ZETA FUNCTIONS AND TRANSFER OPERATORS FOR MULTIDIMENSIONAL PIECEWISE AFFINE AND EXPANDING MAPS

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ABSTRACT. Let $X \subset \mathbb{R}^2$ be a finite union of bounded polytopes and let $T : X \to X$ be piecewise affine and eventually expanding. Then the Perron-Probenius operator $\mathcal{L}$ of T is quasicompact as an operator on the space of functions of bounded variation on $\mathbb{R}^2$ and its isolated eigenvalues (including multiplicities) are just the reciprocals of the poles of the dynamical zeta function of T. In higher dimensions the result remains true under an additional generically satisfied transversality assumption.

1. INTRODUCTION

We present a (partial) generalization of the one-dimensional result of Baladi and Keller [1] to the following multidimensional setting. Let $(X, \mathcal{Z}, T)$ be a piecewise affine map of some compact subset $X \subseteq \mathbb{R}^d$. By this we mean that $T : X \to X$ can be described in the following way: $\mathcal{Z}$ is a finite collection of bounded open polytopes Z_i, $i = 1, \ldots, N$, (i.e. finite intersections of open halfspaces) such that X is the closure of $\bigcup_{i=1}^N Z_i$, and there are invertible affine maps $T_i : \mathbb{R}^d \to \mathbb{R}^d$ such that $T|Z_i = T_i|Z_i$ for $i = 1, \ldots, N$. Since these assumptions do not fix the behaviour of T on the boundaries ∂Z_i, we essentially study the dynamical system $T|Y$ where $Y := \bigcap_{n=0}^\infty T^{-n}(\bigcup_{i=1}^N Z_i)$ is a subset of full d-dimensional Lebesgue measure of X.

Denote by $\mathrm{mb}(X)$ the vector space of measurable functions $f : X \to \mathbb{C}$. With the system $(X, \mathcal{Z}, T)$ we associate a transfer operator

$$\mathcal{L} : \mathrm{mb}(X) \to \mathrm{mb}(X)$$

defined by

$$\mathcal{L}f(x) := \sum_{i=1}^N |\det(DT_i)|^{-1} \cdot f(T_i^{-1}x) \cdot 1_{TZ_i}(x) = \sum_{y \in T^{-1}x} |\det(DT)(y)|^{-1} f(y)$$

which, as we shall see later, acts as a bounded linear operator on BV, the space of functions of bounded variation on $\mathbb{R}^d$ with support in X. (This space is defined in some detail below.)

With $(X, \mathcal{Z}, T)$ we also associate the zeta-function

$$\zeta(z) := \exp \sum_{n=1}^\infty \frac{z^n}{n} \zeta_n$$

where

$$\zeta_n := \sum_{\substack{x \in Y \\ T^n x = x}} g_n(x)$$

with

$$g_1(x) := g(x) := |\det(DT)(x)|^{-1}, \qquad g_n(x) := g(T^{n-1}x) \cdot \cdots \cdot g(Tx) \cdot g(x) \ .$$

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Finally we need the following quantity in order to state our main result relating spectral properties of $\mathcal{L} : BV \to BV$ and poles of $\zeta(z)$:

$$\vartheta := \lim_{n\to\infty} \left(\sup_{x\in Y} \|DT^n(x)^{-1}\| \right)^{\frac{1}{n}}$$

ϑ^{-1} is just the minimal asymptotic expansion rate of T. Therefore, T is eventually expanding if and only if $\vartheta < 1$. Our main result is

Theorem 1. *Let $(X, \mathcal{Z}, T)$ be an eventually expanding piecewise affine dynamical system as introduced above. Assume that $d = 2$ or, in any dimension, that T is generic in the sense of eq. (1) below. Then*

a) $\mathcal{L} : BV \to BV$ is a bounded linear operator with spectral radius $r(\mathcal{L}) = 1$ and with essential spectral radius $r_{\mathrm{ess}}(\mathcal{L}) \leq \vartheta < 1$.
b) $\zeta(z)$ is meromorphic in $\{z : |z| < \vartheta^{-1}\}$.
c) $z_0 \in \{z : |z| < \vartheta^{-1}\}$ is a pole of ζ of multiplicity p if and only if z_0^{-1} is an isolated eigenvalue of $\mathcal{L}$ of algebraic multiplicity p, i.e. $p = \dim(\bigcup_{n=1}^{\infty} \ker(\mathrm{Id} - z_0\mathcal{L})^n)$.

We say that T is *generic* if, setting $\mathcal{Z}_n = \{A_0 \cap T^{-1}A_1 \cap \cdots \cap T^{-n+1}A_{n-1} \neq \emptyset : A_0, \dots, A_{n-1} \in \mathcal{Z}\}$, we have

$$0 = h_{\mathrm{mult}}(T, \mathcal{Z}) := \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in X} \#\{Z \in \mathcal{Z}_n : x \in \overline{Z}\}, \tag{1}$$

i.e., if there is no exponential accumulation of discontinuities. This condition, though it is not always satisfied (see [6, section 2.2]), is satisfied in many cases. Especially,

- ▹ for all multi-dimensional β-transformations, i.e., all maps of the form

$$T : [0,1)^d \to [0,1)^d, \quad T(x) = B(x) \mod \mathbb{Z}^d \text{ with } B : \mathbb{R}^d \to \mathbb{R}^d \text{ affine},$$

  by [6, Lemma 1].
- ▹ for almost all piecewise affine maps by [10, Theorem 1.1].

Remark 1. $\mathcal{L}$ is just the classical Perron-Frobenius operator for T. As T is eventually expanding, $r_{\mathrm{ess}}(\mathcal{L}) \leq \vartheta < 1 = r(\mathcal{L})$ so that the spectrum of $\mathcal{L}$ consists of isolated eigenvalues of finite multiplicity in $\{\lambda : r_{\mathrm{ess}}(\mathcal{L}) < |\lambda| \leq 1\}$ and the essential spectrum is contained in $\{\lambda : |\lambda| \leq \vartheta\}$.

Remark 2. Statement a) of Theorem 1 is only included for completeness. For $d = 2$ it is a special case of a result which was proved independently in [21] and, in a slightly modified form that we will refer to in Theorem 2, in [8]. If $d = 2$ and if all T_i are similarities it is contained in the results of [15]. The generic case, though not considered in [21], follows easily from the proofs there.

Remark 3. Under a more restrictive expansion condition M. Mori [18, 19] characterizes the isolated eigenvalues of $\mathcal{L}$ acting on an invariant subspace of BV as zeroes of a symbolically defined Fredholm determinant associated with the map. In fact, the action of $\mathcal{L}$ on this space is close to the action of our matrix K in Theorems 5-7.

In the next section we state a number of auxiliary results which lead in the end to the proof of this theorem. The proofs of these auxiliary results are given in sections 3–7.

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2. Outline of the proof of the main theorem

Let $(X, \mathcal{Z}, T)$ be an eventually expanding piecewise affine map as in the statement of Theorem 1, i.e., either plane or generic. For $f \in L^1(\mathbb{R}^d, \mathbb{R})$ (the space of equivalence classes of Lebesgue integrable real valued functions on $\mathbb{R}^d$) let

$$\mathrm{var}(f) := \sup \left\{ \int f \cdot \mathrm{div}\, \varphi \, dm : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\} .$$

Here m denotes Lebesgue measure on $\mathbb{R}^d$, $C_c^1(\mathbb{R}^d,\mathbb{R}^d)$ is the space of continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ with compact support, $\operatorname{div}\varphi=\sum_{i=1}^d \frac{\partial\varphi_i}{\partial x_i}$ denotes the divergence of the vector field φ, and $\|\varphi\|_\infty := \sup\{|\varphi(x)|_2 : x\in\mathbb{R}^d\}$ where $|.|_2$ is the Euclidean norm on $\mathbb{R}^d$. Let

$$BV := \{f\in L^1(X,\mathbb{R}) : \operatorname{var}(f)<\infty\}\ .$$

(Here we identify $f\in L^1(X,\mathbb{R})$ with its extension by zero to all of $\mathbb{R}^d$.) Since $\operatorname{var}(f)$ measures the variation of the zero extension of f to $\mathbb{R}^d$, the trace theorem for variation shows that $\operatorname{var}(f)$ is the sum of the variation of f inside X and the "jumps" of $|f|$ on ∂X. For details see [12, 22, 11].

Remark 4. As a space of real valued functions, BV is a real vector space and must be complexified in order to admit the usual notion of spectrum of an operator $\mathcal{L}: BV\to BV$. Since this procedure is absolutely standard, we will not mention it explicitly and also refrain from introducing an extra notation for the complexification. Spectral theoretic statements for operators on other spaces of real valued functions have to be interpreted in the same way.

There is another approach to the concept of variation which at first sight might seem just less elegant, but which in some settings is technically more suitable for investigating dynamical systems, see e.g. [3, 4, 16, 20]. In our case, Saussol's [20] version of this variation (which is equivalent to that of Blank [4]) fits favourably into our scheme of proofs. Here is its definition:

Let $\epsilon_0>0$ be small enough (see the proof of Theorem 3). For $f\in L^1(\mathbb{R}^d,\mathbb{R})$ and a Borel subset S of $\mathbb{R}^d$ define the oscillation of f on S by

$$\operatorname{osc}(f,S) := \operatorname*{ess\,sup}_{x\in S} f(x) - \operatorname*{ess\,inf}_{x\in S} f(x)\ .$$

Denote by $B_\epsilon(x)$ the open ball of radius ϵ centered at x. As $x\mapsto \operatorname{osc}(f,B_\epsilon(x))$ is lower semi-continuous and hence measurable, one can define

$$\mathfrak{var}(f) := \sup_{0<\epsilon\le\epsilon_0} \epsilon^{-1}\int_{\mathbb{R}^d} \operatorname{osc}(f,B_\epsilon(x))\,dx\ .$$

Let

$$\mathfrak{BV} := \left\{f\in L^1(X,\mathbb{R}) : \mathfrak{var}(f)<\infty\right\}\ .$$

(Again we identify $f\in L^1(X,\mathbb{R})$ with its extension by zero to $\mathbb{R}^d$.) $\mathfrak{BV}$ is a proper subspace of BV, namely

Lemma 1. *$\mathfrak{BV}\subseteq BV\cap L^\infty(X)$, $\|f\|_\infty \le \mathrm{const}_d\cdot\epsilon_0^{1-d}\cdot\mathfrak{var}(f)$, and $\operatorname{var}(f)\le 2d^{\frac{3}{2}}\cdot\mathfrak{var}(f)$ for all $f\in\mathfrak{BV}$ (whereas $BV\not\subseteq L^\infty(X)$ in dimension $d\ge 2$).*

This domination of $\mathfrak{var}(f)$ over $\|f\|_\infty$ will allow us to use the thermodynamical estimate below, something which we could not do with the more usual variation $\operatorname{var}(f)$.

Although for bounded X $\operatorname{var}(f)=0$ if and only if $f=0$ m-a.e., and the same holds for $\mathfrak{var}(f)$, it is convenient to introduce the following norms on BV and $\mathfrak{BV}$ respectively:

$$\|f\|_{BV} := \operatorname{var}(f)+\|f\|_1,\qquad \|f\|_{\mathfrak{BV}} := \mathfrak{var}(f)+\|f\|_1\ .$$

With these norms, both spaces are Banach spaces, see the above cited literature.

The first step in the proof of Theorem 1 is to show that it suffices to prove a $\mathfrak{BV}$-version of this result:

Theorem 2. *a) Theorem 1a) holds also for $\mathcal{L}:\mathfrak{BV}\to\mathfrak{BV}$.*
b) Let λ be an isolated eigenvalue of $\mathcal{L}: BV\to BV$, $|\lambda|>\vartheta$, and denote by $\pi_\lambda: BV\to BV$ its associated spectral projector. Then $\pi_\lambda(BV)\subseteq\mathfrak{BV}$.

Together with Lemma 1 this allows the following conclusion:

Corollary 1. *Let $\lambda\in\mathbb{C}$, $|\lambda|>\vartheta$. Then λ is an eigenvalue of $\mathcal{L}: BV\to BV$ with algebraic multiplicity k if and only if λ is an eigenvalue of $\mathcal{L}:\mathfrak{BV}\to\mathfrak{BV}$ with algebraic multiplicity k.*

The following thermodynamical estimate will be used extensively. Consider the boundaries generated by $(X, \mathcal{Z}, T)$:

$$\mathcal{B}(T) = \{\partial T^n Z : n \geq 0 \text{ and } Z \in \mathcal{Z}_{n+1}\},$$

the topological boundary ∂ being understood w.r.t. $\mathbb{R}^d$ (*not* to X). The corresponding uniform topological pressure is defined as

$$P_g(\mathcal{B}(T), T) = \lim_{n\to\infty} \frac{1}{n} \max_{B\in\mathcal{B}(T)} \log \sum_{\substack{Z\in\mathcal{Z}_n \\ \overline{Z}\cap B\neq\emptyset}} g^{(n)}(Z).$$

Here g is the piecewise constant function defined by $g(x) = 1/|\det DT(x)|$ and $g^{(n)}(Z) = g|Z \cdot g|TZ \dots g|T^{n-1}Z$ with $g|T^kZ$ the constant value of that function on T^kZ. (if $S \subset X$, $P_g(S,T) := P_g(\{S\}, T)$ is the usual pressure adapted to our piecewise setting.) The announced estimate is

Proposition 1. *If T is plane* [9, Theorem 6.1 and Remark 6.9] *or if T is generic* [9, Proposition 5.2]*, then, for $g = 1/|\det DT|$,*

$$\exp P_g(\mathcal{B}(T), T) \leq \vartheta.$$

Remark 5. We do not know whether such a uniform estimate holds outside the piecewise affine setting.

