigma \|W\nabla_1 \phi\|_r + \kappa' r |V\nabla_2 \phi|_r \ge \kappa' r \|\phi\|'_r ,
\end{aligned}
\tag{3.5}
$$

which yields the bound

$$
\begin{aligned}
\bigl|\{H, \phi\}\bigr|_r &\ge \bigl|\{h_\omega, \phi\}\bigr|_r - \bigl|\{H - h_\omega, \phi\}\bigr|_r \\
&\ge \bigl(\kappa' r - |H - h_\omega|'_r\bigr)\|\phi\|'_r \ge (\kappa'\rho' - \kappa\rho)\|\phi\|'_r ,
\end{aligned}
\tag{3.6}
$$

for all $H \in B$. This shows that $\mathbb{I}^-$ satisfies a non–resonance condition (with respect to the set B), as defined in Theorem 1.2, for some constants $a, b, c > 0$. The same resonance condition remains satisfied if we replace B by the set $\mathcal{H}$ of all Hamiltonians $H \in \mathcal{A}_\rho$ that lie within a small distance $\varepsilon > 0$ of B. If necessary, we decrease $\varepsilon > 0$ such that $(H \circ \mathcal{U}_H)_{0,(1,0,\ldots,0)}$ is bounded away from zero, for all $H \in \mathcal{H}$. This is possible since $|H_{0,(1,0,\ldots,0)}| > 1 - \sigma > 0$, for all $H \in B$. Now Theorem 1.2 and Lemma 1.3 imply that (1.10) defines a compact analytic map $\mathcal{R}_\mu$ from $\mathcal{H}$ to $\mathcal{A}_\rho$, provided that μ satisfies the second inequality in (1.9).

By using the analyticity improving property of the map described in Lemma 1.3, we obtain the same result for $\mathcal{S}_z \circ \mathcal{R}_\mu$ and $\mathcal{R}_\mu \circ \mathcal{S}_z$, uniformly in z, for all z in some open neighborhood of the unit circle in $\mathbb{C}$. In order to prove that $\mathcal{S}_z \circ \mathcal{R}_\mu = \mathcal{R}_\mu \circ \mathcal{S}_z$, as claimed, it suffices to consider $|z| = 1$. In this case, it is straightforward to check that $\mathcal{S}_z$ "commutes through" each of the steps used in the proof of Theorem 1.2, yielding

$$
\mathcal{U}_{\mathcal{S}_z H} \circ S_z = S_z \circ \mathcal{U}_H ,
\tag{3.7}
$$

where $S_z(q,p) = (q, zp)$. This can be done by using the fact that $\mathcal{S}_z \mathbb{I}^- = \mathbb{I}^- \mathcal{S}_z$ and $\{\mathcal{S}_z H, \mathcal{S}_z \phi\} = \mathcal{S}_z\{H, \phi\}$, which implies that $U_{\mathcal{S}_z \phi} \circ S_z = S_z \circ U_\phi$. The details of this computation are left to the reader; see also [11].

■

Remarks.

∘ Due to the symmetry $\mathcal{S}_z \circ \mathcal{R}_\mu = \mathcal{R}_\mu \circ \mathcal{S}_z$, the transformation $\mathcal{R}_\mu$ maps the scaling orbit $z \mapsto \mathcal{S}_z H$ of a Hamiltonian H to the corresponding scaling orbit for $\mathcal{R}_\mu(H)$. Thus, in situations where these orbits are non–degenerate, a normalization condition can be used to pick an arbitrary representative from each of them. This leads naturally to a transformation (1.10) where the scaling μ is chosen to depend on H, in such a way that the given normalization is preserved.

$\circ$ A relation analogous to (3.7) holds if the scaling transformations S_z and $\mathcal{S}_z$ are replaced by translations of the angles, given by $J_\gamma(q,p) = (q-\gamma,p)$ and $\mathcal{J}_\gamma H = H \circ J_\gamma$, respectively. The corresponding symmetry $\mathcal{R}_\mu \circ \mathcal{J}_\gamma = \mathcal{J}_{T^{-1}\gamma} \circ \mathcal{R}_\mu$ can be related to observations for certain periodic orbits [10].

$\circ$ As was mentioned in the introduction, if H is an even function of the angle variable q, then the same is true for $\mathcal{R}_\mu(H)$. This can be seen easily from our construction of $\mathcal{U}_H$ in the proof of Theorem 1.2.

$\circ$ Another invariance property of $\mathcal{R}_\mu$ is the following. Given $v \in \mathbb{C}^d$, denote by $\mathcal{A}(v)$ the set of Hamiltonians H (in the appropriate domain) for which $v \cdot \nabla_2 H$ is constant. Our proof of Theorem 1.2 shows that $H \circ \mathcal{U}_H$ belongs to $\mathcal{A}(v)$ whenever H does. Thus, $\mathcal{R}_\mu$ maps $\mathcal{A}(v)$ to $\mathcal{A}((T^*)^{-1}v)$. The same holds if we define $\mathcal{A}(v)$ by the condition $v \cdot \nabla_2 H = 0$.

$\circ$ The symmetry properties mentioned above, either separately or combined, can be used e.g. to restrict the search for nontrivial RG fixed points (or invariant families) to appropriate invariant subspaces. Such a fixed point (or family) may be relevant only for Hamiltonians that share all of its symmetries. But as the trivial fixed point h_ω and other examples show, the "domain of relevance" (universality class) may actually be larger; see also [10,17].

