Universal correlations of one-dimensional interacting electrons in the gas phase

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We consider dynamical correlation functions of short range interacting electrons in one dimension at finite temperature. Below a critical value of the chemical potential there is no Fermi surface anymore, and the system can no longer be described as a Luttinger liquid. Its low temperature thermodynamics is that of an ideal gas. We identify the impenetrable electron gas model as a universal model for the gas phase and present exact and explicit expressions for the asymptotics of correlation functions at small temperatures, in the presence of a magnetic field.

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There are two different phases of a one-dimensional system of interacting electrons. In order to explain the difference, let us first consider zero temperature, $T = 0$:

At zero temperature there exists a critical value μ_c of the chemical potential. If $\mu > \mu_c$, the density of electrons in the ground state is positive. This is the familiar Luttinger liquid phase [1]. Correlations are dominated by fluctuations around the Fermi surface. Conformal field theory [2] describes their power law decay.

If $\mu < \mu_c$, the density at $T = 0$ is zero. This is a special case of the electron gas phase, which will become less trivial at positive temperatures.

What happens at small positive temperatures, $T > 0$?

For $\mu > \mu_c$ the Luttinger liquid phase persists. Now correlations decay exponentially. In order to describe correlations at small temperature one has to employ a conformal mapping from the complex plane to a strip of finite width. As a result the rate of exponential decay is defined by conformal dimensions.

Similarly, the gas phase persists for $\mu < \mu_c$ and $T > 0$. The density of electrons is now positive, but typically exponentially small at small temperatures. The ideal gas law holds. This suggested the name 'gas phase'. Correlations in the gas phase behave essentially different compared to those in the Luttinger liquid phase. This is the subject of this paper.

We shall argue below that the low temperature behaviour of correlation functions in the gas phase is universal for different models. The universal model which determines the asymptotics of correlation functions of one-dimensional electrons with short-range repulsive interaction is the impenetrable electron gas, which is the infinite coupling limit of the electron gas with repulsive delta interaction. The universality class of the impenetrable electron gas comprises a large number of physically interesting systems, most notably the Hubbard model and its non-solvable generalizations.

The impenetrable electron gas model is solvable by Bethe ansatz [3,4]. Based on the Bethe ansatz solution a determinant representation for the dynamical two-point Green functions was obtained in [5], and, subsequently, their asymptotics was calculated in [6,7]. The case of static correlations was treated earlier in [8].

I. THE HUBBARD MODEL IN THE GAS PHASE

In order to get a better understanding of the gas phase let us start with the important example of the Hubbard model,

$$H_H = -\sum_{j=1}^{L}(c_{j+1,\sigma}^+ c_{j,\sigma} + c_{j,\sigma}^+ c_{j+1,\sigma}) + U\sum_{j=1}^{L} n_{j\uparrow}n_{j\downarrow} -\mu\sum_{j=1}^{L}(n_{j\uparrow} + n_{j\downarrow}) \ . \tag{1}$$

Here the canonical Fermi operators $c_{j,\sigma}^+$, $c_{j,\sigma}$ are creation and annihilation operators of electrons at site j of a one-dimensional periodic lattice of length L, and $n_{j,\uparrow}$ and $n_{j,\downarrow}$ are the corresponding particle number operators. U is the strength of the Hubbard repulsion, μ is the chemical potential.

The Hubbard model was solved [9] by means of the nested Bethe ansatz. This allows us to test our ideas about the gas phase quantitatively. The energy levels for the N electron system are

$$E = 2\sum_{j=1}^{N}(1 - \cos k_j) - (\mu + 2)N \ , \tag{2}$$

where the charge momenta k_j are solutions of the Lieb-Wu equations [9]. Clearly the first term on the right hand side of (2) is non-negative. Hence, if $\mu < \mu_c = -2$, the energy of all eigenstates becomes non-negative, and the absolute ground state is the empty lattice. For $\mu > -2$, on the other hand, the energy can be lowered by filling states with small k's. Since $k_{j+1} - k_j \sim 1/L$, this leads to a finite density of electrons in the ground state as $L \to \infty$. We conclude that the Hubbard model is in the gas phase for $\mu < -2$ and in the Luttinger liquid phase else. The asymptotics of correlation functions of the Hubbard model in the Luttinger liquid phase was obtained in [10].

