TWO-BODY SHORT-RANGE SYSTEMS IN A TIME-PERIODIC ELECTRIC FIELD

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ABSTRACT. We apply a method developed by J.Howland and K.Yajima in conjunction with ideas from the analysis of Hamiltonians with constant electric fields to obtain absence of bound states and asymptotic completeness for 2-body short-range systems in an external time-periodic electric field.

Section 1: Introduction.

In the present paper we treat the scattering problem for two ν-dimensional particles interacting through a short-range potential and placed in an external time-periodic electric field. The Hamiltonian for such a system is

$$H(t) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - q_1\mathcal{E}(t)\cdot x_1 - q_2\mathcal{E}(t)\cdot x_2 + v(x_2 - x_1) \quad \text{on} \quad L^2(\mathbb{R}^{2\nu}). \tag{1.1}$$

Here m_i and q_i, $i \in \{1,2\}$ are the masses and the charges of the two particles and x_1, x_2 denotes their position. The electric field $\mathcal{E}$ is periodic with some period $T > 0$, that is $\mathcal{E}(t+T) = \mathcal{E}(t)$ almost everywhere. The short-range potential v will be allowed to have an explicit time-dependence as long as this dependence is periodic with the same period as the field.

Recently asymptotic completeness for many-body systems in constant electric fields has been proved for large classes of potentials, see [AT], [HMS1] and [HMS2]. For a treatment of propagation estimates for such systems see [A]. All these results rely on well known techniques which uses local commutator estimates to obtain

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spectral and scattering information. By controlling the energy along the time evolution one can apply some of these techniques to time-dependent problems. This has been done in [SS] and applied in [Si1]. In [G1] and [Z] time-boundedness of the kinetic energy plays an essential role and in [HL] the problem of bounding the kinetic energy is treated for repulsive potentials using positive commutator techniques.

In the present problem however one readily observes that the energy is generally not bounded in time. In fact for very simple examples like $\mathcal{E}(t) = \frac{1}{2} + \cos(t)$ ($\nu = 1$ and $v = 0$) one finds that the expectation value of the energy oscillates with an amplitude that grows like t^2. On the other hand the expectation of $x_2 - x_1$ grows like $(\frac{q_2}{m_2} - \frac{q_1}{m_1})t^2$ as one would expect from the constant field problem. Consequently it is natural to suggest that completeness and absence of bound states (the meaning of which will be discussed later) hold here as well provided the particles have different charge to mass ratio. In order to ensure the growth of $x_2 - x_1$ we make the crucial assumption that the field has non-zero mean.

In order to circumvent the problem of controlling the energy we adopt a method due to Howland, [H1], which is motivated by a similar procedure in Hamiltonian mechanics, where one also faces the problem of non-conservation of energy in time-dependent problems. The idea is to include time as a space variable and introduce a new momentum variable τ, conjugate to time. One defines a new time-independent Hamiltonian $\hat{H} = \tau + H$ as a function on $\mathbb{R}^\nu_q \times \mathbb{R}_t \times \mathbb{R}^\nu_p \times \mathbb{R}_\tau$, from the original Hamiltonian, H, to obtain the following system

$$
\begin{aligned}
\dot{q}(s) &= \nabla_p \hat{H}(t(s), \tau(s), q(s), p(s)), & q(0) &= q_0 \\
\dot{p}(s) &= -\nabla_q \hat{H}(t(s), \tau(s), q(s), p(s)), & p(0) &= p_0 \\
\dot{t}(s) &= 1, & t(0) &= t_0 \\
\dot{\tau}(s) &= -\partial_t \hat{H}(t(s), \tau(s), q(s), p(s)), & \tau(0) &= \tau_0.
\end{aligned}
$$

Obviously the (q, p) part of the solution to this extended system is nothing more than time-translates of the solution to the physical system, with the initial value t_0. In [H1] the scattering problems of the quantized versions of H and $\hat{H}$ are related and in [Ya1] and [H2] the idea is applied to periodic systems. It is shown that the ranges of wave operators are connected in the same way the spectral subspaces of the monodromy operator $U(T, 0)$ are connected with those of $\hat{H}$. This observation makes it possible to transfer asymptotic completeness statements between the two settings. (The monodromy operator is the unitary operator that evolves the physical system through one period.)

Assuming periodicity enables one to compactify the extra space variable. The resulting Hamiltonian $\hat{H}$ is called the Floquet Hamiltonian. Compactification makes it possible to show that the potential is relatively compact with respect to the Floquet Hamiltonian, see Lemma 4.6. This point was used first by Yajima in [Ya1] to treat two-body systems with time-periodic potentials and pointed out to the author by his supervisor E.Skibsted. It can be used to obtain positive commutators locally in energy. This idea has recently been applied in [Yo] to obtain propagation estimates for the Floquet Hamiltonian of time-periodic two-body Schrödinger operators.

The idea of this paper is thus to treat the spectral and scattering problems for the time-independent Floquet Hamiltonian where one can apply local commutator estimates to obtain results and then in the end make the connection to the physical problem. More precisely we will prove that the Floquet Hamiltonian has no bound states which due to an argument by Yajima given in [Ya1] implies that the monodromy operator has empty point spectrum. This is what we mean by absence of bound states. We furthermore prove that the wave operators exist and are unitary. The results hold under some regularity assumptions on the potential which do not include singularities. We do however obtain obtain some partial results for potentials with weak singularities.

For a treatment of the usual two-body problem with time-dependent potentials that does not use local commutator methods see [KY] and for two-body systems in a constant electric field see [G2] and [JY].

In the papers [Ya1] and [H2] the authors did not work explicitly with the Floquet Hamiltonian but rather with its resolvent which can be expressed in terms of the physical flow. In this paper however we wish to utilize specific properties of the Stark Hamiltonian in our analysis of its Floquet Hamiltonian.

In Section 2 we phrase the assumptions which we will impose on the potential and the electric field and state the main results of this paper. In Section 3 we elaborate on the work done by Howland in [H2] and Yajima in [Ya1] and prove some abstract results on the structure of Floquet Hamiltonians. Some preliminary results are derived in Section 4 and in Section 5 we apply a combination of ideas used in [Si2] and [HMS1] to obtain absence of bound states for the Floquet Hamiltonian. In Section 6 we prove a Mourre estimate and apply it to obtain absence of singular continuous spectrum for the Floquet Hamiltonian as well as a pointwise propagation estimate for the momentum operator which we use to get a minimal acceleration estimate. Finally in Section 7 we prove existence and completeness of wave operators for the Floquet Hamiltonian and argue as in [Ya1] and [H2] to obtain the main result. Appendix A is devoted to a treatment of absolutely continuous vector-valued functions and the derivative on an interval and in Appendix B the time-dependent Schödinger equation for Hamiltonians defined almost everywhere is dicussed.

The central ideas of this paper was developed during a stay at University of Tokyo and the author wishes to thank his host Professor H.Kitada for discussions on the absence of bound states. The abstract theory and the removal of the time-dependency from the electric field was done during a stay at University of Virginia. A discussion the author had with his host, I.Herbst, inspired the transformation

which is used to move the time-dependency. The treatment of singularities was added when the paper was included in the authors Ph.D. thesis and comments from the committee led to the two appendices. Finishing touches were made at Universitè Paris-Sud. Finally the author would like to thank his supervisor E.Skibsted and the committee, H.H.Andersen, J.Dereziǹski and A.Jensen, for corrections and suggestions.

Section 2: Assumptions, Notations and Results.

We will work within the following framework. Let X be a real, finite dimensional vector space equipped with an inner product. We denote by x the operator of multiplication with the identity function on X and p is the momentum operator $\frac{1}{i}\nabla_x$. We write $\langle x\rangle$ for $(1+|x|^2)^{\frac{1}{2}}$ and we will use the same abbreviation for parameters as well as for other self adjoint operators on $\mathcal{H}=L^2(X)$. Let $\nu=\dim X$. In this Section $T>0$ will denote a common period for the field and the potential.