4. Periodic Orbits

We begin by showing that under a certain condition on $w \in \mathbb{R}^d$, every Hamiltonian H near h_w determines a "counterterm" $\Phi(H)$, within a d–dimensional subspace of $\mathcal{B}_\rho$ that is roughly parallel to $\mathcal{W}^u$, such that a constant multiple of $H + \Phi(H)$ has a periodic orbit of the type described in Definition 1.7. The subsequent proof of Theorem 1.8 is based on identifying $\Sigma(w)$ locally with $\Phi^{-1}(0)$. A proof of Theorem 1.9 is given at this end of this section, after some results on d–parameter families.

Given any $r > 0$, define A_r to be the Banach space of all analytic functions g on the strip $|\mathrm{Im} z| < r$, with finite norm $|g|_r = \sum |g_n| \exp(r|n|)$, where $g_0, g_{\pm 1}, g_{\pm 2}, \ldots$ are the Fourier coefficients of g. In other words, A_r is the one–variable analogue of $\mathcal{A}_r$. On the product space A_r^d we use norms $|.|_r$ and $\|.\|_r$ analogous to those introduced in Definition 1.1.

Theorem 4.1. *Let δ, r, ρ be positive real numbers satisfying $\delta \le r \le 1$ and $r + \delta \le \rho/2$. Let w be a nonzero vector in $\mathbb{R}^d$ such that $\tau w \in \mathbb{Z}^d$, $\tau \ge 4$, and assume that $|V(w-\omega)| \le 2^{-4}$. Define $b = 2^{-5}\delta^2/\tau$. Then for every Hamiltonian H in $B = \{H \in \mathcal{B}_\rho : |H - h_w|_\rho < b\}$, there exist two complex numbers ξ and E, and a vector u perpendicular to $v = V^*Vw$, such that the Hamiltonian $\xi(H + h_u) + E$ has a periodic orbit (1.19) at energy zero, with $K(\gamma) = 0$, $Q_0 = 0$, $P_0 = kv$ for some $k \in \mathbb{C}$, and with Q (odd) and P (even) belonging to A_r^d. The quantities ξ, u, E, k, Q, P satisfy the bounds*

$$\begin{aligned} &|\xi - 1| \le \delta'/\delta\,, \qquad |Vu| \le \delta'/\delta\,, \qquad |E| \le \delta'\,, \\ &|k| \le \delta'\,, \qquad |VQ|_{r/\tau} \le \tau\delta'/\delta\,, \quad \|WP\|_{r/\tau} \le \tau\delta'/\delta\,, \end{aligned} \tag{4.1}$$

with $\delta' = 4|H - h_w|_\rho$, and they are uniquely determined if we require that (4.1) holds for $\delta' = 4b$. Furthermore, the dependence of ξ, u, E, k, Q, P on the Hamiltonian $H \in B$ is analytic.

Proof. Given $\xi \in \mathbb{C}$, and $u \in \mathbb{C}^d$ satisfying $u \cdot \overline{v} = 0$, we define $x = (\xi - 1)w + \xi u$. If ξ is bounded away from zero, as is the case for the parameter values considered here, the map $(\xi, u) \mapsto x$ is one-to-one. Thus, we shall use x as a parameter and consider ξ, u to be functions of x.

Let now $H = h_w + h$ be a Hamiltonian in $\mathcal{B}_\rho$, with $|h|_\rho \leq 2b$, and consider the family $(x, E) \mapsto G$ defined by

$$G = \xi(H + h_u) + E = h_w + \xi h + h_x + E\,. \tag{4.2}$$

Let $r' = r/\tau$, and denote by D the derivative operator on $A^d_{r'}$. Then the equation for a periodic orbit (1.19) of G, with $Q_0 = 0$ and $P_0 = kv$, can be written as

$$\begin{aligned} DQ &= \tau\bigl(\nabla_2(\xi h + h_x)\bigr)(\zeta + Q, kv + P)\,, \\ DP &= -\tau\bigl(\nabla_1(\xi h + h_x)\bigr)(\zeta + Q, kv + P)\,, \end{aligned} \tag{4.3}$$

where $g(\zeta + Q, kv + P)$ stands for the composition of a given function g with the map $t' \mapsto (t'\tau w + Q(t'), kv + P(t'))$. The variable t' used here is the rescaled time $t' = t/\tau$.

Denote by $\mathbb{E}f$ the average of a periodic function f. It is convenient to split the first equation in (4.3) into two equations, one for the average, and one for the remaining zero-average part. The result can be written in the form $x = \widetilde{x}$ and $Q = \widetilde{Q}$, where

$$\begin{aligned} \widetilde{x} &= -\xi\mathbb{E}(\nabla_2 h)(\zeta + Q, kv + P)\,, \\ \widetilde{Q} &= \tau\xi D^{-1}(\mathrm{I} - \mathbb{E})(\nabla_2 h)(\zeta + Q, kv + P)\,. \end{aligned} \tag{4.4}$$

Here, D^{-1} denotes the antiderivative operator on $(\mathrm{I}-\mathbb{E})A^d_{r'}$. Similarly, the second equation in (4.3) is equivalent to $P = \widetilde{P}$, with

$$\widetilde{P} = -\tau\xi D^{-1}(\nabla_1 h)(\zeta + Q, kv + P)\,. \tag{4.5}$$

Notice that in this equation, the function to the right of D^{-1} has automatically a zero average, due to the parity conditions on Q, P, and h.

