On the spectral gap of Kawasaki dynamics under a mixing condition revisited

N. Cancrini [1], and F. Martinelli [2]

[1] Dipartimento di Energetica, Università dell'Aquila, Italy and INFM Unità di Roma "La Sapienza"
e-mail: nicoletta.cancrini@roma1.infn.it

[2] Dipartimento di Matematica, Università di Roma Tre, Italy
e-mail: martin@mat.uniroma3.it

Abstract. We consider a conservative stochastic spin exchange dynamics which is reversible with respect the canonical Gibbs measure of a lattice gas model. We assume that the corresponding grand canonical measure satisfies a suitable strong mixing condition. We give an alternative and quite natural, from the physical point of view, proof of the famous Lu–Yau result which states that the relaxation time in a box of side L scales like L^2. We then show how to use such an estimate to prove a decay to equilibrium for local functions of the form $\frac{1}{t^{\alpha-\epsilon}}$ where ϵ is positive and arbitrarily small and $\alpha = \frac{1}{2}$ for $d = 1$, $\alpha = 1$ for $d \geq 2$.

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1. Introduction

The problem of computing the relaxation time of stochastic Monte Carlo algorithms for models of classical spin models in $\mathbb{Z}^d$ has attracted in the last years considerable attention and many new rigorous techniques have been developed giving rise to nice progresses in probability theory and statistical mechanics. If for simplicity we confine ourselves to ± 1 (or 0-1 in the lattice gas picture) spins, the two most studied random dynamics have been non– conservative Glauber type algorithms, in which a spin at a time flips its value with a rate satisfying the detailed balance condition w.r.t. the grand canonical Gibbs measure, and conservative dynamics in which nearest neighbors spins exchange theirs values with a rate satisfying the detailed balance condition w.r.t. to the canonical Gibbs measure.

It turns out that the conservation of the particle number (in the lattice gas picture, or of the magnetization in the usual ± 1 spin variables) makes the analysis of the relaxational properties of conservative dynamics much more difficult than in the non–conservative case with interesting analogies with the problem of the Goldstone mode in quantum mechanics [A].
For Glauber dynamics the general picture is relatively clear for a wide class of models both in the one phase and in the phase coexistence region with the notable exception of the critical point (see e.g. [M1] and references therein). In particular, for the two dimensional Ising case with zero external field, the spectral gap of the generator of a Glauber dynamics does not go to zero in the thermodynamic limit for any temperature above the critical one, while below the critical temperature the spectral gap in a box of side L and free boundary conditions becomes exponentially small in L with a precise rate related to the surface tension.

In the conservative case, instead, the fundamental results of [LY], [Y] on the spectral gap and logarithmic Sobolev inequality of Kawasaki dynamics in the one phase region state that, under a suitable mixing condition on the grand canonical Gibbs measure which for the two dimensional Ising model holds for any temperature above the critical one, the spectral gap in a box of side L shrinks like L^{-2}, while in the phase coexistence region, and at least for the two dimensional Ising model with periodic or free boundary condition, it becomes exponentially small in the side of the box [CCM]. The diffusive scaling L^2 for the relaxation time (in what follows identified with the inverse of the spectral gap of the generator) of Kawasaki dynamics proved in [LY] is a key stone in the study of the hydrodynamical limit of the Ising model [VY], and its proof required the development of a rather sophisticated (and intricate) technology which posed new, non trivial, problems on the theory of canonical Gibbs measures and their detailed equivalence to grand canonical ones (see also [BZ1], [BZ2], [CM1], [BCO] and [CZ]).

Unfortunately [LY] and particularily [Y] are quite difficult to study and the application of their techniques to other related problems, for example lattice gases with random interactions in the so–called Griffiths phase, seems to require a considerable effort. With this motivation we decided to reprove the result of [LY] by different means and in way that looks, at least to us, intuitively more appealing and natural to apply to other contexts [CM2]. Although our proof would have never found its way without some very nice ideas that we found in [LY] and [Y], we have tried to be completely selfcontained apart the necessary results on finite volume canonical Gibbs measures and on the equivalence of ensembles that we have developed in a separate paper [CM1]. We have also added to the original result a simple proof of the power law relaxation to equilibrium of strictly local (*i.e.* whose support is independent of the total volume) observables which, at least in one and two dimensions, is arbitrarily close to the expected $t^{-d/2}$ result (see also [BZ2]).

Let us now explain in simple terms the strategy behind our proof by firstly reviewing the way in which one can prove that the spectral gap for non–conservative Glauber dynamics does not shrink to zero with the volume in the high temperature phase.

Take a cube Q_{2L} of side $2L$ and divide it into two equal rectangles $R_i(L, 2L)$. Then, if truncated correlations in the grand canonical Gibbs measure decay fast enough (typically exponentially fast), it is not difficult to see that the relaxation time for the original square Q_{2L} is not larger than $(1+\epsilon_L)\times$(relaxation time in each rectangle), $\epsilon_L = O(L^{-\frac{1}{2}})$, in strict analogy with what happens at infinite temperature where the Glauber evolution becomes a product dynamics. Thus the relaxation time only increases by a factor $(1+\epsilon_L)^2$ when we double the scale and the result follows (see section 4 of [M1] for details).

For a conservative dynamics like the Kawasaki dynamics such a reasoning does not apply because the conservation of the number of particles introduces a global constraint in the system and even at infinite temperature the dynamics does not factorize into independent components. More precisely, the relaxation time in Q_{2L} is related to the relaxation time of the modified dynamics in which the two rectangles do not exchange particles but feel each other only through the transition rates *and* to the relaxation time of the process of exchange of particles between the two halves of Q_{2L}. Such a simple observation suggests to try to separate the two effects which are, apriori, strongly interlaced and to analyze them separately. Technically this is possible and it can be achieved if one plugs into the variational characterization of the spectral gap, gap $= \inf_f \frac{\mathcal{E}(f,f)}{\mathrm{Var}(f)}$, the formula of "conditional variance"

$$\mathrm{Var}(f) = E\big(\mathrm{Var}(f \,|\, n)\big) + \mathrm{Var}\big(E(f \,|\, n)\big) \tag{1.1}$$

where $\mathrm{Var}(\cdot)$, $\mathcal{E}(\cdot,\cdot)$ and $E(\cdot)$ denote respectively the variance, the Dirichlet form (quadratic form of the generator) and expectation w.r.t. the canonical Gibbs measure in Q_{2L} with e.g. N particles, while $\mathrm{Var}(\cdot \,|\, n)$ denotes the variance conditioned to have n of particles in e.g. the upper half of Q_{2L}. The latter should now be controled by an argument similar to that used in the non–conservative case although special care needs to be taken because covariances in a multicanonical Gibbs measure are quite different from their grand canonical analogues.

In order to control the effects produced by the extra conservation law, *i.e.* to be able to bound the last term in the r.h.s. of (1.1), one is led naturally to study the distribution of the number of particles in half cube under the canonical Gibbs distribution in Q_{2L} and in particular to prove a Poincaré inequality for it of the form $\mathrm{Var}(g(n)) \le k(N) E\big((\frac{d}{dn} g)^2\big)$ for any g, $k(N) = O(N)$, where $\frac{d}{dn}$ is the discrete derivative. Notice that $\frac{d}{dn} g$ measure the effects on the function g of the exchange of one particle between the two rectangles. Once such a step has been carried out it is not too difficult to complete the scale reduction from $2L$ to L. The final result is that the relaxation time for the original square Q_{2L} is not larger than $(1+\epsilon_L)\times$(relaxation time in each rectangle) plus const.$\times L^2$. This last term, which was absent in the analysis of the Glauber dynamics, comes precisely from the second term in the r.h.s. of (1.1). The diffusive limit now follows at once.

Let us conclude with a short roadmap of the paper.

- In section 2 we define the setting, the mixing condition we need and state the two main results.
- In section 3 we collect several technical results that will be needed later on. A detailed description of these results is given at the beginning of the section.
- In section 4 we study in some details the distribution of the number of particles in an atom of a given partition of a finite set Λ under a multicanonical measure.
- In section 5 we prove recursively theorem 2.3 on the diffusive scaling of the spectral gap.
- In section 6 we prove theorem 2.4 on the power law relaxation to equilibrium of local observables.
- Finally in the appendix we reprove with some simplifications the famous two–blocks estimate of [LY].

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2. Notation and results

In this section we first define the setting in which we will work (spin model, Gibbs measure, dynamics), then we define the basic mixing condition on the Gibbs measure and subsequently state the main theorem on this work.

2.1 The lattice and the configuration space

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = (x_1, \dots, x_d)$ and norms

$$|x|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p} \quad p \geq 1 \quad \text{and} \qquad |x| = |x|_\infty = \max_{i \in \{1,\dots,d\}} |x_i| \, .$$

The associated distance functions are denoted by $d_p(\cdot,\cdot)$ and $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball (w.r.t $d(\cdot,\cdot)$) of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. $[x, y]$ is the *closed segment* with endpoints x and y. The *edges* of $\mathbb{Z}^d$ are those $e = [x,y]$ with x, y nearest neighbors in $\mathbb{Z}^d$. The *boundary of an edge* $e = [x,y]$ is $\delta e = \{x,y\}$ The *boundary of a set of edges* α is the set $\delta\alpha$ of all sites that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ and by $\bar{\mathcal{E}}_\Lambda$ the set of all edges with at least one endpoint in Λ. Viceversa, for a set of edges X, $\mathcal{V}(X)$ stands for the set of all sites which are endpoints of at least one edge in X.

Given $\Lambda \subset \mathbb{Z}^d$ we define its interior and exterior boundaries as respectively, $\partial^-\Lambda = \{x \in \Lambda : d(x, \Lambda^c) \le 1\}$ and $\partial^+\Lambda = \{x \in \Lambda^c : d(x,\Lambda) \le 1\}$, and more generally we define the boundaries of width n as $\partial_n\Lambda = \{x \in \Lambda : d(x,\Lambda^c) \le n\}$, $\partial_n^+\Lambda = \{x \in \Lambda^c : d(x,\Lambda) \le n\}$. For $\Lambda \subset \mathbb{Z}^d$ we also let

$$\delta\Lambda = \{\, e^* : e = [x,y],\ x \in \Lambda,\ y \in \Lambda^c,\ |x-y|_1 = 1\,\} \qquad (2.1)\,.$$

Regular sets. A finite subset Λ of $\mathbb{Z}^d$ is said to be *l–regular*, $l \in \mathbb{Z}_+$, if Λ is the union of a finite number of cubes $Q_l(x^i)$ where $x^i \in l\mathbb{Z}^d$. We denote the class of all such sets by $\mathbb{F}_l$. Notice that any set is 1–regular *i.e.* $\mathbb{F}_{l=1} = \mathbb{F}$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{0,1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U\tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *number of particles* $N_V : \Omega \mapsto \mathbb{N}$ as

$$N_V(\sigma) = \sum_{x \in V} \sigma(x) \qquad (2.2)$$

while the *density* is given by $\rho_V = N_V/|V|$.

If f is a function on Ω, Δ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. f is called *local* if Δ_f is finite. The *l–support* of a function $\Delta_f^{(l)}$, $l \in \mathbb{Z}_+$, is the smallest l–regular set V such that $\Delta_f \subset V$. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{-1,1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. Finally $\mathrm{Osc}(f) = \sup_{\sigma,\eta} |f(\sigma) - f(\eta)|$.

2.2 The interaction and the Gibbs Measures.

Definition 2.1. *A finite range, translation–invariant potential $\{\Phi_\Lambda\}_{\Lambda\in\mathbb{F}}$ is a collection of real, local functions on Ω with the following properties*

(1) $\Phi_\Lambda = \Phi_{\Lambda+x}$ for all $\Lambda\in\mathbb{F}$ and all $x\in\mathbb{Z}^d$

(2) For each Λ the support of Φ_Λ coincides with Λ

(3) There exists $r>0$ such that $\Phi_\Lambda = 0$ if $\operatorname{diam}\Lambda > r$. r is called the range of the interaction.

(4) $\|\Phi\| := \sum_{\Lambda\ni 0}\|\Phi_\Lambda\|_\infty < \infty$

Given a collection of real numbers $\underline{\lambda} = \{\lambda_x\}_{x\in\mathbb{Z}^d}$ and a *potential* Φ we define $\Phi^{\underline{\lambda}}$ as

$$\Phi^{\underline{\lambda}}_\Lambda(\sigma) = \begin{cases} (h+\lambda_x)\sigma(x) & \text{if } \Lambda=\{x\} \\ \Phi_\Lambda(\sigma) & \text{otherwise} \end{cases}$$

where h is the chemical potential (one body part of Φ).
Given a *potential* Φ ($\Phi^{\underline{\lambda}}$) and $V\in\mathbb{F}$, we define the Hamiltonian $H^\Phi_V:\Omega\mapsto\mathbb{R}$ by

$$H^\Phi_V(\sigma) = -\sum_{\Lambda:\,\Lambda\cap V\neq\emptyset}\Phi_\Lambda(\sigma)$$

For $\sigma,\tau\in\Omega$ we also let $H^{\Phi,\tau}_V(\sigma) = H^\Phi_V(\sigma_V\tau_{V^c})$ and τ is called the *boundary condition*. For each $V\in\mathbb{F}$, $\tau\in\Omega$ the (finite volume) conditional Gibbs measure on $(\Omega,\mathcal{F})$, are given by

$$d\mu^{\Phi,\tau}_V(\sigma) = \begin{cases} \left(Z^{\Phi,\tau}_V\right)^{-1}\exp[-H^{\Phi,\tau}_V(\sigma)] & \text{if } \sigma(x)=\tau(x) \text{ for all } x\in V^c \\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

where $Z^{\Phi,\tau}_V$ is the proper normalization factor called partition function. Notice that in (2.3) we have adsorbed in the interaction Φ the usual inverse temperature factor β in front of the Hamiltonian. In most notation we will drop the superscript Φ if that does not generate confusion. Moreover, whenever we consider $\Phi^{\underline{\lambda}}$ instead of Φ, we will write $H^{\tau,\underline{\lambda}}_V$ for the finite volume Hamiltonian and $\mu^{\tau,\underline{\lambda}}_V$ for the corresponding finite volume Gibbs measure.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the *function* $\sigma\mapsto\mu^\sigma_V(f)$ where $\mu^\sigma_V(f)$ is just the average of f w.r.t. μ^σ_V. Analogously, for any event X, $\mu^\tau_V(X) := \mu^\tau_V(\mathbb{I}_X)$, where $\mathbb{I}_X$ is the characteristic function of X. $\mu^\tau_V(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ^τ_V) of f and g. The set of measures (2.3) satisfies the DLR compatibility conditions

$$\mu^\tau_\Lambda(\mu_V(X)) = \mu^\tau_\Lambda(X) \qquad \forall X\in\mathcal{F} \qquad \forall V\subset\Lambda\subset\subset\mathbb{Z}^d \tag{2.4}$$

Definition 2.2. *A probability measure μ on $(\Omega,\mathcal{F})$ is called a Gibbs measure for Φ if*

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall X\in\mathcal{F} \qquad \forall V\in\mathbb{F} \tag{2.5}$$

see e.g. [G].

We introduce the *canonical Gibbs measures* on $(\Omega,\mathcal{F})$ defined as

$$\nu^\tau_{\Lambda,N} = \mu^\tau_\Lambda(\cdot\mid N_\Lambda = N) \qquad N\in\{0,1,\dots,|\Lambda|\} \tag{2.6}$$

where N_Λ is the number of particles (*i.e.* spins equal to $+1$) in Λ.

2.3 The dynamics

We consider the so–called Kawasaki dynamics in which particles (spins with $\sigma(x) = +1$) can jump to nearest neighbor empty ($\sigma(x) = 0$) locations, keeping the total number of particles constant. For $\sigma \in \Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$ The stochastic dynamics we want to study is determined by the Markov generators L_V, $V \subset\subset \mathbb{Z}^d$, defined by

$$(L_V f)(\sigma) = \sum_{[x,y]\in\mathcal{E}_V} c_{xy}(\sigma)\,(\nabla_{xy}f)(\sigma) \qquad \sigma\in\Omega\,, \quad f:\Omega\mapsto\mathbb{R} \tag{2.7}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The nonnegative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range.* $c_{xy}(\sigma)$ depends only on the spins $\sigma(z)$ with $d(\{x,y\},z) \le r$

(2) *Detailed balance.* For all $\sigma\in\Omega$ and $[x,y]\in\mathcal{E}_{\mathbb{Z}^d}$

$$\exp\bigl[-H_{\{x,y\}}(\sigma)\bigr]\,c_{xy}(\sigma) = \exp\bigl[-H_{\{x,y\}}(\sigma^{xy})\bigr]\,c_{xy}(\sigma^{xy}) \tag{2.8}$$

(3) *Positivity and boundedness.* There exist positive real numbers $c_m(\beta)$ $c_M(\beta)$ such that

$$c_m \le c_{xy}(\sigma) \le c_M \qquad \forall x,y\in\mathbb{Z}^d\,,\sigma\in\Omega\,. \tag{2.9}$$

We denote by $L^\tau_{V,N}$ the operator L_V acting on $L^2(\Omega,\nu^\tau_{V,N})$ (this amounts to choosing τ as the boundary condition and N as the number of particles). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L^\tau_{V,N}$, and whose semigroup we denote by $(T_t^{V,N,\tau})_{t\ge0}$. $L^\tau_{V,N}$ is a bounded operator on $L^2(\Omega,\nu^\tau_{V,N})$ and $\nu^\tau_{V,N}$ is its unique invariant measure. Moreover $\nu^\tau_{V,N}$ is *reversible* with respect to the process, *i.e.* $L^\tau_{V,N}$ is self–adjoint on $L^2(\Omega,\nu^\tau_{V,N})$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\operatorname{gap}(L^\tau_{V,N}) = \inf\operatorname{spec}\,(-L^\tau_{V,N}\restriction\mathbb{1}^\perp)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega,\nu^\tau_{V,N})$ orthogonal to the constant functions. We let $\mathcal{E}$ be the Dirichlet form associated with the generator $L^\tau_{V,N}$,

$$\mathcal{E}^\tau_{V,N}(f,f) = \langle f,\,-L^\tau_{V,N}f\rangle_{L^2(\Omega,\nu^\tau_{V,N})} = \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_V}\nu^\tau_{V,N}\left[\,c_{xy}\,(\nabla_{xy}f)^2\,\right] \tag{2.10}$$

and $\operatorname{Var}^\tau_{V,N}$ is the variance relative to the probability measure $\nu^\tau_{V,N}$. The gap can also be characterized as

$$\operatorname{gap}(L^\tau_{V,N}) = \inf_{\substack{f\in L^2(\Omega,\nu^\tau_{V,N}),\\ \operatorname{Var}^\tau_{V,N}(f)\neq0}} \frac{\mathcal{E}^\tau_{V,N}(f,f)}{\operatorname{Var}^\tau_{V,N}(f)}\,. \tag{2.11}$$

2.4 Definition of the mixing condition and main results.

In order to formulate our basic mixing condition on the interaction Φ we fix positive numbers C, m, l with $l \in \mathbf{N}$. We then say that a collection of real numbers $\underline{\lambda} := \{\lambda_x\}_{x\in\mathbb{Z}^d}$ is l–regular if, for all $i \in \mathbb{Z}^d$ and all $x \in Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$, $\lambda_x = \lambda_{x^i}$.