Assumption 2.1. *The potentials $V_t\in C^2(X)$ form a family of real-valued functions, periodic with period T. The family and its distributional derivative with respect to t satisfy*

$$\sup_{t\in\mathbb{R}}|V_t(x)|+\sup_{t\in\mathbb{R}}|\nabla_x V_t(x)|=o(1)\ \text{ and }\ \sup_{t\in\mathbb{R}}|\partial_t V_t(x)|+\sup_{t\in\mathbb{R}}|\partial_x^\alpha V_t(x)|=O(1),$$

for $|\alpha|=2$.

We say that V_t is short-range if it satisfies Assumption 2.1, and the decay assumption

$$\sup_{t\in\mathbb{R}}|V_t(x)|=O(\langle x\rangle^{-\frac{1}{2}-\epsilon}),$$

for some $\epsilon>0$. (The symbol ϵ will be used in this connection only.)

Assumption 2.2. *The potentials $V_t=V_{\rm reg}^t+V_{\rm sing}^t$ are real valued, $V_{\rm reg}^t$ satisfies Assumption 2.1 and $V_{\rm sing}^t=0$ if $\nu<3$. The singular part is periodic with period T, $\cup_{t\in\mathbb{R}}\operatorname{supp}(V_{\rm sing}^t)\subset X$ is compact and there exist $p>\nu$ such that $V_{\rm sing}^t$ and its first order distributional derivatives satisfy*

$$\sup_{t\in\mathbb{R}}\|V_{\rm sing}^t\|_{L^p(X)}+\sup_{t\in\mathbb{R}}\|\partial_t V_{\rm sing}^t\|_{L^q(X)}+\sup_{t\in\mathbb{R}}\|\nabla_x V_{\rm sing}^t\|_{L^q(X)}<\infty,$$

where $q=\frac{2\nu p}{\nu+4p}$ if $\nu\geq 5$ and $q>\frac{2p}{p+1}$ if $\nu\in\{3,4\}$.

The specific form of this assumption is chosen such that the result on existence of evolutions in [Ya2] applies and such that the class of potentials satisfying Assumption 2.2 is invariant under the type of transformations given in Lemma 4.4.

Assumption 2.3. *The electric field* $E \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, $E(t+T) = E(t)$ *a.e. and* $\int_0^T E(t) \neq 0$.

We consider the Hamiltonians

$$H_0(t) = p^2 - E(t) \cdot x \quad \text{and} \quad H(t) = H_0(t) + V_t. \tag{2.1}$$

In Section 4 we prove that under Assumptions 2.2 and 2.3 the time-dependent Schrödinger equation corresponding to $H_0(t)$ and $H(t)$ can be solved uniquely in the sense of Definition B.7 (with $\mathcal{H}_1 = \mathcal{D}(p^2) \cap \mathcal{D}(\langle x \rangle)$). We write $U_0(t,s)$ and $U(t,s)$ for the solutions. The main results of this paper are

Theorem 2.4 (Absence of bound states). *Assume* V_t *satisfies Assumption 2.1 and* E *satisfies Assumption 2.3. Then the monodromy operator* $U(T,0)$ *has purely absolutely continuous spectrum.*

Under Assumption 2.2 we prove that the eigenfunctions of the Floquet Hamiltonian vanishes in a half-space determined by the field.

Theorem 2.5 (Asymptotic Completeness). *Assume* V_t *is short-range and* E *satisfies Assumption 2.3. Then the wave operators*

$$W_\pm(s) = \mathrm{s} - \lim_{t \to \pm\infty} U^*(t,s) U_0(t,s) \tag{2.2}$$

exist for all $s \in \mathbb{R}$ *and are unitary. Furthermore*

$$W_\pm(s+T) = W_\pm(s) \quad \textit{and} \quad U(s+T,s) W_\pm(s) = W_\pm(s) U_0(s+T,s) \tag{2.3}$$

for all $s \in \mathbb{R}$.

The Hamiltonian presented in (1.1) takes the form (2.1) in the center of mass frame. The configuration space becomes $X = \{x \in \mathbb{R}^{2\nu} : m_1 x_1 + m_2 x_2 = 0\}$ with the inner product $x \cdot y = 2m_1 x_1 \cdot y_1 + 2m_2 x_2 \cdot y_2$. The orthogonal projection onto X is

$$\pi = \frac{1}{m_1 + m_2} \begin{pmatrix} m_2 I_\nu & -m_2 I_\nu \\ -m_1 I_\nu & m_1 I_\nu \end{pmatrix},$$

where I_ν is the $\nu \times \nu$ identity matrix, and we find

$$\begin{aligned} E(t) &= \pi \begin{pmatrix} \frac{q_1}{2m_1} \mathcal{E}(t) \\ \frac{q_2}{2m_2} \mathcal{E}(t) \end{pmatrix} \\ &= \frac{1}{2(m_1+m_2)} \left(\frac{q_1}{m_1} - \frac{q_2}{m_2} \right) \begin{pmatrix} m_2 \mathcal{E}(t) \\ -m_1 \mathcal{E}(t) \end{pmatrix}. \end{aligned}$$

We thus have the following corollary

Corollary 2.6. *Assume $\frac{q_1}{m_1} \neq \frac{q_2}{m_2}$, the potential v is short-range (with $X = \mathbb{R}^\nu$) and the electric field $\mathcal{E}$ satisfies Assumption 2.3 (with $X = \mathbb{R}^\nu$). Then the wave operators* (2.2) *exist and they are unitary. Furthermore the periodicity and intertwining relation* (2.3) *is satisfied.*

Section 3: The Floquet Hamiltonian.

Let $\{U(t,s)\}_{t,s\in\mathbb{R}}$ be a family of strongly continuous unitary operators on a separable Hilbert space $\mathcal{H}$. Assume the family satisfies the Chapman-Kolmogorov equations

$$U(t,r)U(r,s) = U(t,s), \quad \text{for all} \quad t,r,s, \in \mathbb{R} \tag{3.1}$$

and is periodic, i.e. it satisfies the periodicity condition

$$U(t+T, s+T) = U(t,s) \quad \text{for all} \quad t,s \in \mathbb{R} \tag{3.2}$$

for some $T > 0$ which we call the period. We note that $U(t,t) = I$ and $U^*(t,s) = U(s,t)$ for all $t,s \in \mathbb{R}$.

For any unitary operator V on $\mathcal{H}$ we define the set

$$\mathcal{D}_V = \{\psi \in AC^2([0,T];\mathcal{H}) : \psi(0) = V\psi(T)\}$$

and the selfadjoint operator τ_V as $\frac{1}{i}\frac{d}{dt}$ with domain $\mathcal{D}_V$. See Appendix A for a discussion of absolutely continuous functions and the derivatives τ_V.

We define a family of operators pointwisely on $C^0([0,T];\mathcal{H})$ by

$$[\hat{U}(s)\psi](t) = U(t, t-s)\psi(t-s-[t-s]), \tag{3.3}$$

where $[s]$ denotes the largest multiple of T smaller than or equal to s. By boundedness we extend to a family of unitary operators on

$$\hat{\mathcal{H}} = L^2([0,T];\mathcal{H}).$$

It is easily verified that $\{\hat{U}(s)\}_{s\in\mathbb{R}}$ is a strongly continuous group with $\hat{U}(0) = I$. We have

Proposition 3.1 (The Floquet Hamiltonian). *The self adjoint operator that generates $\hat{U}(s)$ is $\hat{H} = U\tau_{U(T,0)}U^*$ with domain $U\mathcal{D}_{U(T,0)}$, where U is the unitary operator defined pointwisely by $[U\psi](t) = U(t,0)\psi(t)$. For $\lambda \in \mathbb{C}$, operatornameIm$\lambda \neq$ 0, the resolvent of $\hat{H}$ is given by*

$$\begin{aligned}[(\hat{H}-\lambda)^{-1}\psi](t) =& iU(t,0)\{\int_0^t e^{i(t-s)\lambda}U(0,s)\psi(s)ds \\ &+ [e^{-i\lambda T}U(0,T) - I]^{-1}\int_0^T e^{i(t-s)\lambda}U(0,s)\psi(s)ds\}\end{aligned}$$

Proof. Compute for $f \in C^0([0,T];\mathcal{H})$ using (3.1-3)

$$\begin{aligned}(\hat{U}(s)f)(t) &= U(t,0)U(0,t-s)f(t-s-[t-s]) \\ &= U(t,0)U(-[t-s],0)(U^*f)(t-s-[t-s]) \\ &= U(t,0)U(0,T)^{\frac{[t-s]}{T}}(U^*f)(t-s-[t-s]).\end{aligned}$$

By Proposition A.8 we find that $\hat{U}(s) = \exp(-isU\tau_{U(T,0)}U^*)$, which shows that the generator, $\hat{H}$, of $\hat{U}(s)$ equals $U\tau_{U(T,0)}U^*$. The resolvent formula is now a consequence of (A.2). □

Let $S(t)$ be a strongly continuous periodic family of unitary operators on $\mathcal{H}$. Then $S(t)U(t,s)S^*(s)$ satisfies (3.1-2) and has a Floquet Hamiltonian. We say that $S(t)$ is a periodic change of coordinates.