The condition that the integral (1.16) along the periodic orbit γ be zero, will be written as $k = \widetilde{k}$, where (omitting the argument t')

$$\begin{aligned} \widetilde{k} &= k - \frac{1}{2\pi\tau c}\int_0^{2\pi}(kv + P)\cdot(\tau w + DQ)\,dt' = -\frac{1}{2\pi\tau c}\int_0^{2\pi} P\cdot DQ\,dt' \\ &= -\frac{\xi}{2\pi c}\int_0^{2\pi} P\cdot(\nabla_2 h)(\zeta + Q, kv + P)\,dt'\,, \end{aligned} \tag{4.6}$$

with $c = v \cdot w$. In addition, we impose the condition that $G = 0$ on the orbit γ, or equivalently (if $k = \widetilde{k}$), that the integral of $Gdt - p \cdot dq$ along γ be zero. This condition can be written in the form $E = \widetilde{E}$, where (omitting the argument t')

$$\begin{aligned}\widetilde{E} &= E - \frac{1}{2\pi}\int_0^{2\pi} G(\zeta+Q, kv+P)\,dt' + \frac{1}{2\pi\tau}\int_0^{2\pi}(kv+P)\cdot(\tau w + DQ)\,dt' \\ &= -\frac{\xi}{2\pi}\int_0^{2\pi}\Big[h(\zeta+Q,kv+P) - P\cdot(\nabla_2 h)(\zeta+Q,kv+P)\Big]\,dt' - kv\cdot x\,.\end{aligned} \tag{4.7}$$

The problem of finding a pair (x, E), such that the Hamiltonian (4.2) has an orbit γ with the desired properties, has now been reduced to a fixed point problem for the map $F : (x, E, k, Q, P) \mapsto (\widetilde{x}, \widetilde{E}, \widetilde{k}, \widetilde{Q}, \widetilde{P})$ defined by the equations (4.4) ... (4.7). Denote by $\mathcal{X}$ the Banach space of all quintuples $X = (x, E, k, Q, P)$ in $\mathbb{C}^d \times \mathbb{C} \times \mathbb{C} \times A^d_{r'} \times A^d_{r'}$, whose components Q and P have average zero, equipped with the norm

$$\|X\| = \max\bigl\{\tau|Vx|,\ \delta^{-1}\tau|E|,\ \delta^{-1}\tau|k|,\ |VQ|_{r'},\ \|WP\|_{r'}\bigr\}\,. \tag{4.8}$$

We note that the conditions (4.1), with $\delta' = 4b$, imply that $\|X\| < \delta/2$. To see this, one first proves e.g. that $|Vx| \le \frac{17}{16}|\xi - 1| + |\xi||Vu|$. Let us now assume that $\|X\| \le \delta$. Then, by using part (b) of Proposition 2.1, we obtain

$$\bigl|g(\zeta+Q, kv+P)\bigr|_{r'} \le |g|_{\rho-\delta}\,, \qquad g \in \mathcal{A}_{\rho-\delta}\,, \tag{4.9}$$

and this bound can be used to show that

$$\|F(X)\| \le \frac{2\tau}{\delta}|h|_\rho + \frac{2}{\tau}\|X\|^2 < \frac{\delta}{4}\,. \tag{4.10}$$

The proof of the first inequality in (4.10) is mildly tedious, but straightforward. This estimate, together with Cauchy's formula, can also be used to show that the derivative of F, at every point in the ball $\|X\| \le \delta/2$, is of norm less than $\frac{1}{2}$. Thus, by the contraction mapping principle, F has a unique fixed point X' in this ball. From the bound (4.10), it follows that

$$\|X'\| \le \frac{16\tau}{7\delta}|h|_\rho\,. \tag{4.11}$$

By combining this with the two estimates $|\xi - 1| \le \frac{8}{7}|Vx|$ and $|Vu| \le \frac{4}{3}|Vx|$, we obtain the inequalities (4.1) with $\delta' = 4|h|_\rho$. The analytic dependence of ξ, u, E, k, Q, P on $H \in B$ follows from the uniform convergence of $F^n(0) \to X'$ as $n \to \infty$.

∎

Proof of Theorem 1.8. We will use the notation and assumptions made in Theorem 4.1. In addition, we assume that h_w lies on $\mathcal{W}^u$, and that $\delta > 0$ is sufficiently small such that B is contained in the domain of $\mathcal{R}_\mu$.

Denote by X the subspace of $\mathcal{B}_\rho$, consisting of all Hamiltonians of the form $ch_w + f$, with $c \in \mathbb{C}$, and with f a function in $\mathcal{B}_\rho$, whose Fourier–Taylor coefficients $f_{\nu,\alpha}$ are zero whenever $\nu = 0$ and $|\alpha| < 2$. In addition, let Y be the space of all Hamiltonians of the form $h_y + C$, with C a constant function, $y \in \mathbb{C}^d$, and $y \cdot \overline{v} = 0$. Then we can identify $\mathcal{B}_\rho$ with $X \oplus Y$.

Define a map ϕ from $X \cap B$ to Y, by setting $\phi(H) = h_u + E/\xi$, with $H \mapsto (\xi, u, E)$ as described in Theorem 4.1. The graph Σ' of this map ϕ is clearly an analytic manifold of codimension d in $\mathcal{B}_\rho$. In addition, Σ' intersects $\mathcal{W}^u$ transversally at h_w. This follows essentially from the fact that $\phi(ch_w) = 0$: Due to the identity $D\phi(h_w)h_w = 0$, it suffices to verify the transversality property in the $(d+1)$–dimensional subspace of all Hamiltonians of the form $(q,p) \mapsto x \cdot p + C$, where it is trivial.

In order to compare Σ' with the set $\Sigma(w)$ defined in Definition 1.7, consider two different choices (δ_1, r_1) and (δ_2, r_2) for the parameters (δ, r) in Theorem 4.1, with $\delta_1 < \delta_2 < r_2 < r_1$. Denote by B_1 and B_2 the corresponding balls B, and by Σ_1' and Σ_2' the corresponding manifolds Σ'. Then, by the uniqueness part of Theorem 4.1, the intersection of Σ_2' with B_1 is equal to Σ_1'. Furthermore, if we set $\rho' = r_1$ and choose $\varepsilon > 0$ sufficiently small, then $\Sigma(w) \cap B_1$ is contained in Σ_2'. And by choosing $\delta_1 > 0$ sufficiently small, we also have $\Sigma_1' \subset \Sigma(w)$. Thus, $\Sigma(w)$ agrees with Σ_1' in the ball B_1.