The thermodynamics of the Hubbard model was first considered by Takahashi [11]. He expressed the Gibbs free energy $\omega = -P$ (P pressure) in terms of the dressed energies $\kappa(k)$, $\varepsilon_n(\Lambda)$, $\varepsilon'_n(\Lambda)$, of elementary excitations at finite temperature. $\kappa(k)$ is the dressed energy of particle (or hole) excitations, $\varepsilon_n(\Lambda)$ describes spin excitations and $\varepsilon'_n(\Lambda)$ k-Λ strings. All k-Λ strings are gapped [11]. They do not contribute to the low temperature thermodynamic properties of the Hubbard model [12] and drop out of the equation for the pressure, which simplifies to

$$P = \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln\left(1+e^{-\frac{\kappa(k)}{T}}\right) \ . \qquad (3)$$

Similarly, the integral equations for the dressed energies at low temperatures become

$$\kappa(k) = -\mu - 2\cos k - T\sum_{n=1}^{\infty}\left([n]\ln\left(1+e^{-\frac{\varepsilon_n}{T}}\right)\right)(\sin k) \ , \qquad (4)$$

$$\ln\left(1+e^{\frac{\varepsilon_n}{T}}\right) = -\int_{-\pi}^{\pi} dk\, \cos k\, a_n(\Lambda - \sin k)\ln\left(1+e^{-\frac{\kappa(k)}{T}}\right) + \sum_{m=1}^{\infty}\left(A_{nm}\ln\left(1+e^{-\frac{\varepsilon_n}{T}}\right)\right)(\Lambda) \ , \qquad (5)$$

where $n = 1, 2, 3, \ldots$ in equation (5), and

$$a_n(\Lambda) = \frac{nU/4\pi}{(nU/4)^2 + \Lambda^2} \ . \qquad (6)$$

$[n]$ and A_{nm} are integral operators defined by

$$([0]f)(\Lambda) = f(\Lambda) \ , \qquad (7)$$

$$([n]f)(\Lambda) = \int_{-\infty}^{\infty} d\Lambda'\, a_n(\Lambda-\Lambda')f(\Lambda') \ , \ n = 1,2,\ldots \qquad (8)$$

$$A_{nm} = \sum_{j=1}^{\min\{n,m\}}\left(\left[|n-m|+2(j-1)\right] + \left[|n-m|+2j\right]\right) . \qquad (9)$$

The gas phase is characterized by the absence of a Fermi surface for $\kappa(k)$. Thus $\kappa(k)$ is positive in the zero temperature limit, and the first term on the right hand side of (5) becomes exponentially small in T. Dropping this term, the equations (5) decouple from (4). Since the equations become independent of Λ, it is not hard to solve them. The solution is the same as in the infinite coupling limit $U \to \infty$ (cf. e.g. [4]), $\exp\{\varepsilon_n(\Lambda)/T\} = n(n+2)$. Inserting this solution into (4) we obtain

$$\kappa(k) = -\mu - 2\cos k - T\ln 2 \ . \qquad (10)$$

Our initial assumption that $\lim_{T\to 0}\kappa(k) > 0$ for all k is self-consistent, if $\mu + 2 < 0$, which is precisely the condition for being in the gas phase stated above. With (10) the low temperature expression for the pressure becomes

$$P = \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln\left(1+2e^{\frac{\mu+2\cos k}{T}}\right) \approx \sqrt{\frac{T}{\pi}}\, e^{\frac{\mu+2}{T}} \ , \qquad (11)$$

and we see that density $D = \partial P/\partial\mu$ and pressure P are related by the ideal gas law,

$$P = TD \ . \qquad (12)$$

There are two important lessons to learn from our simple calculation. First, the low temperature limit in the gas phase works the same way as the strong coupling limit at finite temperatures. Second, the low temperature Gibbs free energy $\omega = -P$ in the gas phase shows no signature of the lattice. It is the same as for the impenetrable electron gas (see below), which is a continuum model. This fits well with our intuitive understanding of the gas phase: The mean free path of the electrons is large compared to the lattice spacing Δ, their momentum is small. Their kinetic energy is of the order T. Hence, the effective repulsion is large for $T \ll U$.