Lemma 3.2. *Let $U_1(t,s)$ and $U_2(t,s)$ be periodic families of unitary operators. Suppose there exists a periodic change of coordinates, $S(t)$, such that $U_1(t,s) = S(t)U_2(t,s)S^*(s)$. Then the Floquet Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ satisfy $\hat{H}_1 = S\hat{H}_2S^*$, where the unitary operator S is given by $[Sf](t) = S(t)f(t)$.*

Proof. We write S_0 for the unitary operator on $\hat{\mathcal{H}}$ given by $(S_0f)(t) = S(0)f(t)$ and compute using the assumption, Proposition 3.1 and (A.3)

$$\begin{aligned}S\hat{H}_2S^* &= SU_2\tau_{U_2(T,0)}U_2S^* \\ &= \{SU_2S_0^*\}\{S_0\tau_{U_2(T,0)}S_0^*\}\{S_0U_2S^*\} \\ &= U_1\tau_{S(0)U_2(T,0)S(0)^*}U_1 \\ &= \hat{H}_1.\end{aligned}$$

□

We now present a result on the spectral structure of the Floquet Hamiltonian, which was observed by Yajima in [Ya1]. It follows from Propositions 3.1 and A.9.

Proposition 3.3. *Let $U(t,s)$ be a periodic family of unitary operators. The Floquet Hamiltonian satisfies*

$$\begin{aligned}\lambda \in \sigma_{\mathrm{pp}}(\hat{H}) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{pp}}(U(T,0)), \quad \mathcal{H}_{\mathrm{pp}}(\hat{H}) = UL^2([0,T];\mathcal{H}_{\mathrm{pp}}(U(T,0)), \\ \lambda \in \sigma_{\mathrm{ac}}(\hat{H}) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{ac}}(U(T,0)), \quad \mathcal{H}_{\mathrm{ac}}(\hat{H}) = UL^2([0,T];\mathcal{H}_{\mathrm{ac}}(U(T,0)), \\ \lambda \in \sigma_{\mathrm{sc}}(\hat{H}) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{sc}}(U(T,0)), \quad \mathcal{H}_{\mathrm{sc}}(\hat{H}) = UL^2([0,T];\mathcal{H}_{\mathrm{sc}}(U(T,0)).\end{aligned}$$

In the following $H(t)$ will denote a time-periodic family of Hamiltonians which fits the requirements given in Definition B.7 (for some $\mathcal{H}_1 \subset \mathcal{H}$). Time-periodicity means that the following identy holds a.e.

$$H(t+T)\varphi = H(t)\varphi, \quad \text{for all} \quad \varphi \in \mathcal{H}_1. \tag{3.4}$$

Suppose we have a family of unitary operators $U(t,s)$, which solves the time-dependent Schrödinger equation in the sense of Definition B.7. For such a solution (3.1) is given by Corollary B.6 and (3.2) follows from (3.4) and Theorem B.4. See Appendix B for details.

We introduce some assumptions which will enable us to interpret $\hat{H}$ as the operator sum of τ_I and $H(t)$.

Assumption 3.4. *Let $H(t)$ be a time-periodic family of Hamiltonians. Suppose there exists a self-adjoint operator B on $\mathcal{H}$ and a family of unitary operators, $U(t,s)$, which solves the time-dependent Schrödinger equation whith $\mathcal{H}_1 = \mathcal{D}(B)$ and suppose furthermore that $H(t)(B-i)^{-1} \in L^2([0,T];\mathcal{B}(\mathcal{H}))$.*

Proposition 3.5. *Let $H(t)$ be a time-periodic family of Hamiltonians. Suppose Assumption 3.4 is satisfied for some B. Then*

$$\mathcal{D}_0 = \{\psi \in \mathcal{D}_I : \psi([0,T]) \subset \mathcal{D}(B) \text{ and } \sup_{t\in[0,T]} \|B\psi(t)\| < \infty\}$$

is a core for $\hat{H}$. The operator $\tau_I + H$ defined on $\mathcal{D}_0$ by $[(\tau_I + H)\psi](t) = [\tau_I\psi](t) + H(t)\psi(t)$ is essentially self-adjoint and its closure equals $\hat{H}$.

Proof. Consider the set $\mathcal{S} = \operatorname{span}\{e^{im\frac{2\pi}{T}}\psi : m \in \mathbb{Z}, \psi \in \mathcal{D}(B)\}$ which is dense in $\hat{\mathcal{H}}$. Since $\mathcal{S} \subset \mathcal{D}_0$ we find that $\mathcal{D}_0$ is dense.

By (3.1), $U(t,t-s) = U(t,0)U(0,t-s)$ and this together with Assumption 3.4 and (B.3) shows that the operator $(B+i)U(t,t-s)(B+i)^{-1}$ is bounded uniformly in $s,t \in [0,T]$. This implies by (3.3) that

$$\sup_{s,t\in[0,T]} \|[B\hat{U}(s)\psi](t)\| < \infty, \tag{3.5}$$

for $\psi \in \mathcal{D}_0$. By (3.1) and Proposition B.3 ii) and iii) we find that

$$t \to U(t,t-s)\psi(t-s-[t-s]) \in AC([0,T];\mathcal{H})$$

for $\psi \in \mathcal{D}_0$ and furthermore we have

$$\begin{aligned}
[U(t,t-s) - &U(s,0)]\psi(t-s-[t-s]) \\
&= \int_s^t i[U(r,r-s)H(r-s) - H(r)U(r,r-s)]\psi(t-s-[t-s]) \\
&\qquad + U(t,t-s)(\partial\psi)(t-s-[t-s])dr.
\end{aligned}$$

The last part of Assumption 3.4 and (3.5) now combines to prove that the derivative of $\hat{U}(s)\psi$ is square-integrable and we thus get $\hat{U}(s)\mathcal{D}_0 \subset \mathcal{D}_0$. This inclusion shows that $\hat{H}_{|\mathcal{D}_0}$ has no other self-adjoint extension than $\hat{H}$ and is therefore essentially self-adjoint with closure equals $\hat{H}$. We compute

$$[\hat{H}\psi](t) = H(t)\psi(t) + [\tau_I\psi](t),$$

for $\psi \in \mathcal{D}_0$, which concludes the proof. $\square$

Example 3.6 If $H(t) = H$ is independent of time one readily finds Assumption 3.4 satisfied with $\mathcal{D} = \mathcal{D}(B) = \mathcal{D}(H)$ and $B = H$. In this case the conclusion of Proposition 3.5 is trivial since $\hat{H}$ equals the closure of $H \otimes I + I \otimes \tau_I$ on $\mathcal{D}(H) \otimes \{\psi \in AC([0,T];\mathbb{C}) : \psi(0) = \psi(T)\}$.

Example 3.7 For $\alpha < 1$ we consider $H(t) = (t - [t])^{-\alpha}$, where $[t]$ is the integer part of t, as an operator on $\mathcal{H} = \mathbb{C}$ ($H(0) = 0$). For $s,t \in [0,1]$ we find $U(t,s) = e^{i\frac{s^{1-\alpha} - t^{1-\alpha}}{1-\alpha}}$ with $\mathcal{D} = \mathbb{C}$. For $\alpha < \frac{1}{2}$ we find Assumption 3.4 satisfied with $B = 1$. When $\alpha \geq \frac{1}{2}$ we find that any domain on which we should be able to write $\hat{H}$ as a sum of $H(t)$ and τ_I has to have its functions vanish at zero at some slow polynomial rate. By the periodicity requirements for τ_I this will in turn fix functions in such a domain to be zero in the endpoints. This is the classical example of an operator with deficiency indices equal to 1 and a whole range of self adjoint extensions indexed by the unitary operators on $\mathbb{C}$. See also Proposition A.7.