We note that the same choice of parameters works for all rotation vectors $w \in \mathbb{R}\mathbb{Z}^d$ that are sufficiently close to ω. (This fact is not used later on.) The reason for this is that our bounds depend on w only through the constant $\tau = \tau(w)$.

■

Our proof of Theorem 1.9 is based on the graph transform method; see e.g. [24]. We start by introducing some notation. Denote by $\mathcal{X}$ and $\mathcal{Y}$ the stable and unstable subspaces, respectively, of the linearized RG transformation $D\mathcal{R}_\mu(h_\omega)$, restricted to $\mathcal{B}_\rho$. Denote by ψ the (analytic) function, defined on an open neighborhood of h_ω in $\mathcal{X}$, with values in $\mathcal{Y}$, whose graph is the local stable manifold $\mathcal{W}^s$ of $\mathcal{R}_\mu$ at h_ω. As was mentioned in the introduction, $\mathcal{Y}$ is spanned by the $d-1$ functions listed in (1.13), together with the constant function. The canonical projections onto $\mathcal{X}$ and $\mathcal{Y}$ will be denoted by $\mathbb{P}_s$ and $\mathbb{P}_u$, respectively. Due to our choice of norm on $\mathcal{B}_\rho$, both $\mathbb{P}_s$ and $\mathbb{P}_u$ have operator norm 1.

Consider the transformation $\mathcal{N}_\mu$,

$$\mathcal{N}_\mu = \Psi^{-1} \circ \mathcal{R}_\mu \circ \Psi\,, \qquad \Psi = \mathbb{I} + \psi \circ \mathbb{P}_s\,, \tag{4.12}$$

defined on a neighborhood of h_ω in $\mathcal{B}_\rho$. Notice that $\mathcal{N}_\mu$ and $\mathcal{R}_\mu$ have the same derivatives at the fixed point h_ω, and the same local unstable manifolds. But the local stable manifold of $\mathcal{N}_\mu$ is trivial; that is, it agrees with $\mathcal{X}$ near h_ω. The largest (in modulus) contracting eigenvalue of $D\mathcal{N}_\mu(h_\omega)$ is $\lambda = \mu\vartheta_1\vartheta_d^{-2}$. Let θ be some fixed real number larger than $\vartheta_1|\vartheta_d|^{-2}$. In the remaining part of this section, we restrict the possible choices of μ by imposing the condition $\theta\mu < |\lambda_2|^{-1}$.

Since $D\mathcal{N}_\mu(h_\omega)$ is compact, we can choose a norm $\|.\|$ on $\mathcal{X}$ that is equivalent to $\|.\|_\rho$, but for which the restriction of $D\mathcal{N}_\mu(h_\omega)$ to $\mathcal{X}$ has (operator) norm less than $\theta\mu$. Assume now that such a norm has been chosen. The extension to $\mathcal{B}_\rho = \mathcal{X} \oplus \mathcal{Y}$ is defined by setting $\|x + y\| = \|x\| + \|y\|_\rho$, for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Given any $\delta > 0$, let $D_\delta = \{y \in \mathcal{Y} : \|y\| < \delta\}$, and define $\mathcal{F}_\delta$ to be the Banach space of all analytic d–parameter families $F : D_\delta \to \mathcal{B}_\rho$, that extend continuously to the boundary of D_δ and satisfy $\mathbb{P}_u F(0) = 0$, equipped with the norm

$$\|F\|_\delta = \sup_{y \in D_\delta} \|F(y)\| \,. \tag{4.13}$$

Of particular interest is the family F^*, defined by $F^*(y) = h_\omega + y$, as it parametrizes the local unstable manifold of $\mathcal{R}_\mu$.

Proposition 4.2. *If $\delta > 0$ is sufficiently small, then the equation*

$$\mathcal{M}_\mu(F) = \mathcal{N}_\mu \circ F \circ Y_F \,, \qquad Y_F = (\mathbb{P}_u \circ \mathcal{N}_\mu \circ F)^{-1} \,, \tag{4.14}$$

defines an analytic contraction mapping $\mathcal{M}_\mu$, on some open neighborhood B of F^* in $\mathcal{F}_\delta$, with fixed point F^*, and contraction rate less than $\theta\mu$.

The proof of this proposition is straightforward: $\mathbb{P}_u \circ \mathcal{N}_\mu \circ F^*$ agrees with the restriction of $D\mathcal{R}_\mu$ to $\mathcal{Y}$, so that by the inverse function theorem, $(F, y) \mapsto Y_F(y)$ is well defined and analytic near $(F^*, 0)$ in $\mathcal{F}_r \times \mathcal{Y}$; and the asserted contraction property of $\mathcal{M}_\mu$ follows from the identity

$$\big(D\mathcal{M}_\mu(F^*) f\big)(y) = \mathbb{P}_s D\mathcal{N}_\mu\big(F^*(Y_{F^*}(y))\big) f(Y_{F^*}(y)) \,, \qquad f \in \mathcal{F}_r \,,\ y \in D_r \,, \tag{4.15}$$

which can be verified by an explicit computation.

Notice that Y_{F^*} is linear, and that $Y_F(0) = 0$ for any F. The following proposition will be used to estimate the composition of maps $f_n = Y_{F_n}$, associated with an orbit $(F_0, F_1, F_2, \ldots)$ of $\mathcal{M}_\mu$.

