ENTROPY REGULAR SETS

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Abstract. One of the objects of geometric measure theory is to derive global geometric structures from local properties (densities with respect to the s–dimensional Hausdorff measure). In the framework of dynamical system it is more natural to consider entropy measures instead of Hausdorff measures. Our aim is to show that *regular* subshifts (with respect to the entropy measure) necessarily have a special rigid structure. Moreover, their entropy has to be the logarithm of an integer. This parallels the well known fact that regular sets (with respect to the Hausdorff measure) have to have integral dimension.

1. Introduction

A classical subject of geometric measure theory is the investigation of s–sets. These are subsets $F \subset \mathbb{R}^n$ which have non-zero finite s–dimensional Hausdorff measure, $0 < \mathcal{H}^s(F) < \infty$. Clearly, these sets have Hausdorff dimension s. The densities w.r.t. the s–dimensional Hausdorff measure play a major role in these studies. These densities are generalizations of Lebesgue densities. We call an s–sets *regular* if these densities exist $\mathcal{H}^s$-a.e. and are equal to 1. It is well-known (for example see [1, 2]) that regular s–sets must have integral dimension, i.e. if an s–set F is regular then $s \in \mathbb{N}$. Moreover, if F is a regular s–set then it is contained (up to $\mathcal{H}^s$–measure zero) in the countable union of s–dimensional rectifiable hyperplanes. The latter means that a regular set has a rigid geometrical structure close to classical geometric objects.

In this paper we want to develop a similar theory for invariant sets of symbolic dynamical systems. The natural class of measures to be considered are *entropy measures* $\mathcal{E}^s$ instead of Hausdorff measures since the metric structure is substituted by the complexity of the system. In section 2 we introduce these measures and define the corresponding notion of densities. In analogy *entropy s–sets* are those sets which

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have nonzero finite s–dimensional entropy measure. These sets have topological entropy s. Entropy *regular* sets are entropy s–sets for which the densities exist $\mathcal{E}^s$-a.e. and equal 1. We will prove that an entropy regular subshift has entropy $s = \log k$ with $k \in \mathbb{N}$. Moreover, regular entropy subshifts have a rigid dynamical structure. They are (up to $\mathcal{E}^s$–measure zero) *equivariant* subshifts of finite type. This class of subshifts exhibits a very rich dynamical structure and has nice ergodic properties.

Although the statements of our results parallel the results from classical geometric measure theory the methods of proof are different. Instead of geometric arguments we use essentially the invariance of the sets and their ergodic properties.

This paper indicates that beyond classical ergodic theory the finer study of the entropy measures gives rise to deeper insight in the structure of the symbolic dynamical system.

2. Definitions and notations

Let $N \in \mathbb{N}$ and $\Sigma_N = \{0, \ldots, N-1\}^{\mathbb{Z}}$ denote the set of all bi–infinite sequences of symbols $0, \ldots, N-1$. An element of Σ_N will be denoted by $\underline{x} = \cdots x_{-1}x_0x_1 \cdots$ We endow this set with the Tychonov product topology generated by the discrete factor $\{0, \ldots, N-1\}$.

A *cylinder* of *length* $|C_n^m| = m - n + 1$, $m, n \in \mathbb{Z}$, $m \geq n$ is the set

$$C_n^m = C_n^m(i_n, \ldots, i_m) := \big\{\underline{x} \in \Sigma_N : x_j = i_j\, ;\, n \leq j \leq m\big\}$$

where $i_l \in \{0, \ldots, N-1\}$; $n \leq l \leq m$. A cylinder of the form C_{-n}^n will be called a *symmetric* cylinder. For $\underline{x} \in \Sigma_N$ we also write $C_n^m(\underline{x})$ for the cylinder $C_n^m(x_n, \ldots, x_m)$. Obviously, cylinder sets constitute a basis for the topology of Σ_N and also form a basis for the Borel σ–algebra of Σ_N.

On Σ_N we define the following transformation $\sigma\colon \Sigma_N \to \Sigma_N$ via $(\sigma(\underline{x}))_i = x_{i+1}$. This map is a homeomorphism of Σ_N and is called the (left) *shift* on Σ_N.