II. SCALING

The above arguments suggest that an effective Hamiltonian that describes the Hubbard model in the gas phase can be obtained as the continuum limit of the Hubbard Hamiltonian. Let us introduce the continuous length $\ell = \Delta L$ and coordinates $x = \Delta n$ connected with the nth lattice site. In the continuum limit, $\Delta \to 0$ for fixed ℓ, we obtain canonical field operators $\Psi_\sigma(x)$ for electrons of spin σ as

$$\Psi_\sigma(x) = \lim_{\Delta\to 0}\frac{c_{n,\sigma}}{\sqrt{\Delta}} \ . \qquad (13)$$

Let us perform the rescaling

$$T_H = \Delta^2 T \quad , \quad \mu_H + 2 = \Delta^2\mu \quad , \quad k_H = \Delta k \ , \qquad t_H = t/\Delta^2 \quad , \quad B_H = \Delta^2 B \ , \qquad (14)$$

where k denotes the momentum, t the time and B the magnetic field, which we shall incorporate below. The index 'H' refers to the Hubbard model. Then, in the limit $\Delta \to 0$, we find

$$H_H/T_H = H/T \ . \qquad (15)$$

Here H is the Hamiltonian for continuous electrons with delta interaction,

$$H = \int_{-\ell/2}^{\ell/2} dx\Big\{(\partial_x\Psi_\alpha^+(x))\partial_x\Psi_\alpha(x) + \frac{U}{\Delta}\Psi_\uparrow^+(x)\Psi_\downarrow^+(x)\Psi_\downarrow(x)\Psi_\uparrow(x) - \mu\Psi_\alpha^+(x)\Psi_\alpha(x)\Big\}. \qquad (16)$$

Note that the coupling $c_1 = U/\Delta$ of the continuum model goes to infinity! This is a peculiarity of the one-dimensional system. The effective interaction in the low density phase becomes large. Similar scaling arguments lead to an effective coupling $c_2 = U$ in two dimensions and to $c_3 = \Delta U$ in three dimensions, i.e. unlike one-dimensional electrons three-dimensional electrons in the gas phase are free.

III. UNIVERSALITY

What happens to more general Hamiltonians in the continuum limit? Let us consider Hamiltonians of the form $H_G = H_H + V$, where H_H is the Hubbard Hamiltonian and V contains additional short range interactions. We shall assume that V is a sum of local terms V_j which preserve the particle number. Then V_j contains as many creation as annihilation operators, and the number of field operators in V_j is even. We shall further assume that V_j is hermitian and space parity invariant.

According to equation (13) every field $c_{j,\sigma}$ on the lattice contributes a factor of $\Delta^{1/2}$ in the continuum limit. One factor of Δ is absorbed by the volume element $dx = \Delta$, when turning from summation to integration. Thus, if V_j contains 8 or more fields, then $V \sim \Delta^3$ and V/T_H vanishes. If V_j contains 6 fields, then at least two of the creation operators and two of the annihilation operators must belong to different lattice sites, since otherwise $V_j = 0$. A typical term is e.g. $V_j = c^+_{j,\uparrow} c^+_{j,\downarrow} c^+_{j+1,\uparrow} c_{j+1,\uparrow} c_{j,\downarrow} c_{j,\uparrow}$. In the continuum limit we have $c_{j+1,\uparrow} = \Delta^{1/2}\Psi_\uparrow(x) + \Delta^{3/2}\partial_x\Psi_\uparrow(x) + O(\Delta^{5/2})$. Hence, the leading term vanishes due to the Pauli principle. The next to leading term acquires an additional power of Δ. We conclude that $V \sim \Delta^3$ and thus $V/T_H \to 0$.

If V_j contains 4 fields, then

$$V \sim \Delta^2 \Psi^+_\uparrow(x)\Psi^+_\downarrow(x)\Psi_\downarrow(x)\Psi_\uparrow(x) + O(\Delta^4) \ . \qquad (17)$$

Here the first term on the right hand side is the density-density interaction of the electron gas. In order to arrive at the impenetrable electron gas model the coefficient in front of this term has to be positive. Note that there are no terms of the order of Δ^3 on the right hand side of (17) and thus no other terms than the first one in the continuum limit. Terms of the order of Δ^3 would contain precisely one spatial derivative. They are ruled out, since they would break space parity.