Proposition 4.3. *Let U, V be normed linear spaces, and let Z be an open ball in $U \oplus V$, centered at zero. Let L be a bounded linear operator on $U \oplus V$ that commutes with the projection $(u, v) \mapsto (u, 0)$, and that satisfies*

$$\|L(u, 0)\| = a\|u\| \,, \quad \|L(0, v)\| \le b\|v\| \,, \qquad u \in U,\ v \in V, \tag{4.16}$$

with $0 < b < a$ fixed. Let $f_0, f_1, \ldots, f_{n-1}$ be Lipschitz maps on Z, that leave the origin fixed. Then

$$\pm a^{-n} \|f_0(f_1(\cdots f_{n-1}(u, v) \cdots))\| \le \pm\|u\| + e^{c_0} c_0 \|u\| + (b/a)^n e^{c_0 + c_1} \|v\| \,, \tag{4.17}$$

where

$$c_m = \frac{1}{a} \sum_{j=0}^{n-1} \Big(\frac{a}{b}\Big)^{m(j+1)} \sup_{z \neq 0} \frac{\|f_j(z) - Lz\|}{\|z\|} \,, \qquad m = 0, 1 \,. \tag{4.18}$$

Proof. Denote by s_j the value of the supremum in (4.18). Let (u_0, v_0) be an arbitrary point in Z. For $k = 1, 2, \ldots, n$, define $(u_k, v_k) = L^k(u_0, v_0)$ and

$$r_k = \big(f_{n-k} \circ f_{n-k+1} \circ \ldots \circ f_{n-1}\big)(u_0, v_0) - (u_k, v_k) \,. \tag{4.19}$$

By (4.16) we have $\|u_k\| = a^k\|u_0\|$ and $\|v_k\| \le b^k\|v_0\|$. If we set $r_0 = 0$, then

$$\begin{aligned}
\|r_k\| &= \bigl\|f_{n-k}\bigl((u_{k-1}, v_{k-1}) + r_{k-1}\bigr) - L(u_{k-1}, v_{k-1})\bigr\| \\
&\le \|Lr_{k-1}\| + s_{n-k}\|(u_{k-1}, v_{k-1}) + r_{k-1}\| \\
&\le (a + s_{n-k})\|r_{k-1}\| + s_{n-k}a^{k-1}\|u_0\| + s_{n-k}b^{k-1}\|v_0\|\,,
\end{aligned} \tag{4.20}$$

for all positive $k \le n$. Define $p_k = \prod_{j=n-k}^{n-1}(1 + s_j/a)$. Then, by applying the bound (4.20) recursively, we obtain

$$\begin{aligned}
\frac{1}{a^k p_k}\|r_k\| &\le \frac{1}{a^{k-1}p_{k-1}}\|r_{k-1}\| + \frac{s_{n-k}}{a}\|u_0\| + \left(\frac{b}{a}\right)^n \left(\frac{a}{b}\right)^{n-k+1} \frac{s_{n-k}}{a}\|v_0\| \\
&\le \sum_{j=n-k}^{n-1} \frac{s_j}{a}\|u_0\| + \left(\frac{b}{a}\right)^n \sum_{j=n-k}^{n-1} \left(\frac{a}{b}\right)^{j+1} \frac{s_j}{a}\|v_0\|\,.
\end{aligned} \tag{4.21}$$

In particular, $a^{-n}\|r_n\| \le p_n c_0\|u_0\| + (b/a)^n p_n c_1\|v_0\|$. The assertion follows by combining this inequality with the trivial bounds $p_n \le e^{c_0}$ and $1 + p_n c_1 \le e^{c_0+c_1}$.

■

Proposition 4.3 can be used to prove uniform upper and lower bounds on $\beta_n - \beta'$, which we will need in order to estimate the accumulation rate of periodic orbits, as described in Theorem 1.9. The following proposition is the first step toward this goal. We note that a general result of this type was proved in [25], but it is not sufficient for our purpose.

Proposition 4.4. *If $\delta > 0$ is sufficiently small, and if $w \in \mathbb{RZ}^d$ is sufficiently close to ω and normalized, such that $h_w - h_\omega$ belongs to $D_{\delta/2}$, then there exists an open neighborhood B' of F^* in $\mathcal{F}_\delta$, and constants $k_1, k_2, N > 0$, such that the following holds. For every $F \in B'$, and for every non–negative integer n, the condition $\Psi(F(y)) \in \Sigma_n(w)$ defines a unique parameter value $y = y_n$ in D_δ, and this value satisfies the bound*

$$k_1|\lambda_2|^{-n} < \|y_n\| < k_2|\lambda_2|^{-n}\,, \qquad n \ge N\,. \tag{4.22}$$

Proof. Under the given conditions on δ and w, Theorem 1.8 guarantees that F^* intersects the manifold $\Psi^{-1}(\Sigma_0(w))$ transversally, and in a single point. Since transversality is preserved under small perturbations, the same holds for every family F in some open ball $B' \subset \mathcal{F}_\delta$ centered at F^*, and the intersection parameter y_0 depends analytically on F. Denote by Z the map $F \mapsto y_0$, and by r' the radius of B'. In addition, let $z = Z(F^*)$ and $L = DY_{F^*}(0)$.

Assume that $\delta, r' > 0$ have been chosen sufficiently small, such that the restriction of $\mathcal{M}_\mu$ to B' has the properties described in Proposition 4.2, and such that the derivatives of $F \mapsto Y_F$ and $\mathcal{M}_\mu$ are uniformly bounded on B'. Given any $F_0 \in B'$, we set $F_n = \mathcal{M}_\mu^n(F_0)$, for $n = 1, 2 \ldots$. By the definition of $\mathcal{M}_\mu$, the family F_n intersects $\Psi^{-1}(\Sigma_n(w))$ at a single point y_n, given by the equation

$$y_n = \bigl(Y_{F_0} \circ Y_{F_1} \circ \ldots \circ Y_{F_{n-1}}\bigr)(z_n)\,, \qquad z_n = Z(F_n)\,. \tag{4.23}$$

By Proposition 4.2, the norm of $F_n - F^*$ is bounded by $(\theta\mu)^n\|F_0 - F^*\|$, for all $n \geq 0$. By analyticity, analogous bounds hold for $\|z_n - z\|$ and $\|DY_{F_n}(y) - L\|$, up to constant factors that we can choose to be independent of n, $F \in B$, and $y \in D_{\delta/2}$. Notice that L is the inverse of the restriction of $D\mathcal{R}_\mu(h_\omega)$ to Y. All eigenvalues of L are either of modulus a, or of modulus less than b, with $\theta\mu < b < a$.