Definition of property $USMT(C,m,l)$. *For any l–regular set Λ and any pair of bounded local functions f and g*

$$\sup_{\substack{\underline{\lambda} \\ \underline{\lambda}\,\mathrm{is}\,l-\mathrm{regular}}} \sup_{\tau} |\mu_\Lambda^{\tau,\underline{\lambda}}(f,g)| \le C \sup_{\tau} \mu_\Lambda^{\tau,\underline{\lambda}}(|f|)\, \mu_{\Lambda\setminus\Delta_f^{(l)}}^{\tau,\underline{\lambda}}(|g|) \sum_{x\in\partial_r^-\Delta_f^{(l)}} \sum_{y\in\partial_r^-\Delta_g^{(l)}} e^{-m|x-y|}$$

provided that $d(\Delta_f^{(l)}, \Delta_g^{(l)}) \ge l$.

Remark. The expert reader may have noticed that our condition is different, and in principle stronger, than the one used in [LY] and [Y] because we require the exponential decay of cavariances uniformly in the chemical potential even when the latter *varies* over the atoms of a partition of Λ while in the above mentioned papers the chemical potential is assumed to be constant over Λ. In two dimension, followig the ideas of [MOS], one can prove [BCO] that the two conditions are equivalent. In higher dimension, contrary to what claimed in section 8 of [Y], one can construct examples in which a kind of phase transition occurs along the interface between two different atoms even if for all l–regular sets Λ the covariances decay exponentially fast uniformly w.r.t. to constant chemical potentials. We remark that the above mixing condition plays an important role also in other contexts like in the analysis of renormalization group pathologies [BCO].

We are finally in a position to formulate the main results of this paper on the spectral gap of the generator of Kawasaki dynamics in a finite volume.

Theorem 2.3. *Assume that there exist positive numbers C, m, l, with $l \in \mathbf{N}$, such that property $USMT(C,m,l)$ holds. Then there exists positive constants c_1, c_2 such that*

$$c_1 L^{-2} \le \min_{N,\tau} \mathrm{gap}(L^\tau_{Q_L,N}) \le \max_{N,\tau} \mathrm{gap}(L^\tau_{Q_L,N}) \le c_2 L^{-2} \tag{2.12}$$

A nice consequence of the above estimate is an inverse polynomial bound on the time decay to equilibrium in $L^2\left(d\nu^\tau_{\Lambda,N}\right)$ of local observables.

Theorem 2.4. *Assume that there exist positive numbers C, m, l, with $l \in \mathbf{N}$, such that property $USMT(C,m,l)$ holds. Then for any $\epsilon \in (0,1)$ and any local function f with $0 \in \Delta_f$ there exists a positive constant $C_{f,\epsilon}$ such that for any integer L multiple of l and any integer $N \in \{1, \ldots, (2L)^d\}$*

$$\mathrm{Var}^\tau_{\Lambda,N}\left(e^{tL^\tau_{\Lambda,N}} f\right) \le C_{f,\epsilon}\, \frac{1}{t^{\alpha-\epsilon}}$$

where $\Lambda := B_L$ and $\alpha = \frac{1}{2}$ in $d = 1$, $\alpha = 1$ for $d > 1$.

Remark. In the infinite temperature case (simple exclusion model) (see [BZ1]) the time evolution of a local linear function of the form $f(\sigma) = \sum_x a_x \sigma(x)$, where $\{a_x\}_{x\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$, can be computed exactly: $e^{tL_\Lambda}f = \sum_x \left(e^{t\Delta_\Lambda}a\right)_x \sigma(x)$ where Δ_Λ is the discrete Laplacian on Λ with the appropriate boundary conditions. In particular one gets in this case that $\|e^{tL_\Lambda}f\|_2^2 \le C_f t^{-d/2}$. Our exponent is thus arbitrarily close to the correct one only for $d = 1, 2$.

3. Preliminary results

Here we collect several preliminary results that will be used in the future. Although the reader may skip this section during a first reading and come back when these results are needed, we still think useful to give a short roadmap of the section.

- In paragraphs §3.1 and §3.2 we collect the necessary results on the comparison between finite volume multicanonical Gibbs measures, namely grand canonical Gibbs measures conditioned to have a specified number of particles in the atoms of a given partition of a finite set Λ, and the corresponding unconditioned measures.
- In paragraph §3.3, in strict analogy with a similar result for non conservative dynamics (see section 4 of [M1]), we prove a simple but important result whose physical meaning is roughly the following. Consider the Kawasaki dynamics in a a box Λ that is covered by two rectangles, Λ_1 and Λ_2, each of size $\approx \frac{|\Lambda|}{2}$ but such that their overlap is a long and thin strip whose size also scales like $|\Lambda|$. Besides the usual constraint that the total number of particles is conserved impose the additional constraint that the number of particles in each rectangle *and* in their overlap does not change with time. Then the relaxation time of the dynamics is not larger than a constant close to one times the largest among the relaxation times of the simple Kawasaki dynamics in each of the two rectangles.
- In paragraph §3.4 we begin to discuss an important topic of our approach to the computation of the spectral gap of Kawasaki dynamics, namely the distribution of the number of particles in an atom of a given partition of a finite set Λ under a multicanonical measure. In this paragraph we only give two simple results and we postpone a more detailed analysis to section 4.
- In paragraph §3.5 we show how to prove a sharp Poincarè inequality for a symmetric, finite volume, one dimensional random walk whose invariant measure is kind of bell–shaped around the mean. The reason of our interest in such a topic is that, as we will show in section 4, the distribution of the number of particles in an atom of a given partition of a finite set Λ under a multicanonical measure has exactly the above property and the Poincarè inequality is an effective tool to bound from above the variance of a random variable.
- In paragraph §3.6 we recall and slightly extend a key result of [LY] related to the so–called "two block estimate". More precisely the result states that the covariance between an arbitrary function f and the spatial average of translationally covariant local functions can be bounded from above by the inverse of the volume times an arbitrarily small constant times the variance of f plus a large constant times the Dirichlet form of f (see the appendix for more details).

- Finally, in paragraphs §3.7, 3.8, 3.9, we first show how to compute and then how to estimate quantities like $\nu^{\tau}_{\Lambda,\mathbf{N}}\left(\left[\frac{d}{dn}\nu^{\tau}_{\Lambda,\mathbf{N}}(f \mid N_V = n)\right]^2\right)$, where $\nu^{\tau}_{\Lambda,\mathbf{N}}$ is a multi canonical measure over the atoms $\{\Lambda_i\}_{i=1}^k$ of a partition of a finite set Λ, V is a subset of a given atom Λ_j and $\frac{d}{dn}$ is the discrete derivative. As explained in section 5, terms like the above one naturally appear in the recursive bound of the spectral gap and, roughly speaking, they measure the influence on the relaxation time of Kawasaki dynamics due to the exchange of particles between different atoms of the partition.

3.1 Existence of the tilting fields

We begin by recalling the following quite general result on the relationship between particle numbers and the chemical potential (see the appendix in [CM1]).

Let $\Lambda = \cup_{i=1}^k \Lambda_i$, where the atoms $\Lambda_1, \dots, \Lambda_k$ are pairwise disjoint and such that there exists $x_1, \dots, x_k$, $x_i \in \Lambda_i \quad i = 1, \dots, k$, with $\min_{i \neq j} d(x_i, x_j) \geq 2r$. Then, for any possible values $N_1, \dots, N_k$ of the particle numbers in $\Lambda_1, \dots, \Lambda_k$ and for any boundary condition τ, there exists a unique choice of $(\lambda_1, \dots, \lambda_k)$ such that $\mu^{\tau,\underline{\lambda}}_{\Lambda}(N_{\Lambda_i}) = N_i, \quad i = 1, \dots, k$, where the chemical potential $\underline{\lambda}$ is constant and equal to λ_i inside Λ_i.

Next we give a simple result on the dependence on the average particle number and on the boundary condition of the chemical potential.

Lemma 3.1. *Assume property $USMT(C, m, l)$. Let f be a bounded local function and $\lambda = \lambda(\tau, n)$ the chemical potential such that the grand canonical Gibbs state on $\Lambda = Q_L$, L a multiple of the basic scale l, satisfies $\mu^{\tau,\lambda}_{\Lambda}(N_\Lambda) = n$ with $n = (0, |\Lambda|)$. Let $\alpha \in (0,1)$. Then there exists a constant k independent on L such that for any L large enough*

(1) For any function f such that $\Delta_f \subset \Lambda$ and $d(\Delta_f, \partial^+_r \Lambda) \geq L^\alpha$

$$i) \qquad \sup_{y \in \partial^+_r \Lambda} \|\nabla_y \mu^{\tau,\lambda}_{\Lambda}(f)\|_\infty \leq k \|f\|_\infty \frac{|\Delta_f|}{|\Lambda|}$$

$$ii) \qquad \sup_{y \in \partial^+_r \Lambda} \|\nabla_y \frac{d}{dn} \mu^{\tau,\lambda}_{\Lambda}(f)\|_\infty \leq k \|f\|_\infty \frac{1}{n} \frac{|\Delta_f|}{|\Lambda|}$$

(2) For any function f such that $\Delta_f \subset \Lambda$

$$i) \qquad \|\frac{d}{dn} \mu^{\tau,\lambda}_{\Lambda}(f)\|_\infty \leq k \|f\|_\infty \frac{|\Delta_f|}{|\Lambda|}$$

$$ii) \qquad \|\frac{d^2}{dn^2} \mu^{\tau,\lambda}_{\Lambda}(f)\|_\infty \leq k \|f\|_\infty \frac{1}{n} \frac{|\Delta_f|}{|\Lambda|}$$

Proof. Parts $i)$ of 1) and 2) follow immediately from the bounds

$$\begin{aligned} &\sup_{y \in \partial^+_r \Lambda} |\lambda(\tau^y, n) - \lambda(\tau, n)| \leq \frac{k_1}{n} \\ &|\frac{d\lambda(\tau, n)}{dn}| = |\frac{1}{\mu^{\tau,\lambda(\tau,n)}_{\Lambda}(N_\Lambda, N_\Lambda)}| \leq \frac{k_1}{n} \end{aligned} \tag{3.1}$$

together with

$$|\frac{d}{d\lambda}\mu_\Lambda^{\tau,\lambda(\tau,n)}(f)| = |\mu_\Lambda^{\tau,\lambda(\tau,n)}\big(N_\Lambda, f\big)| \le k_2\|f\|_\infty \frac{n|\Delta_f|}{|\Lambda|}$$

for suitable constants k_1, k_2 independent of L (see proposition 3.1 and section 7.1 in [CM1]).
Part $ii)$ of 1) is a little bit tedious but it follows without problems from the bound $|\nabla_y \mu_\Lambda^{\tau,\lambda}(N_\Lambda, N_\Lambda)| \le k_3$ which, together with (3.1), implies that $|\nabla_y \frac{d}{dn}\lambda(\tau,n)| \le \frac{k_4}{n^2}$. Part $ii)$ of 2) is straightforward if we observe that

$$|\frac{d^2\lambda(\tau,n)}{dn^2}| = |\frac{\mu_\Lambda^{\tau,\lambda(\tau,n)}\big(N_\Lambda, N_\Lambda, N_\Lambda\big)}{\mu_\Lambda^{\tau,\lambda(\tau,n)}\big(N_\Lambda, N_\Lambda\big)}|\,\big(\frac{d\lambda(\tau,n)}{dn}\big)^2 \le k_5\frac{1}{n^2}$$

and

$$\frac{d^2}{dn^2}\mu_\Lambda^{\tau,\lambda(\tau,n)}(f) = \mu_\Lambda^{\tau,\lambda(\tau,n)}(f, N_\Lambda)\,\frac{d^2\lambda(\tau,n)}{dn^2} + \mu_\Lambda^{\tau,\lambda(\tau,n)}(f, N_\Lambda, N_\Lambda)\,\big(\frac{d\lambda(\tau,n)}{dn}\big)^2$$

(see again proposition 3.1 in [CM1]). $\square$

3.2 Equivalence of ensembles

Here we recall some fine results on the finite volume equivalence of ensembles that will be crucial in most of our future arguments. We refer the reader to sections 5, 7.2 and 7.3 of [CM1].

Fix $\delta, \epsilon \in (0,1)$, $\epsilon \ll 1$, and two integers k, l such that $\delta k < 1$. Let $L_1, \dots, L_k$ be large multiples of the basic length scale l, let $L = \sum_i L_i$ and assume that $L_j \ge \delta L$ for any j. We then choose one coordinate direction, e.g. the d direction, and we take $\Lambda = Q_L$, Λ_1 equal to the first slice of Λ orthogonal to d–direction of width L_1, *i.e.* $\Lambda_1 = \{x \in \Lambda \;:\; 0 \le x_d < L_1\}$, Λ_2 equal to the slice of Λ on top of Λ_1 of width L_2 and so on. Let also $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$ and let us assume, for a given boundary condition τ, that $\underline{\lambda}$ is constant on each set Λ_i and such that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}) = N_i$, $i = 1, \dots, k$. We denote by $\nu_{\Lambda,\mathbf{N}}^\tau$ the multi canonical Gibbs measure $\mu_\Lambda^{\tau,\underline{\lambda}}\big(\cdot\,|\,N_{\Lambda_i} = N_i,\, i = 1, \dots, k\big)$ and by Ω_τ the set of configurations τ' that coincide with τ in the half space $\{x \in \mathbb{Z}^d \;:\; x_d < L\}$.

Next, given $M > 0$ and $\Delta \subset \Lambda$, we say that Δ is good if either

a) Δ together with its $M\log|\Lambda|$ neighborhood is entirely contained in some atom Λ_i with $\rho_i := \frac{N_i}{|\Lambda_i|} \ge |\Lambda|^{-\epsilon}$

or

b) Δ together with its M neighborhood is entirely contained in some atom Λ_i with $\rho_i \le |\Lambda|^{-\epsilon}$.

A set is bad if it is not good. For good sets $\Delta \subset \Lambda_i$, $i = 1, \dots, k$, we define

$$\bar\Delta = \begin{cases} \{x \in \Lambda :\; d(x,\Delta) \le M\log|\Lambda|\,\} & \text{if } \rho_i \ge |\Lambda|^{-\epsilon} \\ \{x \in \Lambda :\; d(x,\Delta) \le M\,\} & \text{if } \rho_i \le |\Lambda|^{-\epsilon} \end{cases}$$

while for bad sets Δ

$$\bar\Delta = \{x \in \Lambda :\; d(x,\Delta) \le M\log|\Lambda|\,\}$$

With these notations the results on the finite volume equivalence of ensembles that will be essential for the rest of this paper read as follows.