Definition 2.1. *A* symbolic dynamical system *or equivalently a* subshift *is a σ–invariant closed subset of Σ_N.*

Definition 2.2. *A subshift S is called of* finite type *if it can be represented (possibly over a different alphabet) by a transition matrix $A = (a_{ij})_{0 \leq i,j \leq N'}$ with $a_{i,j} \in \{0, 1\}$ via*

$$S = \big\{\underline{x} : a_{x_i x_{i+1}} = 1 \quad \textit{for all} \quad i \in \mathbb{Z}\big\}.$$

Definition 2.3. *A subshift of finite type is called* equivariant *if there is an alphabet and a number $l \in \mathbb{N}$ such that the sum of the entries of*

A in each column and each row is either 0 or l. The number l is called the index of the equivariant subshift.

Proposition 2.4. *A subshift S is equivariant if and only if there is a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for each nonempty cylinder C^n_{-n} the number of offspring cylinders is constant, i.e. there is a constant $K \in \mathbb{N}$ such that if $n \geq n_0$ and $C^n_{-n}(i_{-n}, \ldots, i_n) \cap S \neq \emptyset$ then*

$$\#\left\{(j,k) : C^{n+1}_{-n-1}(j, i_{-n}, \ldots, i_n, k) \cap C^n_{-n}(i_{-n}, \ldots, i_n) \cap S \neq \emptyset\right\} = K. \tag{1}$$

Proof. The only if direction is obvious. Throughout the proof we will call a cylinder $C^n_{-n}(i_{-n}, \ldots, i_n)$ nonempty if $C^n_{-n}(i_{-n}, \ldots, i_n) \cap S \neq \emptyset$ and we will assume that $n \geq n_0$. Let

$$T_n = \#\big\{(i_{-n}, \ldots, i_n) : C^n_{-n}(i_{-n}, \ldots, i_n) \text{ is nonempty}\big\}.$$

From equation (1) it is clear that $T_{n+1} = KT_n$. By induction we have for any $l \geq 1$ that $T_{n_0+l} = KT_{n_0+l-1} = K^l T_{n_0}$.

Let the symmetric cylinders of length $2n_0+1$ serve as our new alphabet. We enumerate them $\{l_1, \ldots, l_{T_{n_0}}\}$. Suppose $l_i = C^{n_0}_{-n_0}(i_{-n_0}, \ldots, i_{n_0})$ and $l_j = C^{n_0}_{-n_0}(j_{-n_0}, \ldots, j_{n_0})$. Consider the subshift of finite type S_A defined by the transition matrix $A = (a_{i,j})$ with $a_{i,j} = 1$ exactly when $j_k = i_{k+1}$ for $k = -n_0, \ldots, n_0 - 1$. Then using equation (1) we see that

$$\sum_j a_{i,j} \cdot \sum_k a_{k,i} = K \tag{2}$$

for every i. Equation (2) implies that S_A has exactly $l = \sqrt{K}$ (K must be a perfect square) ones in each row and each column, i.e. S_A is an equivariant subshift with index l. If we view S_A in the old alphabet, then the number of nonempty, symmetric cylinders of length $n_0 + l$ is exactly $K^l T_{n_0}$. Furthermore it is clear that $S \subset S_A$. Since they are additionally both invariant and exactly of the same size they must be equal. □

The next lemma is a simple observation.

Lemma 2.5. *Let S and S' be two equivariant subshifts with the same index. If $S \subset S'$ then each transitive component of S is also a transitive component of S'.*

Proof. We assume that the conclusion of the lemma does not hold. Then there is an $\underline{x} \in S$ such that for all $n \in \mathbb{N}$ the cylinder $C^n_{-n}(\underline{x}) \cap S'$ contains a point $\underline{y} \notin S$. But this means that there is a $k \in \mathbb{N}$ such that

$$\#\left\{C^{n+k}_{-n-k}(\underline{y}) : \underline{y} \in C^n_{-n}(\underline{x}) \cap S'\right\} > \#\left\{C^{n+k}_{-n-k}(\underline{y}) : \underline{y} \in C^n_{-n}(\underline{x}) \cap S\right\}.$$

For large enough n this contradicts the assumption that S and S' have the same index. $\square$

Now we are going to define a family of outer measures on Σ_N. Let $s \in \mathbb{R}$ be a nonnegative real number and $F \subset \Sigma_N$. For $m > 0$ an outer measure is given by

$$\mathcal{E}^s_m(F) := \inf \left\{ \sum_i e^{-s|C_i|} : \bigcup_i C_i \supset F \, ; \, |C_i| \geq m \right\}.$$

Remark 2.6. *In the above definition we used cylinders not necessarily centered at 0. This is due to the convention that the topological entropy of noninvariant sets should be independent of the choice of the origin and the direction of the evolution.*

The *s–dimensional entropy* (outer) *measure* is defined as

$$\mathcal{E}^s(F) := \lim_{m\to\infty} \mathcal{E}^s_m(F).$$

Definition 2.7. *A subset $S \in \Sigma_N$ is called an* entropy s–set *if* $0 < \mathcal{E}^s(S) < \infty$. *If S is in addition a subshift we call it an* entropy s–shift.