Given these properties of the maps Y_{F_n} and L, the bound (4.22) now follows from Proposition 4.3, provided that z has a nonzero component in a spectral subspace of L corresponding to an eigenvalue of modulus a. But $z = h_w - h_\omega$ meets this requirement, since $w = c_1W_1 + \ldots + c_dW_d$, with $c_j \neq 0$ for all j. This follows from the fact that the fields $\mathbb{Q}[\vartheta_j]$ are all isomorphic, and that $\mathbb{Q}[\vartheta_1]$ is spanned by the rationally independent components of ω.

■

Proof of Theorem 1.9. Let w' be a nonzero vector in $\mathbb{R}\mathbb{Z}^d$. In order to prove the assertion for $w = w'$, it suffices to prove it for $w = c(\vartheta^{-1}T)^k w'$, where c can be any nonzero real number, and k any nonnegative integer. Thus, since $c(\vartheta^{-1}T)^k w' \to \omega$ for some value of c, we can, without loss of generality, assume that w is as close to ω as required by Proposition 4.4, after choosing $\delta > 0$ sufficiently small.

In what follows, $c_0, \ldots, c_{18}$ denote positive constants that do not depend on the choice of the Hamiltonian H. But unless stated otherwise, c_i may depend on μ. We will also introduce constants $b_1, \ldots, b_6$ that only depend on the choice of T. We assume that $\mu > 0$ has been chosen sufficiently small such that $b_i\mu < |\lambda_2|^{-1}$.

Since $r > \rho$, there exists an open neighborhood Λ of zero in $\mathbb{C}^d$, such that $(\beta, H) \mapsto H_\beta$ is analytic and has bounded derivatives, as a map from $\Lambda \times \mathcal{B}_r$ to $\mathcal{B}_\rho$. When considering families $\beta \mapsto H_\beta$, we will implicitly assume that $\beta \in \Lambda$.

Let now h be a Hamiltonian in $\mathcal{B}_r$, of the form (1.18), with M as described after (1.18), and assume that h is sufficiently close to h_ω, such that $\mathcal{N}_\mu$ is well defined and analytic on an open ball in $\mathcal{B}_\rho$ containing h and h_ω. Define $Y(\beta) = \mathbb{P}_u\Psi^{-1}(h \circ R_\beta)$. Then $Y(0) = 0$, and our assumption on M implies that Y is invertible as a map from an open neighborhood of zero in $\mathbb{C}^d$, to $\mathcal{Y}$. Define $f_0(y) = \Psi^{-1}(h \circ R_\beta)$, with $\beta = Y^{-1}(y)$. An explicit computation shows that for $n = 1, 2, \ldots$, the equation (4.14) defines a family $f_n = \mathcal{M}_\mu(f_{n-1})$, which belongs to $\mathcal{F}_\delta$ for large n, and that $f_n \to F^*$ in $\mathcal{F}_\delta$, as n tends to infinity. Thus, given any open neighborhood B' of F^* in $\mathcal{F}_\delta$, there exists a positive integer ℓ, such that if $H \in \mathcal{B}_r$ is sufficiently close to h, then the equation

$$F(y) = \Psi^{-1}\bigl(\mathcal{R}_\mu^\ell\bigl(H_{\beta'} \circ R_{Z_\ell(y)}\bigr)\bigr)\,, \qquad Z_\ell = Y^{-1} \circ Y_{f_0} \circ \ldots \circ Y_{f_{\ell-1}}\,, \tag{4.24}$$

defines a family $F \in B'$. Here, β' denotes the parameter value where $\beta \mapsto H_\beta$ intersects $\mathcal{W}^s$. This value is well defined and depends analytically on H, since $\beta \mapsto h_\beta$ intersects $\mathcal{W}^s$ transversally at h. By using Proposition 4.4, and the fact that Z_ℓ is invertible near the origin, we can find constants $k_1', k_2', N' > 0$, such that for all $n \geq N'$, and for all Hamiltonians H in some open neighborhood B of h in $\mathcal{B}_r$, the condition $H_\beta \in \Sigma_n(w)$ defines a unique parameter value $\beta = \beta_n$, and this value satisfies the bound

$$k_1'|\lambda_2|^{-n} < |\beta_n - \beta'| < k_2'|\lambda_2|^{-n}\,. \tag{4.25}$$

In what follows, we assume that n is larger than N'. Define $H_{\beta,m} = \mathcal{R}_\mu^m(H_\beta)$, whenever H_β belongs to the domain of $\mathcal{R}_\mu^m$.