Section 4: Preliminary Results.

In this Section we solve the time-dependent Schrödinger equation for the Hamiltonians $H_0(t)$ and $H(t)$ and prove that multiplication by the indicator function $F(|x| < R)$ is $\hat{H}$-compact. We furthermore construct a unitary transformation which can be used to move the time-dependency of the field into the potential and we end the section by proving that the potential is bounded relatively to the free Floquet Hamiltonian with relative bound zero.

Assumption 2.2 is assumed to be satisfied throughout this section. The periodicity assumption on the potential is not required for existence of evolutions, that is, Lemma 4.1 and Proposition 4.3. For simplicity we take $T = 1$ in the following.

Lemma 4.1. *For $c \in C^1(\mathbb{R}; X)$ we write $G_c(t) = p^2 + W_t^c$, where $W_t^c(x) = V_t(x - c(t))$. There exists a family of unitary operators, $V_c(t,s)$, which solves the time-dependent Schrödinger equation with $\mathcal{H}_1 = \mathcal{D}(p^2)$.*

Proof. Under the given assumptions we find that $G_c(t)$ is self-adjoint on $\mathcal{S} = \mathcal{D}(p^2)$ for all t and by [Ya2, Theorem 1.3] we obtain a solution to (B.2) and (B.4) with the stated property. $\square$

We introduce a time-dependent coordinate change (see Definition B.1 for the set $\mathcal{U}_{1,2}$)

Lemma 4.2. *Assume $E \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$. Let*

$$b(t) = \int_0^t E(r)dr, \quad a(t) = \int_0^t |b(r)|^2 dr \quad \text{and} \quad c(t) = 2\int_0^t b(r)dr,$$

and define a time-dependent family of unitary operators on $\mathcal{H}$ by

$$T(t) = e^{-ia(t)} \exp(ib(t)\cdot x)\exp(-ic(t)\cdot p).$$

Then $T \in \mathcal{U}_{1,2}$, whith $\mathcal{H}_1 = \mathcal{D}(p^2) \cap \mathcal{D}(\langle x \rangle)$. Furthermore

$$\begin{aligned} T^*(t)xT(t) &= x + c(t) \quad \text{and} \quad T^*(t)pT(t) = p + b(t), \\ T(t)xT^*(t) &= x - c(t) \quad \text{and} \quad T(t)pT^*(t) = p - b(t). \end{aligned}$$

Using this coordinate change we get the following generalization of Avron-Herbst formula

Proposition 4.3 (Avron-Herbst). *Assume $E \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$. Then the evolutions*

$$U_0(t,s) = T(t)\exp(i(s-t)p^2)T^*(s) \quad \text{and} \quad U(t,s) = T(t)V_c(t,s)T^*(s)$$

are solutions to the time-dependent Scrödinger equation for $H_0(t)$ and $H(t)$ respectively (with $\mathcal{H}_1 = \mathcal{D}(p^2) \cap \mathcal{D}(\langle x \rangle)$).

Proof. Let $\mathcal{H}_1 = \mathcal{D}(p^2) \cap \mathcal{D}(\langle x \rangle)$. By Lemma 4.1 we find $V_c(t,s)\mathcal{H}_1 \subset \mathcal{D}(p^2)$ and $V_c(\cdot, s), V_c(s, \cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathcal{D}(p^2))$. Compute for $\psi \in \mathcal{H}_1$

$$V_c^*(t,s)\langle x \rangle V_c(t,s)\psi = \langle x \rangle \psi + \int_s^t V_c^*(r,s)\{\nabla\langle x \rangle \cdot p + p \cdot \nabla \langle x \rangle\} V_c(r,s)\psi dr, \quad (4.1)$$

which shows that $V_c(\cdot, s), V_c(s, \cdot) \in \mathcal{U}_{1,2}$. The result now follows from Lemma 4.2, Proposition B.3 iii) and a simple computation. (See the computation for the free evolution in [CFKS, Theorem 7.5].) □

We remark that one can prove existence of wave operators using the Avron-Herbst formula in conjunction with Cook's method and a simple stationary phase argument. For proofs of existence of long-range wave operators using this transformation see [G2] and [JY].

If E, besides from being $L^1_{\mathrm{loc}}(\mathbb{R}; X)$ is periodic we get, due to Proposition 4.3 and the discussion in Section 3, Floquet Hamiltonians $\hat{H}_0$ and $\hat{H}$ on $\hat{\mathcal{H}} = L^2([0,1]; \mathcal{H})$.

Note that due to periodicity we find (as discussed in Section 3) that $U_0(t,s)$ and $U(t,s)$ satisfy (3.2).

We will now introduce another change of variable which is time-periodic. It was inspired by a discussion the author had with Ira Herbst during a stay at University of Virginia. Let

$$E_0 = \int_0^1 E(t)dt.$$

Lemma 4.4. *Let $E \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ satisfy $E(t+1) = E(t)$ a.e. and $E_0 \neq 0$. We define continuous periodic functions*

$$\begin{aligned}
\tilde{b}(t) &= \int_0^t (E(s) - E_0)ds - \tilde{b}_0, & \tilde{b}_0 &= \int_0^1 \int_0^t (E(s) - E_0)dsdt \\
\tilde{c}(t) &= 2\int_0^t \tilde{b}(s)ds - \tilde{c}_0, & \tilde{c}_0 &= \int_0^1 \left[|\tilde{b}(t)|^2 + 2\int_0^t E_0 \cdot \tilde{b}(s)ds\right] dt \frac{E_0}{|E_0|^2} \\
\tilde{a}(t) &= \int_0^t (|\tilde{b}(s)|^2 + E_0 \cdot \tilde{c}(s))ds
\end{aligned}$$

and a strongly continuous periodic family of unitary operators on $\mathcal{H}$ by

$$\tilde{T}(t) = e^{-i\tilde{a}(t)} \exp(i\tilde{b}(t)\cdot x) \exp(-i\tilde{c}(t)\cdot p).$$

We have $\tilde{T} \in \mathcal{U}_{1,2}$ with $\mathcal{H}_1 = \mathcal{D}(p^2) \cap \mathcal{D}(\langle x \rangle)$.

This change of coordinates can be used to move the oscillating part of the electric field into the potential. We write

$$H_0^{\mathrm{const}} = p^2 - E_0 \cdot x \quad \text{and} \quad H^{\mathrm{const}}(t) = H_0^{\mathrm{const}} + W_t^{\tilde{c}}$$

and present another Avron-Herbst type result. Note that $W_t^{\tilde{c}}$ also satisfies Assumption 2.2 and the time-dependent Schrödinger equation for $H^{\mathrm{const}}(t)$ therefore has a solution $\tilde{U}(t,s)$ by Proposition 4.3.

Proposition 4.5. *Let $E \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ satisfy $E(t+1) = E(t)$ a.e. and $E_0 \neq 0$. Then*

$$U_0(t,s) = \tilde{T}(t) \exp(i(s-t)H_0^{\mathrm{const}})\tilde{T}^*(s) \quad \textit{and} \quad U(t,s) = \tilde{T}(t)\tilde{U}(t,s)\tilde{T}^*(s),$$

We furthermore have

$$\hat{H}_0 = \tilde{T}\hat{H}_0^{\mathrm{const}}\tilde{T}^* \quad \textit{and} \quad \hat{H} = \tilde{T}\hat{H}^{\mathrm{const}}\tilde{T}^*,$$

where $[\tilde{T}\psi](t) = \tilde{T}(t)\psi(t)$.

Proof. The proof of the first part follows from Proposition B.3 iii), Lemma 4.4, Proposition 4.3 and Theorem B.4, noting that the transformations at the end of Lemma 4.2 apply here as well. The last part follows from Lemma 3.2. $\square$

This result will be used to reduce the proof of Theorem 2.4 to the case where the electric field is constant. Notice that $\tilde{T}(0) = \exp(-i\tilde{b}_0 \cdot x)\exp(i\tilde{c}_0 \cdot p)$ is not necessarily the identity operator. The constants $\tilde{b}_0$ and $\tilde{c}_0$ are chosen such that the family $\tilde{T}(t)$ becomes time-periodic.