The next step is to construct the Markov extension of the system $(X, \mathcal{Z}, T)$. This is a non-compact piecewise affine dynamical system $(\hat{X}, \hat{\mathcal{Z}}, \hat{T})$ with a *countable* partition $\hat{\mathcal{Z}}$. $\hat{X}$ is the disjoint union of a certain family of subsets of X. Indeed, in section 3 we construct a countable state graph $(\mathcal{D}, \to)$ and associate to each $D \in \mathcal{D}$ a set $\operatorname{fol}(D) \subseteq X$ in such a way that each set $\operatorname{fol}(D)$ is contained in some element of $\mathcal{Z}$ and $T(\operatorname{fol}(D)) \cap \bigcup_{j=1}^N Z_j = \bigcup_{D\to D'\in\mathcal{D}} \operatorname{fol}(D')$, see (3). Let

$$\hat{X} := \bigcup_{D\in\mathcal{D}} \{D\} \times \operatorname{fol}(D), \quad \hat{D} := \{D\} \times \operatorname{fol}(D) .$$

As each $\hat{D}$ is just a copy of $\operatorname{fol}(D) \subseteq \mathbb{R}^d$ it carries in a natural way an Euclidean metric and Lebesgue measure $\hat{m}_{|\hat{D}}$ which is just the lift of $m_{|\operatorname{fol}(D)}$ to $\hat{D}$. Let $\pi : \mathcal{D} \to \mathcal{Z}$ be the map defined by $\operatorname{fol}(D) \subseteq \pi(D)$. Now T can be lifted to a map $\hat{T} : \hat{X} \to \hat{X}$ which, in fact, is only $\hat{m}$-a.e. defined. Namely, for $x \in \operatorname{fol}(D) \cap T^{-1} \bigcup_{j=1}^N Z_j$,

$$\begin{aligned} \hat{T}(D,x) &:= (D', T_i x) \quad \text{where } i \text{ and } D' \text{ are determined by the rule} \\ &\operatorname{fol}(D) \subseteq \pi(D) =: Z_i \in \mathcal{Z},\ T_i x \in \operatorname{fol}(D'), \text{ and } D \to D' . \end{aligned}$$

By construction, $\hat{T}$ has the Markov property with respect to $\mathcal{D}$, namely

$$\hat{T}(\hat{D}) = \bigcup_{\substack{D'\in\mathcal{D} \\ D\to D'}} \hat{D}' \mod \hat{m} .$$

Remark 6. Since we will investigate only the transfer operator associated to $\hat{T}$ and not its zeta function, we only need to define $\hat{T}$ at $\hat{m}$-a.e. $(D,x) \in \hat{X}$. Periodic orbits of $\hat{T}$ will be studied solely in terms of closed loops of the graph $(\mathcal{D}, \to)$.

The space $L^1(\hat{X}, \mathbb{R})$ is well-defined with

$$\|\hat{f}\|_1 := \int |\hat{f}|\, d\hat{m} = \sum_{D\in\mathcal{D}} \int_{\operatorname{fol}(D)} |\hat{f}(D,x)|\, dx ,$$

and we can introduce on $\hat{X}$ also a notion of variation in the sense of Blank and Saussol. For $\hat{f} \in L^1(\hat{X}, \mathbb{R})$ let

$$\widehat{\mathfrak{var}}(\hat{f}) := \sum_{D\in\mathcal{D}} \mathfrak{var}(\hat{f}(D,.)), \qquad \|\hat{f}\|_{\widehat{\mathfrak{BV}}} := \widehat{\mathfrak{var}}(\hat{f}) + \|\hat{f}\|_1 .$$

It is routine to check that also $\widehat{\mathfrak{BV}} := \{\hat{f} \in L^1(\hat{X}, \mathbb{R}) : \widehat{\mathfrak{var}}(\hat{f}) < \infty\}$ is a Banach space when equipped with the norm $\|.\|_{\widehat{\mathfrak{BV}}}$.

Associated with $\hat{T}$ is a transfer operator $\hat{\mathcal{L}} : \mathrm{mb}(\hat{X}) \to \mathrm{mb}(\hat{X})$,

$$\hat{\mathcal{L}}\hat{f}(D,x) := \sum_{\substack{C\in\mathcal{D}\\ C\to D}} |\det(DT_C)|^{-1}\cdot \hat{f}(\hat{T}_C^{-1}(D,x)) = \sum_{(C,y)\in\hat{T}^{-1}(D,x)} |\det(DT_C)|^{-1}\cdot\hat{f}(C,y)$$

Here $T_C := T_i$ if $\pi(C) = Z_i$ and

$$\hat{T}_C^{-1}(D,x) = (C, T_i^{-1}x)\ .$$

For $\hat{\mathcal{L}}$ we have the following theorem:

Theorem 3. *$\hat{\mathcal{L}} : \widehat{\mathfrak{BV}} \to \widehat{\mathfrak{BV}}$ is a bounded linear operator with*

$$r(\hat{\mathcal{L}}) = 1 \quad \textit{and} \quad r_{\mathrm{ess}}(\hat{\mathcal{L}}) \le \vartheta\ .$$

There is a close relation between eigenfunctions of $\mathcal{L}$ and of $\hat{\mathcal{L}}$:

Theorem 4. *Let $\lambda \in \mathbb{C}$, $|\lambda| > \vartheta$. Then λ is an eigenvalue of $\mathcal{L} : \mathfrak{BV} \to \mathfrak{BV}$ with algebraic multiplicity k if and only if λ is an eigenvalue of $\hat{\mathcal{L}} : \widehat{\mathfrak{BV}} \to \widehat{\mathfrak{BV}}$ with algebraic multiplicity k.*

Since the weight function $g = 1/|\det(DT)|$ is constant on each of the sets $\mathrm{fol}(D)$, eigenfunctions of $\hat{\mathcal{L}}$ for eigenvalues λ with $|\lambda| > \vartheta$ are constant on each $\hat{D}$. This allows to characterize them as left eigenvectors of the following $\mathcal{D}\times\mathcal{D}$ matrix K:

$$K(C,D) = g(C) \quad \text{if } C \to D, \quad K(C,D) = 0 \text{ otherwise.}$$

More precisely:

Theorem 5. *Let $\lambda \in \mathbb{C}$, $|\lambda| > \vartheta$.*
a) If $(\hat{\mathcal{L}} - \lambda)^k \hat{f} = 0$ for some $\hat{f} \in \widehat{\mathfrak{BV}}$ and $k \ge 1$, then $\hat{f}$ is constant on each set $\hat{D}$, $D \in \mathcal{D}$. Denote by u_D the constant value of $\hat{f}$ on $\hat{D}$. Then $u = (u_D)_{D\in\mathcal{D}} \in \ell^1(\mathcal{D})$ and $u(K-\lambda)^k = 0$.
b) If $u(K-\lambda)^k = 0$ for some $u \in \ell^1(\mathcal{D})$ and $k \ge 1$ and if $\hat{f} : \hat{X} \to \mathbb{C}$ is defined by $\hat{f}(D,x) = u_D$, then $\hat{f} \in \widehat{\mathfrak{BV}}$ and $(\hat{\mathcal{L}} - \lambda)^k \hat{f} = 0$.

Summarising Corollary 1 and Theorems 4 and 5 we can state:

(2) Let $\lambda \in \mathbb{C}$, $|\lambda| > \vartheta$. Then λ is an eigenvalue of $\mathcal{L} : BV \to BV$ with algebraic multiplicity k if and only if λ is an $\ell^1(\mathcal{D})$-eigenvalue of the matrix K with algebraic multiplicity k.

At this point we can pass from spectral properties of operators to analyticity properties of determinants and zeta functions. We start by defining formally

$$\det(\mathrm{Id} - zK) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\,\mathrm{tr}(K^n)\right)$$

with a "naïve" definition of the trace as $\mathrm{tr}(L) := \sum_{D\in\mathcal{D}} L(D,D)$. In section 3 we shall see that each K^n has only finitely many nonzero diagonal entries so that $\mathrm{tr}(K^n)$ is well-defined. The following theorem on spectral properties of the matrix K acting by multiplication on $\ell^1(\mathcal{D})$ row vectors is proved there:

Theorem 6. *a) $r(K) = 1$.*
b) $r_{\mathrm{ess}}(K) \le \vartheta$.
c) $\det(\mathrm{Id} - zK)$ is well-defined around zero and can be analytically extended to $\{z : |z| < \vartheta^{-1}\}$.
d) On this disk, $\det(\mathrm{Id} - zK) = 0$ if and only if z^{-1} is an eigenvalue of K. In this case the multiplicity of z as a zero of $\det(\mathrm{Id} - zK)$ and the algebraic multiplicity of z^{-1} as an eigenvalue of K coincide.

Finally we relate periodic orbits for T (described in terms of $\zeta(z)$) and for $\hat{T}$ (described in terms of $\det(\mathrm{Id} - zK)$):

Theorem 7. *$\zeta(z)\cdot\det(\mathrm{Id} - zK)$ is analytic and non-zero in $\{z : |z| < \vartheta^{-1}\}$.*

Now Theorem 1b) and c) follow from the spectral properties collected in (2) and from the two last theorems upon observing the bound $r_{\text{ess}} \leq \vartheta$, see also Remark 2.

The proofs of the auxiliary theorems are given in the remaining sections. We have tried to write each step in as general a setting as our techniques have allowed, in order to prepare the generalization of our Main theorem to, say, generic piecewise expanding and C^{1+1} dynamics.

3. The matrix operator and its functional determinant: Proof of Theorem 6

We prove an abstract version of Theorem 6 for matrix operators K defined by Markov diagrams of rather general (not necessarily affine) piecewise invertible dynamical systems with arbitrary piecewise constant weights.

A *piecewise invertible* dynamical system on $\mathbb{R}^d$ is a triple $(X, \mathcal{Z}, T)$ with:

1. $\mathcal{Z}$ a finite collection of open and bounded subsets of $\mathbb{R}^d$ which form a partition of X in the sense that they are pairwise disjoint and that their union is a dense subset of X;
2. $T : \bigcup_{Z\in\mathcal{Z}} Z \to X$ satisfying: each restriction $T|Z$ coincides with the restriction of a homeomorphism T_Z between neighborhoods in $\mathbb{R}^d$ of the closures of Z and $T(Z)$.

$(X, \mathcal{Z}, T)$ is *piecewise* C^{1+1} if each homeomorphism T_Z is in fact a diffeomorphism such that $x \mapsto |\det DT_Z(x)|$ is Lipschitz. The iterate $(X, \mathcal{Z}_n, T^n)$ is again a piecewise invertible system.

If $(X, \mathcal{Z}, T)$ is piecewise affine, then the cylinders $[A_0 \dots A_n] := A_0 \cap T^{-1}A_1 \cap \dots \cap T^{-n}A_n \in \mathcal{Z}_{n+1}$, $n \geq 0$, are convex. In more general settings, they need not even be connected. However the following property is satisfied in many cases (in particular for generic piecewise expanding and C^{1+1} maps).

Definition 1. *We say that $\mathcal{Z}$ separates cylinder components, if, for every $L < \infty$, there exists $N = N(L) < \infty$ such that each connected component of the closure of any L-cylinder is a union of N-cylinders, modulo m.*

We equip $(X, \mathcal{Z}, T)$ with a piecewise constant weight g, i.e., g is constant on each $Z \in \mathcal{Z}$. Recall that $\mathcal{B}(T)$ is the set of boundaries of T and that $P_g(\mathcal{B}(T), T)$ is its topological pressure (if $g \geq 0$).

The Markov diagram $\mathcal{D}$ defined below is a directed graph with a countable state set endowed with a natural projection to $\mathcal{Z}$. $\mathcal{D}$ is defined so as to ensure, among other things, that at most $\#\mathcal{Z}$ arrows leave from any given vertex, that each periodic orbit has at most one lift and that its behavior at infinity is controlled by the boundary of the partition $\mathcal{Z}$, especially by $P_{|g|}(\mathcal{B}(T), T)$.

$\mathcal{D}$ and g define a countable matrix $K : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ according to: $K(C, D) = 0$ if there is no arrow from C to D, and $K(C, D) = g|\pi(C)$ otherwise. K acts by right multiplication on $\ell^1(\mathcal{D}) = \{u : \mathcal{D} \to \mathbb{C} : \|u\|_1 < \infty\}$ with $\|u\|_1 = \sum_{D\in\mathcal{D}} |u_D|$,

$$(K(u))_D = (uK)_D = \sum_{E\in\mathcal{D}} u_E \cdot K(E, D).$$

In fact, $\|K\|_1 \leq \#\mathcal{Z} \cdot \sup |g|$. Let us stress that the norm $\|u\|_1$ is *not* the $L^1(m)$-norm (for some $\frac{1}{g}$-conformal measure m) of the piecewise constant function on the Markov extension associated with u. But it is well-suited for our thermodynamical approach.

The determinant of $\mathrm{Id} - zK$ is defined, according to [17], to be the formal power series:

$$\det(\mathrm{Id} - zK) = \exp\left(-\sum_{n\geq 1} \frac{z^n}{n} \operatorname{tr}(K^n)\right),$$

with $\operatorname{tr}(L) = \sum_{D\in\mathcal{D}} L(D, D)$.

This is obviously the same thing as the (inverse of the) zeta function of the Markov diagram (more accurately, of the topological Markov chain defined by it with respect to the obvious weight).

Theorem 8. *Let K be the matrix ℓ^1-operator defined as above by a piecewise invertible dynamical system $(X, \mathcal{Z}, T)$ with piecewise constant complex weight g. Let R be given by the uniform pressure of the boundaries of T:*

$$R = \exp P_{|g|}(\mathcal{B}(T), T).$$

Assume that $\mathcal{Z}$ separates cylinder components. Then:

1. *The spectral radius of K satisfies: $r(K) \leq \exp P_{|g|}(X,T)$ with equality if $g \geq 0$.*
2. *The essential spectral radius of K satisfies: $r_{\mathrm{ess}}(K) \leq R$.*
3. $\det(\mathrm{Id} - zK)$ *is well-defined around zero and can be analytically extended to* $\{z : |z| < R^{-1}\}$, *and*
4. *on this disk,* $\det(\mathrm{Id} - zK) = 0$ *iff z^{-1} is in the spectrum of K. Moreover, z^{-1} is then an eigenvalue of K with algebraic multiplicity given by the multiplicity of z as a zero of* $\det(\mathrm{Id} - zK)$.