Remark 2.8. *The topological entropy $h_{top}(F)$ of a subset $F \subset \Sigma_N$ (w.r.t. σ) can be defined as $h_{top}(F) := \sup\{s : \mathcal{E}^s(F) = \infty\} = \inf\{s : \mathcal{E}^s(F) = 0\}$ (for example see* [3]*). Hence, entropy s–sets have topological entropy s. If we define the entropy of a (finite) measure μ to be $h_\mu := \inf\{h_{top}(F) : \mu(\Sigma_N \setminus F) = 0\}$ then the restriction of $\mathcal{E}^s$ to an entropy s–set has entropy s. For an equivariant subshift of index l the number of cylinders of length n is up to a constant l^n. Hence, its topological entropy equals* $\log l$.

The *upper* and *lower densities* of an entropy s–set S at a point $\underline{x} \in \Sigma_N$ are defined as

$$\overline{D}^s(S, \underline{x}) := \limsup_{n\to\infty} \frac{\mathcal{E}^s(S \cap C^n_{-n}(\underline{x}))}{e^{-s(2n+1)}}$$

and

$$\underline{D}^s(S, \underline{x}) := \liminf_{n\to\infty} \frac{\mathcal{E}^s(S \cap C^n_{-n}(\underline{x}))}{e^{-s(2n+1)}}$$

respectively. If $\overline{D}^s(S, \underline{x}) = \underline{D}^s(S, \underline{x})$ we say that the density of S at $\underline{x}$ exists and write $D^s(S, \underline{x})$ for the common value.

Definition 2.9. *A* regular point *of an entropy s–set is a point at which $\overline{D}^s(S, \underline{x}) = \underline{D}^s(S, \underline{x}) = 1$. An entropy s–set is called* regular *if $\mathcal{E}^s$-a.e. of its points is regular.*

3. A density lemma

This section contains a density lemma which is standard in $\mathbb{R}^n$. The proof in our situation is much simpler since the corresponding version of a Besicovitch covering result is almost trivial.

Lemma 3.1. *Let μ be a finite Borel measure on Σ_N, $f \in L^1(\Sigma_N, \mu)$. Then for μ-a.e. point $\underline{x}$*

$$\lim_{n\to\infty} \frac{1}{\mu(C^n_{-n}(\underline{x}))} \int_{C^n_{-n}(\underline{x})} f\, d\mu = f(\underline{x}).$$

Proof. Let $g \in L^1$ and $\hat{g}(\underline{x}) := \sup_{n\in\mathbb{N}} \frac{1}{\mu(C^n_{-n}(\underline{x}))} \int_{C^n_{-n}(\underline{x})} g\, d\mu$. We claim the following maximal inequality holds:

$$\mu\left\{\underline{x} : \hat{g}(\underline{x}) > r\right\} < \frac{1}{r} \int_{\Sigma_N} g\, d\mu.$$

For this let $B := \{\underline{x} : \hat{g}(\underline{x}) > r\}$. For $\underline{x} \in B$ there is an $n(\underline{x})$ with $\frac{1}{\mu(C^{n(\underline{x})}_{-n(\underline{x})}(\underline{x}))} \int_{C^{n(\underline{x})}_{-n(\underline{x})}(\underline{x})} g\, d\mu > r$. We can cover B by disjoint cylinders $\left\{C^{n(\underline{y}_i)}_{n(\underline{y}_i)}(\underline{y}_i) : i \in \mathbb{N}\right\}$. Hence,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(C^{n(\underline{y}_i)}_{-n(\underline{y}_i)}(\underline{y}_i)) < \sum_{i=1}^{\infty} \frac{1}{r} \int_{C^{n(\underline{y}_i)}_{-n(\underline{y}_i)}(\underline{y}_i)} g\, d\mu \leq \frac{1}{r} \int_{\Sigma_N} g\, d\mu.$$