Proposition 3.2. *Assume condition $USMT(C,m,l)$. Then, for any l, M large enough and ϵ small enough independent of $\{\rho_i\}_{i=1}^k$, there exist constants $C' = C'(C,m,\|\Phi\|,l,\delta,M,\epsilon)$, $L_0 = L_0(C,m,\|\Phi\|,l,M,\delta,\epsilon)$ such that, if $L \ge L_0$,*

(1) *Assume $k=1$. Then for all local functions f,g with l–support contained in Λ such that $|\Delta_f^{(l)}| \le |\Lambda|^{1-\epsilon}$ and similarly for g*

$$|\,\nu^\tau_{\Lambda,N}(f,g)\,| \le C(f,g)\rho\left[\frac{1}{|\Lambda|} + e^{-md(\Delta_f^{(l)},\Delta_g^{(l)})}\right]$$

where $C(f,g) = C'\|f\|_\infty\,\|g\|_\infty|\Delta_f^{(l)}|^2\,|\Delta_g^{(l)}|^2$. Moreover

$$|\,\nu^\tau_{\Lambda,N}(f) - \mu^{\tau,\lambda}_\Lambda(f)\,| \le C'\|f\|_\infty\frac{|\Delta_f|}{|\Lambda|}$$

(2) *Assume $k \ge 2$. Then for all local functions f,g with l–support contained in Λ such that $|\Delta_f^{(l)}| \le |\Lambda|^{1-\epsilon}$ and similarly for g*

$$|\,\nu^\tau_{\Lambda,\mathbf{N}}(f,g)\,| \le C(f,g)A_\Lambda(f,g)\begin{cases}\min\{\nu^\tau_{\Lambda,\mathbf{N}}(|f|),\nu^\tau_{\Lambda,\mathbf{N}}(|g|)\} & \text{if } \Delta_f^{(l)} \text{ and } \Delta_g^{(l)} \text{ are both good or bad}\\ \nu^\tau_{\Lambda,\mathbf{N}}(|f|) & \text{if } \Delta_f^{(l)} \text{ is bad}\\ \nu^\tau_{\Lambda,\mathbf{N}}(|g|) & \text{if } \Delta_g^{(l)} \text{ is bad}\end{cases}$$

where $C(f,g) = C'\|f\|_\infty\,\|g\|_\infty|\Delta_f^{(l)}|\,|\Delta_g^{(l)}|$ and

$$A_\Lambda(f,g) = \begin{cases}\frac{1}{|\Lambda|} + e^{-md(\Delta_f^{(l)},\Delta_g^{(l)})} & \text{if } \Delta_f^{(l)} \text{ or } \Delta_g^{(l)} \text{ is good}\\ \frac{1}{|\Lambda|}\left(\frac{|\bar\Delta_g^{(l)}|}{|\Delta_g^{(l)}|}\right)^2\left(\frac{|\bar\Delta_f^{(l)}|}{|\Delta_f^{(l)}|}\right)^2 + e^{-md(\Delta_f^{(l)},\Delta_g^{(l)})} & \text{otherwise}\end{cases}$$

(3) *Assume $k \ge 2$ and $\Delta \equiv \Delta_f^{(l)} \subset \Lambda_n$, $n \le k$. Then*

$$\sup_{\tau'\in\Omega_\tau}|\nu^\tau_{\Lambda,\mathbf{N}}(f) - \nu^{\tau'}_{\Lambda,\mathbf{N}}(f)| \le C'\|f\|_\infty\begin{cases}|\Delta|\left[\frac{1}{|\Lambda|} + \frac{1}{L^{k-n+1}}\right] & \text{if } \Delta \text{ is good}\\ \frac{|\Delta|}{|\Lambda|}\left(\frac{|\bar\Delta|}{|\Delta|}\right)^2 + \max_{j=n,n\pm1}\left[\frac{|\bar\Delta\cap\Lambda_j|^{\frac12}}{L^{k-j+1}}\right] & \text{if } \Delta \text{ is bad}\end{cases}$$

(4) *Assume $k \ge 2$ and $\Delta_f^{(l)} \subset \partial_r\Lambda_n \cap \partial_r^+\Lambda_{n-1}$. Let $\bar n = \lfloor|\frac{d-1}{2} - 1|\rfloor + 1$ and $\gamma(L) = \left[\frac{(\log L)^2}{L} + \frac{(\log L)^{\frac12}}{L^{\bar n+1-(d-1)/2}}\right]$. Then*

$$\sup_{\tau'\in\Omega_\tau}|\nu^\tau_{\Lambda,\mathbf{N}}(f) - \nu^{\tau'}_{\Lambda,\mathbf{N}}(f)| \le C'\mathrm{Osc}(f)\,\gamma(L)^{\lfloor\frac{k-n+1}{\bar n+1}\rfloor}$$

where $Osc(f) := \max_{\sigma,\eta}|f(\sigma) - f(\eta)|$.

Remark. Actually the first part of the proposition holds in a much more general geometric context (see §7.3 of [CM1]).

For future purposes the next result is stated in a slighlty more general form.

Proposition 3.3. *Assume condition $USMT(C,m,l)$. Fix $\delta \in (0,1)$, $\Lambda \in \mathbb{F}_l$ and a partition of Λ into l–regular sets $\Lambda_1,\dots,\Lambda_k$. Let f be such that $|\Lambda_j \setminus \Delta_f| \geq \delta|\Lambda_j|$ for any $j=1,\dots k$. Then for any l large enough there exists a constant A depending only on $C,m,\|\Phi\|,l,k,\delta$ such that*

$$\nu^\tau_{\Lambda,\mathbf{N}}(|f|) \leq A\,\mu^{\tau,\underline{\lambda}}_\Lambda(|f|)$$

where $\mathbf{N}$ and $\underline{\lambda}$ are as in the beginning of this paragraph. In particular

$$\nu^\tau_{\Lambda,\mathbf{N}}(f,f) \leq A\,\mu^{\tau,\underline{\lambda}}_\Lambda(f,f)$$

Proof. Following [CM1] (see formula (5.3) there) we write

$$\nu^\tau_{\Lambda,\mathbf{N}}(|f|) = \frac{\int\limits_{-T_1}^{T_1} dt_1 \dots \int\limits_{-T_k}^{T_k} dt_k\, \mu^{\tau,\underline{\lambda}}_\Lambda\left(e^{i\sum_j \frac{t_j}{\sigma_i}(N_{\Lambda_j}(\eta)-N_j)}|f|\right)}{\int\limits_{-T_1}^{T_1} dt_1 \dots \int\limits_{-T_k}^{T_k} dt_k\; F(t_1,\dots,t_k)} \tag{3.2}$$

where $\sigma_j^2 := \mu^{\tau,\underline{\lambda}}_\Lambda\big(N_{\Lambda_j},N_{\Lambda_j}\big)$, $T_j := \pi\sigma_j$ and $F(t_1,\dots,t_k) := \mu\big(e^{i\sum_j \frac{t_j}{\sigma_j}(N_{\Lambda_j}(\eta)-N_j)}\big)$.
With our assumptions the denominator of (3.2) is bounded from below by a suitable constant depending only on $C,m,\|\Phi\|,l,k$ (see lemma 5.2 in [CM1]). In order to bound from above the numerator we observe that

$$\mu^{\tau,\underline{\lambda}}_\Lambda\left(e^{i\sum_j \frac{t_j}{\sigma_j}(N_{\Lambda_j}(\eta)-N_j)}|f|\right) = \mu^{\tau,\underline{\lambda}}_\Lambda\left(\mu^{\tau,\underline{\lambda}}_\Lambda\left(e^{i\sum_i \frac{t_j}{\sigma_j}(N_{\Lambda_j}(\eta)-N_j)}\,|\,\mathcal{F}_{\Delta_f}\right)|f|\right) \tag{3.3}$$

and that, because of proposition 3.1, 3.3 of [CM1], for any choice of $t_j \in [-T_j,T_j]$, $j=1,\dots,k$ we have $\|\mu^{\tau,\underline{\lambda}}_\Lambda\left(e^{i\sum_j \frac{t_j}{\sigma_j}(N_{\Lambda_j}(\eta)-N_j)}\,|\,\mathcal{F}_{\Delta_f}\right)\|_\infty \leq e^{-\alpha\sum_{j=1}^k t_j^2}$ for a suitable constant α depending only on $C,m,\|\Phi\|,l,k,\delta$. Notice that it is here that the hypothesis $|\Lambda_j \setminus \Delta_f| \geq \delta|\Lambda_j|$ for any $j=1,\dots k$ is being used. The first statement of the proposition now follows at once. To prove the statement about the variance we simply observe that

$$\nu^\tau_{\Lambda,\mathbf{N}}(f,f) \leq \nu^\tau_{\Lambda,\mathbf{N}}\left(\left(f-\mu^{\tau,\underline{\lambda}}_\Lambda(f)\right)^2\right)$$

□

3.3 A block dynamics bound

Here we give a result that is a key step in our recursive bound of the spectral gap of Kawasaki dynamics. For simplicity we discuss our estimate in two dimensions and at the end we explain how to generalize it to higher dimensions.
Fix $\delta \in (0,\frac14)$ and an integer l. Let $L \in \mathbb{Z}_+$ be a multiple of l and let Λ denote the rectangle

$$\Lambda = \big\{(x_1,x_2) \in \mathbb{Z}^2;\ 0 \leq x_1 \leq l_1-1,\ 0 \leq x_2 \leq l_2-1\ \big\}\,, \qquad l_1,l_2 \in \mathbb{Z}_+$$

with $\min\{l_1,l_2\} \geq 0.1\max\{l_1,l_2\}$ and $\max\{l_1,l_2\} \leq L$.
Let $\Lambda_1 = \big\{(x_1,x_2) \in \Lambda;\ 0 \leq x_2 \leq (\frac12+2\delta)\,l_2\ \big\}$, $\Lambda_2 = \big\{(x_1,x_2) \in \Lambda;\ (\frac12+\delta)\,l_2 \leq x_2 \leq l_2\ \big\}$, and let $\Lambda_3 = \Lambda_1 \cap \Lambda_2$.
Let finally N_i be possible values of the number of particles in Λ_i, $i=1,2,3$, and let $\nu^\tau_{\Lambda,\mathbf{N}}$ be the multi canonical Gibbs measure $\mu^\tau_\Lambda\big(\cdot\,|\,N_{\Lambda_i}=N_i,\ i=1,\dots,3\big)$. Then we have

Proposition 3.4. *Assume condition $USMT(C,m,l)$. Then, for any $\epsilon > 0$ there exist $L_0 = L_0(\epsilon, C, m, \|\Phi\|, l, \delta)$ such that, if $L \geq L_0$,*

$$\nu^{\tau}_{\Lambda,\mathbf{N}}(f,f) \leq (1+\epsilon)\nu^{\tau}_{\Lambda,\mathbf{N}}\left(\mathrm{Var}^{\eta}_{\Lambda_1,\mathbf{N}_1}(f) + \mathrm{Var}^{\eta}_{\Lambda_2,\mathbf{N}_2}(f)\right)$$

where $\mathrm{Var}^{\eta}_{\Lambda_i,\mathbf{N}_i}(f)$, $i=1,2$, denotes the variance of f w.r.t. the multicanonical measure on Λ_i with N_i particles, N_3 of which are in Λ_3, and boundary condition η on $\partial^+_r\Lambda_i$.

Proof. Fix $\epsilon \in (0,1)$ and consider the continuous time reversible w.r.t. $\nu^{\tau}_{\Lambda,\mathbf{N}}$ Markov chain on Ω_Λ (*block dynamics*) in which, with rate one, either Λ_1 or Λ_2 is chosen and there the "old" configuration σ is replaced by a "new" one σ' distributed according to the multicanonical Gibbs measure of the chosen block with total number of particles equal to either N_1 or N_2, N_3 particles in Λ_3 and boundary condition σ outside. The above proposition simply says that the spectral gap of our chain is larger than $\frac{1}{1+\epsilon}$ provided that L is large enough.
It is in fact easy to check that the associated Dirichlet form is given by

$$\mathcal{E}_{\mathrm{block}}(f,f) = \nu^{\tau}_{\Lambda,\mathbf{N}}\left(\mathrm{Var}^{\eta}_{\Lambda_1,\mathbf{N}_1}(f) + \mathrm{Var}^{\eta}_{\Lambda_2,\mathbf{N}_2}(f)\right)$$

and that the action of the corresponding semigroup $T_{\mathrm{block}}(t)$ is given by (see e.g. section 3 of [M1])

$$T_{\mathrm{block}}(t)f = \sum_{n=0}^{\infty}\frac{(2t)^n}{n!}e^{-2t}\frac{1}{2^n}\sum_{X\in\{1,2\}^n}\nu_{X_1}\cdots\nu_{X_n}(f) \tag{3.4}$$

where $(\nu_i f)(\sigma) := \nu^{\sigma}_{\Lambda_i,N_i}(f \mid N_{\Lambda_3} = N_3)$, $i=1,2$. Since $(\nu_i)^2 = \nu_i$ the last summation (over X) in (3.4) can be written as

$$\sum_{k=0}^{n-1}\binom{n-1}{k}(\hat{A}_{k+1} + \hat{B}_{k+1})f \tag{3.5}$$

where

$$\hat{A}_k = (\nu_1\circ\nu_2)^{\lfloor k/2\rfloor}\circ\nu_1^{k-2\lfloor k/2\rfloor} \qquad \hat{B}_k = (\nu_2\circ\nu_1)^{\lfloor k/2\rfloor}\circ\nu_2^{k-2\lfloor k/2\rfloor}$$

If now f is an arbitrary bounded measurable function on Ω_Λ, such that $\nu^{\tau}_{\Lambda,\mathbf{N}}(f) = 0$, we get

$$\|\nu_1\,\nu_2\,\nu_1 f\|_\infty \leq \|\nu^{\tau}_{\Lambda,\mathbf{N}}\,\nu_2\,\nu_1 f\|_\infty + \|\nu^{\tau}_{\Lambda,\mathbf{N}}\,\nu_2\,\nu_1\,f - \nu_1\,\nu_2\,\nu_1\,f\|_\infty \tag{3.6}$$

By construction the first term on the RHS of (3.6) is equal to $\nu^{\tau}_{\Lambda,\mathbf{N}}(f) = 0$. Furthermore, since the interaction has range r and the number of particles is fixed in each set Λ_i, $i=1,2,3$, the function $h \equiv \nu_2\,\nu_1 f$ is $\mathcal{F}_{\Lambda^c\cup\partial^+_r\Lambda_2}$ measurable. This fact implies that

$$\|\nu^{\tau}_{\Lambda,\mathbf{N}}\,\nu_2\,\nu_1\,f - \nu_1\,\nu_2\,\nu_1\,f\|_\infty \leq \sup_{\sigma,\eta\in\Omega_{\partial^+_r\Lambda_1}}|\nu_1(h)(\sigma) - \nu_1(h)(\eta)| \tag{3.7}$$

Thanks to (2) of proposition 3.2, the r.h.s. of (3.7) is smaller than $C'\|h\|_\infty\left(\frac{\log L}{L}\right)^{\frac{1}{2}}$ which is smaller than $\left(\frac{\epsilon}{1+\epsilon}\right)^3\|f\|_\infty$ if L is large enough. Iterating this inequality we get

$$\|\hat{A}_k f\|_\infty \leq \left(\frac{\epsilon}{1+\epsilon}\right)^{3\lfloor\frac{k}{3}\rfloor}\|f\|_\infty \qquad \|\hat{B}_k f\|_\infty \leq \left(\frac{\epsilon}{1+\epsilon}\right)^{3\lfloor\frac{k}{3}\rfloor}\|f\|_\infty \tag{3.8}$$

Thus, the sup norm of (3.5) is not greater than

$$\|f\|_\infty 2(1+\frac{\epsilon}{1+\epsilon})^{n-1}$$

which, inserted back into (3.4) yields

$$\|T_{\text{block}}(t)f\|_\infty \le \|f\|_\infty 4e^{-(1-\frac{\epsilon}{1+\epsilon})t}$$

In other words the spectral gap of the block dynamics is larger than $(1-\frac{\epsilon}{1+\epsilon})$ so that

$$\nu^\tau_{\Lambda,\mathbf{N}}(f,f) \le (1-\frac{\epsilon}{1+\epsilon})^{-1}\mathcal{E}_{\text{block}}(f,f) = (1+\epsilon)\nu^\tau_{\Lambda,\mathbf{N}}\left(\mathrm{Var}^\eta_{\Lambda_1,\mathbf{N}_1}(f) + \mathrm{Var}^\eta_{\Lambda_2,\mathbf{N}_2}(f)\right) \quad \square$$

Remark. The restriction of $d=2$ comes from (2) of proposition 3.2. In fact, in e.g. three dimensions, the r.h.s. of (3.7) would have been bounded by only $\|h\|_\infty\sqrt{\log L}$ which is a completely useless bound ! The way out is to divide the "safety belt" Λ_3 into several layers (just two in $d=3$) each one of width proportional to δL, fix the number of particles in each one of them and then use (3) of proposition 3.2. In other words, in higher dimensions the block dynamics has a fast relaxation only if we prevent the exchange of particles also inside a certain number of layers of Λ_3.

3.4 On the distribution of the particle number

Here we provide some simple results on the distribution of the particle numbers in the atoms of a partition of a given set Λ. Throughout this subsection the setting will be as follows.

Let $l\in\mathbb{Z}_+$, $\delta,\epsilon\in(0,1)$, and let us consider $\Lambda\in\mathbb{F}_l$, $\Lambda=\cup_{i\in I}Q_l(x^i)$, such that $|\partial^-_{l-1}\Lambda|\le|\Lambda|^{\frac{d-1+\epsilon}{d}}$. Let $I_1\dots I_k$ be a partition of I, let $\Lambda_i=\cup_{j\in I_i}Q_l(x^j)$ and assume that $0<\delta\le\frac{|\Lambda_i|}{|\Lambda|}$ and $|\partial^-_{l-1}\Lambda_i|\le|\Lambda|^{\frac{d-1+\epsilon}{d}}$ for all $i=1,\dots,k$.

Let also $\mathbf{N}=\{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda:=\{N_{\Lambda_i}\}_{i=1}^k$. Given a boundary condition τ, let $\underline{\lambda}$ be the chemical potential, constant on each atom, such that $\mu^{\tau,\underline{\lambda}}_\Lambda(\mathbf{N}_{\boldsymbol{\Lambda}})=\mathbf{N}$ (see above for the existence of $\underline{\lambda}$). Then we have (see Lemma 5.2 in [CM1])

Proposition 3.5. *Assume property $USMT\,(C,m,l)$. Let $\sigma_i^2:=\mu^{\tau,\underline{\lambda}}_\Lambda(N_{\Lambda_i},N_{\Lambda_i})$ $i=1,\dots,k$. Then*

$$\frac{1}{C'}\frac{1}{\prod_i\sigma_i}\le\mu^{\tau,\underline{\lambda}}_\Lambda(\mathbf{N}_\Lambda=\mathbf{N})\le C'\frac{1}{\prod_i\sigma_i}$$

for a suitable constant $C'=C'(C,m,l,\|\Phi\|_\infty)>1$.

The next result concerns the way particles distribute inside one block of the partition. More precise results in this direction are given in the next section.

Pick $j\in[1,\dots,k]$ and divide Λ_j into two subsets V, W. Then we have

Proposition 3.6. *For all $n \in [0, |V|-1]$*

$$\frac{(|V|-n)(N_j-n)}{(n+1)(|W|-N_j+n+1)}\, e^{-2\|\Phi\|} \le \frac{\nu^{\tau}_{\Lambda,\mathbf{N}}\,(N_V=n+1)}{\nu^{\tau}_{\Lambda,\mathbf{N}}\,(N_V=n)} \le e^{2\|\Phi\|}\, \frac{(|V|-n)(N_j-n)}{(n+1)(|W|-N_j+n+1)}$$

where $\nu^{\tau}_{\Lambda,\mathbf{N}}(\cdot) := \mu^{\tau,\underline{\lambda}}_{\Lambda}\big(\cdot \,|\, \mathbf{N}_\Lambda = \mathbf{N}\big)$.