We introduce some notation. We write $F(s < R)$ for the characteristic function of the set $(-\infty, R)$. We denote by $\chi(s < R)$ a smooth version of $F(s < R)$. More precisely $\chi : \mathbb{R} \to [0,1]$ is smooth, $\chi(s < R) = 0$ for $s > 2R$ and $\chi(s < R) = 1$ for $s < R$. Furthermore we write $F(s \geq R) = 1 - F(s < R)$ and $\chi(s > R) = 1 - \chi(s < R)$.

Lemma 4.6. *Suppose* $E \in L^1_{\text{loc}}(\mathbb{R}; X)$ *and* $E(t+1) = E(t)$ *a.e. Then the operator* $F(|x| < R)(\hat{H}_0 - \lambda)^{-1}$ *is compact for any* $R > 0$ *and* $\lambda \in \mathbb{C}$ *with* $operatornameIm\lambda \neq 0$.

Proof. The idea is to use the resolvent formula of Proposition 3.1, Avron-Herbst formula and the known integral kernel for the free propagator to approximate the operator

$$\chi(|x| < R)(\hat{H}_0 - \lambda)^{-1}\chi(|x| < R)$$

with Hilbert-Schmidt operators times bounded operators. This implies the result by the first resolvent formula. It is sufficient to consider $operatornameIm\lambda < 0$.

For the first term in the formula for the resolvent given in Proposition 3.1 we get

$$\begin{aligned}
&i\chi(|x| < R)U_0(t,0)\int_0^t e^{i(s-t)\lambda}U_0(s,0)^*\chi(|x| < R)f(s)ds \\
&= iT(t)\chi(|x + c(t)| < R)\int_0^t e^{i(s-t)\lambda}\exp(i(s-t)p^2)\chi(|x + c(s)| < R)T^*(s)f(s)ds \\
&= iT(t)\int_0^1\int_X F(s \in [0, t-\delta])\chi(|x + c(t)| < R)\times \\
&\qquad (4\pi(t-s))^{-\frac{\nu}{2}}e^{i(s-t)\lambda + i\frac{|x-y|^2}{4(t-s)}}\chi(|y + c(s)| < R)[T^* f](s;y)dsdy + O_\delta(t).
\end{aligned}$$

Here O_δ denotes a function in $\hat{\mathcal{H}}$ satisfying that $\|O_\delta\| \leq \sqrt{\delta}e^{-operatornameIm(\lambda)\delta}\|f\|$ and T^* is the adjoint of the operator T given by $[Tf](t) = T(t)f(t)$. Since T is unitary and δ is arbitrary we conclude compactness for the first of the two terms.

For the second term we use the Neumann series to write

$$[e^{-i\lambda}U_0^*(1,0)-I]^{-1}=-\sum_{n=0}^{\infty}[e^{-i\lambda}U_0^*(1,0)]^n,$$

which is convergent by the choice of λ. We can use this together with (3.1-2) to write the second term in the resolvent formula as

$$-i\sum_{n=0}^{N(\delta)}F(|x|<R)\int_0^1 e^{i(t-s-n)\lambda}U_0(t+n,s)F(|x|<R)f(s)ds+O_\delta(t).$$

To treat this term we cut out a δ piece as for the first term, apply Avron-Herbst formula and use the integral kernel for the free propagator to rewrite the remaining part. As before we find that each term in the sum above is compact. This concludes the proof. $\square$

If E is $L^2_{\mathrm{loc}}(\mathbb{R};X)$ and periodic we find by (4.1), Lemma 4.1 and Proposition 4.3 that the operators $H_0(t)$ and $H(t)$ satisfy Assumption 3.4 with $B=p^2+\langle x\rangle$. Proposition 3.5 thus implies that the operators $\hat{H}_0$ and $\hat{H}$ are essentially self-adjoint on the domain

$$\mathcal{D}_0=\{\psi\in\mathcal{D}_I:\psi([0,1])\subset\mathcal{D}(\langle x\rangle)\cap\mathcal{D}(p^2)\text{ and }\sup_{0\le t\le 1}\|(p^2+\langle x\rangle)\psi(t)\|<\infty\}.$$

Let V_{reg}, V_{sing} and V denote the multiplication operators defined by

$$[V_{\mathrm{reg}}\psi](t)=V^t_{\mathrm{reg}}\psi(t),\quad [V_{\mathrm{sing}}\psi](t)=V^t_{\mathrm{sing}}\psi(t)\quad\text{and}\quad [V\psi](t)=V_t\psi(t),$$

for $\psi\in\hat{\mathcal{H}}$. Note that $V=V_{\mathrm{reg}}+V_{\mathrm{sing}}$. As operators on $\mathcal{D}_0$ we have $\hat{H}_0=\tau+p^2-E(t)\cdot x$ and $\hat{H}=\hat{H}_0+V$.

Lemma 4.7. *Suppose $E\in L^2_{\mathrm{loc}}(\mathbb{R};X)$ and $E(t+1)=E(t)$ a.e. Then the operator V_{sing} is $\hat{H}_0$ bounded with relative bound zero.*

Proof. The proof is very similar to a proof given by Yajima in [Ya1, Lemma 3.1] and we therefore skip some of the details. Notice that by assumption $\mathcal{D}(p^2)\subset\mathcal{D}(V_{\mathrm{sing}})$, for all t, and hence $\mathcal{D}_0\subset\mathcal{D}(V_{\mathrm{sing}})$. We compute for $\psi\in\mathcal{D}_0$ and $operatorname Im\lambda>0$ (a.e with respect to t)

$$\begin{aligned}[V_{\mathrm{sing}}(\hat{H}_0-\lambda)^{-1}V_{\mathrm{sing}}\psi](t)&=i\int_0^\infty e^{is\lambda}[V_{\mathrm{sing}}\hat{U}_0(s)V_{\mathrm{sing}}\psi](t)ds\\&=i\int_{-\infty}^t e^{i(t-s)\lambda}V^t_{\mathrm{sing}}U_0(t,s)V^s_{\mathrm{sing}}\psi(s-[s])ds\\&=I_1(t)+I_2(t),\end{aligned}\tag{4.2}$$

where

$$I_1(t) = i \sum_{n=-\infty}^{-2} \int_0^1 e^{i(t-s-n)\lambda} V_{\text{sing}}^t U_0(t, s+n) V_{\text{sing}}^s \psi(s) ds$$

$$I_2(t) = i \int_{-1}^t e^{i(t-s)\lambda} V_{\text{sing}}^t U_0(t,s) V_{\text{sing}}^s \psi(s-[s]) ds.$$

At this point we use Avron-Herbst formula to substitute $U_0(t,s) = T(t)\exp(i(s-t)p^2)T^*(s)$ and rewrite

$$I_1(t) = iT(t) \sum_{n=-\infty}^{-2} \int_0^1 e^{i(t-s-n)\lambda} V_{\text{sing}}^{0,t} \exp(i(s+n-t)p^2) V_{\text{sing}}^{n,s} T^*(s)\psi(s) ds$$

$$I_2(t) = iT(t) \int_{-1}^t e^{i(t-s)\lambda} V_{\text{sing}}^{0,t} \exp(i(s-t)p^2) V_{\text{sing}}^{0,s} T^*(s)\psi(s) ds,$$

where $V_{\text{sing}}^{k,t}(x) = V_{\text{sing}}^t(x - c(t+k))$ for $k \le 0$. Note that $\|V_{\text{sing}}^{k,t}\|_{\tilde{p}} = \|V_{\text{sing}}^t\|_{\tilde{p}} < \infty$ for all $2 \le \tilde{p} \le p$ and $k \le 0$. One can now proceed exactly as in [Ya1] and use a result by Kato, namely

$$\|f \exp(-itp^2) g u\| \le (4\pi|t|)^{-\frac{\nu}{\tilde{p}}} \|f\|_{\tilde{p}} \|g\|_{\tilde{p}} \|u\|,$$

for $f, g \in L^{\tilde{p}}(X)$, $\tilde{p} \ge 2$, and $u \in \mathcal{D}(g)$. The unitary transformation $T(t)$ dissapears after an application of this inequality to I_1 and I_2. The problem is now exactly the same as the one considered in [Ya1] and we omit the remaining steps. We thus obtain $I_1, I_2 \in \hat{\mathcal{H}}$ and

$$\|I_1\|^2 + \|I_2\|^2 \le C(operatorname Im \lambda)\|f\|^2,$$

where $C(r) \to 0$ as $|r| \to \infty$. The same procedure applies to the case $operatorname Im\lambda < 0$. The result now follows from (4.2), Proposition 3.5 and the first resolvent formula.
□

We are now in a position where we can begin the spectral analysis of the Floquet Hamiltonian.