Now let $f \in L^1$ and $\epsilon > 0$. Then we can find a continuous function h_ϵ with $\int_{\Sigma_N} |f - h_\epsilon|\, d\mu < \epsilon^2$ and $h_\epsilon(\underline{x}) = f(\underline{x})$ outside a set of measure ϵ. Since $g := |f - h_\epsilon|$ is in L^1 the maximal inequality implies that $\mu\left(\{\underline{x} : \lim_{n\to\infty} \frac{1}{\mu(C^n_{-n}(\underline{x}))} \int_{C^n_{-n}(\underline{x})} |f - h_\epsilon|\, d\mu > \epsilon\}\right) < \epsilon$. But for a continuous function the statement of the lemma holds trivially. Therefore, letting ϵ tend to zero in the last inequality yields the assertion of the lemma. $\square$

4. Basic properties of entropy densities

In this section we state simple facts that correspond to well–known facts for the Hausdorff measure. The proofs are based on exactly the same arguments and therefore are omitted.

Lemma 4.1. *$\mathcal{E}^s$ is an outer measure whose σ–algebra of measurable sets include the Borel sets. Moreover, $\overline{D}^s(S, \underline{x})$ and $\underline{D}^s(S, \underline{x})$ are Borel–measurable functions.*

Proposition 4.2. *Let S be an entropy s–set. Then $\overline{D}^s(S, \underline{x}) = 0$ for all $\underline{x} \in \Sigma_N \setminus S$ and $\overline{D}^s(S, \underline{x}) = 1$ for $\mathcal{E}^s$-a.e. $\underline{x} \in S$.*

Proof. Both statements are proved in exactly the same way as theorem 2.3. in [1]. □

Corollary 4.3. *If F is a measurable subset of an entropy s–set S then* $\overline{D}^s(F,\underline{x}) = \overline{D}^s(S,\underline{x})$ *and* $\underline{D}^s(F,\underline{x}) = \underline{D}^s(S,\underline{x})$ *at* $\mathcal{E}^s$*-a.e.* $\underline{x} \in F$.

Corollary 4.4. *Any subset of positive measure of an entropy regular set is regular.*

5. Properties of entropy measures

In this section we derive the preparatory results needed for the proof of our main theorem. These results are different from those for the Hausdorff measure. They reflect the dynamical nature of the entropy measure. The following proposition is an immediate consequence of the definition of entropy measures.

Proposition 5.1. *Let S be an entropy s–subshift. Then the restriction of $\mathcal{E}^s$ to S is invariant.*

Proof. Fix a subset $F \subset S$. We have to show that $\mathcal{E}^s(F) = \mathcal{E}^s(\sigma(F))$. By definition for $\epsilon > 0$ there is an arbitrary large m and a cover $\{C_1, C_2, \dots\}$ of F of cylinders of length at least m such that

$$\sum_i e^{-s|C_i|} - \epsilon \leq \mathcal{E}^s(F) \leq \sum_i e^{-s|C_i|} + \epsilon.$$

Now $\{\sigma(C_1), \sigma(C_2), \dots\}$ covers $\sigma(F)$ with $\sum_i e^{-s|C_i|} = \sum_i e^{-s|\sigma(C_i)|}$. Hence, $\mathcal{E}^s(\sigma(F)) \leq \sum_i e^{-s|C_i|} \leq \mathcal{E}^s(F) + \epsilon$. Letting $\epsilon \to 0$ we obtain $\mathcal{E}^s(\sigma(F)) \leq \mathcal{E}^s(F)$. The other inequality is similarly derived. Finally, the invariance of S implies $\mathcal{E}^s(S) = \mathcal{E}^s(\sigma(S))$. □

Since the topological entropy of an entropy s–subshift S equals the entropy of $\mathcal{E}^s|_S$ equals s the following corollary is immediate.

Corollary 5.2. *The restriction of $\mathcal{E}^s$ to an entropy s–shift S is a measure of maximal entropy.*

6. Properties of entropy regular sets

In this section we investigate regular entropy s–subshifts. We will prove that these subshifts have a very special structure.

We need the following lemma.

Lemma 6.1. *Let S be a regular entropy s–subshift. Then for every $k \in \mathbb{N}$ at $\mathcal{E}^s|_S$-a.e. $\underline{x} \in S$ the limit*

$$\lim_{n\to\infty} \frac{\mathcal{E}^s(S \cap C^n_{-n}(\underline{x}))}{\mathcal{E}^s(S \cap C^{n+k}_{-n-k}(\underline{x}))}$$

exists and equals e^{2sk}.