Theorem 6 will follow from this one using Proposition 1 and remarking that $r(K) = 1$ follows directly from $r_{\mathrm{ess}}(K) \leq \vartheta < 1$ because the transfer operator preserves Lebesgue measure.

Theorem 8 will be a consequence of a result from [17] recalled as Theorem 9 below, once we shall have estimated the spectral properties of K at infinity in terms of the boundary pressure.

Quasi-nuclear spectral theory. Denote by $|K|$ the $\mathcal{D} \times \mathcal{D}$ matrix $|K|(C,D) = |K(C,D)|$. Recall that both, K and $|K|$ act by right multiplication as bounded linear operators on $\ell^1(\mathcal{D})$. In the following we identify $\mathcal{D}$ with $\mathbb{N}$.

For $0 \leq i \leq j \leq \infty$, let $\pi_{i,j}$ be the projector defined by $\pi_{i,j}(u) = 1_{[i,j]} \cdot u$, i.e., $\pi_{i,j}$ forgets components with indices outside the interval $[i,j]$.

The asymptotic spectral radius is:

$$r_{\mathrm{as}}(K) = \lim_{i\to\infty} r(\pi_{i,\infty}|K|\pi_{i,\infty})$$

and the asymptotic trace norm is:

$$t_{\mathrm{as}}(K) = \lim_{i\to\infty} \limsup_{k\to\infty} \left|\mathrm{tr}((\pi_{i,\infty}|K|\pi_{i,\infty})^k)\right|^{1/k}.$$

The following statement is an immediate corollary of Theorem 2.5 of [17].

Theorem 9. *Let K be an arbitrary bounded linear operator on $\ell^1(\mathbb{N})$ satisfying:*

(QN1) *the series* $\mathrm{tr}(K^n) := \sum_{i\geq 0} K^n(i,i)$ *is absolutely convergent (and thus well-defined) for each* $n \geq 1$.

(QN2) $\mathrm{tr}(|K|^n) \leq S^n$ *for all $n \geq 1$ and some constant $S < \infty$.*

(QN3) *for each $i \geq 0$ and $n \geq 1$, only finitely many integers j appear in loops of length n based at i (i.e., in sequences $i_1 = i, i_2, \ldots, i_n = i$ with $K(i_s, i_{s+1}) \neq 0$ for all $1 \leq s < n$).*

Then conclusions (2)-(4) of Theorem 8 above hold with $R = \max(r_{\mathrm{as}}(K), t_{\mathrm{as}}(K))$.

The Markov Diagram. We give a self-contained presentation of the construction of [5], which generalizes the work of F. Hofbauer [13]. (Remark 7 explains why Hofbauer's construction is not convenient for multi-dimensional systems.)

Let $\mathcal{Z}^* = \{A_0 \ldots A_{n-1} \in (\mathcal{Z})^n : n \geq 1 \text{ and } [A_0 \ldots A_{n-1}] \neq \emptyset\}$ be the set of non-empty words on $\mathcal{Z}$. The Markov Diagram $\mathcal{D}$ is then the quotient of $\mathcal{Z}^*$ by the following equivalence relation: Two finite words $A_{-m} \ldots A_0$ and $B_{-n} \ldots B_0$ (we translate the indices for convenience) are equivalent iff there exists $0 \leq k \leq \min(m,n)$ such that:

$$\begin{aligned} T^k[A_{-k} \ldots A_0] &= T^m[A_{-m} \ldots A_0] \\ T^k[B_{-k} \ldots B_0] &= T^n[B_{-n} \ldots B_0] \\ A_{-k} \ldots A_0 &= B_{-k} \ldots B_0 \end{aligned}$$

i.e., iff the two words coincide on some piece which determines already their follower sets [1] defined by:

$$\mathrm{fol}(A_{-m} \ldots A_0) := T^m[A_{-m} \ldots A_0] \subset A_0.$$

We remark that these follower sets depend only on the equivalence class. An immediate consequence of this definition is:

[1]This is expressed by the third requirement on k which distinguishes the present approach from Hofbauer's one.

$$T(\mathrm{fol}(A_{-m}\dots A_0))\cap Z_j=\mathrm{fol}(A_{-m}\dots A_0Z_j)\quad \text{for each } Z_j\in\mathcal{Z}\ . \tag{3}$$

We write $\langle A_{-m}\dots A_0\rangle$ for the equivalence class of $A_{-m}\dots A_0$. It has a canonical representative: $A_{-k}\dots A_0$ with k minimal as above, which we call the significant part of $A_{-m}\dots A_0$, or of the equivalence class.

We have the obvious projection $\pi:\mathcal{D}\to\mathcal{Z}$ defined by $\pi(\langle A_{-m}\dots A_0\rangle)=A_0$.

Finally, $\mathcal{D}$ is endowed with the following arrows: $\alpha\to\beta$ iff there is *some* $A_{-n}\dots A_0A_1\in\mathcal{Z}^*$ such that $\alpha=\langle A_{-n}\dots A_0\rangle$ and $\beta=\langle A_{-n}\dots A_0A_1\rangle$. In fact we have the following stronger and very useful property:

Lemma 2. *If $\alpha\to\beta$ and $A_1=\pi(\beta)$ then $\beta=\langle A_{-m}\dots A_0A_1\rangle$ for* any *$A_{-m}\dots A_0$ ($m\geq 0$) such that $\alpha=\langle A_{-m}\dots A_0\rangle$.*

Proof. As $\alpha\to\beta$, there exists $B_{-n}\dots B_0A_1\in\mathcal{Z}^*$ such that: $\alpha=\langle B_{-n}\dots B_0\rangle$ and $\beta=\langle B_{-n}\dots B_0A_1\rangle$ and clearly $A_1=\pi(\beta)$. Thus, $A_{-m}\dots A_0$ and $B_{-n}\dots B_0$ are equivalent so that they have the same significant part: $B_{-k}\dots B_0=A_{-k}\dots A_0$, for some k. In particular,

$$T^n[B_{-n}\dots B_0]=T^k[B_{-k}\dots B_0]. \tag{4}$$

But,

$$T^{n-k}[B_{-n}\dots B_0]=T^{n-k}[B_{-n}\dots B_{-k-1}]\cap[B_{-k}\dots B_0]$$

as $T^{n-k}|[B_{-n}\dots B_{-k-1}]$ is one-to-one. Hence eq. (4) implies:

$$T^{n-k}[B_{-n}\dots B_{-k-1}]\supset[B_{-k}\dots B_0]=[A_{-k}\dots A_0]\supset[A_{-k}\dots A_0A_1].$$

Thus,

$$\begin{aligned}T^{n+1}[B_{-n}\dots B_0A_1]&=T^{k+1}(T^{n-k}[B_{-n}\dots B_{-k-1}]\cap[A_{-k}\dots A_0A_1])\\&=T^{k+1}[A_{-k}\dots A_0A_1].\end{aligned}$$

Therefore $A_{-k}\dots A_0A_1$ is equivalent to $B_{-n}\dots B_0A_1$ and represents β.

Similarly, one proves that $A_{-k}\dots A_0A_1$ is equivalent to $A_{-m}\dots A_0A_1$. The lemma follows. □

We shall use the following properties:

Lemma 3. *A finite or infinite path $D_0\to D_1\to\dots$ on $\mathcal{D}$ is determined by its starting point and its π-projection.*

Proof. Apply the previous lemma to see that, if $D_0=\langle A_{-m}\dots A_0\rangle$ and if $A_k=\pi(D_k)$ for $k\geq 0$, then $D_n=\langle A_{-m}\dots A_n\rangle$ for all $n\geq 0$. □

For a loop, even the starting point is not needed:

Lemma 4. *A loop $D_0\to D_1\to\dots\to D_p=D_0$ on $\mathcal{D}$ is determined by its π-projection. More precisely, for all large enough m, for all $0\leq n<p$, $D_n=\langle A_{-m}\dots A_n\rangle$ if we write $A_k=\pi(D_{k \bmod p})$ for $k\in\mathbb{Z}$.*

Moreover $A_0A_1\dots$ does not admit a smaller period than $D_0D_1\dots$.

Remark 7. Let us pause to remark that such a property does not hold for Hofbauer's Markov Diagram (i.e., the family of follower sets endowed with arrows $F\to F'\iff F'=T(F)\cap Z$ for some $Z\in\mathcal{Z}$). Indeed, it easy to construct in dimension 2, resp. 3, examples with periodic orbits having an arbitrary number of, resp. infinitely many, lifts.

Proof. It is enough to establish the formula for D_0, the general case following then from Lemma 2. The assertion about the period of $A_0A_1\dots$ is an obvious consequence of these formulas. We prove that $D_0=\langle A_{-m}\dots A_0\rangle$ for all large enough m. Let $B_{-M}\dots B_0\in\mathcal{Z}^*$ be the significant part of D_0. By Lemma 2, for all $r\geq 0$,

$$D_0=\langle B_{-M}\dots B_0(A_1\dots A_p)^r\rangle\,. \tag{5}$$

For $r \geq M/p$, this implies that the significant part of D_0, $B_{-M} \dots B_0$, equals $A_{p-M} \dots A_p = A_{-M} \dots A_0$. This proves the formula for $m = M$. Let $m > M$. Obviously, if $r \geq (m-M)/p$,

$$T^{M+rp}[A_{-M} \dots A_0(A_1 \dots A_p)^r] \subset T^m[A_{-m} \dots A_0] \subset T^M[A_{-M} \dots A_0].$$

The two extreme terms are equal by eq. (5), hence all three terms are equal. But this is precisely to say that $A_{-m} \dots A_0$ is equivalent to $A_{-M} \dots A_0$: $D_0 = \langle A_{-m} \dots A_0 \rangle$ for all $m \geq M$. □

Lemma 5. *Let $D_0 \to D_1 \to \dots \to D_n$ be some path on $\mathcal{D}$ with π-projection $A_0 \dots A_n$. If $[A_0 \dots A_n] \subset \mathrm{fol}(D_0)$, then $D_n = \langle A_0 \dots A_n \rangle$.*

Proof. By Lemma 2, $D_n = \langle A_{-k} \dots A_n \rangle$ if $A_{-k} \dots A_0$ denotes the significant part of D_0. But:

$$\begin{aligned} T^{k+n}[A_{-k} \dots A_n] &= T^n(T^k[A_{-k} \dots A_0] \cap [A_0 \dots A_n]) \\ &= T^n(\mathrm{fol}(D_0) \cap [A_0 \dots A_n]) = T^n[A_0 \dots A_n] \end{aligned}$$

under our hypothesis. But this is to say that $A_0 \dots A_n$ is equivalent to $A_{-k} \dots A_n$: $D_n = \langle A_0 \dots A_n \rangle$. □

Finally we have the following control over loops on the higher parts of the diagram. Let $\mathcal{D}_N$ be the set of equivalence classes generated by words of length at most N. $\mathcal{D}$ is the increasing union of the finite sets $\mathcal{D}_N$, $N \geq 1$. Also any arrow starting from within $\mathcal{D}_N$ ends in $\mathcal{D}_{N+1}$.

Lemma 6. *Let $D_0 \to D_1 \to \dots \to D_p = D_0$ be a loop on $\mathcal{D} \setminus \mathcal{D}_N$, for some N. Then there exists a path $E_0 \to E_1 \to \dots E_p$ on $\mathcal{D} \setminus \mathcal{D}_N$ such that $E_0 \in \mathcal{D}_{N+1} \setminus \mathcal{D}_N$ and $\pi(E_k) = \pi(D_{k+q \bmod p})$ for all $0 \leq k < p$ and some q.*

Again, this does not hold for Hofbauer's Markov Diagram in dimension 2 or higher.

Proof. We consider the path $F_0 \to F_1 \to \dots$ defined by $F_k = \langle A_0 \dots A_k \rangle$, where $A_k = \pi(D_{k \bmod p})$. Clearly $F_k \in \mathcal{D}_{k+1}$. By Lemma 4, for k large enough, $F_k = D_k$ so that $F_k \notin \mathcal{D}_N$. Hence we can take $E_k = F_{k+d}$ with d maximal such that $F_{d-1} \in \mathcal{D}_N$. □

Lemma 7. *Let $x \in \mathrm{fix}(T^n)$. $\mathrm{fix}(T^n) \subset Y$ so that x has a well-defined itinerary $A \in \mathcal{Z}^{\mathbb{N}}$ (i.e., $T^k x \in A_k$). Assume that $\mathrm{diam}([A_0 \dots A_k]) \to 0$ as $k \to \infty$.*

Then there exists a unique loop $(D_i)_{i \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ such that $\pi(D_i) = A_i$ for all $i \in \mathbb{N}$.

Proof. $x = T^n x$ is in the open set $T^n[A_0 \dots A_n]$. By the assumption, $[(A_0 \dots A_{n-1})^k]$ is a small neighborhood of x for large k. Thus, for k large enough:

$$[(A_0 \dots A_{n-1})^k] \subset T^n[A_0 \dots A_n].$$

This implies that $\left\langle (A_0 \dots A_{n-1})^k \right\rangle = \left\langle (A_0 \dots A_{n-1})^{k+1} \right\rangle$. Setting $D_j = \left\langle (A_0 \dots A_{n-1})^k A_1 \dots A_j \right\rangle$ it follows that $D_0 \dots D_{n-1}$ is a loop on $\mathcal{D}$ projecting to x. The uniqueness follows from Lemma 4. □

Lemma 8. *Let L be some positive integer. Assume that $\mathcal{Z}$ separates cylinder components and $N = N(L)$ is as in Definition 1. For any path $D_0 \to D_1 \to \dots \to D_N$ with $D_N \in \mathcal{D} \setminus \mathcal{D}_N$ and with projection $A_i = \pi(D_i)$ $(i \in \mathbb{N})$,*

$$\overline{[A_1 \dots A_L]} \cap \partial(T\,\mathrm{fol}(D_0)) \neq \emptyset.$$

Proof. We claim that if $[A_1 \dots A_N]$ meets some $S \subset \mathbb{R}^d$ but is not contained in it, then $\overline{[A_1 \dots A_L]}$ meets the boundary of S. Indeed, by the choice of N, $[A_1 \dots A_N]$ is contained in a connected component of $\overline{[A_1 \dots A_L]}$. This connected component must intersect S without being contained in it. Therefore it must intersect the boundary of S, proving the claim.