Section 5: Absence of bound states.

In this Section we prove that the monodromy operator $U(1,0)$ has no bound states (for non-singular potentials). This is equivalent to proving absence of bound states for $\hat{H}$ as noted by Yajima in [Ya1, Section 4], see Proposition 3.3. We restrict attention to constant electric fields, $E(t) = E_0$, in view of Proposition 4.5 and use the notation $p_0 = \omega_0 \cdot p$, where $\omega_0 = \frac{E_0}{|E_0|}$. We work under the assumption that V_t satisfies Assumption 2.2.

For $\theta > 0$ we write $\chi_\theta(r) = \chi(\frac{r}{\theta} < 1)$ and define

$$G_\theta(r) = \int_0^r \chi_\theta(s) ds.$$

This function will be used to reguralize an exponential weight.

The fact that $\langle x \rangle^{-\frac{1}{2}} p$ is relatively bounded with respect to the Hamiltonian $H(t)$ is used in both [Si2] and [HMS1] to prove absence of bound states in the case of constant fields. This is however not true for the Floquet Hamiltonian. We introduce the following operator family in order to circumvent the technical problems arising from this

$$P_R = i(\frac{p_0}{R} + i)^{-1}, \quad R > 0,$$

which satisfies

$$\mathrm{s} - \lim_{R \to \infty} P_R = I \quad \text{and} \quad \mathrm{s} - \lim_{R \to \infty} \frac{p_0}{R} P_R = 0. \tag{5.1}$$

Pick $d > 0$ such that

$$\sup_{0 \le t \le T} |\nabla V_{\mathrm{reg}}^t(x)| < \frac{|E_0|}{2} \quad \text{and} \quad x \notin \bigcup_{0 \le t \le T} \mathrm{supp}(V_{\mathrm{sing}}^t) \quad \text{for} \quad \omega_0 \cdot x > d.$$

We will in the following write, unless otherwise noted, $\chi_\theta = \chi_\theta(\omega_0 \cdot x - d)$ and $G_\theta = G_\theta(\omega_0 \cdot x - d)$ and introduce

$$A_\theta = \frac{1}{2}(\chi_\theta p_0 + p_0 \chi_\theta).$$

We write $\hat{H}_1 = \hat{H}_0 + V_{\mathrm{reg}}$ and compute the following commutators (as forms on $\mathcal{D}_0$)

$$i[\hat{H}_1, P_R] = \frac{i}{R} P_R \omega_0 \cdot (E_0 - \nabla V_{\mathrm{reg}}) P_R. \tag{5.2}$$

and

$$i[\hat{H}_1, e^{\alpha G_\theta}] = \{2\alpha A_\theta + i\alpha^2 \chi_\theta^2\} e^{\alpha G_\theta}. \tag{5.3}$$

Lemma 5.1. *Suppose V_t satisfies Assumption 2.2. Let ψ be an eigenfunction for $\hat{H}$. For all $\alpha > 0$ we have $e^{\alpha \omega_0 \cdot x} \psi \in \hat{\mathcal{H}}$.*

Proof. The proof is inspired by ideas of Sigal used in [Si2]. Assume there exists $\alpha > 0$ such that $e^{\alpha \omega_0 \cdot x} \psi \notin \hat{\mathcal{H}}$.

For functions $\varphi \in \hat{\mathcal{H}}$ we abbreviate

$$\varphi_{\theta,R} = \frac{e^{\alpha G_\theta} P_R \varphi}{\|e^{\alpha G_\theta} P_R \varphi\|}, \quad \text{and} \quad \varphi_\theta = \frac{e^{\alpha G_\theta} \varphi}{\|e^{\alpha G_\theta} \varphi\|}.$$

For expectation values we write

$$\langle A \rangle_\varphi = \langle \varphi, A\varphi \rangle.$$

Using the choice of d we estimate for $\varphi \in \mathcal{D}_0$

$$\langle i[\hat{H}_1, p_0] \rangle_{\varphi_{\theta,R}} \geq \frac{|E_0|}{2} - \|\nabla V_{\text{reg}}\|_\infty \frac{\|\varphi\|^2}{\|e^{\alpha G_\theta} P_R \varphi\|^2}. \tag{5.4}$$

On the other hand we can use (5.3) to compute

$$\begin{aligned}\langle i[\hat{H}_1, p_0] \rangle_{\varphi_{\theta,R}} &= -4\alpha \operatorname{Re}\{\langle A_\theta p_0 \rangle_{\varphi_{\theta,R}}\} + \alpha^2 \langle i[\chi_\theta^2, p_0] \rangle_{\varphi_{\theta,R}} \\ &\quad - 2\|e^{\alpha G_\theta} P_R \varphi\|^{-1} \operatorname{Re}\{\langle i e^{\alpha G_\theta}([\hat{H}_1, P_R] + P_R\{\hat{H} - V_{\text{sing}}\})\varphi, p_0 \varphi_{\theta,R} \rangle\}.\end{aligned}$$

To estimate the first term we notice that by construction

$$A_\theta = p_0 \chi_\theta + \frac{i}{2} \chi_\theta'$$

which in turn yields

$$2 \operatorname{Re}\{A_\theta p_0\} = 2 p_0 \chi_\theta p_0 - \frac{1}{2} \chi_\theta''.$$

This gives the identity

$$\begin{aligned}\langle i[\hat{H}_1, p_0] \rangle_{\varphi_{\theta,R}} &= -4\alpha \langle p_0 \chi_\theta p_0 \rangle_{\varphi_{\theta,R}} + \langle \alpha \chi_\theta'' - 2\alpha^2 \chi_\theta \chi_\theta' \rangle_{\varphi_{\theta,R}} \\ &\quad - 2\|e^{\alpha G_\theta} P_R \varphi\|^{-1} \operatorname{Re}\{\langle i e^{\alpha G_\theta}([\hat{H}_1, P_R] + P_R\{\hat{H} - V_{\text{sing}}\})\varphi, p_0 \varphi_{\theta,R} \rangle\}.\end{aligned} \tag{5.5}$$

We rewrite the term containing V_{sing} using $V_{\text{sing}} = \chi_\theta V_{\text{sing}}$

$$\langle e^{\alpha G_\theta} P_R V_{\text{sing}} \varphi, p_0 \varphi_{\theta,R} \rangle = \langle e^{\alpha G_\theta} P_R V_{\text{sing}} \varphi, \chi_\theta p_0 \varphi_{\theta,R} \rangle + \langle B_\theta(R) V_{\text{sing}} \varphi, \frac{1}{R} p_0 \varphi_{\theta,R} \rangle,$$

where $B_\theta(R)$ denote bounded operators satisfying $\sup_{R \geq 1} \|B_\theta(R)\| < \infty$. We estimate the first term using Cauchy-Schwartz inequality and $\chi_\theta^2 < \chi_\theta$

$$\begin{aligned}|\langle e^{\alpha G_\theta} P_R V_{\text{sing}} \varphi, p_0 \varphi_{\theta,R} \rangle| &\leq \frac{\|e^{\alpha G_\theta} P_R V_{\text{sing}} \varphi\|^2}{2\alpha \|e^{\alpha G_\theta} P_R \varphi\|} + 2\alpha \|e^{\alpha G_\theta} P_R \varphi\| \langle p_0 \chi_\theta p_0 \rangle_{\varphi_{\theta,R}} \\ &\quad + \langle B_\theta(R) V_{\text{sing}} \varphi, \frac{1}{R} p_0 \varphi_{\theta,R} \rangle.\end{aligned}$$