Proof. Fix $\epsilon > 0$ and let

$$n_1(\underline{x}) := \inf \left\{ n \in \mathbb{N} : \quad 1 - \epsilon < \frac{\mathcal{E}^s(S \cap C^m_{-m}(\underline{x}))}{e^{-s(2m+1)}} < 1 + \epsilon, \quad \forall m \geq n \right\}.$$

We note that at $\mathcal{E}^s|_S$-a.e. $\underline{x} \in S$ the number $n_1(\underline{x})$ is finite. Hence, for a.e. $\underline{x} \in S$ and for $n > n_1(\underline{x})$

$$\left(\frac{1-\epsilon}{1+\epsilon}\right) e^{2sk} < \frac{\mathcal{E}^s(S \cap C^n_{-n}(\underline{x}))}{\mathcal{E}^s(S \cap C^{n+k}_{-n-k}(\underline{x}))} < \left(\frac{1+\epsilon}{1-\epsilon}\right) e^{2sk}.$$

Since ϵ was arbitrary our proof is complete. □

Lemma 6.2. *Given* $\epsilon > 0$ *there is a subset* $\Lambda \subset S$ *with* $\mathcal{E}^s|_S(S \setminus \Lambda) = 0$ *and a measurable function* $n_2(\underline{x}, \epsilon)$ *such that for all* $\underline{x} \in \Lambda$ *and all* $n > n_2(\underline{x}, \epsilon)$, $k \in \mathbb{N}$ *the following holds*

$$(1-\epsilon)\mathcal{E}^s(S \cap C^n_{-n}(\underline{x})) < \sum_{Y_{n,k}(\underline{x})} \mathcal{E}^s(S \cap C^{n+k}_{-n-k}(\underline{y})) < \mathcal{E}^s(S \cap C^n_{-n}(\underline{x}))$$

where $Y_{n,k}(\underline{x}) := \{C^{n+k}_{-n-k}(\underline{y}) : \underline{y} \in C^n_{-n}(\underline{x}) \, , \, n_1(\underline{y}) < n_2(\underline{x}, e)\}$

Proof. We define $\Omega_M := \{\underline{x} \in S : n_1(\underline{x}) \leq M\}$. Since S is a regular entropy s–set we have that $\lim_{M\to\infty} \mathcal{E}^s(S \setminus \Omega_M) = 0$. By the density lemma 3.1 a.e. point in Ω_M is a density point (choose f to be the indicator function of Ω_M), i.e. for a.e. point $\underline{x} \in \Omega_M$ there is a number $n_2(\underline{x}, \epsilon) \geq M$ such that $\mathcal{E}^s(S \cap C^m_{-m}(\underline{x}) \cap \Omega_M) > (1 - \epsilon)\mathcal{E}^s(S \cap C^m_{-m}(\underline{x}))$ for $m \geq n_2(\underline{x}, \epsilon)$. Fix $n > n_2(\underline{x}, \epsilon)$. Cylinder sets which are centered at 0 are either disjoint or one is contained in the other. Thus for given $k \in \mathbb{N}$ there exists $L = L(\underline{x}, k)$ such and points $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_L \in \Omega_M \cap C^n_{-n}(\underline{x})$ with $C^{n+k}_{-n-k}(\underline{y}_i)$ pairwise disjoint and $\mathcal{E}^s(S \cap \Omega_M \cap C^n_{-n}(\underline{x})) = \sum_{i=1}^L \mathcal{E}^s(S \cap \Omega_M \cap C^{n+k}_{-n-k}(\underline{y}_i))$. We note that all the cylinders $C^{n+k}_{-n-k}(\underline{y}_i) \in Y_{n,k}$ since $n_1(\underline{y}_i) \leq M \leq n_2(\underline{x}, \epsilon)$. Hence,

$$\begin{aligned}
(1-\epsilon)\mathcal{E}^s(S \cap C^n_{-n}(\underline{x})) &< \mathcal{E}^s(S \cap \Omega_M \cap C^n_{-n}(\underline{x})) \\
&= \sum_{i=1}^{L} \mathcal{E}^s(S \cap \Omega_M \cap C^{n+k}_{-n-k}(\underline{y}_i)) \\
&\leq \sum_{Y_{n,k}(\underline{x})} \mathcal{E}^s(S \cap C^{n+k}_{-n-k}(\underline{y})) \\
&< \mathcal{E}^s(S \cap C^n_{-n}(\underline{x})).
\end{aligned}$$

This proves the assertion for $\underline{x} \in \Omega_M$. The lemma follows by letting M tend to ∞. □

Now we are ready to prove the next proposition which will be improved later.