To deduce the lemma from the claim observe that $[A_1 \dots A_N]$ meets $T\,\mathrm{fol}(D_0)$, because $\mathrm{fol}(D_N) = T^N(\mathrm{fol}(D_0) \cap [A_0 \dots A_N]) \neq \emptyset$, but it cannot be contained in $T\,\mathrm{fol}(D_0)$ by Lemma 5, as $D_N \notin \mathcal{D}_N$. □

Proof of Theorem 8: We begin with the proof of conclusion (1) about the spectral radius $r(K)$. By definition,

$$\begin{aligned} r(K) &= \lim_{n\to\infty} \|K^n\|_1^{1/n} \le \lim_{n\to\infty} \sup_{D\in\mathcal{D}} \||K|^n(1_D)\|_1^{1/n} \\ &= \lim_{n\to\infty} \sup_{D\in\mathcal{D}} \left(\sum_{\substack{\eta\in\mathcal{Z}_n \\ \text{s.t. } \eta\cap\mathrm{fol}(D)\neq\emptyset}} |g^{(n)}(\eta)| \right)^{1/n} \\ &\le \lim_{n\to\infty} \left(\sum_{\eta\in\mathcal{Z}_n} |g^{(n)}(\eta)| \right)^{1/n} = \exp P_{|g|}(X,T). \end{aligned}$$

If $g \ge 0$, then, considering the *finite* "base" B of $\mathcal{D}$, $B = \{\langle A\rangle : A \in \mathcal{Z}\} \subset \mathcal{D}$, we have:

$$r(K) \ge \lim_{n\to\infty} \|K^n(1_B)\|_1^{1/n} = \lim_{n\to\infty} \left(\sum_{\eta\in\mathcal{Z}_n} g^{(n)}(\eta) \right)^{1/n} = \exp P_g(X,T).$$

Hence we have equality in this case.

To get conclusions (2)-(4), it will be enough to apply Theorem 9. We identify $\mathcal{D}$ to $\mathbb{N}$ in a way which preserves the filtration, i.e., $i(\mathcal{D}_{N+1} \setminus \mathcal{D}_N)$ is an interval for each N. We abbreviate $\pi_{i(\mathcal{D}_N)}$ as $\pi_{\mathcal{D}_N}$, etc.

We check conditions (QN1)-(QN3).

(QN1)-(QN2) follow from Lemma 4 as it implies that there are at most $(\#\mathcal{Z})^n$ n-loops so that we can take: $S = \#\mathcal{Z} \cdot \sup|g|$.

(QN3) follows from the fact that from $\mathcal{D}_N$ a path of length k can reach only states in $\mathcal{D}_{N+k}$, a finite set for each $k \ge 0$.

It remains to bound $r_{\mathrm{as}}(K)$ and $t_{\mathrm{as}}(K)$ by $\exp P_{|g|}(\mathcal{B}(T),T)$. Let $\epsilon > 0$ and choose L so large that, for all $n \ge L$

$$\sup_{S\in\mathcal{B}(T)} \sum_{\substack{Z\in\mathcal{Z}_n \\ \overline{Z}\cap S\neq\emptyset}} |g^{(n)}(Z)| \le e^{(P_{|g|}(\mathcal{B}(T),T)+\epsilon/2)n}.$$

Let $N(L)$ be given by the assumption that $\mathcal{Z}$ separates cylinder components, see Definition 1. Assume that $N \ge N(L)$.

Consider $r_{\mathrm{as}}(K)$. We have to prove that, for all n large enough,

$$\sup_{D_*} \sum_{D_0=D_*\to\cdots\to D_{n-1}} |g^{(n)}(D_0\ldots D_{n-1})| \le \exp[(P_{|g|}(\mathcal{B}(T),T)+\epsilon)n] \ , \tag{6}$$

the sum being restricted by the additional condition

$$D_{kL+N} \notin \mathcal{D}_N \text{ for } 0 \le k < q := [(n-(N-L))/L] \ .$$

(This weakening of $D_k \notin \mathcal{D}_N$, $k = 0,\ldots,n-1$, will be used in the proof of Theorem 3).

Let $D_* \in \mathcal{D}$. By Lemma 3, each path $D_0 \to \cdots \to D_{n-1}$ starting from D_* is determined by its π-projection. Thus we can write the sum in eq. (6) as a sum over these projections. But each such projection $A_0 \ldots A_{n-1}$ can be divided (up to the first symbol A_0 and also a final segment of length $< N$) into segments $A_{kL+1} \ldots A_{(k+1)L}$, $0 \le k < q$. The sum over each given segment is bounded by $\exp((P_{|g|}(\mathcal{B}(T),T) + \frac{\epsilon}{2})L)$ using Lemma 8 and the following remarks: first, D_{kL+1} is determined by D_* and $A_0 \ldots A_{kL+1}$ according again to Lemma 3; second, $D_{kL+N} \notin \mathcal{D}_N$; third, the boundaries of follower sets are in $\mathcal{B}(T)$. Hence

$$\begin{aligned}
&\sum_{D_0=D_*\to\cdots\to D_{n-1}} |g^{(n)}(D_0\dots D_{n-1})| \\
&\quad\le |g(A_0)| \sum_{\substack{A_1\dots A_L\\ \overline{[A_1\dots A_L]}\cap\partial(T\,\mathrm{fol}(D_0))\neq\emptyset}} |g^{(L)}(A_1\dots A_L)| \sum_{\substack{A_{L+1}\dots A_{2L}\\ \overline{[A_{L+1}\dots A_{2L}]}\cap\partial(T\,\mathrm{fol}(D_L))\neq\emptyset}} |g^{(L)}(A_{L+1}\dots A_{2L})|\dots \\
&\qquad\cdots\times \sum_{A_{qL+1}\dots A_{n-1}} |g^{(n-qL-1)}(A_{qL+1}\dots A_{n-1})| \\
&\quad\le \sup|g|\cdot e^{(P_{|g|}(\mathcal{B}(T),T)+\epsilon/2)qL}\times[\#\mathcal{Z}\cdot\sup|g|]^{n-qL-1} \\
&\quad\le \max(\sup|g|,1)^N\cdot\#\mathcal{Z}^N\times e^{(P_{|g|}(\mathcal{B}(T),T)+\epsilon/2)n}.
\end{aligned}$$

For n large enough, the constant to the left of $\times$ is less than $e^{(\epsilon/2)n}$ and the bound on $r_{\mathrm{as}}(K)$ is proved.

We have to prove the same bound for $t_{\mathrm{as}}(K)$. We claim that, for all $n\ge 1$:

$$\left|\mathrm{tr}\left((\pi_{\mathcal{D}\setminus\mathcal{D}_N}|K|\pi_{\mathcal{D}\setminus\mathcal{D}_N})^n\right)\right|\le n\cdot\#(\mathcal{D}_{N+1}\setminus\mathcal{D}_N)\|(\pi_{\mathcal{D}\setminus\mathcal{D}_N}|K|\pi_{\mathcal{D}\setminus\mathcal{D}_N})^n\|_1.$$

As $\mathcal{D}_{N+1}$ is finite, this will imply that $t_{\mathrm{as}}(K)\le r_{\mathrm{as}}(K)$ completing the proof of the theorem.

We prove the claim. Observe first that by Lemma 4, it is enough to sum over the π-projections of the loops. Now Lemma 6 implies that for each loop over $\mathcal{D}\setminus\mathcal{D}_N$ there is a path on $\mathcal{D}\setminus\mathcal{D}_N$ starting from $\mathcal{D}_{N+1}\setminus\mathcal{D}_N$ which has the same projection as the loop, except for a possible shift $0\le q<n$. Thus, the sum on the n-loops on $\mathcal{D}\setminus\mathcal{D}_N$ is bounded by n times the sum on the n-paths on $\mathcal{D}\setminus\mathcal{D}_N$ starting from $\mathcal{D}_{N+1}\setminus\mathcal{D}_N$. This implies $t_{\mathrm{as}}(K)\le r_{\mathrm{as}}(K)$ as claimed. □

4. Proof of Lemma 1 and some auxiliary estimates

Throughout this section we denote by $\mathcal{W}_\epsilon$ the partition of $\mathbb{R}^d$ into cubes of size ϵ whose corners have coordinates in $\epsilon\mathbb{Z}$. The conditional expectation of $f\in L^1(\mathbb{R}^d,\mathbb{R})$ with respect to Lebesgue measure and the partition $\mathcal{W}_\epsilon$ is denoted by $\mathcal{E}_\epsilon f$, i.e.

$$\mathcal{E}_\epsilon f(x)=\sum_{W\in\mathcal{W}_\epsilon} 1_W(x)\,m(W)^{-1}\int_W f\,dm\ .$$

(In order to have a partition in the strict sense one may take the cubes to be "left open, right closed", but this does not really matter.)

Proof of Lemma 1: An elementary calculation shows that $\|f\|_\infty\le\mathrm{const}_d\cdot\epsilon_0^{1-d}\cdot\mathfrak{var}(f)$, see also [20, Proposition 2.3]. So it remains to show that f belongs to BV.

For $f\in L^1(\mathbb{R}^d,\mathbb{R})$ and $i=1,\dots,d$ let

$$\mathrm{var}^{(i)}(f):=\sup\left\{\int f\cdot\frac{\partial\varphi}{\partial x_i}\,dx:\varphi\in C_c^1(\mathbb{R}^d,\mathbb{R}),\|\varphi\|_\infty\le 1\right\}\ .$$

(This is the variation of f in x_i-direction, see also (8) below.) Note that $\mathrm{var}(f)\le\sum_{i=1}^d\mathrm{var}^{(i)}(f)$. We apply this estimate to $\mathcal{E}_{2\epsilon}f$. Since $\mathcal{E}_{2\epsilon}f$ is constant on each cube $W\in\mathcal{W}_{2\epsilon}$, it is readily checked that

$$\mathrm{var}^{(i)}(\mathcal{E}_{2\epsilon}f)\le(2\epsilon)^{-1}\int_{\mathbb{R}^d}\mathrm{osc}(\mathcal{E}_{2\epsilon}f,B_\epsilon(x))\,dx\le(2\epsilon)^{-1}\int_{\mathbb{R}^d}\mathrm{osc}(f,B_{4\sqrt{d}\epsilon}(x))\,dx$$

Therefore, if $0<4\sqrt{d}\epsilon\le\epsilon_0$,

$$\mathrm{var}(\mathcal{E}_{2\epsilon}f)\le\sum_{i=1}^d\mathrm{var}^{(i)}(\mathcal{E}_{2\epsilon}f)\le 2d^{\frac32}(4\sqrt{d}\epsilon)^{-1}\int_{\mathbb{R}^d}\mathrm{osc}(f,B_{4\sqrt{d}\epsilon}(x))\,dx\le 2d^{\frac32}\,\mathfrak{var}(f)\ .$$

As $\lim_{\epsilon\to0}\|f-\mathcal{E}_{2\epsilon}f\|_1=0$, it follows that

$$\mathrm{var}(f)\le\liminf_{\epsilon\to0}\mathrm{var}(\mathcal{E}_{2\epsilon}f)\le 2d^{\frac32}\,\mathfrak{var}(f)\ ,\tag{7}$$

see e.g. [11, Theorem 5.2.1]. □

Corollary 2. *For $\hat{f} \in \widehat{\mathfrak{BV}}$ we have*

$$\|\hat{f}\|_{\widehat{\mathfrak{BV}}} \leq (m(X)^{\frac{1}{d}} 2d^{\frac{3}{2}} + 1)\, \widehat{\mathfrak{var}}(\hat{f}) \ .$$

Proof. For each $D \in \mathcal{D}$ we apply Sobolev's inequality [11, Theorem 5.6.1] and Lemma 1 to obtain

$$\|\hat{f}(D,.)\|_1 \leq \|1_{\mathrm{fol}(D)}\|_d \cdot \|\hat{f}(D,.)\|_{\frac{d}{d-1}} \leq m(X)^{\frac{1}{d}} \operatorname{var}(\hat{f}(D,.)) \leq m(X)^{\frac{1}{d}} 2d^{\frac{3}{2}}\, \mathfrak{var}(\hat{f}(D,.)) \ .$$

(Note that the additional constant C_1 in [11, Theorem 5.6.1] equals 1 in our case.) Now

$$\|\hat{f}\|_1 = \sum_{D \in \mathcal{D}} \|\hat{f}(D,.)\|_1 \leq m(X)^{\frac{1}{d}} 2d^{\frac{3}{2}} \sum_{D \in \mathcal{D}} \mathfrak{var}(\hat{f}(D,.)) = m(X)^{\frac{1}{d}} 2d^{\frac{3}{2}}\, \widehat{\mathfrak{var}}(\hat{f}) \ .$$

□

For convex $C \subseteq \mathbb{R}^d$, $f \in L^1(C,\mathbb{R})$ and $i = 1, \dots, d$ let

$$\operatorname{var}_C^{(i)}(f) := \sup\left\{ \int f \cdot \frac{\partial \varphi}{\partial x_i}\, dx : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}), \|\varphi\|_\infty \leq 1, \varphi(x) = 0\ \forall x \in \mathbb{R}^d \setminus C \right\}$$

$$\operatorname{var}_C(f) := \sup\left\{ \int f \cdot \operatorname{div} \varphi\, dx : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \leq 1, \varphi(x) = 0\ \forall x \in \mathbb{R}^d \setminus C \right\}$$

Both expressions account for the variation of f inside C (neglecting boundary contributions). Obviously $\operatorname{var}_C^{(i)}(f) \leq \operatorname{var}_C(f)$.