By this inequality, (5.2) and Lemma 4.7 we can estimate (5.5)

$$\begin{aligned}\langle i[\hat{H}_1, p_0]\rangle_{\varphi_{\theta,R}} \leq & \langle \alpha\chi_\theta'' - 2\alpha^2\chi_\theta\chi_\theta'\rangle_{\varphi_{\theta,R}} \\ & - 2\|e^{\alpha G_\theta}P_R\varphi\|^{-1}\operatorname{Re}\{\langle ie^{\alpha G_\theta}P_R\hat{H}\varphi, p_0\varphi_{\theta,R}\rangle\} \\ & + \frac{\|e^{\alpha G_\theta}P_R V_{\text{sing}}\varphi\|^2}{\alpha\|e^{\alpha G_\theta}P_R\varphi\|^2} + \langle B_\theta(R)(\hat{H}_0 + i)\varphi, \frac{1}{R}p_0\varphi_{\theta,R}\rangle.\end{aligned}$$

First we combine this estimate with (5.4) and take the limit $\varphi \to \psi$ in the graph norm of $\hat{H}$. This is possible due to the fact that $\mathcal{D}_0$ is core for $\hat{H}$ and by assumption the limit of $\operatorname{Re}\{\langle ie^{\alpha G_\theta}P_R\hat{H}\varphi, p_0\varphi_{\theta,R}\rangle\}$ is equal to zero. Secondly we use both limits in (5.1) to take R to infinity. In this way we obtain the inequality

$$\frac{|E_0|}{2} \leq \langle \alpha\chi_\theta'' - 2\alpha^2\chi_\theta\chi_\theta'\rangle_{\psi_\theta} + \frac{\|\nabla V_{\text{reg}}\|_\infty\|\psi\|^2 + \frac{1}{\alpha}\|V_{\text{sing}}\psi\|^2}{\|e^{\alpha G_\theta}\psi\|^2}.$$

We have thus obtained a contradiction since $|\chi_\theta^{(k)}| = O(\theta^{-k})$ and $\|e^{\alpha G_\theta}\psi\| \to \infty$ as $\theta \to \infty$ by assumption. □

Lemma 5.2. *Let $\alpha > 0$ and $R > 2\alpha$. The following holds:*

(1) *For any $\varphi \in \hat{\mathcal{H}}$ we have $\|e^{\alpha G_\theta}P_R\varphi\| \leq \frac{1}{1-\frac{\alpha}{R}}\|e^{\alpha G_\theta}\varphi\|$.*

(2) *Let $\psi \in \hat{\mathcal{H}}$ satisfy $e^{2\alpha\omega_0\cdot x}\psi \in \hat{\mathcal{H}}$. Then*

$$e^{\alpha\omega_0\cdot x}P_R\psi \in \hat{\mathcal{H}} \quad \textit{and} \quad \lim_{\theta\to\infty} e^{\alpha G_\theta}P_R\psi = e^{\alpha(\omega_0\cdot x - d)}P_R\psi.$$

Proof. We first prove i). Compute the commutator

$$i[e^{\alpha G_\theta}, P_R] = \frac{-i\alpha}{R}P_R\chi_\theta e^{\alpha G_\theta}P_R. \tag{5.7}$$

This gives the estimate

$$\|e^{\alpha G_\theta}P_R\varphi\| \leq \|e^{\alpha G_\theta}\varphi\| + \frac{\alpha}{R}\|e^{\alpha G_\theta}P_R\varphi\|,$$

for any $\varphi \in \hat{\mathcal{H}}$. Rearrangement gives i).

In order to prove ii) we first show that the family $\{\psi_\theta\}_{\theta>0} = \{e^{\alpha G_\theta}P_R\psi\}_{\theta>0}$ is Cauchy. Secondly we verify that its limit is as expected.

We use (5.7) again to estimate

$$\|\psi_{\theta_1} - \psi_{\theta_2}\| \leq \|(e^{\alpha G_{\theta_1}} - e^{\alpha G_{\theta_2}})\psi\| + \frac{\alpha}{R}\{\|\psi_{\theta_1} - \psi_{\theta_2}\| + \|(\chi_{\theta_1} - \chi_{\theta_2})\psi_{\theta_1}\|\}.$$

We rewrite this to obtain

$$\|\psi_{\theta_1} - \psi_{\theta_2}\| \le \frac{1}{1-\frac{\alpha}{R}}\{\|(e^{\alpha G_{\theta_1}} - e^{\alpha G_{\theta_2}})\psi\| + \frac{\alpha}{R}\|(\chi_{\theta_1} - \chi_{\theta_2})\psi_{\theta_1}\|\}.$$

We can estimate the second term using the Cauchy-Schwarz inequality and the proof of i) with G_θ replaced by $2G_{\theta_1}$

$$\|(\chi_{\theta_1} - \chi_{\theta_2})\psi_{\theta_1}\|^2 \le \frac{1}{1-\frac{2\alpha}{R}}\|(\chi_{\theta_1} - \chi_{\theta_2})^2 P_R\psi\|\|e^{2\alpha G_{\theta_1}}\psi\|.$$

The fact that the sets $\operatorname{supp}(\chi_{\theta_1} - \chi_{\theta_2})$ and $\operatorname{supp}(e^{\alpha G_{\theta_1}} - e^{\alpha G_{\theta_2}})$ are contained in $\{x : \omega_0 \cdot x \ge \min\{\theta_1, \theta_2\} + d\}$ together with the two bounds

$$|\chi_{\theta_1} - \chi_{\theta_2}| \le 2 \quad \text{and} \quad |(e^{\alpha G_{\theta_1}} - e^{\alpha G_{\theta_2}})e^{-\alpha\omega_0\cdot x}| \le 2e^{\alpha d}$$

shows in conjunction with the choice of ψ that $\{\psi_\theta\}_{\theta>0}$ is Cauchy.

The limit $\lim_{\theta\to\infty}\psi_\theta$ thus exists. The Lebesgue Theorem on monotone convergence yields $e^{\alpha\omega_0\cdot x}P_R\psi \in \hat{\mathcal{H}}$ and a simple argument completes the proof of ii). □

We now apply Lemma 5.1 using an idea from [HMS1] to obtain absence of bound states.

Proposition 5.3. *Let ψ be an eigenfunction for $\hat{H}$. Then ψ vanishes on the set $\{x \in X : \omega_0 \cdot x > d\}$. If $V_{\text{sing}} = 0$ then $\sigma_{\text{pp}}(\hat{H}) = \emptyset$.*

Proof. Let λ be an eigenvalue for $\hat{H}$ with corresponding eigenfunction ψ. First we compute as in [HMS1] for $\phi \in P_R\mathcal{D}_0$ using (5.3)

$$\begin{aligned}\|e^{\alpha G_\theta}(\hat{H}_0 - \lambda)\phi\|^2 &= \|2\alpha A_\theta e^{\alpha G_\theta}\phi\|^2 + \|[\hat{H}_0 - (\lambda + \alpha^2\chi_\theta^2)]e^{\alpha G_\theta}\phi\|^2 \\ &\quad + 2\alpha\langle i[p^2 - E_0\cdot x - \alpha^2\chi_\theta^2, A_\theta]\rangle_{e^{\alpha G_\theta}\phi} \\ &\ge 2\alpha|E_0|\langle\chi_\theta\rangle_{e^{\alpha G_\theta}\phi} + \langle 4\alpha^3\chi_\theta^2\chi_\theta' - \alpha\chi_\theta^{(3)}\rangle_{e^{\alpha G_\theta}\phi} \\ &\quad + 2\alpha\langle p_0\chi_\theta' p_0\rangle_{e^{\alpha G_\theta}\phi}. \end{aligned} \tag{5.8}$$

On the other hand substitute $\phi = P_R\varphi$, $\varphi \in \mathcal{D}_0$ and compute

$$\|e^{\alpha G_\theta}(\hat{H}_0 - \lambda)P_R\varphi\|^2 = \|e^{\alpha G_\theta}\{[\hat{H}_1, P_R] - V_{\text{reg}}P_R + P_R(\hat{H} - \lambda - V_{\text{sing}})\}\varphi\|^2.$$