Remark. It is important that the error is $e^{2\|\Phi\|}$ and not $e^{2\|\Phi^{\underline{\lambda}}\|}$.

Proof. We write

$$\begin{aligned}
&\mu^{\tau,\underline{\lambda}}_{\Lambda}\,(N_V = n+1\,;\, \mathbf{N}_\Lambda = \mathbf{N}) \\
&= \frac{1}{(n+1)(|W|-N_j+n+1)} \sum_{\substack{x\in V\\ y\in W}} \mu^{\tau,\underline{\lambda}}_{\Lambda}\left(\mathbb{1}_{\{\mathbf{N}_\Lambda=\mathbf{N}\}}\mathbb{1}_{\{N_V=n+1\}}\sigma(x)\,(1-\sigma(y))\right) \\
&= \frac{1}{(n+1)(|W|-N_j+n+1)} \sum_{\substack{x\in V\\ y\in W}} \mu^{\tau,\underline{\lambda}}_{\Lambda}\left(e^{-(\nabla_{xy}H^{\tau,\underline{\lambda}}_{\Lambda})(\sigma)}\,(1-\sigma(x))\,\sigma(y)\mathbb{1}_{\{\mathbf{N}_\Lambda=\mathbf{N}\}}\mathbb{1}_{\{N_V=n\}}\right) \\
&\le \frac{(|V|-n)(N_j-n)}{(n+1)(|W|-N_j+n+1)} e^{2\|\Phi\|}\mu^{\tau,\underline{\lambda}}_{\Lambda}\,(N_V = n\,;\, \mathbf{N}_\Lambda = \mathbf{N})
\end{aligned}$$

where we used the change of variable $\sigma \mapsto t_{xy}\sigma$ to obtain the second equality and $\|e^{-\nabla_{xy}H^{\tau,\underline{\lambda}}_{\Lambda}}\|_\infty \le e^{2\|\Phi\|}$ for the last inequality. The lower bound is analogous. $\square$

3.5 Spectral gap of one dimensional discrete random walks.

Proposition 3.7. *Let γ be a positive probability measure on the integers $\Omega = \{N_1, N_1+1, \dots, N_2\}$ and let $N^* \in \Omega$ be the largest integer such that $\sum_{n\le N^*}\gamma(n) \le \frac{1}{2}$. Then, for all functions f on Ω we have*

$$\mathrm{Var}(f) \le C_\gamma \sum_{n=N_1+1}^{N_2} (\gamma(n)\wedge\gamma(n-1))\,\big[f(n)-f(n-1)\big]^2$$

where

$$C_\gamma = 4\,\max\left[\left(\sup_{n\le N^*+1}\sum_{j\le n}\frac{\gamma(j)}{\gamma(n)}\right)^2,\ \left(\sup_{n\ge N^*}\sum_{j\ge n}\frac{\gamma(j)}{\gamma(n)}\right)^2\right]$$

Proof. We consider a continuous time Markov chain with transition rates

$$c(n,j) = \begin{cases} \frac{\gamma(j)}{\gamma(n)}\wedge 1 & \text{if } j = n\pm 1 \\ 0 & \text{otherwise.}\end{cases} \tag{3.9}$$

Since the rates satisfy the detailed balance condition

$$\gamma(n)\, c(n,j) = \gamma(j)\, c(j,n) \tag{3.10}$$

the probability measure γ is reversible with respect to the chain. The associated Dirichlet form is given by

$$\begin{aligned}\mathcal{E}_\gamma(f,f) &= \sum_{n=1}^{N} \gamma(n) c(n,n-1)\, [f(n-1) - f(n)]^2 = \\ &= \sum_{n=1}^{N} (\gamma(n) \wedge \gamma(n-1))\, \big[f(n) - f(n-1)\big]^2 .\end{aligned}$$

If we denote by λ the spectral gap of the generator of the chain, we have

$$\mathrm{Var}_\gamma(f) \le \frac{1}{\lambda}\, \mathcal{E}_\gamma(f,f) = \frac{1}{\lambda} \sum_{i=1}^{N} (\gamma(n) \wedge \gamma(n-1))\, \big[f(n) - f(n-1)\big]^2 .$$

To conclude the proof we need a lower bound for λ. Cheeger's inequality (see Theorem 2.1 in [LS]) states that

$$\lambda \ge \frac{I^2}{8M}$$

where $M = \sup_i\, (c(n,n+1) + c(n,n-1))$ and

$$I = \min_{A \subset \Omega} \frac{\sum_{(j,k) \in A \times A^c} c(j,k) \gamma(j)}{\gamma(A)\, (1 - \gamma(A))}$$

With the choice (3.9) $M \le 2$. As the state space Ω is countable and connected, by Corollary 4.4 in [LS], the minimum can be taken over all subsets $A \subset \Omega$ such that A and A^c are connected. Using the symmetry between A and A^c we can also impose $\gamma(A) \le 1/2$. We can thus write, using (3.10) and (3.9),

$$\begin{aligned}\frac{2}{I} &\le \max\left[\sup_{n \le N^*} \frac{\sum_{j \le n} \gamma(j)}{\gamma(n) \wedge \gamma(n+1)} \,,\, \sup_{n \ge N^*+1} \frac{\sum_{j \ge n} \gamma(j)}{\gamma(n) \wedge \gamma(n-1)} \right] \\ &\le \max\left[\left(\sup_{n \le N^*+1} \sum_{j \le n} \frac{\gamma(j)}{\gamma(n)} \right) \,,\, \left(\sup_{n \ge N^*} \sum_{j \ge n} \frac{\gamma(j)}{\gamma(n)} \right) \right]\end{aligned} \tag{3.11}$$

□

Proposition 3.8. *In the same hypotheses of proposition 3.7 assume that for some* $N_1 \le \bar{N} \le N_2$ *and* $\alpha > 0$ $\frac{\gamma(j-1)}{\gamma(j)} \le e^{-\alpha(\bar{N}-j)}$ *for all* $j \le \bar{N}$ *and* $\frac{\gamma(j+1)}{\gamma(j)} \le e^{-\alpha(j-\bar{N})}$ *for all* $j \ge \bar{N}$*. Then for all functions* f *on* Ω *we have*

$$\mathrm{Var}(f) \le C \max\{\frac{1}{\alpha}, 1\} \sum_{n=N_1+1}^{N_2} (\gamma(n) \wedge \gamma(n-1))\, \big[f(n) - f(n-1)\big]^2$$

for a suitable numerical constant C independent of f and all the other parameters.

Proof. Without loss of generality we can assume $N^* \le \bar N$. Using the hypothesis, for any $n \le \bar N$ we have

$$\begin{aligned}\sum_{j\le n}\frac{\gamma(j)}{\gamma(n)} &= 1+\sum_{j<n}\prod_{z=j+1}^{n}\frac{\gamma(z-1)}{\gamma(z)}\\ &\le 1+\sum_{j<n}\prod_{z=j+1}^{n}e^{-\alpha(\bar N - z)}\\ &\le 1+\sum_{j<n}e^{-\frac{\alpha}{2}(i-j)^2}\\ &\le 1+\frac{c}{\sqrt{\alpha}}\end{aligned}$$

for a suitable numerical constant c. Similarly for $\sum_{j\ge n}\frac{\gamma(j)}{\gamma(n)}$, $n \ge \bar N$.
Therefore, for any $n \in \Omega\cap[N^*+1,\bar N]$, $\gamma(n) \ge \frac{\sqrt{\alpha}}{2(c+\sqrt{\alpha})}$ so that

$$\sup_{n\le N^*+1}\sum_{j\le n}\frac{\gamma(j)}{\gamma(n)} \le c' \max\{\frac{1}{\sqrt{\alpha}},1\}$$

for a suitable numerical constant c'. Analogously

$$\begin{aligned}\sup_{n\ge N^*}\sum_{j\ge n}\frac{\gamma(j)}{\gamma(n)} &\le 1+\frac{c}{\sqrt{\alpha}}+\sup_{n\in\Omega\cap[N^*+1,\bar N]}\frac{1}{\gamma(n)}\\ &\le c'' \max\{\frac{1}{\sqrt{\alpha}},1\}\end{aligned}$$

The thesis now follows from proposition 3.7. $\square$

3.6 A KEY BOUND ON SPECIAL COVARIANCES.

Assume the same setting of paragraph §3.4. For any local function f and for any $x \in \mathbb{Z}^d$, let $\Delta_f(x) := \Delta_f + x$ and $f_x(\sigma) := f\left(\sigma_{\Delta_f(x)}\right)$. Then one has the following interesting result.

Lemma 3.9. *Assume condition $USMT(C,m,l)$ and fix $i,j \in \{1,\dots,k\}$ with $i \ne j$. Let g,h be two local functions with support containing the origin and of diameter smaller than $2r$, r being the range of the interaction. Then for any $\epsilon > 0$ there exists C_ϵ such that for any f*

$$\nu^\tau_{\Lambda,\mathbf{N}}\big(f,\frac{1}{|\Lambda_i||\Lambda_j|}\sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}} g_x h_z\big)^2 \le \frac{C_\epsilon}{|\Lambda|}\,\nu^\tau_{\Lambda,\mathbf{N}}\left(\sum_{[x,z]\in\mathcal{E}_\Lambda} c_{xz}(\nabla_{xz} f)^2\right) + \frac{\epsilon}{|\Lambda|}\,\mathrm{Var}^\tau_{\Lambda,\mathbf{N}}(f)$$

provided that $|\Lambda|$ *is larger than* C_ϵ.

Remark. When $k=1$, namely the partition of Λ is the trivial one, and $g(\eta)\equiv 1$ the above result is nothing but the so called "two–blocks estimate" of lemma 4.4. of [LY].

Proof. Fix $\epsilon>0$. In view of the above remark we can assume, without loss of generality, that $\nu^\tau_{\Lambda,\mathbf{N}}(g_x)=0$ for any $x\in\Lambda_i$.
Let for notation simplicity $G:=\frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i}g_x$, $\mathcal{H}=\frac{1}{|\Lambda_j|}\sum_{z\in\Lambda_j}h_z$ and let us write $\mathcal{H}=\mathcal{H}^{\rm in}+\mathcal{H}^{\rm ext}$ where $\mathcal{H}^{\rm in}$ is the sum over those z's in Λ_j such that $\Delta_h(z)\subset\Lambda_j$ and $\mathcal{H}^{\rm ext}$ the rest. Then, using the formula relating the covariance of two functions f and g w.r.t. the measure $\nu^\tau_{\Lambda,\mathbf{N}}$ to the covariance w.r.t. the same measure conditioned to a sub σ-algebra, we get

$$
\begin{aligned}
\nu^\tau_{\Lambda,\mathbf{N}}\Bigl(f,\frac{1}{|\Lambda_i||\Lambda_j|}\sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}}g_xh_z\Bigr)^2 &= \nu^\tau_{\Lambda,\mathbf{N}}\bigl(f,G\mathcal{H}\bigr)^2\\
&\le 2\|\mathcal{H}^{\rm ext}\|^2_\infty\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(G)\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f)+4\|\mathcal{H}^{\rm in}\|^2_\infty\nu^\tau_{\Lambda,\mathbf{N}}\left(\Bigl[\nu^\tau_{\Lambda,\mathbf{N}}\Bigl(f,G\,|\,\mathcal{F}_{\Lambda_i^c}\Bigr)\Bigr]^2\right)\\
&\quad+4\Bigl[\nu^\tau_{\Lambda,\mathbf{N}}\Bigl(f,\,\nu^\tau_{\Lambda,\mathbf{N}}\Bigl(G\,|\,\mathcal{F}_{\Lambda_i^c}\Bigr)\mathcal{H}^{\rm in}\Bigr)\Bigr]^2\\
&\le \frac{1}{|\Lambda|}\left[C'\left(\frac{|\partial_r^+\Lambda_j|}{|\Lambda_j|}\right)^2+\frac{\epsilon}{2}\right]\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f)+\frac{C_\epsilon}{|\Lambda|}\nu^\tau_{\Lambda,\mathbf{N}}\left(\sum_{[x,z]\in\mathcal{E}_\Lambda}c_{xz}(\nabla_{xz}f)^2\right)\\
&\quad+4\bigl[\nu^\tau_{\Lambda,\mathbf{N}}\bigl(f,\,\nu^\tau_{\Lambda,\mathbf{N}}(G\,|\,\mathcal{F}_{\Lambda_i^c})\mathcal{H}^{\rm in}\bigr)\bigr]^2
\end{aligned}
\tag{3.12}
$$

where we have used the hypothesis $\frac{|\Lambda_i|}{|\Lambda|}\ge\delta$ together with lemma 4.4. of [LY] (see proposition A.1 for a simpler proof) to bound the term $\Bigl[\nu^\tau_{\Lambda,\mathbf{N}}\Bigl(f,G\,|\,\mathcal{F}_{\Lambda_i^c}\Bigr)\Bigr]^2$ and proposition 3.3 to get $\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(G)\le C''/|\Lambda|$.
The third term in the.r.h.s of (3.12) can be bounded from above by

$$
\|\nu^\tau_{\Lambda,\mathbf{N}}(G\,|\,\mathcal{F}_{\Lambda_i^c})\|^2_\infty\ \operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(\mathcal{H}^{\rm in})\ \operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f)
\tag{3.13}
$$

In turn, the second factor in the r.h.s. of (3.13), using proposition 3.3 together with the mixing condition, is bounded from above by $\frac{C_1}{|\Lambda|}$. The first factor in the r.h.s. of (3.13), thanks to the hypothesis $\nu^\tau_{\Lambda,\mathbf{N}}(g_x)=0$, to a simple telescopic argument and to part 1) of proposition 3.2, is bounded from above by

$$
\begin{aligned}
&\Bigl[\sup_{\tau,\tau'\in\Omega_{\partial_r^+\Lambda_i}}\frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i}|\nu^\tau_{\Lambda_i,N_i}(g_x)-\nu^{\tau'}_{\Lambda_i,N_i}(g_x)|\Bigr]^2\\
&\qquad\le\left[\sup_{\tau\in\Omega_{\partial_r^+\Lambda_i}}\frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i}\sum_{y\in\partial_r^+\Lambda_i}e^{2\|\Phi\|}|\nu^\tau_{\Lambda_i,N_i}(g_x,e^{-\nabla_yH^\tau_{\Lambda_i}})|\right]^2\le C'\left[\frac{|\partial_r^+\Lambda_i|}{|\Lambda_i|}\right]^2
\end{aligned}
$$

In conclusion, for any $\epsilon > 0$, the first and third term in the.r.h.s of (3.12) can be bounded from above by $\frac{\epsilon}{2|\Lambda|} \operatorname{Var}^{\tau}_{\Lambda,\mathbf{N}}(f)$ provided that $|\Lambda|$ is large enough. The proof is complete. $\square$

3.7 Computing the gradient with respect to the particle number.

Assume the same setting of paragraph §3.4 and let V and W be such that $V \cap W = \emptyset$ and $\Lambda_j = V \cup W$ for some $j \in \{1, \dots, k\}$. Here we give a result that allows us to compute for any function f the gradient w.r.t. to n of $\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n)$ and to bound it in terms of the Dirichlet form and the variance of f.

For $x, z \in \mathbb{Z}^d$, we define the events

$$E_{xz} = \{\sigma \in \Omega : \sigma(x) = 1,\ \sigma(z) = 0\}\,. \tag{3.14}$$

Then we have

Proposition 3.10. *Let V and W be such that $V \cap W = \emptyset$ and $\Lambda_j = V \cup W$ for some $j \in \{1, \dots, k\}$. Let $\gamma(n) = \nu^{\tau}_{\Lambda,\mathbf{N}}\{N_V = n\}$. Let also $c_n = n(|W| - N_j + n)$, that is number of particles in $V \times$ number of holes in W, and let $c'_n = n(|V| - N_j + n)$. Then, for all functions f on Ω we have*

(1)

$$\begin{aligned}
&\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n) - \nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n-1) = \\
&\qquad \frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu^{\tau}_{\Lambda,\mathbf{N}}\left[(\nabla_{zx} f)\mathbb{1}_{E_{zx}} e^{-\nabla_{xz} H_\Lambda} \,|\, N_V = n-1\right] \\
&\qquad + \frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu^{\tau}_{\Lambda,\mathbf{N}}\left[\left(e^{-\nabla_{xz} H_\Lambda} - 1\right)\mathbb{1}_{E_{zx}}, f \,|\, N_V = n-1\right]
\end{aligned}$$

(2)

$$\begin{aligned}
&\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n) - \nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n-1) = \\
&\qquad - \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu^{\tau}_{\Lambda,\mathbf{N}}\left[(\nabla_{xz} f)\mathbb{1}_{E_{xz}} e^{-\nabla_{xz} H_\Lambda} \,|\, N_V = n\right] - \\
&\qquad - \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu^{\tau}_{\Lambda,\mathbf{N}}\left[\left(e^{-\nabla_{xz} H_\Lambda} - 1\right)\mathbb{1}_{E_{xz}}, f \,|\, N_V = n\right]
\end{aligned}$$

Remark. A similar statement is contained in Lemma 3.1 in [LY].

Before the proof we note that given Λ_j, V, W, N_j as above, let $n_{\max}$, $n_{\min}$ be the maximum and minimum value of the particle number in V under the constraint that $N_{\Lambda_i} = N_i, \quad \forall\, i = 1, \dots, k$.