We represent $x \in \mathbb{R}^d$ as $x = (x_i, x_i')$ where $x_i' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$. An alternative expression for $\operatorname{var}_C^{(i)}(f)$ is

$$\operatorname{var}_C^{(i)}(f) = \int_{\mathbb{R}^{d-1}} \operatorname{ess} \mathrm{V}_{a(x_i')}^{b(x_i')}(x_i \mapsto f(x_i, x_i'))\, dx_i' \tag{8}$$

where $a(x_i') = \inf\{x_i : (x_i, x_i') \in C\}$ and $b(x_i') = \sup\{x_i : (x_i, x_i') \in C\}$ are the endpoints of the interval $\{x_i : (x_i, x_i') \in C\}$, and

$$\operatorname{ess} \mathrm{V}_a^b(h) := \sup\left\{ \sum_{j=1}^k |h(t_j) - h(t_{j-1})| : a < t_0 \leq \dots \leq t_k < b, h \text{ essentially continuous at } t_j \right\}$$

for $h : (a,b) \to \mathbb{R}$. See [11, Section 5.10].

Lemma 9. *Let $C \subseteq \mathbb{R}^d$ be a convex set which is contained in a cube of size ϵ (parallel to the coordinate axes), and let $f \in L^1(C,\mathbb{R})$ with $\int_C f\, dm = 0$. Then*

$$\int_C |f|\, dm \leq d\,\epsilon \cdot \operatorname{var}_C(f) \ .$$

Proof. For $k = 1, \dots, d$ and $x \in \mathbb{R}^d$ let

$$C_k(x) := \{(y_{d-k+1}, \dots, y_d) : (x_1, \dots, x_{d-k}, y_{d-k+1}, \dots, y_d) \in C\} \ .$$

Given $f \in L^1(C,\mathbb{R})$ set $f_0 := f \cdot 1_C$ and define $f_1, \dots, f_d \in L^1(C,\mathbb{R})$ by

$$f_k(x) := 1_C(x) \cdot m_k(C_k(x))^{-1} \int_{C_k(x)} f(x_1, \dots, x_{d-k}, y_{d-k+1}, \dots, y_d)\, dy_{d-k+1} \dots dy_d$$

Observe that $C_d(x) = C$ so that $f_d = 0$ by assumption. Therefore $f = \sum_{k=1}^d (f_{k-1} - f_k)$ where $f_k(x)$ depends on x only through the variables $x_1, \dots, x_{d-k}$. Since, by construction,

$$\int f_{k-1}(x)\, \frac{m_{k-1}(C_{k-1}(x))}{m_k(C_k(x))}\, dx_{d-k+1} = f_k(x) \quad \text{for all } x$$

(i.e. f_k is a conditional expectation of f_{k-1}), it follows in view of the convexity of C that

$$\int |f_{k-1}(x) - f_k(x)|\, dx_{d-k+1} \leq \epsilon \cdot \operatorname{ess} \mathrm{V}_{a(x_{d-k+1}')}^{b(x_{d-k+1}')}(x_{d-k+1} \mapsto f_{k-1}(x_{d-k+1}, x_{d-k+1}'))$$

(see (8) for the notational conventions). Integrating this inequality over the remaining $d-1$ variables yields

$$\int_C |f_{k-1} - f_k|\, dm \le \epsilon \cdot \mathrm{var}_C^{(d-k+1)}(f_{k-1})\ .$$

To continue, let $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{R})$, $\|\varphi\|_\infty \le 1$. Define φ_k and $\left(\frac{\partial\varphi}{\partial x_{d-k+1}}\right)_k$ w.r.t. φ and $\frac{\partial\varphi}{\partial x_{d-k+1}}$ in the same way as f_k was defined w.r.t. f. Then

$$\begin{aligned}\int f_{k-1}(x)\cdot\frac{\partial\varphi}{\partial x_{d-k+1}}\, dm(x) &= \int f_{k-1}(x)\cdot\left(\frac{\partial\varphi}{\partial x_{d-k+1}}\right)_{k-1} dm(x)\\ &= \int f_0(x)\cdot\left(\frac{\partial\varphi}{\partial x_{d-k+1}}\right)_{k-1} dm(x)\\ &= \int f_0(x)\cdot\frac{\partial\varphi_{k-1}}{\partial x_{d-k+1}}\, dm(x)\end{aligned}$$

with $\varphi_{k-1} \in C_c^1(\mathbb{R}^d, \mathbb{R})$, $\|\varphi_{k-1}\|_\infty \le 1$. Hence

$$\mathrm{var}_C^{(d-k+1)}(f_{k-1}) \le \mathrm{var}_C^{(d-k+1)}(f_0) = \mathrm{var}_C^{(d-k+1)}(f)$$

so that

$$\int_C |f_{k-1} - f_k|\, dm \le \epsilon \cdot \mathrm{var}_C^{(d-k+1)}(f) \le \epsilon \cdot \mathrm{var}_C(f)\ .$$

Summing this estimate over $k = 1, \ldots, d$ finishes the proof of the lemma. □

Corollary 3. *For each $f \in BV$*

$$\|f - \mathcal{E}_\epsilon f\|_1 \le d\,\epsilon \cdot \|f\|_{BV}\ .$$

Proof. For each $W \in \mathcal{W}_\epsilon$ we have $\int_W |f - \mathcal{E}_\epsilon f|\, dm \le d\,\epsilon \cdot \mathrm{var}_W(f)$ by Lemma 9. Therefore

$$\|f - \mathcal{E}_\epsilon f\|_1 \le d\,\epsilon \cdot \sum_{W \in \mathcal{W}_\epsilon} \mathrm{var}_W(f) \le d\,\epsilon \cdot \mathrm{var}(f) \le d\,\epsilon \cdot \|f\|_{BV}\ .$$

□

Lemma 10. *For $f \in BV$ and $\epsilon > 0$,*

$$\|\mathcal{E}_\epsilon f\|_{BV} \le d \cdot \|f\|_{BV}\ .$$

Proof. $\|\mathcal{E}_\epsilon f\|_1 \le \|f\|_1$ since $\mathcal{E}_\epsilon$ is a conditional expectation operator. For the variation part of the $\|.\|_{BV}$-norm we show $\mathrm{var}^{(i)}(\mathcal{E}_\epsilon f) \le \mathrm{var}(f)$ for all i.

For $W \in \mathcal{W}_\epsilon$ write $W = W_i \times W_i'$ such that $x_i \in W_i$, $x_i' \in W_i'$ if $x = (x_i, x_i') \in W$. Let

$$\begin{aligned}\mathcal{E}_{\epsilon,i} f(x) &:= \sum_{W_i}\sum_{W_i'} 1_W(x)\frac{1}{m_1(W_i)}\int_{W_i} f(\,.\,, x_i')\, dm_1\\ \mathcal{E}_{\epsilon,i}' f(x) &:= \sum_{W_i}\sum_{W_i'} 1_W(x)\frac{1}{m_{d-1}(W_i')}\int_{W_i'} f(x_i,\,.\,)\, dm_{d-1}\end{aligned}$$

$\mathcal{E}_{\epsilon,i}$ and $\mathcal{E}_{\epsilon,i}'$ are conditional expectation operators (in particular they are self-adjoint), and since m_d is the product of m_1 and m_{d-1}, it is not hard to check that $\mathcal{E}_\epsilon = \mathcal{E}_{\epsilon,i}' \circ \mathcal{E}_{\epsilon,i}$.

Let $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{R})$, $\|\varphi\|_\infty \le 1$, and $f \in BV$. Then

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{E}_\epsilon f \cdot \frac{\partial \varphi}{\partial x_i}\, dm_d &= \int_{\mathbb{R}^d} \mathcal{E}'_{\epsilon,i}\mathcal{E}_{\epsilon,i} f \cdot \frac{\partial \varphi}{\partial x_i}\, dm_d \\ &= \int_{\mathbb{R}^d} \mathcal{E}'_{\epsilon,i}\mathcal{E}_{\epsilon,i} f \cdot \mathcal{E}'_{\epsilon,i}\frac{\partial \varphi}{\partial x_i}\, dm_d \\ &= \int_{\mathbb{R}^d} \mathcal{E}_{\epsilon,i} f \cdot \mathcal{E}'_{\epsilon,i}\frac{\partial \varphi}{\partial x_i}\, dm_d \\ &= \int_{\mathbb{R}^d} \mathcal{E}_{\epsilon,i} f \cdot \frac{\partial}{\partial x_i}\mathcal{E}'_{\epsilon,i}\varphi\, dm_d \\ &= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \mathcal{E}_{\epsilon,i} f(x_i, x'_i) \cdot \frac{\partial}{\partial x_i}\mathcal{E}'_{\epsilon,i}\varphi(x_i, x'_i)\, dx_i \right) dx'_i \\ &\le \mathrm{var}^{(i)}(f)\ .\end{aligned}$$

For the last estimate we used (8) and the fact that $x_i \mapsto \mathcal{E}'_{\epsilon,i}\varphi(x_i, x'_i)$ is continuously differentiable and $\|\mathcal{E}'_{\epsilon,i}\varphi(x_i, x'_i)\|_\infty \le \|\varphi\|_\infty \le 1$ so that

$$\int_{\mathbb{R}} \mathcal{E}_{\epsilon,i} f(x_i, x'_i) \cdot \frac{\partial}{\partial x_i}\mathcal{E}'_{\epsilon,i}\varphi(x_i, x'_i)\, dx_i \le \mathrm{ess}\, \mathrm{V}_{-\infty}^{\infty}(x_i \mapsto \mathcal{E}_{\epsilon,i} f(x_i, x'_i)) \le \mathrm{ess}\, \mathrm{V}_{-\infty}^{\infty}(x_i \mapsto f(x_i, x'_i))\ .$$

(The last inequality reflects an elementary one-dimensional fact.) □

5. Proof of Theorem 2

Proof of Theorem 2: Assertion a), i.e. the estimates on $r(\mathcal{L}|\mathfrak{BV})$ and $r_{\mathrm{ess}}(\mathcal{L}|\mathfrak{BV})$, follow from [8] combined with [20]. As noted in [21], $\mathcal{L}$ satisfies a Lasota-Yorke type inequality on BV, namely for each $\alpha > \vartheta$ there are $K_1, K_2 > 0$ such that

$$\|\mathcal{L}^n f\|_{BV} \le K_1 \alpha^n \|f\|_{BV} + K_2 \|f\|_1 \qquad \forall f \in BV, n \in \mathbb{N}\ . \tag{9}$$

(The references deal with the arbitrary plane case, but the generic case (1) is easily deduced from the proofs there.)

We turn to the proof of assertion b). Let λ be an isolated eigenvalue of $\mathcal{L} : BV \to BV$, $|\lambda| > \vartheta$, and denote by $\pi_\lambda : BV \to BV$ its associated spectral projector. We must show that $\pi_\lambda(BV) \subseteq \mathfrak{BV}$. To this end it suffices to prove the following two claims:

i) $\pi_\lambda(\mathfrak{BV}) = \pi_\lambda(BV)$,
ii) $\pi_\lambda(\mathfrak{BV}) \subseteq \mathfrak{BV}$.

Proof of Claim i): Denote by $\overline{\mathfrak{BV}}$ the $\|.\|_{BV}$-closure of $\mathfrak{BV}$ in BV. (We do not know whether $\overline{\mathfrak{BV}} = BV$.) Our first step is to show that

$$\pi_\lambda(BV) \subseteq \overline{\mathfrak{BV}}\ . \tag{10}$$

So let $0 \ne f \in \pi_\lambda(BV)$, i.e. there is $k > 0$ such that $(\lambda - \mathcal{L})^k f = 0$ and $(\lambda - \mathcal{L})^{k-1} f \ne 0$. Let $h := (\lambda - \mathcal{L}) f$ and suppose that $h \in \overline{\mathfrak{BV}}$. Then

$$h_n := f - (\lambda^{-1}\mathcal{L})^n f = \lambda^{-1} \sum_{j=0}^{n-1} (\lambda^{-1}\mathcal{L})^j h \in \overline{\mathfrak{BV}}$$

since $\mathcal{L}(\mathfrak{BV}) \subseteq \mathfrak{BV}$ implies $\mathcal{L}(\overline{\mathfrak{BV}}) \subseteq \overline{\mathfrak{BV}}$. Let $\epsilon = \alpha^n$ and recall the conditional expectation operator $\mathcal{E}_\epsilon$ from the preceding section.

$$\begin{aligned}\|f - (\lambda^{-1}\mathcal{L})^n \mathcal{E}_\epsilon f - h_n\|_{BV} &= \|(\lambda^{-1}\mathcal{L})^n (f - \mathcal{E}_\epsilon f)\|_{BV} \\ &\le K_1 \left(\frac{\alpha}{|\lambda|}\right)^n \|f - \mathcal{E}_\epsilon f\|_{BV} + K_2 |\lambda|^{-n} \|f - \mathcal{E}_\epsilon f\|_1\end{aligned}$$

by (9). Remark that the support of $\mathcal{E}_\epsilon f$ is contained in an ϵ-neighborhood of X, but not necessarily in X. However $\mathcal{L}$ extends naturally to functions defined on this larger domain (one keeps the same formula for the definition of $\mathcal{L}$, it just forgets the function outside X) and (9) is easily seen to remain true. Since $\max_{Z\in\mathcal{Z}_n}\operatorname{diam}(Z)\le \text{const}\cdot\alpha^n$, Corollary 3 and Lemma 10 imply

$$\|f-(\lambda^{-1}\mathcal{L})^n\mathcal{E}_\epsilon f-h_n\|_{BV}\le K_3\left(\frac{\alpha}{|\lambda|}\right)^n\|f\|_{BV}$$

for a suitable constant $K_3>0$. As $|\lambda|>\vartheta$, we can choose $\alpha\in(\vartheta,|\lambda|)$ and conclude that f can be approximated in $\|.\|_{BV}$ by elements from $\mathfrak{BV}$. Hence $f\in\overline{\mathfrak{BV}}$. This proves (10) if $k=1$ since in that case $(\lambda-\mathcal{L})f=0\in\overline{\mathfrak{BV}}$. For $k>1$ one proceeds by induction using that $(\lambda-\mathcal{L})^{k-1}f\in\overline{\mathfrak{BV}}$.