Consider now a fixed but arbitrary Hamiltonian $H \in B$. Given that $H_{\beta_n,n}$ lies on $\Sigma(w)$, a constant multiple of this Hamiltonian has a periodic orbit with rotation vector w, given by an analytic curve g_n in $\mathcal{D}_\rho$ with fixed domain, as described in Definition 1.7. By the definition of $\mathcal{R}_\mu$, the Hamiltonian $H_{\beta_n,n}$ is related to H_{β_n} by a canonical change of coordinates homotopic to T_1^n, and a scaling. Formally, a constant multiple of H_{β_n} has a periodic orbit G_n with rotation vector $w_n = (\vartheta^{-1}T)^n w$, and this orbit is given by the equation

$$\begin{aligned} G_n &= \mathcal{U}_{H_{\beta_n,0}} \circ T_\mu \circ \mathcal{U}_{H_{\beta_n,1}} \circ T_\mu \circ \ldots \circ \mathcal{U}_{H_{\beta_n,n-1}} \circ T_\mu \circ g_n \circ \Theta^{-n} \\ &= \mathcal{V}_n(H_{\beta_n}) \circ T_\mu^n \circ g_n \circ \Theta^{-n}, \end{aligned} \tag{4.26}$$

where $\Theta(t) = \vartheta t$ for all t, and where $\mathcal{V}_n(H_{\beta_n})$ is the transformation given by (1.14). But in order to establish the existence of such an orbit G_n, we need to show that the maps in equation (4.26) can be composed as indicated.

By construction, the size of the domain (width of the strip in the angle variable, and diameter of the ball in the action variable) of the transformation $\mathcal{V}_n(H_{\beta_n})$ decreases exponentially with n. But the rate of decrease is independent of μ: Due to the identity (3.7), the transformations $V_k(H_{\beta_n})$, defined in (1.14), are not only canonical, but also independent of the choice of μ. Thus, the same is true for $\mathcal{V}_n(H_{\beta_n})$. On the other hand, the nonlinear part of g_n is bounded by a constant times $|H_{\beta_n,n} - h_w|_\rho$, as was shown in Theorem 4.1. And this norm is less than $c_0(\theta\mu)^n$. This follows from Proposition 4.2, and from the fact that $\Sigma(w)$ intersects F^* transversally at h_w. Thus, if $\mu > 0$ is chosen sufficiently small, we find that the range of $T_\mu^n \circ g_n$ is contained in the domain of $\mathcal{V}_n(H_{\beta_n})$, for large n.

A more detailed discussion of the transformations $\mathcal{V}_n(H)$ can be found in [11]. The estimates obtained there are formulated for $H = H_{\beta'}$ only, but they are easy to adapt to the Hamiltonians considered here. To be more precise, the starting point for these estimates is a bound

$$\|\phi(H_{\beta',m})\|'_{r_0} \le c_2 \|\mathbb{I}^- H_{\beta',m}\|_\rho \le c_2 c_1 (b_1\mu)^m \|H_{\beta'} - h_\omega\|_\rho, \tag{4.27}$$

where $\phi(H_{\beta',m})$ is the generating function of the canonical transformation $\mathcal{U}_{H_{\beta',m}}$. The constants c_2 and b_1, and the parameter r_0 defining the domain of $\phi_m(H_{\beta',m})$, are independent of μ. We have omitted an additional factor $(b_1\mu)^m$ that appears in equation (5.11) of [11], since it is not used or needed. Concerning the replacement of $H_{\beta'}$ by H_{β_n}, we note that the first inequality in (4.27) is a general bound on ϕ that applies directly to $H_{\beta_n,m}$. The second inequality in (4.27) carries over as well. In fact, it can be improved if H is close to h:

$$\|\mathbb{I}^- H_{\beta,m}\|_\rho \le c_3(\theta\mu)^m \|H - h\|_r, \qquad H \in B, \tag{4.28}$$

for all β in some open set $\Lambda_m \subset \mathbb{C}^d$ containing β' and β_n. This inequality is trivial for $m \le \ell$, and if $m = \ell + k$ with k positive, it follows from the bound

$$\begin{aligned} \big\|\mathbb{I}^-\Psi\big((\mathcal{M}_\mu^k(F))(y)\big)\big\|_\rho &= \big\|\mathbb{I}^-\Psi\big((\mathcal{M}_\mu^k(F))(y)\big) - \mathbb{I}^-\Psi\big((\mathcal{M}_\mu^k(f_\ell))(y)\big)\big\|_\rho \\ &\le c_4\big\|\mathcal{M}_\mu^k(F) - \mathcal{M}_\mu^k(f_\ell)\big\| \le c_4(\theta\mu)^k\|F - f_\ell\| \\ &\le c_5(\theta\mu)^k\|H - h\|_r. \end{aligned} \tag{4.29}$$

Here, F is the family defined in (4.24), and y is an arbitrary point in D_δ.

We note that for $m = \ell + k$, the abovementioned set Λ_m can be taken to be the image of D_δ under the map $Z_\ell \circ Y_{F_0} \circ \ldots \circ Y_{F_{k-1}}$, where $F_n = \mathcal{M}_\mu^n(F)$. Since $Y_{F_n} \to Y_{F^*}$ uniformly on D_δ, there exists a universal constant $b > 0$, such that Λ_m contains a union of balls of radii $c_6 b^m$, whose centers trace out a path of length $\leq c_7 b^{-m}|\beta_n - \beta'|$, connecting β_n and β'. Thus, by using (4.29), together with Cauchy's formula to estimate the derivative of $\beta \mapsto \mathbb{I}^- H_{\beta,m}$ along the path from β_n and β', we obtain the first of the following two bounds:

$$\begin{aligned} \|\mathbb{I}^- H_{\beta_n,m} - \mathbb{I}^- H_{\beta',m}\|_\rho &\leq c_8 (b_2\mu)^m |\beta_n - \beta'|\, \|H - h\|_r\,, \\ \|H_{\beta_n,m} - H_{\beta',m}\|_\rho &\leq c_9 b_3^m |\beta_n - \beta'|\,. \end{aligned} \tag{4.30}$$

The second bound is obtained similarly.