Let $u = \lfloor \rho_j |V| \rfloor$ where $\rho_j = \frac{N_j}{|\Lambda_j|}$ and let, for $n \in [n_{\min}, n_{\max}]$,

$$A(n) = \begin{cases} \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[(\nabla_{zx} f) \mathbb{1}_{E_{zx}} e^{-\nabla_{xz} H_\Lambda} \,|\, N_V = n-1 \right] & \text{if } n \le u \\ \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[(\nabla_{xz} f) \mathbb{1}_{E_{xz}} e^{-\nabla_{xz} H_\Lambda} \,|\, N_V = n \right] & \text{otherwise} \end{cases}$$

$$B(n) = \begin{cases} \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[\left(e^{-\nabla_{xz} H_\Lambda} - 1 \right) \mathbb{1}_{E_{zx}}, f \,|\, N_V = n-1 \right] & \text{if } n \le u \\ \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[\left(e^{-\nabla_{xz} H_\Lambda} - 1 \right) \mathbb{1}_{E_{xz}}, f \,|\, N_V = n \right] & \text{otherwise} \end{cases} \tag{3.15}$$

With this definition we have immediately

Corollary 3.11. *In the same setting of proposition 3.10*

$$\sum_i (\gamma(n) \wedge \gamma(n-1)) \left[\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n) - \nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n-1) \right]^2 \le \sum_i (\gamma(n) \wedge \gamma(n-1)) \left[A(n)^2 + B(n)^2 \right]$$

Proof of the proposition. Adding and subtracting $T_{xz} f$ we can write

$$\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n) = \frac{1}{c_n} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[(f - T_{xz} f) \mathbb{1}_{E_{xz}} \,|\, N_V = n \right] + \frac{1}{c_n} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[(T_{xz} f) \mathbb{1}_{E_{xz}} \,|\, N_V = n \right] \tag{3.16}$$

After the change of variable $\sigma \mapsto \sigma^{xz}$ and using the equality $\nu^{\tau}_{\Lambda,\mathbf{N}}(fg) = \nu^{\tau}_{\Lambda,\mathbf{N}}(f,g) + \nu^{\tau}_{\Lambda,\mathbf{N}}(f)\nu^{\tau}_{\Lambda,\mathbf{N}}(g)$, we can write the first term in the r.h.s. of (3.16) as

$$\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[(\nabla_{zx} f) \mathbb{1}_{E_{zx}} e^{-\nabla_{xz} H_\Lambda} \,|\, N_V = n-1 \right]$$

while the second term becomes

$$\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[\left(e^{-\nabla_{xz} H_\Lambda} - 1 \right) \mathbb{1}_{E_{zx}}, f \,|\, N_V = n-1 \right] + \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x\in V \\ z\in W}} \nu^{\tau}_{\Lambda,\mathbf{N}} \left[e^{-\nabla_{xz} H_\Lambda} \mathbb{1}_{E_{zx}} \,|\, N_V = n-1 \right] \; \nu^{\tau}_{\Lambda,\mathbf{N}} \left(f \,|\, N_V = n-1 \right) \tag{3.17}$$

where we have exploit the fact that $\nu^{\tau}_{\Lambda,\mathbf{N}}(\cdot \,|\, N_V = n-1)$–almost surely $\sum_{\substack{x\in V \\ z\in W}} \mathbb{1}_{E_{zx}} = c'_{N-n+1}$ in order to substract 1 from the $e^{-\nabla_{xz} H_\Lambda}$ term. Taking $f = 1$ in equation (3.16) we obtain that the term multiplying $\nu^{\tau}_{\Lambda,\mathbf{N}}(f \,|\, N_V = n-1)$ in (3.17) is equal to one and the result is obtained.

The second equality follows from the first by interchanging V and W $\square$

3.8 Bound on $\sum_i (\gamma(n) \wedge \gamma(n-1))\, A(n)^2$.

We now show how to bound from above the term $\sum_i (\gamma(n) \wedge \gamma(n-1))\, A(n)^2$ of corollary 3.11. We first need the following definition.

Definition 3.12. Given a finite connected subset Λ of $\mathbb{Z}^d$ a *path choice in* Λ is a collection $\lambda = \{\lambda_{xz} : (x,z) \in \Lambda \times \Lambda\}$ such that λ_{xz} is a self–avoiding path from x to z inside Λ.

Given a path choice λ in Λ_j and V, W as above, we let

$$\mathcal{G}_V(\lambda) = \max_{e \in \mathcal{E}_\Lambda} \#\{(x,z) \in V \times W : \lambda_{xz} \sqsupset e\}$$

$$\mathcal{D}_V(\lambda) = \max_{(x,z) \in V \times W} |\lambda_{xz}|$$

With this notation we have the following result

Proposition 3.13. *There exists some C depending only on $\|\Phi\|$ such that*

$$\sum_{n=n_{\min}}^{n_{\max}} (\gamma(n) \wedge \gamma(n-1))\, A(n)^2 \le C \left[\frac{\mathcal{G}_V(\lambda)\mathcal{D}_V(\lambda)}{\rho_j(1-\rho_j)\,|V|\,|W|} \sum_{e \sqsubset \mathcal{E}_\Lambda} \nu^\tau_{\Lambda,\mathbf{N}}[(\nabla_e f)^2] \right]$$

Proof of Proposition 3.13. Assume for simplicity that $n \le u$. Then we observe that

$$\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu^\tau_{\Lambda,\mathbf{N}} \left[e^{-\nabla_{xz} H_\Lambda} \mathbb{1}_{E_{zx}} \,|\, N_V = n-1 \right] = 1$$

(see right after (3.17)) so that

$$e^{-2\|\Phi\|} \frac{1}{c'_{N_j - n+1}} \le \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \le e^{2\|\Phi\|} \frac{1}{c'_{N_j - n+1}}$$

Moreover

$$\frac{1}{c'_{N_j-n+1}} \sum_{(x,z) \in V \times W} \nu^\tau_{\Lambda,\mathbf{N}} \left[E_{zx} \,|\, N_V = n-1 \right] = 1$$

We can thus use the Schwarz inequality and obtain

$$\begin{aligned} A(n)^2 &= \\ &= \Big[\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z) \in V \times W} \nu^\tau_{\Lambda,\mathbf{N}} \left[e^{-\nabla_{xz} H} \nabla_{xz} f \,|\, N_V = n-1, E_{zx} \right] \nu^\tau_{\Lambda,\mathbf{N}} \left[E_{zx} \,|\, N_V = n-1 \right] \Big]^2 \\ &\le C \frac{1}{c'_{N_j-u+1}} \sum_{(x,z) \in V \times W} \nu^\tau_{\Lambda,\mathbf{N}} \left[(\nabla_{xz} f)^2 \,|\, N_V = n-1, E_{zx} \right] \nu^\tau_{\Lambda,\mathbf{N}} \left[E_{zx} \,|\, N_V = n-1 \right] \\ &\le C \frac{1}{c'_{N_j-u+1}} \sum_{(x,z) \in V \times W} \nu^\tau_{\Lambda,\mathbf{N}} \left[(\nabla_{xz} f)^2 \,|\, N_V = n-1 \right] \end{aligned} \tag{3.18}$$

since c'_n is increasing in n. Similarily, for $n > u$, we obtain

$$A(n)^2 \leq C\frac{1}{c_u} \sum_{(x,z)\in V\times W} \nu^\tau_{\Lambda,\mathbf{N}}\left[\,(\nabla_{xz} f)^2 \,|\, N_V = n\right] \tag{3.19}$$

Notice that $c_u \geq c'\rho_j(1-\rho_j)\,|V|\,|W|$ and similarly for $c'_{N_j-u+1} \geq C'\rho_j|W|\,((1-\rho_j)|V|)$ where c' is some suitable constant. Thus, by (3.18) and (3.19), we obtain

$$\sum_{n=n_{\min}}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))\,A(n)^2 \leq C'' \frac{1}{\rho_j(1-\rho_j)\,|V|\,|W|} \sum_{(x,z)\in V\times W} \nu^\tau_{\Lambda,\mathbf{N}}\left[(\nabla_{xz} f)^2\right] \tag{3.20}$$

Let now λ be any path choice. Thanks to Lemma 4.3 in [Y] we get that there exists $C_3(\|\Phi\|)$ such that

$$\nu^\tau_{\Lambda,\mathbf{N}}\left[\,(\nabla_{xz} f)^2\,\right] \leq C_3(\beta)\,|\lambda_{xz}| \sum_{e\sqsubset\lambda_{xz}} \nu^\tau_{\Lambda,\mathbf{N}}[(\nabla_e f)^2]$$

which, together with (3.20) and the definition of $\mathcal{G}_V(\lambda)$ and $\mathcal{D}_V(\lambda)$, finishes the proof. $\square$

3.9 Bound on $\sum_n (\gamma(n)\wedge\gamma(n-1))\,B(n)^2$.

Here we give an estimate of the B–term in corollary 3.11 under a regularity assumption on the sets V and W. We refer to §3.7 for all the necessary notation.

Proposition 3.14. *Assume property $USMT(C,m,l)$ and, without loss of generality, $\rho_j \leq 1/2$. Fix $\epsilon > 0$ and assume that $|\partial_r^+ V \cap \Lambda_j| \leq \epsilon|\Lambda|$.*

(1) There exists a constant K independent of f such that

$$\sum_{n=n_{min}}^{n_{max}} (\gamma(n)\wedge\gamma(n-1))\,B(n)^2 \leq \frac{K}{|\Lambda|}\,\mathrm{Var}^\tau_{\Lambda,\mathbf{N}}(f)$$

(2) There exists C_ϵ independent of f and C_0 independent of ϵ and f such that

$$\sum_{n=n_{min}}^{n_{max}} (\gamma(n)\wedge\gamma(n-1))\,B(n)^2 \leq \frac{C_\epsilon}{\rho_j^2|\Lambda|} \sum_{e\sqsubset\mathcal{E}_\Lambda} \nu^\tau_{\Lambda,\mathbf{N}}[(\nabla_e f)^2] + \frac{C_0\epsilon}{\rho_j^2|\Lambda|}\,\mathrm{Var}^\tau_{\Lambda,\mathbf{N}}(f)$$

Proof.
(1) Let $n \leq u$ where $u = \lfloor\rho_j|V|\rfloor$. Then, as in the proof of proposition 3.13,

$$B(n)^2 \leq e^{4\|\Phi\|_\infty}\,\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, \frac{1}{c'_{N_j-u+1}} \sum_{\substack{x\in V\\ z\in W}} \big(e^{-\nabla_{xz}H_\Lambda}-1\big)\,\mathbb{1}_{E_{zx}} \,|\, N_V = n-1\Big)^2 \tag{3.21}$$

Notice that $(e^{-\nabla_{xz}H_\Lambda}-1)\,\mathbb{1}_{E_{zx}} = 0$ unless there are at least two particles in an r–neighborhood of $\{x\}\cup\{z\}$. Therefore $\nu^\tau_{\Lambda,\mathbf{N}}\big(|(e^{-\nabla_{xz}H_\Lambda}-1)\,\mathbb{1}_{E_{zx}}|\,|\,N_V = n-1\big) \leq c\rho_j^2$ for a suitable constant *c.* Schwartz inequality, part 2) of proposition 3.2 together with the bound $c_{N_j-u+1} \geq c'(\delta)\rho_j(1-\rho_j)\,|\Lambda_j|^2$ (see the proof of proposition 3.13) show that the r.h.s. of (3.21) can be bounded from above by $\frac{K'}{|\Lambda_j|}\,\nu^\tau_{\Lambda,\mathbf{N}}\big(f,f\,|\,N_V=n-1\big)$. Thus

$$\sum_{n\leq u}(\gamma(n)\wedge\gamma(n-1))\,B(n)^2 \leq \frac{K'}{|\Lambda_j|}\nu_{\Lambda,\mathbf{N}}(f,f) \leq \frac{K'}{\delta|\Lambda|}\nu_{\Lambda,\mathbf{N}}(f,f)$$

because of the assumption $|\Lambda_j| \geq \delta|\Lambda|$. Similarly one proceeds for $n>u$.

(2) Let $g_x(\sigma) := e^{-\nabla_x H^\tau_\Lambda(\sigma)}(1-\sigma(x))$, $h_z(\sigma) := e^{-\nabla_z H^\tau_\Lambda(\sigma)}\sigma(z)$. Notice that $e^{-\nabla_{xz}H_\Lambda}\,\mathbb{I}_{E_{zx}} - g_x\,h_z = 0$ unless $d(x,z)\leq r$. Therefore

$$\begin{aligned}
B(n)^2 &\leq\; e^{4\|\Phi\|_\infty}\,\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, \frac{1}{\rho_j|V||W|}\sum_{\substack{x\in V\\ z\in W}}\big(e^{-\nabla_{xz}H_\Lambda}-1\big)\,\mathbb{I}_{E_{zx}}\,|\,N_V=n-1\Big)^2 \\
&\leq 2\,e^{4\|\Phi\|_\infty}\,\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, \frac{1}{\rho_j|V||W|}\sum_{\substack{x\in V\\ z\in W}}\big(e^{-\nabla_{xz}H_\Lambda}\,\mathbb{I}_{E_{zx}} - g_xh_z\big)\,|\,N_V=n-1\Big)^2 \\
&\quad + 2\,e^{4\|\Phi\|_\infty}\,\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, \frac{1}{\rho_j|V||W|}\sum_{\substack{x\in V\\ z\in W}} g_xh_z\,|\,N_V=n-1\Big)^2
\end{aligned} \tag{3.22}$$

where we have dropped the minus one from $\big(e^{-\nabla_{xz}H_\Lambda}-1\big)$ because $\sum_{\substack{x\in V,\\ z\in W}}\mathbb{I}_{E_{zx}}$ is constant under $\nu^\tau_{\Lambda,\mathbf{N}}(\cdot\,|\,N_V=n-1)$.
The first term in the r.h.s. of (3.22), using Schwartz inequality together with proposition 3.3, condition $USMT$ and the assumption $|\partial^-_r V\cap\Lambda_j| \leq \epsilon|\Lambda|$, is bounded from above by

$$\frac{C''\epsilon}{|\Lambda|}\nu^\tau_{\Lambda,\mathbf{N}}\Big(f,f\,|\,N_V=n-1\Big)$$

for a suitable constant C''.
The second term in the r.h.s. of (3.22), using lemma 3.9, is bounded from above by

$$\frac{C_\epsilon}{\rho_j^2|\Lambda|}\sum_{e\sqsubset\mathcal{E}_\Lambda}\nu^\tau_{\Lambda,\mathbf{N}}\big([\nabla_e f]^2\,|\,N_V=n-1\big) + \frac{\epsilon}{\rho_j^2|\Lambda|}\,\nu^\tau_{\Lambda,\mathbf{N}}\Big(f,f\,|\,N_V=n-1\Big)$$

Thus

$$\sum_{n\leq u}(\gamma(n)\wedge\gamma(n-1))\,B(n)^2 \leq \frac{C_\epsilon}{\rho_j^2|\Lambda|}\sum_{e\sqsubset\mathcal{E}_\Lambda}\nu^\tau_{\Lambda,\mathbf{N}}\big([\nabla_e f]^2\big) + \frac{C_0\epsilon}{\rho_j^2|\Lambda|}\,\nu^\tau_{\Lambda,\mathbf{N}}(f,f)$$

for a suitable constants C_0 independent of ϵ. Similarly one proceeds for the case $n>u$. $\square$

4. More on the distribution of the number of particles inside one block.

Fix $\delta \in (0,1)$ and an integer $k \geq 2$ with $\delta k < 1$. Let $L_1, \dots, L_k$ be large multiples of the basic length scale l, let $L = \sum_i L_i$ and assume that $L_j \geq \delta L$ for any j. We then choose one coordinate direction, e.g. the d direction, and we take $\Lambda = Q_L$, Λ_1 equal to the first slice of Λ orthogonal to the d–axis of width L_1, *i.e.* $\Lambda_1 = \{x \in \Lambda \ : \ 0 \leq x_d < L_1\}$, Λ_2 equal to the slice of Λ on top of Λ_1 of width L_2 and so on. Let also $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$ and let us assume, for a given boundary condition τ, that $\underline{\lambda}$ is constant on each block Λ_i and such that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}) = N_i$, $i = 1, \dots, k$. We denote by $\nu_{\Lambda,\mathbf{N}}^\tau$ the multi canonical Gibbs measure $\mu_\Lambda^{\tau,\underline{\lambda}}\big(\cdot \,|\, N_{\Lambda_i} = N_i, \, i = 1, \dots, k\big)$.

Pick now $j \in \{1, \dots, k\}$ and consider the new finer partition of Λ obtained from the previous one by splitting Λ_j into two slices orthogonal to the d–direction, V and W, in such a way that $\delta \leq |V|/|W| \leq 1-\delta$ and V and W are still multiples of the basic length scale l. Assume without loss of generality that $\frac{N_j}{|\Lambda_j|} \leq \frac{1}{2}$ and denote by N^* the average number of particles in V according to $\mu_\Lambda^{\tau,\underline{\lambda}}$. Let also $n_{\min} = \max\{0, N_j - |W|\}$ and $n_{\max} = \min\{|V|, N_j\}$ be the smallest and the largest value of $N_V(\sigma)$ under the constraint that $N_{\Lambda_j}(\sigma) = N_j$. It is easy to check that $c(\delta)^{-1} \leq \frac{n_{\max} - N^*}{N^*} \leq c(\delta)$ for a suitable constant $c(\delta) \geq 1$ and similarily for $\frac{N^* - n_{\min}}{N^*}$.