Now $\pi_\lambda(\mathfrak{BV})$ is a linear subspace of $\pi_\lambda(BV)$, and as $\dim(\pi_\lambda(BV))<\infty$, also $\dim(\pi_\lambda(\mathfrak{BV}))<\infty$. Therefore $\pi_\lambda(\mathfrak{BV})$ is closed. As $\pi_\lambda:BV\to BV$ is continuous, it follows that $\pi_\lambda(\overline{\mathfrak{BV}})=\pi_\lambda(\mathfrak{BV})$, so that in view of (10)

$$\pi_\lambda(BV)=\pi_\lambda(\pi_\lambda(BV))\subseteq\pi_\lambda(\overline{\mathfrak{BV}})=\pi_\lambda(\mathfrak{BV})\subseteq\pi_\lambda(BV)$$

This proves Claim i).

Proof of Claim ii): For this proof denote $\mathcal{L}:BV\to BV$ simply by $\mathcal{L}$ and $\mathcal{L}:\mathfrak{BV}\to\mathfrak{BV}$ by $\tilde{\mathcal{L}}$. Fix $\delta>0$ so small that $B_\delta(\lambda)\cap\sigma(\mathcal{L})=\{\lambda\}$ and $|\lambda|-\delta>\vartheta$. Then $B_\delta(\lambda)\cap\sigma(\tilde{\mathcal{L}})\subseteq\{\lambda\}$, because each $\tilde\lambda\in B_\delta(\lambda)\cap\sigma(\tilde{\mathcal{L}})$ must be an isolated eigenvalue of $\tilde{\mathcal{L}}$ by assertion a) of this theorem and hence also an eigenvalue of $\mathcal{L}$. Therefore,

$$\tilde\pi_\lambda:=\int_{|z-\lambda|=\delta}(z-\tilde{\mathcal{L}})^{-1}\,dz$$

is well-defined as a continuous projection from $\mathfrak{BV}\to\mathfrak{BV}$. As $\|.\|_{BV}\le\text{const}\cdot\|.\|_{\mathfrak{BV}}$ by Lemma 1, $\tilde\pi_\lambda f=\pi_\lambda f$ for $f\in\mathfrak{BV}$. Hence $\pi_\lambda(\mathfrak{BV})=\tilde\pi_\lambda(\mathfrak{BV})\subseteq\mathfrak{BV}$. □

Remark 8. The proof above would apply to any subspace S of BV endowed with a norm stronger than the BV-norm and such that:
$\mathcal{L}:S\to S$ is well-defined and bounded with essential spectral radius at most ϑ.
$\mathcal{E}_\epsilon(BV)\subset S$ for each $\epsilon>0$.

6. The transfer operator on the extension: Proofs of Theorems 3, 4 and 5

Proof of Theorem 3: We begin with the bound on the spectral radius.

First, $r(\hat{\mathcal{L}})\ge 1$ as $\|\hat{\mathcal{L}}^n(\hat h)\|_{\widehat{\mathfrak{BV}}}\ge\|\hat{\mathcal{L}}^n(\hat h)\|_1=\|\hat h\|_1>0$ for any positive function $\hat h\in\widehat{\mathfrak{BV}}$ and all $n=1,2,\dots$. Assuming the second assertion of the theorem, i.e., $r_{\text{ess}}(\hat{\mathcal{L}})\le\vartheta<1$, we see that $r(\hat{\mathcal{L}})>r_{\text{ess}}(\hat{\mathcal{L}})$ so that $r(\hat{\mathcal{L}})$ is the largest modulus of the eigenvalues of $\hat{\mathcal{L}}$ in $\widehat{\mathfrak{BV}}$. But $\widehat{\mathfrak{BV}}\subset L^1(\hat X)$, hence $\widehat{\mathfrak{BV}}$-eigenfunctions are $L^1(\hat X)$-eigenfunctions and $r(\hat{\mathcal{L}})\le r(\hat{\mathcal{L}}|L^1(\hat X))=1$: $r(\hat{\mathcal{L}})=1$ is proved.

We turn to the bound on the essential spectral radius. It will follow from Nussbaum's formula:

$$r_{\text{ess}}(\hat{\mathcal{L}})=\lim_{n\to\infty}\inf\{\|\hat{\mathcal{L}}^n-Q\|^{1/n}_{\widehat{\mathfrak{BV}}}:Q\text{ compact}\},$$

if, for each $\Theta>\vartheta$, we can find compact operators $\hat{\mathcal{L}}_1,\hat{\mathcal{L}}_2,\dots$ such that, for all large n:

$$\|\hat{\mathcal{L}}^n-\hat{\mathcal{L}}_n\|_{\widehat{\mathfrak{BV}}}\le\Theta^n. \tag{11}$$

Recall that the $\widehat{\mathfrak{BV}}$-norm depends on a parameter $\epsilon_0>0$ which we still have to choose (independently of n). We fix $\Theta\in(\vartheta,1)$ (and will increase it by an arbitrarily small amount, a finite number of times).

Set $\hat P(D,x)=(\mathcal{Z}(x),x)$ and $(\hat P_*\hat h)(\hat x)=\sum_{\hat y\in\hat P^{-1}(\hat x)}\hat h(\hat y)$. Let $p(D,x)=x$ and $(p_*\hat h)(x)=\sum_{\hat x\in p^{-1}(x)}\hat h(\hat x)$. Denote by $\mathcal{E}_Mf$ the conditional expectation of $f\in L^1(\mathbb{R}^d,\mathbb{R})$ with respect to Lebesgue measure and the partition $\mathcal{Z}_M$.

We are going to prove that there is an integer M such that $\hat{\mathcal{L}}_n = \hat{\mathcal{L}}^n \hat{P}_* - \hat{Q}_n$ gives the required approximation for all large n with

$$\hat{Q}_n \hat{h} = \hat{\mathcal{L}}^r \sum_{Z_1,\dots,Z_m \in \hat{\mathcal{Z}}_M} 1_{\hat{D}} \cdot \mathcal{L}^M(\mathrm{Id} - \mathcal{E}_M) 1_{Z_m} \mathcal{L}^M(\mathrm{Id} - \mathcal{E}_M) 1_{Z_{m-1}} \dots \mathcal{L}^M(\mathrm{Id} - \mathcal{E}_M) 1_{Z_1} p_* \hat{h}$$

where $n = mM + r$, $0 \leq r < M$, and $D = \langle Z_1 \dots Z_m \rangle$. Here 1_{Z_j} is to be interpreted as a multiplication operator on $\mathfrak{BV}$, and $1_{\hat{D}}$ denotes the operator that lifts a function $f \in \mathfrak{BV}$ with support in $\mathrm{fol}(D)$ to a function $1_{\hat{D}} f \in \widehat{\mathfrak{BV}}$ with support in $\hat{D}$.

To begin with:

(12) each $\hat{\mathcal{L}}_n$, $n \geq 0$, is a compact, even finite-dimensional, operator: it is a sum of terms of the form $A \circ \mathcal{E}_M \circ B$, where A and B are bounded.

Indeed, $\hat{Q}_n \hat{h}$ is equal, modulo such terms, to:

$$\hat{\mathcal{L}}^r \sum_{Z_1,\dots,Z_m \in \mathcal{Z}_M} 1_{\hat{D}} \mathcal{L}^M 1_{Z_m} \dots \mathcal{L}^M 1_{Z_1} p_* \hat{h} = \hat{\mathcal{L}}^n \hat{P}_* \hat{h}.$$

It remains to bound $\|\hat{\mathcal{L}}^n - \hat{\mathcal{L}}_n\|_{\widehat{\mathfrak{BV}}} \leq \|\hat{\mathcal{L}}^n - \hat{\mathcal{L}}^n \hat{P}_*\|_{\widehat{\mathfrak{BV}}} + \|\hat{Q}_n\|_{\widehat{\mathfrak{BV}}}$ for a suitable choice of the parameters M and ϵ_0.

Step 1: Specific estimates for piecewise affine or plane maps.

Lemma 11. *Let $(X, \mathcal{Z}, T)$ be a piecewise affine map with a piecewise constant weight g. Set, for each $n \geq 1$,*

$$\Delta_n := \sup_{Z \in \mathcal{Z}_n} \sup_{D \in \mathcal{D}} \mathfrak{var} \left(\frac{1}{g^{(n)}(Z)} \mathcal{L}^n 1_{Z \cap p(\hat{D})} \right).$$

Then

$$\limsup_{n \to \infty} \frac{1}{n} \log \Delta_n = 0.$$

Proof. We prove the stronger statement: $\sup_{n \geq 1} \Delta_n < \infty$.

g is piecewise constant so that $g^{(n)}(Z)^{-1} \cdot \mathcal{L}^n 1_{Z \cap p(\hat{D})} = 1_{T^n(Z \cap p(\hat{D}))}$. Thus it suffices to find a uniform bound for $\mathfrak{var}\,(1_C)$ for convex $C \subset X$. (Remark that $T^n(Z \cap p(\hat{D}))$ is a polytope but with an unbounded number of faces.)

By definition: $\mathfrak{var}\,(1_C) = \sup_{0<\epsilon<\epsilon_0} \mathrm{vol}(B(\partial C, \epsilon))/\epsilon$.

Now for every point $x \in C$ there is a face $F \subset \partial C$ such that the orthogonal projection of x onto the hyperplane supporting F lies in F. This implies that, on the one hand, $\mathrm{vol}(B(\partial C, \epsilon) \cap C)$ is bounded by ϵ times the area (i.e., the $d-1$-dimensional measure) of ∂C. But this last quantity is uniformly bounded (see, e.g., [2, 12.3.7]).

On the other hand, $\mathrm{vol}(B(\partial C, \epsilon) \setminus C)$ is a polynomial in ϵ with uniformly bounded coefficients (see, e.g., [2, 12.3.6 and 12.3.7]). This proves the lemma. □

The following result controls the local complexity of the iterates of T. Recall that Saussol's G function is

$$G(T, \epsilon, \epsilon_0) = \sup_{x \in X} \sum_{Z \in \mathcal{Z}} \frac{m(T_Z^{-1}(B(\partial T Z, \epsilon)) \cap B(x, \frac{1}{2}\epsilon_0))}{m(B(x, \frac{1}{2}\epsilon_0))}.$$

Proposition 2. *If $(X, \mathcal{Z}, T)$ is piecewise affine and either plane or generic in the sense of eq. (1), then for all large n, there exists $\epsilon_*(n) > 0$ such that:*

$$\frac{\epsilon_0}{\epsilon} G(T^n, \epsilon, \epsilon_0) \leq \Theta^n \qquad \forall 0 < \epsilon \leq \epsilon_0 \leq \epsilon_*(n).$$

Proof. The plane case is proved in [8]. The generic case is an easy consequence of the proofs there. □

Finally, by Proposition 1, $\exp P_g(\mathcal{B}(T), T) \leq \vartheta$ for these maps.

We return to a general, piecewise expanding and C^{1+1} system $(X, \mathcal{Z}, T)$ with weight $g = 1/|\det DT| \geq 0$ satisfying:

1. $\mathcal{Z}$ separates cylinder components.
2. the conclusions of the above Lemma 11 and Propositions 1 and 2 hold.

Step 2: Exponential contraction.

We claim that $\|\hat{Q}_n\|_{\widehat{\mathfrak{BV}}} \leq \Theta^n$ for all large n, provided that the parameters ϵ_0 and M are respectively small and large enough. We first establish contraction for $\mathcal{L}$ by proving a refinement of [20, Lemma 3]. The proof makes use of the following fact: Since some iterate of T is uniformly expanding, there is $\epsilon_1 > 0$ such that $(T^n{}_Z)^{-1}$ is defined on a ϵ_1-neighbourhood of $T^n Z$ for all $n \geq 0$ and all $Z \in \mathcal{Z}_n$.