Proposition 6.3. *Let S be a regular entropy s–subshift. Then $e^{2s} \in \mathbb{N}$.*

Proof. Fix $\epsilon > 0$ and choose $\underline{x} \in S$ such that the conclusions of lemma 6.2 hold. Suppose $n \geq \max(n_1(\underline{x}), n_2(\underline{x}, \epsilon))$. Then

$$(1-\epsilon)\mathcal{E}^s(S \cap C^n_{-n}(\underline{x})) < \sum_{Y_{n,1}(\underline{x})} \mathcal{E}^s(S \cap C^{n+1}_{-n-1}(\underline{y})) < \mathcal{E}^s(S \cap C^n_{-n}(\underline{x})).$$

Now, noting that $C^n_{-n}(\underline{y}) = C^n_{-n}(\underline{x})$ lemma 6.1 implies

$$(1-\epsilon)^3 e^{-s(2n+1)} < \#Y_{n,1} e^{-s(2n+3)} < (1+\epsilon)^2 e^{-s(2n+1)}$$

or $\#Y_{n,1} \in [(1-\epsilon)^3 e^{2s}, (1+\epsilon)^2 e^{2s}]$ for all $n \geq \max(n_1(\underline{x}), n_2(\underline{x}, \epsilon))$. Since $\#Y_{n,k}$ is always an integer we have that $[(1-\epsilon)^3 e^{2s}, (1+\epsilon)^2 e^{2s}] \cap \mathbb{N} \neq \emptyset$. Letting ϵ approach zero we conclude that $e^{2s} \in \mathbb{N}$. □

This enables us to prove the final step in our main result.

Theorem 6.4. *Let S be a regular entropy s–subshift. Then S is an equivariant subshift of index e^s up to $\mathcal{E}^s$ measure zero, i.e. there is an equivariant subshift S' with $\mathcal{E}^s(S' \triangle S) = 0$..*

Proof. Let Λ be the set defined in lemma 6.2. For $\epsilon > 0$ define

$$\Lambda_\epsilon := \{\underline{x} \in \Lambda : n_2(\underline{x}, \epsilon) \leq n(\epsilon)\}$$

where $n(\epsilon)$ is an increasing function of ϵ so that $\Lambda_{\epsilon'} \supset \Lambda_\epsilon$ for $0 < \epsilon' < \epsilon$ and is chosen so large that because of lemma 6.2 $\mathcal{E}^s(\Lambda_\epsilon) > \mathcal{E}^s(S) - \epsilon$ holds. Let $S_\epsilon := \bigcup_{j \in \mathbf{Z}} \sigma^j(\Lambda_\epsilon)$ be the invariant hull of Λ_ϵ. Note that $S_\epsilon \subset S$ since S is invariant. The proof of proposition 6.3 implies that if ϵ is sufficiently small then for $\underline{x} \in \Lambda_\epsilon$ and $n > n(\epsilon)$ the number of cylinders in $Y_{n,1}$ is e^{2s}. This statement immediately implies that S_ϵ fulfills the assumptions of proposition 2.4 and hence is an equivariant subshift of index e^s. The same holds for $S_{\epsilon'}$ and moreover $S_\epsilon \subset S_{\epsilon'}$ if $\epsilon > \epsilon' > 0$. Hence, by lemma 2.5 $S_{\epsilon'}$ is the disjoint union of S_ϵ and some transitive components.

Now we need to estimate the entropy measure of a single transitive component. Let R be a transitive component of an equivariant subshift of index l. Then R itself is a (transitive) equivariant subshift of index l. Let n_0 be the number introduced in proposition 2.4 for the subshift R. The proof of proposition 6.3 can be trivially modified to show that for fixed k if ϵ is sufficiently small then for $\underline{x} \in \Lambda_\epsilon$ and $n > n(\epsilon)$ the number of cylinders in $Y_{n,k}$ is e^{2sk}. Then for $n > n_0$, $k = n - n_0$ and