In what follows we will consider the distribution of the number of particles in V under the measure $\nu_{\Lambda,\mathbf{N}}^\tau$. More precisely we define $\gamma = \{\gamma(n)\}$ to be the probability measure on $I = \{n \in [n_{\min}, n_{\max}] :$ n is an integer $\}$, given by

$$\gamma(n) := \nu_{\Lambda,\mathbf{N}}^\tau \left(N_V = n\right)$$

In order to obtain sharp bounds on $\gamma(n)$, $n \in I$, we modify the chemical potential $\underline{\lambda}$ in an n–dependent way in such a way that the value n becomes equal to the average of N_V. More precisely, given $n \in [n_{\min}, n_{\max}]$, let $\underline{\lambda}(n) = \{\lambda_1, \dots, \lambda_{j-1}, \lambda_V, \lambda_W, \lambda_{j+1}, \dots, \lambda_k\}$ be a new chemical potential constant on the atoms of the new partition and such that

$$\begin{aligned} \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_i}) &= N_i, \quad i = 1, \cdots, k \\ \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V) &= n \\ \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_W) &= N_j - n \end{aligned} \tag{4.1}$$

It is then easy to check that γ can be written in the Gibbsian form, $\gamma(n) = e^{-H(n)}\varphi(n)$, where

$$\begin{aligned} H(n) &:= \sum_{i \neq j} \left(\lambda_i(n) - \lambda_i\right) N_i + \lambda_V \, n + \lambda_W (N_j - n) - \lambda_j N_j - \log\left(\frac{Z_\Lambda^{\tau,\underline{\lambda}(n)}}{Z_\Lambda^{\tau,\underline{\lambda}}}\right) \\ \varphi(n) &:= \frac{\mu_\Lambda^{\tau,\underline{\lambda}(n)}\left(\mathbf{N}_\Lambda = \mathbf{N}\, ; \, N_V = n\right)}{\mu_\Lambda^{\tau,\underline{\lambda}}\left(\mathbf{N}_\Lambda = \mathbf{N}\right)} \end{aligned} \tag{4.2}$$

Finally, given $\epsilon \in (0,1)$, we consider for technical reasons the "ϵ–regularization" of γ defined by

$$\tilde{\gamma}(n) := \begin{cases} \frac{1}{Z} e^{-H(n)} & \text{if } n \in I_\epsilon \equiv [N^* + \epsilon(n_{\min} - N^*), \, N^* + \epsilon(n_{\max} - N^*)] \cap I \\ \gamma(n) & \text{otherwise} \end{cases} \tag{4.3}$$

where $Z := \frac{\sum_{n\in I_\epsilon} e^{-H(n)}}{\sum_{n\in I_\epsilon}\gamma(n)}$.

The following lemma shows that the relative density between γ and its regularization is bounded uniformly in the size of I.

Lemma 4.1. *Assume property $USMT(C,m,l)$. Then there exists $C' = C'(C,m,l,\delta,\epsilon,\|\Phi\|_\infty) \geq 1$ such that*

$$(C')^{-1} \leq \inf_{n\in I}\frac{\gamma(n)}{\tilde\gamma(n)} \leq \sup_{n\in I}\frac{\gamma(n)}{\tilde\gamma(n)} \leq C'$$

Proof. By the definition (4.3) of $\tilde\gamma$ we immediately have

$$\frac{\gamma(n)}{\tilde\gamma(n)} \leq \max\left[\sup_{n,n'\in I_\epsilon}\frac{\varphi(n')}{\varphi(n)},\, 1\right]$$

Thus we must prove

$$\sup_{n,n'\in I_\epsilon}\frac{\varphi(n')}{\varphi(n)} \leq C' \tag{4.4}$$

Thanks to proposition 3.5 and by the definition of $\varphi(n)$ we have

$$\frac{\varphi(n')}{\varphi(n)} \leq C_1\,\frac{\sigma_V(n)}{\sigma_V(n')}\,\frac{\sigma_W(n)}{\sigma_W(n')}\prod_{i\neq j}\frac{\sigma_i(n)}{\sigma_i(n')}$$

for a suitable constant C_1, where $\sigma_i^2(n) := \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_i}, N_{\Lambda_i})$ and similarily for $\sigma_V^2(n)$, $\sigma_W^2(n)$.
It is one of the result of proposition 3.1 of [CM1] that, under our mixing assumption, the variance of the number of particles in each atom can be bounded from above and below by a suitable constant times its average. Therefore

$$\sup_{n,n'\in I_\epsilon}\frac{\varphi(n')}{\varphi(n)} \leq C_2\left[\sup_{n,n'\in I_\epsilon}\frac{n(N_j-n)}{n'(N_j-n')}\right]^{\frac{1}{2}} \leq C'$$

for a suitable constant C' because of the definition of I and the assumption on the ratio $\frac{|V|}{|W|}$. Similarily one proves the lower bound. $\square$

The next lemma proves that the tails of γ are at least exponential.

Lemma 4.2. *There exists a positive constant $\epsilon_0 = \epsilon_0(\delta,\|\Phi\|_\infty)$ such that $\forall\epsilon\in(\epsilon_0,1)$*

$$\frac{\gamma(n+1)}{\gamma(n)} \leq \frac{1}{2} \quad \text{if} \quad n\in[N^*+\epsilon\,(n_{\max}-N^*), n_{\max}]$$
$$\frac{\gamma(n-1)}{\gamma(n)} \leq \frac{1}{2} \quad \text{if} \quad n\in[n_{\min}, N^*+\epsilon\,(n_{\min}-N^*)]$$

Proof. Using the definition of the canonical measure and proposition 3.6 we have

$$\frac{\gamma(n+1)}{\gamma(n)} \leq e^{2\|\Phi\|_\infty}\frac{(|V|-n)(N_j-n)}{(n+1)(|W|-N_j+n+1)} \tag{4.5}$$

Thus the r.h.s. of (4.5) is smaller than $e^{2\|\Phi\|_\infty}\frac{1}{\delta}\frac{1-\epsilon}{\epsilon} \le \frac{1}{2}$ for any $n \in [N^* + \epsilon\,(n_{\max} - N^*), n_{\max}]$, provided that ϵ is sufficiently close to 1, because of our assumption on V, W and the definition of $n_{\max}$. A similar reasoning applies to $\frac{\gamma(n-1)}{\gamma(n)}$ for $n \in [n_{\min}, N^* + \epsilon\,(n_{\min} - N^*)]$ $\square$

Our last result shows that the ϵ-regularized version of γ is bell–shaped around $n = N^*$ with width proportional to $\sqrt{N^*}$.

Lemma 4.3. *Assume property $USMT(C, m, l)$. Then there exists a positive constant $C'' = C''(C, m, l, \epsilon, \|\Phi\|_\infty)$ such that*

$$
\begin{aligned}
H(n+1) - H(n) &\ge C''\,\frac{n - N^*}{N^*} \qquad \text{if} \quad n \in [N^*, N^* + \epsilon\,(n_{\max} - N^*)] \\
H(n-1) - H(n) &\ge C''\,\frac{|n - N^*|}{N^*} \qquad \text{if} \quad n \in [N^* + \epsilon\,(n_{\min} - N^*), N^*)
\end{aligned}
$$

Proof. By the definition (4.2) we have

$$
\frac{dH}{dn} = \lambda_V(n) - \lambda_W(n)
$$

so that we can write

$$
\begin{aligned}
H(n+1) - H(n) &= \int_n^{n+1} dn' \int_0^{n'} dn''\,\frac{d}{dn''}\left(\lambda_V(n'') - \lambda_W(n'')\right) \\
H(n-1) - H(n) &= \int_{n-1}^{n} dn' \int_0^{N'} dn''\,\frac{d}{dn''}\left(\lambda_V(n') - \lambda_W(n'')\right)
\end{aligned}
\tag{4.6}
$$

The integrand in (4.6) can be computed thanks to the following identities valid for any $i \neq j$ that can be easily obtained by taking the derivative w.r.t. n of both sides of (4.1)

$$
\begin{cases}
\sum_{k\neq j} \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_i}, N_{\Lambda_k})\,\frac{d}{dn}\lambda_k(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V, N_{\Lambda_i})\frac{d}{dn}\lambda_V(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_W, N_{\Lambda_i})\frac{d}{dn}\lambda_W(n) = 0 \\
\sum_{k\neq j} \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_k}, N_V)\frac{d}{dn}\lambda_k(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V, N_V)\frac{d}{dn}\lambda_V(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V, N_W)\frac{d}{dn}\lambda_W(n) = 1 \\
\sum_{k\neq j} \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_k}, N_W)\frac{d}{dn}\lambda_j(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V, N_W)\frac{d}{dn}\lambda_V(n) + \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_W, N_W)\frac{d}{dn}\lambda_W(n) = -1
\end{cases}
\tag{4.7}
$$

In order to solve the above linear system we can use the following general bounds for the covariances of the number of particles in the atoms of a partition of Λ (see proposition 3.1 of [CM1])

$$
\begin{aligned}
\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}, N_{\Lambda_i}) &\ge A^{-1}\rho_i|\Lambda_i| \\
|\mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_i}, N_{\Lambda_j}\right)| &\le A\rho_i\rho_j\left(|\partial_r^-\Lambda_i||\partial_r^-\Lambda_j|\right)^{1/2} \\
|\mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_i}, N_{\Lambda_j}\right)| &\le A|\Lambda_i|\,|\Lambda_j|e^{-md(\Lambda_i,\Lambda_j)}
\end{aligned}
\tag{4.8}
$$

where $\rho_i = \frac{\mu_\Lambda^{\tau,\lambda}(N_{\Lambda_i})}{|\Lambda_i|}$.
If we apply the above bounds in our case we get that, for L large enough , we can solve (4.7) by Neumann series and obtain

$$c_1^{-1}\frac{1}{\sigma_V^2(n)} \le \frac{d}{dn}\lambda_V(n) \le c_1\frac{1}{\sigma_V^2(n)}$$
$$c_1^{-1}\frac{1}{\sigma_W^2(n)} \le -\frac{d}{dn}\lambda_W(n) \le c_1\frac{1}{\sigma_W^2(n)}$$

where $c_1 = c_1(C,l,m,r,\|\Phi\|_\infty) > 0$ is a suitable constant.
In particular $\frac{d}{dn}(\lambda_V(n) - \lambda_W(n)) \ge c_1^{-1}\frac{1}{\sigma_V^2(n)}$. Take now $n \in [N^*, N^* + \epsilon(n_{\max} - N^*)]$. If we use once more the fact that $\sigma_V^2(n)$ is comparable to the average number of particles in V, n (see the proof of lemma 4.1), we get from (4.6), the fact that $\delta \le \frac{|V|}{|W|} \le 1-\delta$ and some simple considerations on the size of $n_{\max}$ that

$$H(n+1) - H(n) \ge c_2 \int_n^{n+1} dn' \int_0^{n'} dn'' \frac{1}{N^* + \epsilon(n_{\max} - N^*)} \ge c_3 \frac{n}{N^*}$$

A similar reasoning applies to $H(n-1) - H(n)$ when $n \in [N^* + \epsilon(n_{\min} - N^*), N^*)$. $\square$

We are finally in a position to state the main result of this section.

Theorem 4.4. *Assume property $USMT(C,m,l)$. Then there exists $c_0 = c_0(C,m,\delta,\|\Phi\|)$ such that for all $f : \Omega_\Lambda \mapsto \mathbb{R}$ that depend only on $N_V(\sigma)$ the following Poincarè inequality holds*

$$\nu_{\Lambda,\mathbf{N}}^\tau(f,f) \le c_0 N^* \sum_{n\in I} (\gamma(n) \wedge \gamma(n-1))\,[f(n) - f(n-1)]^2$$

Proof. Using lemma 4.1 it is sufficient to prove the Poincare' inequality for the regularized measure $\tilde\gamma(n)$. Pick ϵ sufficiently close to one in such a way that lemma 4.2 holds. Then, thanks to lemma 4.3 and the fact that $\frac{n_{\max}-N^*}{N^*} \le c(\delta)$, there exists positive constants c_1, c_2 depending on $C,m,\|\Phi\|,\delta$ such that

$$\frac{\tilde\gamma(n+1)}{\tilde\gamma(n)} \le \begin{cases} \exp\left(-c_1\frac{n-N^*}{N^*}\right) & \text{if } n\in I,\ n\ge N^* \text{ and } n \ne \lfloor N^* + \epsilon(n_{\max} - N^*)\rfloor \\ c_2 & \text{if } \ n = \lfloor N^* + \epsilon(n_{\max} - N^*)\rfloor \end{cases} \tag{4.9}$$

and similarly for $\frac{\tilde\gamma(n-1)}{\tilde\gamma(n)}$, $n \le 0$. We can therefore apply proposition 3.8 and get the result. $\square$

Remark. The reader may worry about the fact that (4.9) is not, strictly speaking, identical to the condition in proposition 3.8, because of the "spurious" point $n = \lfloor N^* + \epsilon(n_{\max} - N^*)\rfloor$ where the original distribution γ starts to agree with its regularization $\tilde\gamma$. However, as one can readily check, the presence of this single point only affects the final result by a multiplicative constant depending on $C,m,\|\Phi\|,\delta$.

Remark. It is interesting to notice that, in the same hypotheses of theorem 4.4 and using the discrete version of the arguments described in section 7 of [Le], one can prove [M2] a logarithmic Sobolev inequality of the form

$$\nu^{\tau}_{\Lambda,\mathbf{N}}\left(f^2\log f\right) \le c_0' N^* \sum_{n\in I} (\gamma(n)\wedge\gamma(n-1))\,[f(n)-f(n-1)]^2$$

for any non–negative function f that depends only on $N_V(\sigma)$ and such that $\nu^{\tau}_{\Lambda,\mathbf{N}}\left(f^2\right) = 1$. This observation is useful if one wants to bound from above the logarithmic Sobolev constant of Kawasaki dynamics [CM3] following the same approach described in this paper.

5. Recursive estimate of the spectral gap.

In this section we prove the main result via a recursive analysis on the behaviour of the spectral gap when the linear size of the volume under consideration is doubled.
For simplicity we carry out our analysis in two dimensions but the extension to higher dimension is straightforward.
Let $R(l_1,l_2)$ denote the rectangle

$$R(l_1,l_2) = \left\{(x_1,x_2)\in\mathbb{Z}^2;\ 0\le x_1\le l_1-1,\ 0\le x_2\le l_2-1\ \right\};\qquad l_1,l_2\in\mathbb{Z}_+$$

and let $\mathcal{R}_L$ be the family of "fat" rectangles with "size" smaller than L, namely those rectangles $R(l_1,l_2,x)\equiv R(l_1,l_2)+x$, $x\in\mathbb{Z}^2$, with $l_1\wedge l_2\ge 0.1\,(l_1\vee l_2)$ and $l_1\vee l_2\le L$.
Let also

$$g(L) = \min_{R\in\mathcal{R}_L}\min_{N,\tau}\mathrm{gap}(L^{\tau}_{R,N})$$

where $\mathrm{gap}(L^{\tau}_{R,N})$ has been defined in (2.11).
With the above notation we will prove the following recursive bound.

Theorem 5.1. *Assume $USMT(C,m,l)$. Then there exists a positive constant $k=k(d,r,\|\Phi\|)$ such that*

$$g(2L)^{-1}\le\frac{3}{2}\,g(L)^{-1}+kL^2$$

for all L large enough. In particular $\inf_L g(L)\,L^2>0$.

Proof. Let us consider a rectangle $\Lambda\equiv R(l_1,l_2)\in\mathcal{R}_{2L}$ and assume, without loss of generality, that the longest side is l_2. If $l_2\le L$ then, using the definition of $g(L)$, $\min_{N,\tau}\mathrm{gap}(L^{\tau}_{R,N})\ge g(L)$. Thus we assume $L<l_2\le 2L$.
We fix a small number $\delta\in(0,10^{-2})$ and we set $d=\lfloor\delta L\rfloor$. Given an integer $j\in[1,\lfloor\frac{1}{10\delta}\rfloor]$ we partition Λ into three atoms $\{\Lambda_i\}_{i=1}^3$ as follows (we omit the index j for simplicity)

$$\begin{aligned}\Lambda_1 &= \{x\in\Lambda;\ 0\le x_2\le l_2/2+jd\}\\ \Lambda_2 &= \{x\in\Lambda;\ l_2/2+(j-1)d<x_2\le l_2-1\}\\ \Lambda_3 &= \Lambda_1\cap\Lambda_2\end{aligned}\tag{5.1}$$

Fix now a boundary condition τ outside Λ and a number of particles $N \in [0, \dots, |\Lambda|]$. We will then use twice the formula relating the variance of a function f w.r.t. the measure $\nu^\tau_{\Lambda,N}$ to the variance of f w.r.t. the measure $\nu^\tau_{\Lambda,N}$ conditioned to a sub σ–algebra $\mathcal{F}_0$

$$\nu^\tau_{\Lambda,N}(f,f) = \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \mid \mathcal{F}_0)\right) + \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f \mid \mathcal{F}_0), \nu^\tau_{\Lambda,N}(f \mid \mathcal{F}_0)\right) \tag{5.2}$$

to write

$$\begin{aligned} \nu^\tau_{\Lambda,N}(f,f) &= \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \mid \mathcal{F}_1)\right) + \nu^\tau_{\Lambda,N}(f_1, f_1) \\ &= \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \mid \mathcal{F}_{1,3})\right) + \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f_{1,3}, f_{1,3} \mid \mathcal{F}_1)\right) + \nu^\tau_{\Lambda,N}(f_1, f_1) \end{aligned} \tag{5.3}$$

where $\mathcal{F}_1$ and $\mathcal{F}_{1,3}$ are the σ–algebras generated by N_{Λ_1} and $\{N_{\Lambda_1}, N_{\Lambda_3}\}$ respectively, and $f_1 := \nu^\tau_{\Lambda,N}(f \mid \mathcal{F}_1)$, $f_{1,3} := \nu^\tau_{\Lambda,N}(f \mid \mathcal{F}_{1,3})$. Formula (5.3) will represent our basic starting point. We will now examine separately each term in the r.h.s. of (5.3).