Proposition 3. *Let $(X, \mathcal{Z}, T)$ be a piecewise expanding C^{1+1} system. Let $g = 1/|\det DT|$. Then for all large n, all $h \in \mathfrak{BV}$, and all $0 < \epsilon_0 \leq \epsilon_1$*

$$\sum_{Z \in \mathcal{Z}_n} \mathfrak{var}\left(\mathcal{L}^n(1_Z(h - \mathcal{E}_n h))\right) \leq \text{const} \cdot \left(\Theta^n + \sup_{0<\epsilon\leq\epsilon_0} \frac{\epsilon_0}{\epsilon} \cdot G(T^n, \epsilon, \epsilon_0)\right) \cdot \mathfrak{var}\,(h)\,.$$

Proof. Let $h \in \mathfrak{BV}$ and set $\bar{h} = h - \mathcal{E}_n h$. By the definition of osc(.),

$$\begin{aligned}
\sum_{Z\in\mathcal{Z}_n} \operatorname{osc}(\mathcal{L}^n(1_Z \bar{h}), B(x,\epsilon)) &\leq \sum_{Z\in\mathcal{Z}_n} \operatorname{osc}((\bar{h}\cdot g^{(n)}) \circ T_Z^{-n} \cdot 1_{T^n Z}, B(x,\epsilon)) \\
&\leq \sum_{Z\in\mathcal{Z}_n} \Bigg(\underbrace{\operatorname{osc}(\bar{h}\cdot g^{(n)}, Z \cap T^{-n}B(x,\epsilon))}_{=:R_Z^1(x)} \cdot 1_{T^n Z}(x) \\
&\qquad + \underbrace{2 \operatorname*{ess\,sup}_Z g^{(n)} \cdot \operatorname*{ess\,sup}_{Z\cap T^{-n}B(x,\epsilon)} |\bar{h}| \cdot 1_{B(\partial T^n Z,\epsilon)(x)}}_{=:R_Z^2(x)} \Bigg)\,.
\end{aligned}$$

T is uniformly expanding and g is piecewise Lipschitz and bounded away from zero. Therefore a standard argument gives that: $g^{(n)}(x)/g^{(n)}(y) - 1 \leq \text{const} \cdot d(T^n x, T^n y)$, for all x, y in the same element of $\mathcal{Z}_n$. Recall also that $\operatorname{osc}(uv, B) \leq \operatorname{ess\,sup} u \operatorname{osc}(v, B) + \operatorname{ess\,inf} v \operatorname{osc}(u, B)$. Thus, for $x \in T^n Z$,

$$\begin{aligned}
R_Z^1(x) &\leq \operatorname*{ess\,sup}_Z g^{(n)} \cdot \operatorname{osc}(\bar{h}, Z\cap T^{-n}B(x,\epsilon)) + \operatorname{osc}(g^{(n)}, Z\cap T^{-n}B(x,\epsilon)) \cdot |\bar{h}(T_Z^{-n}x)| \\
&\leq \text{const} \cdot g^{(n)}(T_Z^{-n}x) \cdot \operatorname{osc}(h, B(T_Z^{-n}x, \Theta^n\epsilon)) + \text{const} \cdot \epsilon \cdot g^{(n)}(T_Z^{-n}x)|\bar{h}(T_Z^{-n}x)|,
\end{aligned}$$

hence

$$\sum_{Z\in\mathcal{Z}_n} R_Z^1(x) \cdot 1_{T^n Z}(x) \leq \text{const} \cdot \mathcal{L}^n\left(\operatorname{osc}(h, B(\cdot, \Theta^n\epsilon)) + \epsilon|\bar{h}|\right)(x)$$

and, observing that $|\bar{h}(x)| \leq \operatorname{osc}(h, B(x, \Theta^n \operatorname{diam} X))$,

$$\begin{aligned}
\int_X \sum_{Z\in\mathcal{Z}_n} R_Z^1(x) \cdot 1_{T^n Z}(x)\,dx &\leq \text{const} \cdot \int_X \operatorname{osc}(h, B(x,\Theta^n\epsilon)) + \epsilon \cdot \operatorname{osc}(h, B(x, \Theta^n \operatorname{diam} X))\,dx \\
&\leq \text{const} \cdot \Theta^n \cdot \epsilon \cdot \mathfrak{var}(h).
\end{aligned}$$

For $R_Z^2(x)$ we have (assuming $\epsilon \leq \epsilon_1$ and $\Theta^n \leq 1/2$ and making a change of variables):

$$\begin{aligned}
\int_X \sum_{Z\in\mathcal{Z}_n} R_Z^2(x)\,dx &\leq \text{const} \cdot \sum_{Z\in\mathcal{Z}_n} \int_X 1_{T_Z^{-n}B(\partial T^n Z,\epsilon)}(x) \cdot \operatorname*{ess\,sup}_{Z\cap B(x,\Theta^n\epsilon)} |\bar{h}|\,dx \\
&\leq \text{const} \cdot \sum_{Z\in\mathcal{Z}_n} \int_X dx \frac{1_{T_Z^{-n}B(\partial T^n Z,\epsilon)}(x)}{m(B(x,\frac{1}{2}\epsilon_0))} \cdot \int_{B(x,\frac{1}{2}\epsilon_0)} \operatorname{osc}(h, B(z,\epsilon_0))\,dz.
\end{aligned}$$

So

$$\begin{aligned}\int_X \sum_{Z\in\mathcal{Z}_n} R_Z^2(x)\,dx &\le \text{const}\cdot\int_X dz\,\text{osc}(h,B(z,\epsilon_0))\cdot\int_X\sum_{Z\in\mathcal{Z}_n}\frac{1_{T_Z^{-n}B(\partial T^nZ,\epsilon)}(x)1_{B(z,\frac12\epsilon_0)}(x)}{m(B(x,\frac12\epsilon_0))}\,dx\\ &\le \text{const}\cdot G(T^n,\epsilon,\epsilon_0)\cdot\int_X \text{osc}(h,B(z,\epsilon_0))\,dz\\ &\le \text{const}\cdot G(T^n,\epsilon,\epsilon_0)\cdot\mathfrak{var}(h)\cdot\epsilon_0.\end{aligned}$$

Together we have

$$\sum_{Z\in\mathcal{Z}_n}\frac{1}{\epsilon}\int_X \text{osc}(\mathcal{L}^n(1_Z\cdot\bar h),B(x,\epsilon))\,dx \le \text{const}\cdot\Big(\Theta^n+\frac{\epsilon_0}{\epsilon}\cdot G(T^n,\epsilon,\epsilon_0)\Big)\cdot\mathfrak{var}(h).$$

□

To prove the bound on $\|\hat Q_n\|_{\widehat{\mathfrak{BV}}}$, we pick M and then ϵ_0 so that the two previous Propositions give:

$$\sum_{Z\in\mathcal{Z}_M}\mathfrak{var}\left(\mathcal{L}^M(1_Z(h-\mathcal{E}_Mh))\right)\le\Theta^M\cdot\mathfrak{var}\,(h)\,.$$

Recall that $n=mM+r$ with $0\le r<M$ and that $D=\langle Z_1\dots Z_m\rangle$. $\hat{\mathcal{L}}$ being bounded, it is enough to deal with the case $r=0$.

$$\begin{aligned}\widehat{\mathfrak{var}}(\hat Q_n\hat h) &\le \sum_{Z_1,\dots,Z_m\in\mathcal{Z}_M}\mathfrak{var}\left(1_{\text{fol}(D)}\cdot\mathcal{L}^M(\text{Id}-\mathcal{E}_M)1_{Z_m}\dots\mathcal{L}^M(\text{Id}-\mathcal{E}_M)1_{Z_1}p_*\hat h\right)\\ &\le \sum_{Z_1,\dots,Z_m\in\mathcal{Z}_M}\text{const}_d\cdot\epsilon_0^{1-d}\cdot\mathfrak{var}(1_{\text{fol}(D)})\\ &\qquad\cdot\mathfrak{var}\left(\mathcal{L}^M(\text{Id}-\mathcal{E}_M)1_{Z_m}\dots\mathcal{L}^M(\text{Id}-\mathcal{E}_M)1_{Z_1}p_*\hat h\right)\\ &\le\text{const}_d\cdot\epsilon_0^{1-d}\cdot\Delta_0\cdot\Theta^{mM}\cdot\mathfrak{var}(p_*\hat h),\end{aligned}$$

where we have used that

$$\mathfrak{var}(uv)\le\operatorname{ess\,sup}u\cdot\mathfrak{var}\,v+\mathfrak{var}\,u\cdot\operatorname{ess\,sup}v\le\text{const}_d\cdot\epsilon_0^{1-d}\;\mathfrak{var}\,u\cdot\mathfrak{var}\,v \tag{13}$$

in view of Lemma 1. Finally, observe that $\mathfrak{var}(p_*\hat h)\le\widehat{\mathfrak{var}}(\hat h)$.

This concludes Step 2, using that $\widehat{\mathfrak{var}}$ dominates the $\widehat{\mathfrak{BV}}$-norm (see Corollary 2).

Step 3: We prove that, for all large n: $\|\hat{\mathcal{L}}^n-\hat{\mathcal{L}}^n\circ\hat P_*\|_{\widehat{\mathfrak{BV}}}\le\Theta^n$.

As above, we bound the variation $\mathfrak{var}$. Fix $\epsilon>0$ such that $e^{3\epsilon}\vartheta\le\Theta$.

$\mathcal{Z}$ separates cylinder components. Therefore we can find $L\mapsto N(L)$ non-decreasing and as in Definition 1. Let $L=L(n)$ be maximal under the condition

$$N(L)<\epsilon n/\log(\#Z\cdot\max(|g|,1)). \tag{14}$$

Clearly $L\to\infty$ as $n\to\infty$, so we may assume that L is large.

Let $\hat h\in\widehat{\mathfrak{BV}}$. Setting $\hat U_n=\{\hat x\in\hat X:\hat T^n\hat x\ne\hat T^n\hat P\hat x\}$, notice that

$$\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n\hat h-(\hat{\mathcal{L}}^n\circ\hat P_*)\hat h\right)\le 2\,\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n(1_{\hat U_n}\cdot\hat h)\right).$$

We claim that $\hat{\mathcal{L}}^n(1_{\hat U_n}\cdot\hat h)$ is supported by $\hat X\setminus\hat X_n$. Fix $(D,x)\in\hat U_n$. By Lemma 2,

$$\hat T^n(D,x)\in\hat E\text{ with }E=\langle A_{-q}\dots A_0A_1\dots A_n\rangle$$

if $D=\langle A_{-q}\dots A_0\rangle$ and $[A_0\dots A_n]=\mathcal{Z}_{n+1}(x)$. Likewise,

$$\hat T^n(\mathcal{Z}(x),x)\in\hat F\text{ with }F=\langle A_0A_1\dots A_n\rangle\,.$$

As $E\ne F$, this implies $E\notin\mathcal{D}_n$, proving the claim.

Denoting by $\hat h_D$ the function $x\mapsto\hat h(D,x)$ on $\text{fol}(D)$, and by $A_0\dots A_n$ the projection of $C_0\dots C_n$, we conclude

$$\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n\hat h-(\hat{\mathcal{L}}^n\circ\hat P_*)\hat h\right)\le 2\sum_{D\in\mathcal{D}\setminus\mathcal{D}_n}\;\sum_{C_0\to\dots\to C_n=D}\mathfrak{var}\left(\mathcal{L}^n(1_{[A_0\dots A_n]}\cdot\hat h_{C_0})\right).$$

Now,

$$\mathcal{L}^n(1_{[A_0\ldots A_n]}\cdot\hat{h}_{C_0}) = g^{(n)}\circ T^{-n}_{[A_0\ldots A_n]}\cdot 1_{T^n[A_0\ldots A_n]}\cdot\hat{h}_{C_0}\circ T^{-n}_{[A_0\ldots A_n]}.$$

Hence,

$$\begin{aligned}\mathfrak{var}(\mathcal{L}^n(1_{[A_0\ldots A_n]}\cdot\hat{h}_{C_0})) &\le \operatorname{ess\,sup}\mathcal{L}^n 1_{[A_0\ldots A_n]}\cdot\mathrm{const}\vartheta^n\cdot\mathfrak{var}(\hat{h}_{C_0})+\mathfrak{var}(\mathcal{L}^n 1_{[A_0\ldots A_n]})\cdot\operatorname{ess\,sup}\hat{h}_{C_0}\\ &\le \mathrm{const}\cdot\mathfrak{var}(\mathcal{L}^n 1_{[A_0\ldots A_n]})\cdot\mathfrak{var}(\hat{h}_{C_0})\end{aligned}$$

using eq. (13). Therefore, by Lemma 11,

$$\begin{aligned}\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n\hat{h}-(\hat{\mathcal{L}}^n\circ\hat{P}_*)\hat{h}\right) &\le \mathrm{const}\sum_{D\in\mathcal{D}\setminus\mathcal{D}_n}\ \sum_{C_0\to\cdots\to C_n=D}\mathfrak{var}\left(\mathcal{L}^n 1_{[A_0\ldots A_n]}\right)\cdot\mathfrak{var}\left(\hat{h}_{C_0}\right)\\ &\le \mathrm{const}\sum_{D\in\mathcal{D}\setminus\mathcal{D}_n}\ \sum_{C_0\to\cdots\to C_n=D}\Delta_n\,|g^{(n)}(A_0\ldots A_n)|\cdot\mathfrak{var}\left(\hat{h}_{C_0}\right)\\ &\le \sum_{D\in\mathcal{D}\setminus\mathcal{D}_n}\ \sum_{C_0\to\cdots\to C_n=D} e^{\epsilon n}\cdot|g^{(n)}(A_0\ldots A_n)|\cdot\mathfrak{var}\left(\hat{h}_{C_0}\right)\end{aligned}$$

for all large n. Summing first on $C=C_0$ we get:

$$\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n\hat{h}-(\hat{\mathcal{L}}^n\circ\hat{P}_*)\hat{h}\right)\le e^{\epsilon n}\sum_{C\in\mathcal{D}}\mathfrak{var}(\hat{h}_C)\sum_{\substack{C_0=C\to\cdots\to C_n\\ C_n\notin\mathcal{D}_n}}|g^{(n)}(A_0\ldots A_n)|.$$

The successors of elements of $\mathcal{D}_\ell$ are elements of $\mathcal{D}_{\ell+1}$, therefore $C_n\notin\mathcal{D}_n$ implies that $C_k\notin\mathcal{D}_k$ for each $k\le n$. In particular, $C_{iL+N}\notin\mathcal{D}_N$ for $0\le i<q:=[(n-(N-L))/L]$. Using eq. (6) of section 3, we get

$$\begin{aligned}\widehat{\mathfrak{var}}\left(\hat{\mathcal{L}}^n\hat{h}-(\hat{\mathcal{L}}^n\circ\hat{P}_*)\hat{h}\right) &\le \mathrm{const}\cdot e^{(P_g(\mathcal{B}(T),T)+2\epsilon)n}\sum_{C\in\mathcal{D}}\mathfrak{var}(\hat{h}_C)\\ &\le \mathrm{const}\cdot e^{(P_g(\mathcal{B}(T),T)+2\epsilon)n}\|\hat{h}\|_{\widehat{\mathfrak{BV}}}\\ &\le \mathrm{const}\cdot\Theta^n\|\hat{h}\|_{\widehat{\mathfrak{BV}}}\end{aligned}$$

by Proposition 1.