Considering the case, when V_j contains 2 fields, we find, except for the kinetic energy and the chemical potential term, terms which correspond to coupling to an external magnetic field B_H. For these terms to be finite in the continuum limit we have to rescale the magnetic field as $B_H = \Delta^2 B$ (cf. equation (14)).

Our considerations show that the impenetrable electron gas model with magnetic field,

$$H_B = H + B\int_{-\ell/2}^{\ell/2} dx\ \Psi^+_\alpha(x)\sigma^z_{\alpha\beta}\Psi_\beta(x) \ , \qquad (18)$$

is indeed the universal model (for small T) for the gas phase of one-dimensional lattice electrons with repulsive short-range interaction.

IV. IMPENETRABLE ELECTRONS

The impenetrable electron gas is the infinite coupling limit of the electron gas with repulsive delta interaction ($\Delta \to \infty$ in (18)), which was the first model solved by nested Bethe ansatz [3]. The pressure of the system as a function of T, μ and B is known explicitly [4],

$$P = \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\ \ln\left(1 + e^{\frac{\mu+B-k^2}{T}} + e^{\frac{\mu-B-k^2}{T}}\right) \ , \qquad (19)$$

and may serve as thermodynamic potential. The expression (19) is formally the same as for a gas of free spinless fermions with effective (temperature dependent) chemical potential $\mu_{eff} = \mu + T\ln(2\cosh B/T)$. Hence the Fermi surface vanishes for $\lim_{T\to 0}\mu_{eff} = \mu + |B| < 0$. The finite temperature correlation functions of the impenetrable electron gas depend crucially on the sign of μ_{eff}. This allows us to define the gas phase at finite temperature by the condition $\mu_{eff} < 0$, which is also sufficient for deriving the ideal gas law (12) from the low temperature limit of (19). Note that for zero magnetic field and small temperature equation (19) coincides with the rhs of (11).

The time and temperature dependent (two-point) Green functions are defined as

$$G^+_{\uparrow\uparrow}(x,t) = \frac{\mathrm{tr}\left(e^{-H/T}\,\Psi_\uparrow(x,t)\Psi^+_\uparrow(0,0)\right)}{\mathrm{tr}\left(e^{-H/T}\right)} \ , \qquad (20)$$

$$G^-_{\uparrow\uparrow}(x,t) = \frac{\mathrm{tr}\left(e^{-H/T}\,\Psi^+_\uparrow(x,t)\Psi_\uparrow(0,0)\right)}{\mathrm{tr}\left(e^{-H/T}\right)} \ . \qquad (21)$$

For the impenetrable electron gas these correlation functions were represented as determinants of Fredholm integral operators in [5]. The determinant representation provides a powerful tool to study their properties analytically.

The short distance asymptotics can be obtained 'perturbatively' by expanding the determinant representation for small x and t and fixed ratio $k_0 = x/2t$,

$$G^+_{\uparrow\uparrow}(x,t) = \frac{e^{-\frac{i\pi}{4}}}{2\sqrt{\pi t}} + \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi}}(k_0^2 + \mu - B)\sqrt{t} - \left(1 - \frac{1}{\gamma}\right)\frac{D}{\pi} - \left(1 + \frac{1}{\gamma}\right)\frac{D}{2} + O(t) \ , \qquad (22)$$

$$G^-_{\uparrow\uparrow}(x,t) = D_\uparrow + E_\uparrow\, \mathrm{i}t + O\left(t^{3/2}\right) \ . \qquad (23)$$

Here $\gamma = 1 + e^{B/T}$, $D_\uparrow = \partial P/\partial(\mu - B)$ is the density of up-spin electrons and

$$E_{\uparrow} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{(k^2-\mu+B)e^{-\frac{B}{T}}}{2\cosh(B/T)+e^{\frac{k^2-\mu}{T}}} \tag{24}$$

may be interpreted as the 'energy' of the up-spin electrons. In the gas phase at low temperatures the density D becomes exponentially small and the terms proportional to D may be neglected in (22). Then, to lowest order in D, (22) and (23) are the same as for free fermions.