$\underline{x} \in R$

$$\mathcal{E}^{\log l}(C^{n_0}_{-n_0}(\underline{x})) = \mathcal{E}^{\log l}_n(C^{n_0}_{-n_0}(\underline{x})) = \#Y_{n_0,n-n_0} \cdot e^{-(2n+1)\log l} = l^{-(2n_0+1)} \tag{3}$$

and

$$\mathcal{E}^{\log l}(R) = T_{n_0} l^{-(2n_0+1)}.$$

Let T_{3n+1} be of nonempty cylinders $C^{3n+1}_{-3n-1} \in R$. Clearly T_{3n+1} does not exceed the number of cylinders $C^{3n+1}_{-3n-1}(i_{-3n-1}, \cdots, i_{3n+1})$, where $C^n_{-n}(i_{-3n-1}, \cdots, i_{-n-1})$, $C^n_{-n}(i_{-n}, \cdots, i_n)$ and $C^n_{-n}(i_{n+1}, \cdots, i_{3n+1})$ are nonempty cylinders in R. In particular, $T^3_{n_0} \geq T_{3n_0+1}$. Using the proof of proposition 2.4 (note $K = l^2$) we have $T_{3n_0+1} = l^{2(2n_0+1)} T_{n_0}$. Combining the last two equations yields $T_{n_0} \geq l^{2n_0+1}$. Combining this with equation (3) implies that

$$\mathcal{E}^{\log l}(R) \geq 1.$$

If the difference of two equivariant subshifts is nonempty then it contains an open set U. By lemma 2.5 the orbit of U is disjoint from the smaller subshift, hence it must contain a full transitive component. We apply this reasoning to conclude that $S_{\epsilon'} \setminus S_\epsilon$ contains a transitive component for $\epsilon > \epsilon' > 0$.

If R is a transitive component contained in $S_{\epsilon'} \setminus S_\epsilon$ we have that $\mathcal{E}^{\log l}(S_{\epsilon'} \setminus S_\epsilon) \geq \mathcal{E}^{\log l}(R) \geq 1$, and thus $\mathcal{E}^{\log l}(S_{\epsilon'}) \geq 1 + \mathcal{E}^{\log l}(S_\epsilon)$. Hence the measure of S is bounded below by the number of different S_ϵ. Since the measure of S is bounded there are only finitely many different S_ϵ, i.e. there is an ϵ_0 such that for all $0 < \epsilon < \epsilon_0$ the sets $S_{\epsilon_0} =: S' \subset S$ coincide. Hence $\mathcal{E}(S \setminus S') < \epsilon$ for all $\epsilon > 0$. This completes the proof of the theorem. ☐

Since the index of an equivariant subshift is always a natural number we obtain.

Corollary 6.5. *There is no regular s–shift unless e^s is a natural number.*

Remark 6.6. *In all the proofs we did not need the particular structure of the outer measure $\mathcal{E}^s$. We only used that this measure is invariant, nonzero and finite and has the following scaling property on regular sets:* $\lim_{n\to\infty} \frac{\mathcal{E}^s(C^n_{-n}(\underline{x}))}{\mathcal{E}^s(C^{n+1}_{-n-1}(\underline{x}))}$ *exists a.e. and is constant a.e. Therefore we can derive the following statement by similar proofs.*

Theorem 6.7. *Let S be a subshift and μ an invariant probability measure with the property that the limit* $\lim_{n\to\infty} \frac{\mu(C^n_{-n}(\underline{x}))}{\mu(C^{n+1}_{-n-1}(\underline{x}))} = 2c$ *exists a.e. Then $c \in \mathbb{N}$ and $S_\mu := supp\,\mu$ is an equivariant subshift of index c with measure of maximal entropy μ.*

Proof. The only statement left to prove is that μ is the measure of maximal entropy. If $\underline{x} \in \operatorname{supp}\mu$ then $\lim_{n\to\infty} \frac{\mu(C^{n}_{-n}(\underline{x}))}{\mu(C^{n+1}_{-n-1}(\underline{x}))} = 2c$ implies that

$$\lim_{n\to\infty} -\frac{1}{2n+1} \log \mu(C^{n}_{-n}(\underline{x})) = \log c \quad \text{for all} \quad x \in \operatorname{supp}\mu.$$

By the Shannon–McMillan–Breiman theorem this number equals the metric entropy of μ. On the other hand (copying the previous proofs) c is the index of the equivariant subshift $\operatorname{supp}\mu$ and hence its topological entropy is $\log c$. □

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