This concludes Step 3 and the proof of Theorem 3. □

Proof of Theorem 4: Let $\lambda\in\mathbb{C}$, $|\lambda|>\vartheta$. Suppose first that λ is an eigenvalue of $\mathcal{L}:\mathfrak{BV}\to\mathfrak{BV}$ with algebraic multiplicity k. We will apply Proposition 7.1(b) from [17] to show that λ is also an eigenvalue of $\hat{\mathcal{L}}:\widehat{\mathfrak{BV}}\to\widehat{\mathfrak{BV}}$ with algebraic multiplicity k. To this end we must check a number of assumptions for that proposition. Recall that $p_*:\widehat{\mathfrak{BV}}\to\mathfrak{BV}$ is defined by $(p_*\hat{f})(x):=\sum_{\hat{x}\in p^{-1}x}\hat{f}(\hat{x})$. Then

(A1) $(\widehat{\mathfrak{BV}},\|.\|_{\widehat{\mathfrak{BV}}})$ is a Banach space of L^1-equivalence classes of functions on $\hat{X}$, $\hat{\mathcal{L}}(\widehat{\mathfrak{BV}})\subseteq\widehat{\mathfrak{BV}}$.
(A2) $(\mathfrak{BV},\|.\|_{\mathfrak{BV}})$ is a Banach space of L^1-equivalence classes of functions on X, $\mathcal{L}(\mathfrak{BV})\subseteq\mathfrak{BV}$.
(A3) If $\hat{f}\in\widehat{\mathfrak{BV}}$, then

$$\|p_*\hat{f}\|_{\mathfrak{BV}}\le\sum_{D\in\mathcal{D}}\|\hat{f}(D,.)\|_{\mathfrak{BV}}=\|\hat{f}\|_{\widehat{\mathfrak{BV}}}\ ,$$

so $p_*\hat{f}\in\mathfrak{BV}$.
(A4) $\hat{\mathcal{L}}:\widehat{\mathfrak{BV}}\to\widehat{\mathfrak{BV}}$ is quasicompact, i.e. $r_{\mathrm{ess}}(\hat{\mathcal{L}})<r(\hat{\mathcal{L}})=1$ (Theorem 3).
(A5) $\mathcal{L}:\mathfrak{BV}\to\mathfrak{BV}$ is bounded with spectral radius $r(\mathcal{L})$.
(A6) For $f\in\mathfrak{BV}$ define $\hat{f}$ by $\hat{f}(D,x)=\sum_{Z\in\mathcal{Z}}\delta_{Z,\mathrm{fol}(D)}f(x)$. In view of the construction in section 3, $\hat{f}$ is the "injection" of f to the "base" of the Markov extension. Obviously $\hat{f}\in\widehat{\mathfrak{BV}}$. Hence $f=p_*\hat{f}\in p_*(\widehat{\mathfrak{BV}})$. (See also [17, Remark 7.2].)

Under (A1)–(A6), Proposition 7.1(b) in [17] shows that λ is also an eigenvalue of $\hat{\mathcal{L}}:\widehat{\mathfrak{BV}}\to\widehat{\mathfrak{BV}}$ with algebraic multiplicity k.

Conversely, suppose that λ is an eigenvalue of $\hat{\mathcal{L}}:\widehat{\mathfrak{BV}}\to\widehat{\mathfrak{BV}}$ with algebraic multiplicity k. In order to show that it is an eigenvalue of $\mathcal{L}$ with multiplicity at least k, it suffices to show that if $(\hat{\mathcal{L}}-\lambda\mathrm{Id})^j\hat{f}=0$ for some $0\ne\hat{f}\in\widehat{\mathfrak{BV}}$ and $j>0$, then $p_*\hat{f}\ne0$ and $(\mathcal{L}-\lambda\mathrm{Id})^j p_*\hat{f}=0$. The latter follows from the fact that $p_* \circ \hat{\mathcal{L}} = \mathcal{L} \circ p_*$. For the proof of $p_* \hat{f} \neq 0$ we follow the proof of Lemma 4.3 in [1].

Suppose that $p_* \hat{f} = 0$. We consider the case $j = 1$ first, i.e. we assume that $\hat{f} = \lambda^{-m} \hat{\mathcal{L}}^m \hat{f}$ for all $m > 0$. For $x \in X$ denote by $\mathcal{Z}_m[x]$ that cylinder set in $\mathcal{Z}_m$ which contains x. Since the boundaries of all $Z \in \mathcal{Z}_m$ have Lebesgue measure zero, $\mathcal{Z}_m[x]$ is well defined for Lebesgue-a.e. x.

Let $\hat{F}(D, x) = \hat{f}(D, x)$ if $\mathcal{Z}_m[x] \subset \mathrm{fol}(D)$, zero otherwise. Θ will denote quantities slightly larger than ϑ. Assume m to be large. Observe that

$$\begin{aligned} \|\hat{F} - \hat{f}\|_1 &\leq \sum_{D \in \mathcal{D}} m(\{x \in \mathrm{fol}(D) : \mathcal{Z}_m[x] \not\subset \mathrm{fol}(D)\}) \cdot \|\hat{f}_D\|_\infty \\ &\leq \Delta_0 \cdot \Theta^m \cdot \mathrm{const}_d \cdot \epsilon_0^{1-d} \cdot \|\hat{f}\|_{\widehat{\mathfrak{BV}}} \leq \Theta^m \|\hat{f}\|_{\widehat{\mathfrak{BV}}} \end{aligned}$$

by Lemmas 1 and 11. Also $\hat{\mathcal{L}}^m \hat{F} = \hat{\mathcal{L}}^m \hat{P}_* \hat{F}$ and $\hat{P}_* \hat{f} = 0$ (because $\hat{P}_* \hat{f}(\mathcal{Z}[x], x) = p_* \hat{f}(x) = 0$), so that

$$\|\hat{\mathcal{L}}^m \hat{F}\|_1 = \|\hat{\mathcal{L}}^m \hat{P}_* \hat{F}\|_1 = \|\hat{\mathcal{L}}^m (\hat{P}_* \hat{F} - \hat{P}_* \hat{f})\|_1 \leq \|\hat{P}_* \hat{F} - \hat{P}_* \hat{f}\|_1 \leq \|\hat{F} - \hat{f}\|_1.$$

Thus,

$$\begin{aligned} \|\hat{f}\|_1 &= \|\lambda^{-m} \hat{\mathcal{L}}^m \hat{f}\|_1 = \|\lambda^{-m} \hat{\mathcal{L}}^m (\hat{f} - \hat{F}) + \lambda^{-m} \hat{\mathcal{L}}^m \hat{F}\|_1 \\ &\leq \mathrm{const} \cdot \left(\frac{\Theta}{|\lambda|} \right)^m . \end{aligned}$$

In the limit $m \to \infty$, this implies $\hat{f} = 0$.

As in [1], the case $j > 1$ is easily reduced to the case $j = 1$ by induction. □

Proof of Theorem 5: a) Denote by $\hat{V}$ the closed subspace of $\widehat{\mathfrak{BV}}$ consisting of functions which are constant on each $\hat{D}$ ($D \in \mathcal{D}$). Since the weights of $\hat{\mathcal{L}}$ are constant on each $\hat{D}$, $\hat{\mathcal{L}}(\hat{V}) \subseteq \hat{V}$. We proceed as in the proof of Theorem 2, Claim i). Consider $\hat{h} := (\hat{\mathcal{L}} - \lambda)\hat{f}$ and suppose that $\hat{h} \in \hat{V}$. Then also

$$\hat{h}_n := \hat{f} - (\lambda^{-1}\hat{\mathcal{L}})^n \hat{f} = -\lambda^{-1} \sum_{j=0}^{n-1} (\lambda^{-1}\hat{\mathcal{L}})^j \hat{h} \in \hat{V} \ .$$

Recall that $\hat{\mathcal{L}}_n$ is the finite-dimensional approximation of $\hat{\mathcal{L}}^n$ introduced in the proof of Theorem 3. Since g is piecewise constant, reasoning as in remark (12) shows that $\lambda^{-n} \hat{\mathcal{L}}_n \hat{f} \in \hat{V}$. Let $\Theta \in (\vartheta, |\lambda|)$. Inequality (11) leads to

$$\|\hat{f} - (\lambda^{-n}\hat{\mathcal{L}}_n)\hat{f} - \hat{h}_n\|_{\widehat{\mathfrak{BV}}} = |\lambda|^{-n} \|\hat{\mathcal{L}}^n \hat{f} - \hat{\mathcal{L}}_n \hat{f}\|_{\widehat{\mathfrak{BV}}} \leq \mathrm{const} \left(\frac{\Theta}{|\lambda|} \right)^n \|\hat{f}\|_{\widehat{\mathfrak{BV}}} \ .$$

Letting n tend to infinity, it follows that $\hat{f}$ belongs to the (closed) subspace $\hat{V}$.

Denote by u_D the constant value of $\hat{f}$ on $\hat{D}$. Because of Lemma 1,

$$\widehat{\mathfrak{var}}(\hat{f}(D, .)) \geq \mathrm{const}_d \cdot \epsilon_0^{d-1} \cdot |u_D| \ ,$$

and as each D is a convex set with diameter bounded by that of X, there is a constant const that bounds the $(d-1)$-dimensional surface measure of ∂D for each $D \in \mathcal{D}$ (see [2, 12.3.7]) so that

$$\widehat{\mathfrak{var}}(\hat{f}(D, .)) \leq \mathrm{const} \cdot |u_D| \ . \tag{15}$$

The first of these estimates yields at once that $u \in \ell^1(\mathcal{D})$ if $\hat{f} \in \widehat{\mathfrak{BV}}$, and an elementary calculation based on the definitions of the transition matrix K and the transfer operator $\hat{\mathcal{L}}$ shows that $uK = \lambda u + v$ where v_D is defined as the value of $\hat{h}$ on $\hat{D}$. If $k = 1$, i.e. if $\hat{h} = 0$, this yields at once that $u(K - \lambda) = 0$. For $k > 1$ we argue by induction: Since $(\hat{\mathcal{L}} - \lambda)^{k-1} \hat{h} = (\hat{\mathcal{L}} - \lambda)^k \hat{f} = 0$, we can conclude that $v(K - \lambda)^{k-1} = 0$, and as $v = u(K - \lambda)$, it follows that $u(K - \lambda)^k = 0$.

b) Suppose now that $u \in \ell^1(\mathcal{D})$ and $u(K - \lambda)^k = 0$. Define $\hat{f} \in \hat{V}$ by $\hat{f}(D, .)) = u_D$. Then

$$\widehat{\mathfrak{var}}(\hat{f}) = \sum_{D \in \mathcal{D}} \mathfrak{var}(\hat{f}(D, .)) \leq \sum_{D \in \mathcal{D}} \mathrm{const}_d \cdot |u_D| < \infty$$

by (15) and $((\hat{\mathcal{L}}-\lambda)^k\hat{f})(D,.) = (u(K-\lambda)^k)_D = 0$ for all $D \in \mathcal{D}$ by another elementary calculation. □

7. Periodic orbits: Proof of Theorem 7

Theorem 7 is a consequence of the following more general statement (remark that in the setting of Theorem 1, $P_g(\partial\mathcal{Z},T) \leq P_g(\mathcal{B}(T),T) \leq \log\vartheta$ by Proposition 1).

Theorem 10. *Let $(X,\mathcal{Z},T,g)$ be a piecewise invertible system with piecewise constant weight. Assume that* $\operatorname{diam}([A_0\dots A_n]) \to 0$ *when $n\to\infty$ for any sequence $A \in \mathcal{Z}^{\mathbb{N}}$.*

Then the formal power series:

$$h(z) = \det(\mathrm{Id} - zK)\cdot\zeta_g(z)$$

extends to a non-zero holomorphic function in $\{z : |z| < \exp(-P_{|g|}(\partial\mathcal{Z},T))\}$ with $\partial\mathcal{Z} = \bigcup_{Z\in\mathcal{Z}}\partial Z$.

Proof. Let L_n denote the set of loops of length n on the Markov Diagram $\mathcal{D}$,

$$L_n = \{D \in \mathcal{D}^{\mathbb{N}} : \forall j \geq 0\ D_j \to D_{j+1} \text{ and } D_{j+n} = D_j\}\ .$$

Consider $\iota : \bigcup_{n\geq 1} L_n \to X$ where $\iota(D) = x$ if

$$\{x\} = \bigcap_{k\geq 1} \overline{[A_0\dots A_k]} \text{ with } A = \pi(D).$$

Observe that ι is well-defined by the compactness of X and the assumption and that:

1. $\iota(L_n)\cap Y \subset \operatorname{fix}(T^n) \subset Y$,
2. $D = D'$ if $\iota(D) = \iota(D') \in Y$,
3. every $x \in \operatorname{fix}(T^n)$ admits a lift in L_n.

(1) is obvious, given the assumption. (2) follows from Lemma 4 and (3) from Lemma 7. By (2) and (3) above, we get:

$$h(z) = \exp\left(-\sum_{n\geq 1}\frac{z^n}{n}\sum_{D\in L_n\setminus\iota^{-1}(\operatorname{fix}(T^n))}|g^{(n)}(D)|\right).$$

Let D be as in the sum, $x = \iota(D)$ and let $A \in \mathcal{Z}^{\mathbb{N}}$ be the projection of D. By Lemma 4, the projection $D \mapsto A$ is one-to-one. Hence the sum over D above can be re-written as a sum over some A's.

By (1) above, $x \notin Y$. Hence $T^jx \in \partial\mathcal{Z}$ for some $0 \leq j < n$. Thus, the A's above satisfy

$$\bigcap_{k\geq 1}\overline{[(A_j\dots A_{n-1}A_0\dots A_{j-1})^k]} = T^jx \in \partial\mathcal{Z}.$$

Therefore the sum over D is bounded by

$$n\cdot\sum_{\substack{A\in\mathcal{Z}_n\\ \bar{A}\cap\partial\mathcal{Z}\neq\emptyset}}|g^{(n)}(A)|\ .$$

But this grows like $\exp P_{|g|}(\partial\mathcal{Z},T)n$. □

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