In [6,7] the determinant representation was used to derive a nonlinear classical differential equation, which drives the correlation functions. This equation is closely related to the quantum Hamiltonian (16). It is the separated nonlinear Schrödinger equation. Together with a corresponding Riemann-Hilbert problem it determines the large-time, long-distance asymptotics of the correlators (20), (21). In [6,7] the asymptotics $x, t \to \infty$ was calculated for fixed ratio $k_0 = x/2t$. The crucial parameter for the asymptotics is the average number of particles xD in the interval $[0, x]$. If x is large but $xD \ll 1$ (i.e. T small), an electron propagates freely from 0 to x, and the correlation functions (20), (21) are those of free fermions,

$$G_f^+(x,t) = \frac{e^{-\frac{i\pi}{4}}}{2\sqrt{\pi}}\, t^{-\frac{1}{2}} e^{it(\mu-B)} e^{\frac{ix^2}{4t}} \ , \tag{25}$$

$$G_f^-(x,t) = \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi}}\, e^{\frac{(\mu-B-k_0^2)}{T}}\, t^{-\frac{1}{2}} e^{-it(\mu-B)} e^{-\frac{ix^2}{4t}} \ . \tag{26}$$

The true asymptotic region is characterized by a large number xD of particles in the interval $[0, x]$, specifically, $xD \gg z_c^{-1}$, where $z_c = (T^{3/4} e^{-k_0^2/2T})/(2\pi^{1/4} k_0^{3/2})$. If the latter condition is satisfied, the correlation functions decay due to multiple scattering. The cases $B > 0$ and $B \le 0$ have to be treated separately. For $B > 0$ there is a critical line $x = 4t\sqrt{B}$ in the x-t plane, which separates it into a time and a space like regime. The asymptotics (for small T) in these respective regimes is:

Time like regime ($x < 4t\sqrt{B}$):

$$G_{\uparrow\uparrow}^+(x,t) = G_f^+(x,t)\, \frac{t^{-i\nu(z_c)} e^{-xD_\downarrow}}{\sqrt{4\pi z_c x D_\downarrow}} \ , \tag{27}$$

$$G_{\uparrow\uparrow}^-(x,t) = G_f^-(x,t)\, \frac{t^{i\nu(z_c)} e^{-xD_\downarrow}}{\sqrt{4\pi z_c x D_\downarrow}} \ , \tag{28}$$

where

$$\nu(z_c) = -\frac{2D_\downarrow k_0^{3/2} e^{-k_0^2/2T}}{\pi^{1/4} T^{5/4}} \ , \tag{29}$$

$$D_\downarrow = \frac{1}{2}\sqrt{\frac{T}{\pi}} e^{(\mu+B)/T} \ . \tag{30}$$

$D_\downarrow = \partial P/\partial(\mu+B)$ is the low temperature expression for the density of down-spin electrons.

Space like regime ($x > 4t\sqrt{B}$):

$$G_{\uparrow\uparrow}^+(x,t) = G_f^+(x,t)\, t^{-i\nu(\gamma^{-1})} e^{-xD_\downarrow} \ , \tag{31}$$

$$G_{\uparrow\uparrow}^-(x,t) = G_f^-(x,t)\, t^{i\nu(\gamma^{-1})} e^{-xD_\downarrow} \ , \tag{32}$$

where

$$\nu(\gamma^{-1}) = -\frac{e^{(3B+\mu-k_0^2)/T}}{2\pi} \ . \tag{33}$$

For $B \le 0$ there is no distinction between time and space like regimes. The asymptotics is given by (31), (32).

Equations (27), (28) and (31), (32) are asymptotic expansions in t consisting of an exponential factor, a power law factor and a constant factor. Note that the method employed in [7] allows for a systematic calculation of the next, subleading orders.

The leading exponential factor in (27), (28) and (31), (32) has a clear physical interpretation: Because of the specific form of the infinite repulsion in (16), up-spin electrons are only scattered by down-spin electrons. This is reflected in the fact that the correlation length is $1/D_\downarrow$. The exponential decay means that, due to the strong interaction, an up-spin electron is confined by the gas of surrounding down-spin electrons. Thus we are facing an interesting situation: Although at small distances the electrons look like free fermions, they are confined on a macroscopic scale set by the density $D_\downarrow$.

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