5.1 Analysis of the first term in the r.h.s. of (5.3).

For any small ϵ and large enough L, the first term in the r.h.s. of (5.3) can be bounded from above using proposition 3.4 by

$$\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \mid \mathcal{F}_{1,3})\right) \le (1+\epsilon)\nu^\tau_{\Lambda,N}\left(\nu^\eta_{\Lambda_1,N_1}(f,f) + \nu^\eta_{\Lambda_2,N_2}(f,f)\right) \tag{5.4}$$

where the average is over the random variables η, N_{Λ_1} and N_{Λ_2}. Above we have used the bound $\nu^\tau_{\Lambda,N}\left(\nu^\eta_{\Lambda_1,N_1}(f,f \mid N_{\Lambda_3})\right) \le \nu^\tau_{\Lambda,N}\left(\nu^\eta_{\Lambda_1,N_1}(f,f)\right)$ which follows at once from (5.2).
Let us now examine the spectral gap of the bottom rectangle Λ_1, the reasoning being similar for the top one.
There are two cases to analyze:

a) $l_1 \le \frac{3}{2}L$. In this case one easily verifies that $\Lambda_1 \in \mathcal{R}_{\frac{3}{2}L}$.
b) $l_1 > \frac{3}{2}L$. In this case $\Lambda_1 \in \mathcal{R}_{2L}$ but now the *longest* side is l_1 and the *shortest* one is smaller than $\frac{l_2}{2} + jd + 1$ which in turn is smaller than $1.2L$ since $l_2 \le 2L$ and $jd \le \frac{L}{10}$

Therefore $\min_{\eta,N_1} \text{gap}(L^\eta_{\Lambda_1,N_1}) \ge \min\{g(\frac{3}{2}L), \hat{g}(2L)\}$ where

$$\hat{g}(2L) = \min_{\substack{R \in \mathcal{R}_{2L} \\ l_1 \le 1.2L, \\ l_2 \ge \frac{3}{2}L}} \min_{\tau,N} \text{gap}(L^\tau_{R(l_1,l_2),N})$$

In conclusion we obtain that the r.h.s. of (5.4) is smaller than

$$(1+\epsilon)\max\{\, g(\frac{3}{2}L)^{-1}, \hat{g}(2L)^{-1}\,\}\left[\mathcal{E}^\tau_{\Lambda,N}(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^\tau_{\Lambda,N}\left[c_{xy}\left(\nabla_{xy}f\right)^2\right]\right] \tag{5.5}$$

uniformly in $j \in [1, \frac{1}{10\delta}]$. Notice that the "spurious" term $\frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^\tau_{\Lambda,N}\left[c_{xy}\left(\nabla_{xy}f\right)^2\right]$ comes from the fact that Λ_1 and Λ_2 overlap.

5.2 Analysis of the second and third term in the r.h.s. of (5.3).

Here we bound from above the second and third term in (5.3). The necessary steps are identical in both cases and therefore, for shortness, we treat only the third term which is (notationally speaking) also the simplest. Later on we will state without further comments the analogous result for the second one.

Let $\rho := \frac{N}{|\Lambda|}$ and assume, without loss of generality, that $\rho \leq \frac{1}{2}$. Let also $N_1^* = \mu_\Lambda^{\tau,\lambda}(N_{\Lambda_1})$, $\mu_\Lambda^{\tau,\lambda}$ being the grand canonical ensemble with average particle number equal to N, and let $\gamma(n) := \nu_{\Lambda,N}^\tau(N_{\Lambda_1} = n)$. Then, using theorem 4.4, we can write

$$\begin{aligned} \nu_{\Lambda,N}^\tau(f_1, f_1) &\leq c_0 N_1^* \sum_n (\gamma(n) \wedge \gamma(n-1)) \left[\nu_{\Lambda,N}^\tau(f \mid N_{\Lambda_1} = n) - \nu_{\Lambda,N}^\tau(f \mid N_{\Lambda_1} = n-1)\right]^2 \\ &\leq c_0 N_1^* \sum_n (\gamma(n) \wedge \gamma(n-1)) \left[A(n)^2 + B(n)^2\right] \end{aligned} \tag{5.6}$$

where $A(n)$ and $B(n)$ have been defined in (3.15).
In order to estimate the first term in the r.h.s of (5.6) we first need to fix a *path choice* (see proposition 3.13 and definition 3.12 for the necessary notation).
Our choice is the following. Given $x \in \Lambda_1$ and $z \in \Lambda_2 \setminus \Lambda_1$, start increasing (or decreasing) the first coordinate of x until it is equal to the first coordinate of z. Then adjust the second coordinate until you get to z. With this particular path choice it is easy to see that $\mathcal{G}_V(\lambda) \leq (2L)^3$. Assume in fact that the path λ_{xz} contains the edge $e = [u, v]$ where u and v differ in the j^{th} coordinate. This means that $x_i = u_i$ for all $i > j$ and $z_i = u_i$ for all $i < j$. So the number of possible pairs (x, z) is not greater than $(2L)^j\,(2L)^{2-j+1} = (2L)^3$. Moreover with this geometry $\mathcal{D}_V(\lambda) \leq 4L$.
Thanks to proposition 3.13 first term in the r.h.s. of (5.6) is smaller than

$$C N_1^* \frac{L^4}{\rho(1-\rho)|\Lambda_1||\Lambda_2|}\, \mathcal{E}_{\Lambda,N}^\tau(f, f) \tag{5.7}$$

Notice that $N_1^* \leq C'\rho|\Lambda_1|$ (see proposition 3.1 of [CM1]) so that

$$C N_1^* \frac{L^4}{\rho(1-\rho)|\Lambda_1||\Lambda_2|}\, \mathcal{E}_{\Lambda,N}^\tau(f, f) \leq C' L^2 \mathcal{E}_{\Lambda,N}^\tau(f, f) \tag{5.8}$$

We now turn to the estimate of the second term in the r.h.s. of (5.6). Using proposition 3.14 we get that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that the second term in the r.h.s. of (5.6) is smaller than

$$C(\epsilon)\, \mathcal{E}_{\Lambda,N}^\tau(f, f) + \epsilon\, \nu_{\Lambda,N}^\tau(f, f) \tag{5.9}$$

In conclusion, for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that the third term in the r.h.s. of (5.3) is smaller than

$$\nu_{\Lambda,N}^\tau(f_1, f_1) \leq \left(C'' L^2 + C(\epsilon)\right)\, \mathcal{E}_{\Lambda,N}^\tau(f, f) + \epsilon\, \nu_{\Lambda,N}^\tau(f, f) \tag{5.10}$$

for a suitable constant C''. A similar bound holds for the second term in the r.h.s. of (5.3).

5.3 THE RECURSION COMPLETED.

We are finally in a position to complete the proof of theorem 5.1. If we put together (5.10) and (5.5) we get that, for any $\epsilon \in (0, \frac{1}{2})$

$$\begin{aligned}\text{r.h.s. of (5.3)} &\le (1+\epsilon)\max\{\, g(\frac{3}{2}L)^{-1},\, \hat g(2L)^{-1}\,\}\Big[\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^{\tau}_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right]\Big] \\ &\quad + 2\left(C''L^2 + C(\epsilon)\right)\,\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + 2\epsilon\,\nu^{\tau}_{\Lambda,N}\,(f,f)\end{aligned} \tag{5.11}$$

that is

$$\begin{aligned}\nu^{\tau}_{\Lambda,N}\,(f,f) \le \left(\frac{1+\epsilon}{1-2\epsilon}\right)\,\max\Big\{\, g(\frac{3}{2}L)^{-1},\, \hat g(2L)^{-1}\,\Big\}\Big[\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^{\tau}_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right]\Big] \\ + kL^2\,\mathcal{E}^{\tau}_{\Lambda,N}(f,f)\end{aligned} \tag{5.12}$$

for a suitable constant k.
Finally we average w.r.t. to the index j (see (5.1)) and use the observation that, as j varies in $[1, \frac{1}{10\delta}]$, the strips $\Lambda_3 := \Lambda_3^{(j)}$ are disjoint. In particular

$$\frac{1}{2}\sum_{j\in[1,\frac{1}{10\delta}]}\ \sum_{[x,y]\in\mathcal{E}_{\Lambda_3^{(j)}}} \nu^{\tau}_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right] \le \mathcal{E}^{\tau}_{\Lambda,N}(f,f)$$

so that

$$\begin{aligned}\nu^{\tau}_{\Lambda,N}\,(f,f) \le \\ \left(\frac{1+\epsilon}{1-2\epsilon}\right)(1+\lfloor 10\delta\rfloor)\max\Big\{\, g(\frac{3}{2}L)^{-1},\, \hat g(2L)^{-1}\,\Big\}\,\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + kL^2\,\mathcal{E}^{\tau}_{\Lambda,N}(f,f)\end{aligned} \tag{5.13}$$

In other words

$$\text{gap}(L^{\tau}_{\Lambda,N})^{-1} \le \left(\frac{1+\epsilon}{1-2\epsilon}\right)(1+\lfloor 10\delta\rfloor)\max\Big\{\, g(\frac{3}{2}L)^{-1},\, \hat g(2L)^{-1}\,\Big\} + kL^2 \tag{5.14}$$

Notice that if the original rectangle Λ was such that $l_1 \le 1.2L$ while $l_2 \ge \frac{3}{2}L$, *i.e.* Λ was chosen in the sub–class of $\mathcal{R}_{2L}$ entering in the definition of $\hat g(2L)$, then we would have obtained the inequality (5.14) with the factor $\max\{\, g(\frac{3}{2}L)^{-1},\, \hat g(2L)^{-1}\,\}$ replaced by $g(\frac{3}{2}L)^{-1}$ simply because case *b)* right after (5.4) would have been impossible. Thus

$$\hat g(2L)^{-1} \le \left(\frac{1+\epsilon}{1-2\epsilon}\right)(1+\lfloor 10\delta\rfloor)\, g(\frac{3}{2}L)^{-1} + kL^2 \tag{5.15}$$

If we combine (5.14) with (5.15) we finally get

$$\operatorname{gap}(L^{\tau}_{\Lambda,N})^{-1} \leq \left(\frac{1+\epsilon}{1-2\epsilon}\right)^2 (1+\lfloor 10\delta\rfloor)^2 \, g(\frac{3}{2}L)^{-1} + k' L^2 \tag{5.16}$$

for another constant k'. Thus

$$g(2L)^{-1} \leq \left(\frac{1+\epsilon}{1-2\epsilon}\right)^2 (1+\lfloor 10\delta\rfloor)^2 \, g(\frac{3}{2}L)^{-1} + k' L^2$$

and two more iterations prove the recursive inequality of the theorem provided that the two parameters ϵ, δ were chosen small enough.
Finally the fact that $\min_L \left(g(L)\, L^2\right) > 0$ is a trivial consequence of the recursive bound. $\square$

6. Proof of Theorem 2.4.

Fix a local function f with $0 \in \Delta_f$ and denote by E_λ the spectral projection associated to the interval $[0,\lambda]$ for the selfadjoint operator $-L^{\tau}_{\Lambda,N}$ on $L^2\left(\Omega_\Lambda, d\nu^{\tau}_{\Lambda,N}\right)$. Assume that $\nu^{\tau}_{\Lambda,N}(f) = 0$. Then we will prove that for any $\epsilon \in (0,1)$ there exists a constant $C_{f,\epsilon}$ independent of Λ and N such that

$$\|E_\lambda f\|_2^2 \leq C_{f,\epsilon} \lambda^{\alpha-\epsilon} \tag{6.1}$$

where $\|\cdot\|_2$ denote the $L^2\left(\Omega_\Lambda, d\nu^{\tau}_{\Lambda,N}\right)$–norm and $\alpha = \alpha(d)$ is as in the theorem. It is clear that once such an estimate is available then the result follows by the simple formula

$$\begin{aligned}
\|e^{tL^{\tau}_{\Lambda,N}} f\|_2^2 &\leq \sum_{j=0}^{\infty} e^{-j} \|E_{\frac{j+1}{t}} f - E_{\frac{j}{t}} f\|_2^2 \\
&\leq C_{f,\epsilon} \frac{1}{t^{\alpha-\epsilon}} \sum_{j=0}^{\infty} e^{-j} (j+1)^{\alpha-\epsilon} \\
&\leq C'_{f,\epsilon} \frac{1}{t^{\alpha-\epsilon}}
\end{aligned}$$

Let us prove (6.1). For any integer $L' \leq \frac{1}{2}L$ multiple of l let $\mathcal{F}_{L'} := \mathcal{F}_{\Lambda\setminus B_{L'}}$ and let $f_{L'} := \nu^{\tau}_{\Lambda,N}(f \,|\, \mathcal{F}_{L'})$. If we use proposition 3.3 together with the Glauber spectral gap inequality for the grand canonical Gibbs measure $\mu^{\tau,\lambda}_\Lambda$ (the chemical potential λ here is, as always, such that $\mu^{\tau,\lambda}_\Lambda(N_\Lambda) = N$),

$$\mu^{\tau,\lambda}_\Lambda(g,g) \leq C_1 \, \mu^{\tau,\lambda}_\Lambda\Big(\sum_{x\in\Delta_g} [\nabla_x g]^2\Big)$$

for any g and a suitable constant C_1 depending on $C, m, l, \|\Phi\|$ (see [M1]), we get

$$\operatorname{Var}^{\tau}_{\Lambda,N}(f_{L'}) \leq C_1 \, \mu^{\tau,\lambda}_\Lambda\Big(\sum_{x\in B_{L'}} [\nabla_x f_{L'}]^2\Big) \tag{6.2}$$

It is not difficult to check at this point, using the hypothesis $USMT(C,m,l)$ together with the results on the equivalence of ensembles of [CM1] and lemma 3.1, that uniformly in L, the r.h.s. of (6.2) is bounded from above by $C_f \frac{1}{(L')^d}$ for some constant C_f.
Observe now that, for any function g and for any integer $l_0 \le \frac{1}{2}L$ multiple of the basic length scale l, the formula of the conditional covariance (see e.g. (5.2)) together with the definition of spectral gap and the result of theorem 2.3 give the following inequality

$$\begin{aligned}\nu^\tau_{\Lambda,N}(g,f)^2 &\le 2\,\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}\left(g,f\,|\,\mathcal{F}_{l_0}\right)^2\right) + 2\ \mathrm{Var}^\tau_{\Lambda,N}(g)\ \mathrm{Var}^\tau_{\Lambda,N}\left(f_{l_0}\right)\\ &\le 2\,\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}\left(g,f\,|\,\mathcal{F}_{l_0}\right)^2\right) + 2C_f\,\frac{1}{l_0^d}\ \mathrm{Var}^\tau_{\Lambda,N}(g)\\ &\le C'_f\ \left[l_0^2\,\mathcal{E}^\tau_{\Lambda,N}(g,g) + \frac{1}{l_0^d}\ \mathrm{Var}^\tau_{\Lambda,N}(g)\right]\end{aligned} \tag{6.3}$$

Notice that, if we take $g := E_\lambda f$, then (6.3) gives $\|E_\lambda f\|_2^2 \le C'_f\ \left[l_0^2\,\lambda + \frac{1}{l_0^d}\right]$ which, if we optimize over the free parameter l_0, becomes $\|E_\lambda f\|_2^2 \le C''_f \lambda^{\frac{d}{d+2}}$.
It is important to observe that the optimal choice is $l_0 = O\left(\lambda^{-\frac{1}{d+2}}\right) \ll L$ since $\lambda \ge CL^{-2}$ because of theorem 2.3.
We will now use (6.3) as the starting point of a recursive procedure whose final result will be a bound like (6.3) but with the factor l_0^2 replaced by l_0^γ with $\gamma \le \epsilon$ if $d \ge 2$ and $\gamma \le 1+\epsilon$ in $d = 1$. Clearly such a bound will suffice to prove (6.1) because of the little argument given above.

Lemma 6.1. *Let $\beta_d = 0$ if $d \ge 2$ and $\beta_d = 1$ if $d = 1$. In the same hypotheses of theorem 2.4 assume that for some $\beta \in [\beta_d, 2)$, some constant $C(f,\beta)$, all pairs $l_1 \le \frac{1}{2}\,l_2$ multiples of l and all N the following inequality holds*

$$\nu^\tau_{B_{l_2},N}(g,f)^2 \le C(f,\beta)\ \left[l_1^\beta\,\mathcal{E}^\tau_{B_{l_2},N}(g,g) + \frac{1}{l_1^d}\ \mathrm{Var}^\tau_{B_{l_2},N}(g)\right] \tag{6.4}$$

Then there exists a new constant $C'(f,\beta)$ such that the same inequality holds with β replaced by $\beta' = \frac{2\beta}{d+\beta}$.

Proof. Pick $l_1 \le \frac{1}{2}\,l_2$ and apply (6.3) to $\Lambda := B_{l_2}$ and $l_0 := l_1$ to get

$$\nu^\tau_{\Lambda,N}(g,f)^2 \le 2\,\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}\left(g,f\,|\,\mathcal{F}_{l_1}\right)^2\right) + 2C_f\,\frac{1}{l_1^d}\ \mathrm{Var}^\tau_{\Lambda,N}(g) \tag{6.5}$$

For any $l_3 \le \frac{1}{2}\,l_1$ multiple of l we can bound the first term in the r.h.s. of (6.5) by

$$C(f,\beta)\ \nu^\tau_{\Lambda,N}\left(\left[l_3^\beta\,\mathcal{E}^\tau_{\Lambda,N}\left(g,g\,|\,\mathcal{F}_{l_1}\right) + \frac{1}{l_3^d}\ \mathrm{Var}^\tau_{\Lambda,N}\left(g\,|\,\mathcal{F}_{l_1}\right)\right]\right) \le C_1(f,\beta)\ \left[l_3^\beta + \frac{l_1^2}{l_3^d}\right]\,\mathcal{E}^\tau_{\Lambda,N}(g,g) \tag{6.6}$$

for a suitable constant $C_1(f,\beta)$, where we have used theorem 2.3 to bound $\mathrm{Var}^\tau_{\Lambda,N}\left(g\,|\,\mathcal{F}_{l_1}\right)$ in terms of the Dirichlet form $\mathcal{E}^\tau_{\Lambda,N}\left(g,g\,|\,\mathcal{F}_{l_1}\right)$.

We now chose $l_3 = \frac{1}{2}\,l_1^{\frac{2}{\beta+d}} \le \frac{1}{2}\,l_1$ and get that the r.h.s. of (6.6) is bounded from above by

$$C'(f,\beta)\,l_1^{\frac{2\beta}{d+\beta}}\,\mathcal{E}^\tau_{\Lambda,N}(g,g) = C'(f,\beta)\,l_1^{\beta'}\,\mathcal{E}^\tau_{\Lambda,N}(g,g)$$

and the result follows. $\square$

Notice that the inequality of the lemma holds for $\beta = 2$ because of (6.3). Moreover it is simple to convince oneself that the sequence $\beta_{n+1} = \frac{2\beta_n}{d+\beta_n}$, $\beta_0 = 2$ converges to β_d from above. Therefore, for any $\epsilon \in (0,1)$ there exists a constant $C_{f,\epsilon}$ such that, for all pairs $l_1 \le \frac{1}{2} l_2$ multiples of l and all N

$$\nu^\tau_{B_{l_2},N}(g,f)^2 \le C_{f,\epsilon} \begin{cases} l_1^{1+\epsilon}\, \mathcal{E}^\tau_{B_{l_2},N}(g,g) + \frac{1}{l_1}\, \mathrm{Var}^\tau_{B_{l_2},N}(g) & \text{if } d = 1 \\ l_1^{\epsilon}\, \mathcal{E}^\tau_{B_{l_2},N}(g,g) + \frac{1}{l_1^d}\, \mathrm{Var}^\tau_{B_{l_2},N}(g) & \text{if } d \ge 2 \end{cases} \tag{6.7}$$

The bound (6.1) on $\|E_l f\|_2^2$ then follows and, as a consequence, the statement of the theorem. $\square$

Appendix. Proof of Lu–Yau's two block estimate.