Consider now the curve $\gamma_n = R_{\beta_n} \circ G_n$. Clearly, γ_n is a periodic orbit for a constant multiple of H, with rotation vector w_n. Our next goal is to show that the n–dependence of $\gamma_n(0)$ is mostly due to the translations R_{β_n}. To this end, we split $G_n(0) - \Gamma_{H_{\beta'}}(0)$ into three pieces and use the triangle inequality:

$$\begin{aligned} \bigl|G_n(0) - \Gamma_{H_{\beta'}}(0)\bigr| \leq{}& \bigl|\mathcal{V}_n(H_{\beta_n})\bigl(T_\mu^n(g_n(0))\bigr) - \mathcal{V}_n(H_{\beta_n})(0)\bigr| \\ &+ \bigl|\mathcal{V}_n(H_{\beta_n})(0) - \mathcal{V}_n(H_{\beta'})(0)\bigr| + \bigl|\mathcal{V}_n(H_{\beta'})(0) - \Gamma_{H_{\beta'}}(0)\bigr|\,. \end{aligned} \tag{4.31}$$

The first term on the right hand side of (4.31) can be estimated by using the results from [11], with $H_{\beta'}$ replaced by H_{β_n}, as mentioned above. The relevant fact is that $\mathcal{V}_n(H_{\beta_n})$ is uniformly bounded on a domain whose size decreases exponentially in n, independently of μ, while $|g_n(0)| \leq c_{10}(\theta\mu)^n$, as mentioned earlier. Thus,

$$\bigl|\mathcal{V}_n(H_{\beta_n})\bigl(T_\mu^n(g_n(0))\bigr) - \mathcal{V}_n(H_{\beta_n})(0)\bigr| \leq c_{11}(b_4\mu)^n\,. \tag{4.32}$$

In order to bound the second term on the right hand side of (4.31), we define an interpolating family of transformations $s \mapsto \mathcal{V}_{n,s}(H)$, such that $\mathcal{V}_{n,0}(H) = \mathcal{V}_n(H_{\beta'})$ and $\mathcal{V}_{n,1}(H) = \mathcal{V}_n(H_{\beta_n})$. To obtain $\mathcal{V}_{n,s}(H)$, each of the transformations $\mathcal{U}_{H_{\beta',m}}$ that enter the definition (1.14) of $\mathcal{V}_n(H_{\beta'})$, is replaced by the canonical transformation with generating function

$$\phi_{n,m,s}(H) = \phi(H_{\beta',m}) + s\bigl[\phi(H_{\beta_n,m}) - \phi(H_{\beta',m})\bigr]\,. \tag{4.33}$$

By using the analyticity of the map $H \mapsto \phi(H)$, and the bounds (4.25), (4.29), and (4.30), we find that for $|s| \leq |\lambda_2|^n$,

$$\begin{aligned} \|\phi_{n,m,s}(H)\|'_{r_0} &\leq c_{12}\|\mathbb{I}^- H_{\beta',m}\|_\rho + c_{13}|s|\, \|\mathbb{I}^- H_{\beta_n,m} - \mathbb{I}^- H_{\beta',m}\|_\rho \\ &\quad + c_{14}|s|\, \|\mathbb{I}^- H_{\beta',m}\|_\rho \|H_{\beta_n,m} - H_{\beta',m}\|_\rho \\ &\leq c_{15}(b_5\mu)^m \|H - h\|_r\,. \end{aligned} \tag{4.34}$$

With this bound replacing (4.27), we now obtain the analogue of Lemma 5.5 in [11], which implies that $v(s) = \mathcal{V}_{n,s}(H)(0)$ is bounded in modulus by $c_{16}\|H - h\|_r$, if n is sufficiently large and $|s| \leq |\lambda_2|^n$. Thus, by writing $v(1) - v(0)$ as the integral of v' over $[0,1]$, and estimating v' by using Cauchy's formula with contour $|s| = |\lambda_2|^n$, we obtain the bound

$$\bigl|\mathcal{V}_n(H_{\beta_n})(0) - \mathcal{V}_n(H_{\beta'})(0)\bigr| \leq c_{17}|\lambda_2|^{-n}\|H - h\|_r\,. \tag{4.35}$$

The last term in (4.31) satisfies again a bound of the form (4.32), as was shown in [11]. Putting the pieces together, we now have

$$\left|G_n(0) - \Gamma_{H_{\beta'}}(0)\right| \le c_{17}|\lambda_2|^{-n}\|H-h\|_r + c_{18}(b_6\mu)^n\,, \tag{4.36}$$

provided that H is sufficiently close to h in $\mathcal{B}_r$, and n sufficiently large. Since

$$\pm|\gamma_n(0) - \Gamma'(0)| \le \pm|\beta_n - \beta'| + \left|G_n(0) - \Gamma_{H_{\beta'}}(0)\right|, \tag{4.37}$$

as a result of the identities $\gamma_n(0) = G_n(0) + (0,\beta_n)$ and $\Gamma'(0) = \Gamma_{H_{\beta'}}(0) + (0,\beta')$, the bound (1.21) now follows from (4.25) and (4.36), if H is sufficiently close to h.

■

We conclude with a proof of (1.22), using the same notation as above. By the definition of $\Sigma(w)$, we have $H_{\beta_n,n}\circ g_n = 0$, and thus $H\circ\gamma_n = H_{\beta_n}\circ G_n = a_n H_{\beta_n,n}\circ g_n\circ\Theta^{-n} = 0$, where a_n is some nonzero constant. If U is a canonical transformation with globally defined generating function, or a composition of such transformations, then $K(U\circ\gamma) = K(\gamma)$ for any closed curve γ. The corresponding identity for T_μ is $K(T_\mu\circ\gamma) = \mu K(\gamma)$. As a result, we have $K(G_n) = \mu^n K(g_n) = 0$, and thus $K(\gamma_n) = \tau(w_n)w_n\cdot\beta_n$, which is equivalent to equation (1.22).

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