Here we reprove, for completeness, lemma 4.4 of [LY] which played a key role in the estimates in §3.9. Our proof is close to the original one with some simplification due to the fact that we have at our disposal proposition 3.3.

The setting is the following. We fix $\Lambda = Q_L$, L a multiple of the basic length scale l, an integer $N \in [0, \dots, |\Lambda|]$ and a local function g such that $0 \in \Delta_g$ and $\mathrm{diam}(\Delta_g) \le 2r$. As in lemma 3.9 we let $G = \frac{1}{|\Lambda|}\sum_{x\in\Lambda} g_x$ where g_x is the translate of g by $x \in \Lambda$. Let also $\rho := \frac{N}{|\Lambda|}$.

Proposition A.1. *Assume property $USMT(C,m,l)$. Then for any $\epsilon > 0$ there exists C_ϵ such that for any f*

$$\nu^\tau_{\Lambda,N}\big(f,G\big)^2 \le \frac{C_\epsilon}{|\Lambda|}\, \mathcal{E}^\tau_{\Lambda,N}(f,f) + \frac{\epsilon}{|\Lambda|}\, \mathrm{Var}^\tau_{\Lambda,N}(f)$$

provided that $|\Lambda|$ is larger than C_ϵ.

Proof. Fix $\epsilon > 0$. If $\rho \ll \epsilon$ or $1-\rho \ll \epsilon$ the statement follows at once from the Schwartz inequality together with part 1) of proposition 3.2. We will thus assume, without further notice, that $\rho \in (\epsilon, 1-\epsilon)$.

We define $\{C_\alpha\}_{\alpha\in I}$ to be a collection of cubes of side l_0 multiple of l such that for any $\alpha \ne \beta$ $\mathrm{dist}(C_\alpha, C_\beta) \ge l_0^{\frac{1}{2}}$, $\mathrm{dist}(C_\alpha, \partial\Lambda) \ge L^{\frac{1}{2}}$ and $|\Lambda \setminus \cup_\alpha C_\alpha| \le l_0^{-\frac{1}{2}}|\Lambda|$. Clearly such a construction is always possible.

Next we observe that, without loss of generality, we can replace g_x by $g_x - \delta\sigma(x)$, δ being an arbitrary constant independent of x, because $\sum_{x\in\Lambda}\sigma(x) = N$ almost surely w.r.t. $\nu^\tau_{\Lambda,N}$. Accordingly we define $G_\delta := G - \delta\, N_\Lambda$. Our choice of δ will be made later but we anticipate that it will be almost independent of l_0. We then set

$$G^{\mathrm{int}}_\delta := \frac{1}{|\Lambda|}\sum_{x\in\cup_\alpha C^{\mathrm{int}}_\alpha} \big(g_x - \delta\sigma(x)\big) \quad \text{and} \quad G^{\mathrm{ext}}_\delta := G_\delta - G^{\mathrm{in}}_\delta$$

where $C^{\mathrm{int}}_\alpha = \{x \in C_\alpha \,:\, d(x, C^c_\alpha) \ge l_0^{1/4}\}$. Notice that

$$\mathrm{Var}^\tau_{\Lambda,N}(G^{\mathrm{ext}}_\delta) \le C'\mu^{\tau,\lambda}_\Lambda(G^{\mathrm{ext}}_\delta, G^{\mathrm{ext}}_\delta) \le C''\frac{|\Lambda\setminus\cup_\alpha C_\alpha| + \sum_\alpha |C_\alpha \setminus C^{\mathrm{int}}_\alpha|}{|\Lambda|^2} \le C'' l_0^{-\frac{1}{2}}\frac{1}{|\Lambda|} \tag{A.1}$$

because of proposition 3.3, the mixing condition $USMT$ and the definition of $\{C_\alpha\}_{\alpha\in I}$. In particular, for any given $\epsilon > 0$,

$$\nu^\tau_{\Lambda,N}\big(f, G^{\rm ext}_\delta\big)^2 \le \frac{\epsilon}{|\Lambda|}\,\mathrm{Var}^\tau_{\Lambda,N}(f)$$

provided that $l_0 \gg \epsilon^{-2}$.

We now turn to bound the relevant part $\nu^\tau_{\Lambda,N}\big(f, G^{\rm int}_\delta\big)^2$. Let $\mathcal{F}_0$ be the σ–algebra generated by the the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus\cup_\alpha C_\alpha}$, $\{N_\alpha\}_{\alpha\in I}$, where $N_\alpha(\sigma) := \sum_{x\in C_\alpha}\sigma(x)$. Then we write

$$\begin{aligned}
\nu^\tau_{\Lambda,N}\big(f, G^{\rm int}_\delta\big)^2 &\le 2\,\nu^\tau_{\Lambda,N}\Big(\nu^\tau_{\Lambda,N}\big(f, G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\Big)^2 + 2\,\nu^\tau_{\Lambda,N}\Big(f, \nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\Big)^2 \\
&\le 2\ \mathrm{Var}^\tau_{\Lambda,N}(G^{\rm int}_\delta)\nu^\tau_{\Lambda,N}\Big(\mathrm{Var}^\tau_{\Lambda,N}\big(f \,|\, \mathcal{F}_0\big)\Big) + 2\,\nu^\tau_{\Lambda,N}\Big(f, \nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\Big)^2 \qquad (A.2) \\
&\le \frac{C(l_0)}{|\Lambda|}\mathcal{E}^\tau_{\Lambda,N}(f,f) + 2\ \mathrm{Var}^\tau_{\Lambda,N}(f)\ \mathrm{Var}^\tau_{\Lambda,N}\ \big(\nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\big)
\end{aligned}$$

where we have used (A.1) to bound $\mathrm{Var}^\tau_{\Lambda,N}(G^{\rm int}_\delta)$ by $\frac{C'}{|\Lambda|}$ and the estimate

$$\nu^\tau_{\Lambda,N}\Big(\ \mathrm{Var}^\tau_{\Lambda,N}(f \,|\, \mathcal{F}_0)\Big) \le C(l_0)\,\mathcal{E}^\tau_{\Lambda,N}(f,f)$$

for some constant $C(l_0)$, valid since $\nu^\tau_{\Lambda,N}(\cdot \,|\, \mathcal{F}_0)$ is the product of canonical Gibbs measures over the cubes C_α. Actually, using theorem 2.1 of [CCM], the constant $C(l_0)$ is not larger than $\exp(c\,l_0^{d-1})$.

The key point is now to prove that, for any $\epsilon > 0$, $\mathrm{Var}^\tau_{\Lambda,N}\ \big(\nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\big)$ is smaller than $\frac{\epsilon}{|\Lambda|}$ provided that l_0 is large enough.

Notice that $\nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)(\eta)$ is the sum of local functions

$$\nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)(\eta) = \frac{1}{|\Lambda|}\sum_{\alpha\in I}\nu^\eta_{C_\alpha,N_\alpha(\eta)}\Big(\sum_{x\in C^{\rm int}_\alpha} g_x - \delta\sigma(x)\Big) \equiv \frac{1}{|\Lambda|}\sum_{\alpha\in I} G^\delta_\alpha(\eta)$$

Thus, if we order in an arbitrary way the set I, we can split the above sum into the sum of even and odd α's and apply proposition 3.3 to each term and get

$$\mathrm{Var}^\tau_{\Lambda,N}\ \big(\nu^\tau_{\Lambda,N}\big(G^{\rm int}_\delta \,|\, \mathcal{F}_0\big)\big) \le C'\frac{1}{|\Lambda|^2}\,\mu^{\tau,\lambda}_\Lambda\left(\sum_\alpha G^\delta_\alpha, \sum_\alpha G^\delta_\alpha\right)$$

for some constant C' independent of Λ and l_0.

Let now $\xi^\delta_\alpha(\eta) := \mu^{\eta,\lambda(\eta)}_{C_\alpha}\Big(\sum_{x\in C^{\rm int}_\alpha}[g_x - \delta\sigma(x)]\Big)$, where the chemical potential $\lambda(\eta)$ is such that $\mu^{\eta,\lambda(\eta)}_{C_\alpha}(N_\alpha) = N_\alpha(\eta)$. In the (rare) case in which $N_\alpha(\eta) = 0$ ($N_\alpha(\eta) = l_0^d$) the measure $\mu^{\eta,\lambda(\eta)}_{C_\alpha}$ will simply be the Dirac measure on the constant configuration identically equal to 0 (1). Thanks to *1)* of proposition 3.2 we have $\sup_\eta |G^\delta_\alpha(\eta) - \xi^\delta_\alpha(\eta)| \le C'$.

In particular, using to the mixing condition together with the fact that $\mathrm{dist}(C_\alpha, C_\beta) \ge l_0^{\frac{1}{2}}$ for any $\alpha \neq \beta$, we get

$$\frac{1}{|\Lambda|^2}\mu_\Lambda^{\tau,\lambda}\Big(\sum_\alpha G_\alpha^\delta - \xi_\alpha^\delta, \sum_\alpha G_\alpha^\delta - \xi_\alpha^\delta\Big) \le \frac{C}{|\Lambda|^2}\sum_\alpha \mu_\Lambda^{\tau,\lambda}\left(G_\alpha^\delta - \xi_\alpha^\delta, G_\alpha^\delta - \xi_\alpha^\delta\right)$$
$$\le C'\frac{1}{|\Lambda| l_0^d} \le \frac{\epsilon}{|\Lambda|}$$

for l_0 large enough. It is therefore enough to bound $\mu_\Lambda^{\tau,\lambda}\Big(\sum_\alpha \xi_\alpha^\delta, \sum_\alpha \xi_\alpha^\delta\Big)$ (this is the second point where our version of the two blocks estimate differs from the original one).

If we use the Poincarè inequality $\mu_\Lambda^{\tau,\lambda}(f,f) \le C'\mu_\Lambda^{\tau,\lambda}\left(\sum_{x\in\Lambda}(\nabla_x f)^2\right)$ valid because of the mixing condition $USMT(C,m,l)$ (see e.g. [M1]) we get

$$\mu_\Lambda^{\tau,\lambda}\Big(\sum_\alpha \xi_\alpha^\delta, \sum_\alpha \xi_\alpha^\delta\Big) \le \mu_\Lambda^{\tau,\lambda}\Big(\sum_{y\in\Lambda}\big[\nabla_y \sum_{\alpha\in I}\xi_\alpha^\delta\big]^2\Big) \tag{A.3}$$

Observe now that, by construction, $\nabla_y \xi_\alpha^\delta = 0$ unless $\mathrm{dist}(y, C_\alpha) \le r$. Thus

$$\mu_\Lambda^{\tau,\lambda}\Big(\sum_{y\in\Lambda}\big[\nabla_y \sum_{\alpha\in I}\xi_\alpha^\delta\big]^2\Big) = \sum_{\alpha\in I}\sum_{\substack{y\in\Lambda\\ \mathrm{dist}(y,C_\alpha)\le r}} \mu_\Lambda^{\tau,\lambda}\Big(\big[\nabla_y \xi_\alpha^\delta\big]^2\Big) \tag{A.4}$$

Let us estimate a generic term $\mu_\Lambda^{\tau,\lambda}\left(\big[\nabla_y \xi_\alpha^\delta\big]^2\right)$. It is at this stage that the subtraction with the free parameter δ made at the beginning becomes important. It is in fact clear that without such a subtraction generically one expects $\Big[\nabla_y \xi_\alpha^\delta\Big]^2 = O(1)$ (take for instance the trivial case $g_x(\sigma) = \sigma(x)$) which would imply that the r.h.s. of (A.3) is of the order of $|\Lambda|$. The way out in order to gain a factor of ϵ is to choose δ in such a way that $\mu_\Lambda^{\tau,\lambda}\left(\big[\nabla_y \xi_\alpha^\delta\big]^2\right)$ becomes itself a variance which again can be bounded using the Poincarè inequality above (see [LY]).

Let $\tilde\nabla_y f(\sigma) = (1-\sigma(y))\nabla_y f - \sigma(y)\nabla_y f$ and notice that

$$\big[\nabla_y \xi_\alpha^\delta\big]^2 = \big[\tilde\nabla_y \xi_\alpha^\delta\big]^2 = \big[\tilde\nabla_y \mu_{C_\alpha}^{\eta,\lambda(\eta)}\big(\sum_{x\in C_\alpha^{\rm int}} g_x\big) - \delta\big]^2$$

Lemma A.2. *Assume $USMT(C,m,l)$. Let C_0 be the cube of side l_0 centered at y^*, the center of Λ, and define*

$$\delta^* := \mu_\Lambda^{\tau,\lambda}\Big(\tilde\nabla_{y^*}\mu_{C_0}^{\eta,\lambda(\eta)}\big(\sum_{x\in C_0^{\rm int}} g_x\big)\Big)$$

Then there exists a constants k_1, k_2 independent of L and l_0, and $\alpha > 0$ such that $\delta^* \le k_1$ and

$$\mu_\Lambda^{\tau,\lambda}\left(\big[\nabla_y \xi_\alpha^{\delta^*}\big]^2\right) \le \begin{cases} k_2 & \text{if } y \in \partial_r^+ C_\alpha \\ \frac{k_2}{l_0^d} & \text{if } y \in C_\alpha \end{cases}$$

Proof. The fact that δ^* is bounded from above uniformly in L, l_0 as well as the result for $y \in \partial_r^+ C_\alpha$ follow immediately from 1) of lemma 3.1.

Let us now consider $y \in C_\alpha$.
In this case, under the flip of the variable $\eta(y)$, the value of $\xi_\alpha^{\delta^*}$ changes only because the number of particles of η varies by ± 1. Notice that

$$| \, \mu_\Lambda^{\tau,\lambda}\Big(\tilde\nabla_y \, \mu_{C_\alpha}^{\eta,\lambda(\eta)} \big(\sum_{x \in C_\alpha^{\rm int}} g_x \big) \Big) - \delta^* | \le k_6 e^{-m\sqrt{L}}$$

because of translation invariance, the mixing condition and the assumption $\mathrm{dist}(C_\alpha, \partial^+\Lambda) \ge L^{\frac12}$. Thus

$$\begin{aligned} \mu_\Lambda^{\tau,\lambda}\left([\, \nabla_y \, \xi_\alpha^{\delta^*} \,]^2 \right) &\le \mu_\Lambda^{\tau,\lambda}\left(\tilde\nabla_y \, \xi_\alpha^{\delta^*}, \tilde\nabla_y \, \xi_\alpha^{\delta^*} \right) + k_7 e^{-m\sqrt{L}} \\ &\le \sum_{z \in C_\alpha \cup \partial_r^+ C_\alpha} k_8 \, \mu_\Lambda^{\tau,\lambda}\Big([\, \tilde\nabla_z \tilde\nabla_y \, \xi_\alpha^{\delta^*} \,]^2 \Big) + k_7 e^{-m\sqrt{L}} \end{aligned} \tag{A.5}$$

where we have used once more the Poincarè inequality for the Glauber dynamics.
We are left with the estimate of $\tilde\nabla_z \tilde\nabla_y \, \xi_\alpha^{\delta^*}(\eta)$.
Suppose for definiteness that $\eta(z) = \eta(y) = 0$, call $N_\alpha(\eta) = n$ and let $\lambda(s) = \lambda(\eta, s)$ the chemical potential such that $\mu_{C_\alpha}^{\eta,\lambda(\eta,s)}\big(N_\alpha\big) = s$ with $s \in [0, l_0^d]$. In what follows we will assume without further notice than $0 < n < \frac12 l_0^d$. If instead e.g. $n = 0$ then we will simply bound $|\tilde\nabla_z \tilde\nabla_y \, \xi_\alpha^{\delta^*}|$ by $C l_0^d$ for a suitable constant C.
Let us first consider $z \in C_\alpha$. Then a little computation shows that

$$|\tilde\nabla_z \tilde\nabla_y \, \xi_\alpha^{\delta^*}(\eta)| \le \int_n^{n+1} ds \int_s^{s+1} dt \, | \frac{d^2}{dt^2} \, \mu_{C_\alpha}^{\eta,\lambda(t)} \big(\sum_{x \in C_\alpha^{\rm int}} g_x \big) | \le k_9 \frac{1}{n} \tag{A.6}$$

Let us now take $z \in \partial_r^+ C_\alpha$. Then lemma 3.1 gives

$$|\tilde\nabla_z \tilde\nabla_y \, \xi_\alpha(\eta)| \le \int_n^{n+1} ds \, |\tilde\nabla_z \frac{d}{ds} \, \mu_{C_\alpha}^{\eta,\lambda(s)} \big(\sum_{x \in C_\alpha^{\rm int}} g_x \big) | \le \frac{k_{10}}{n} \tag{A.7}$$

If we now remember that the density ρ belongs to the interval $(\epsilon, 1-\epsilon)$, standard large deviations bounds for $\mu_\Lambda^{\tau,\lambda}$ together with (A.6) and (A.7) imply that $\mu_\Lambda^{\tau,\lambda}\left(\left[\tilde\nabla_z \tilde\nabla_y \, \xi_\alpha^{\delta^*} \right]^2 \right) \le k_{11}(\epsilon) \frac{1}{l_0^{2d}}$. Thus the r.h.s. of (A.5) is smaller than $k_{12} \, \frac{1}{l_0^d}$ and the lemma follows. $\square$

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