We combine this with (5.8), take the limit $\varphi \to \psi$ in the graph-norm of $\hat{H}$ and use (5.2), the choice of d and Lemma 5.2 i)

$$\begin{aligned}&2\left(\frac{\|E_0 - \nabla V_{\text{reg}}\|_\infty}{R-\alpha} + \|V_{\text{reg}}\|_\infty\right)^2\|e^{\alpha G_\theta}P_R\psi\|^2 + \frac{2}{(1-\frac{\alpha}{R})^2}\|V_{\text{sing}}\psi\|^2 \\ &\qquad \ge 2\alpha|E_0|\langle\chi_\theta\rangle_{e^{\alpha G_\theta}P_R\psi} + \langle 4\alpha^3\chi_\theta^2\chi_\theta' - \alpha\chi_\theta^{(3)}\rangle_{e^{\alpha G_\theta}P_R\psi} + 2\alpha\langle p_0\chi_\theta' p_0\rangle_{e^{\alpha G_\theta}P_R\psi}.\end{aligned}$$

We apply Lemma 5.2 ii) and the fact that $|\chi_\theta^{(k)}| = O(\theta^{-k})$ and take the limit $\theta \to \infty$ on both sides to obtain

$$\left(\frac{\|E_0 - \nabla V_{\text{reg}}\|_\infty}{R - \alpha} + \|V_{\text{reg}}\|_\infty \right)^2 \|e^{\alpha\omega_0 \cdot x} P_R \psi\|^2 + \frac{e^{\alpha d}}{(1 - \frac{\alpha}{R})^2} \|V_{\text{sing}} \psi\|^2$$
$$\geq \alpha |E_0| \|e^{\alpha\omega_0 \cdot x} P_R \psi\|^2. \tag{4.8}$$

Here we used that $\|p_0 e^{G_\theta} P_R \psi\|$ is bounded uniformly with respect to θ as can be seen from (5.7), Lemma 5.1 and Lemma 5.2 i). We estimate using Lemma 5.1 and Lemma 5.2 i) and ii)

$$\|e^{\alpha\omega_0 \cdot x}(I - P_R)\psi\| \leq \|(I - P_R)e^{\alpha\omega_0 \cdot x}\psi\| + \frac{\alpha}{R - \alpha}\|e^{\alpha\omega_0 \cdot x}\psi\|$$

which by Lemma 5.1 and (5.1) implies

$$e^{\alpha\omega_0 \cdot x} P_R \psi \to e^{\alpha\omega_0 \cdot x}\psi \quad \text{as} \quad R \to \infty.$$

Taking R to infinity in (5.8) thus gives for all $\alpha > 0$

$$\|V_{\text{reg}}\|_\infty^2 \|e^{\alpha\omega_0 \cdot x}\psi\|^2 + e^{\alpha d}\|V_{\text{sing}}\psi\|^2 \geq \alpha |E_0| \|e^{\alpha\omega_0 \cdot x}\psi\|^2.$$

From this inequality we conclude the statement of the Proposition. □

By Propositions 3.3 and 4.5 this proves that the monodromy operator has no pure point spectrum. If one could prove a unique continuation Theorem for the differential operator $\tau + p^2$, see [ABG], one could conclude absence of bound states for singular potentials as well.

Section 6: Mourre Estimate.

We will write η_δ for any smooth function $\eta : \mathbb{R} \to [0, 1]$ satisfying that $\eta = 0$ on the complement of $[-2\delta, 2\delta]$ and $\eta = 1$ on $[-\delta, \delta]$.

We now combine the result on absence of bound states with Proposition 4.5 to obtain the "squeezing rule" (cf [HMS1, Proposition 3.7] for a corresponding result which played a crucial role in the treatment of the many-body constant field problem). In this Section we work under Assumption 2.1.

Proposition 6.1 (Squeezing rule). *Suppose E satisfies Assumption 2.3. Let $\lambda \in \mathbb{R}$. Then we have for any $R > 0$*

$$\lim_{\delta \to 0} \|F(|x| < R)\eta_\delta(\hat{H} - \lambda)\| = 0.$$

Proof. Let $0 < \delta < \frac{1}{2}$. Since, by Propositions 4.5 and 5.3, $\lambda \in \mathbb{R}$ is not an eigenvalue for $\hat{H}$ we see that $\mathrm{w}-\lim_{\delta\to 0} \eta_\delta(\hat{H}-\lambda) = 0$. This combined with Lemma 4.6 gives

$$F(|x| < R)\eta_\delta(\hat{H}-\lambda) = \{F(|x| < R)\eta_1(\hat{H}-\lambda)\}\eta_\delta(\hat{H}-\lambda) \to 0$$

in norm as $\delta \to 0$. $\square$

Keeping Proposition 4.5 in mind, we will in the remaining part of this Section assume $E(t) = E_0 \neq 0$. We write as in Section 5: $p_0 = \omega_0 \cdot p$, $\omega_0 = \frac{E_0}{|E_0|}$, and compute as a form on $\mathcal{D}_0$

$$i[\hat{H}, p_0] = |\cdot E_0| - \omega_0 \cdot \nabla V.$$

Since $\mathcal{D}_0 \subset \mathcal{D}(p_0) \cap \mathcal{D}(\hat{H})$ we find by Proposition 3.5 that $\mathcal{D}(p_0) \cap \mathcal{D}(\hat{H})$ is dense in $\mathcal{D}(\hat{H})$. Clearly $\exp(isp_0)\mathcal{D}_0 \subset \mathcal{D}_0$ and $\|\hat{H}\exp(isp_0)\psi\| \leq C(\|\hat{H}\psi\| + \|\psi\|)$, uniformly in $|s| \leq 1$, for $\psi \in \mathcal{D}_0$, which implies the same estimate for $\psi \in \mathcal{D}(\hat{H})$. These remarks together with [Mo, Proposition II.1] (applied with $\mathcal{S} = \mathcal{D}_0$) shows that the form $i[\hat{H}, p_0]$ is represented on $\mathcal{D}(\hat{H})$ by the operator on the right-hand side. This argument, Assumption 2.1 and the squeezing rule implies

Proposition 6.2 (Mourre Estimate). *Let $\lambda \in \mathbb{R}$. Then for all $e < |E_0|$ there exists $\delta > 0$ such that*

$$\eta_\delta(\hat{H}-\lambda) i[\hat{H}, p_0]\eta_\delta(\hat{H}-\lambda) \geq e\eta_\delta^2(\hat{H}-\lambda).$$

From the abstract theory of Mourre [Mo] we get the limiting absorption principle which implies the following corollary and by Proposition 4.5 it holds for E satisfying Assumption 2.3 (Note that we have verified above the technical Assumptions used in [Mo]). In conjunction with Proposition 3.3 this completes the proof of Theorem 2.4.

Corollary 6.3. *The spectrum of $\hat{H}$ is purely absolutely continuous.*

In the time-independent case one would now proceed in standard fashion to obtain an integral propagation estimate for p_0 from the limiting absorption principle (local smoothness). Since $\langle x_0\rangle^{-\frac{1}{2}} p_0$ is bounded relative to $p^2 - E_0 \cdot x$ this implies an integral propagation estimate for x_0 which in turn yields an easy proof of asymptotic completeness. In the present case however $\langle x_0\rangle^{-\frac{1}{2}} p_0$ is not $\hat{H}_0$-bounded and instead we choose to proceed via pointwise propagation estimates.

Corollary 6.4. *There exist $\kappa > 0$ and $\rho > 0$ such that*

$$F(\frac{p_0}{s} < \kappa)\exp(-is\hat{H})f(\hat{H})\langle p_0\rangle^{-1} = O(\langle s\rangle^{-\rho}) \quad \textit{as} \quad s \to \pm\infty,$$

for any $f \in C_0^\infty(\mathbb{R})$.

Proof. Let $A(s) = p_0 - es$ for some $0 < e < |E_0|$. Its Heisenberg derivative is

$$\begin{aligned} DA(s) &= i[\hat{H}, A(s)] + \partial_s A(s) \\ &= i[\hat{H}, p_0] - e. \end{aligned}$$

By the Mourre estimate one can apply [Sk, Corollary 2.5] to obtain the result. Notice that as in [HMS2, Appendix A] one has to remove the lower boundedness assumption on the Hamiltonian by making a slight modification of [Sk, Lemma 2.10]. □

Before we present the applications of this propagation estimate we give a technical Lemma, which will be needed

Lemma 6.5. *Let* $f \in C_0^\infty(\mathbb{R})$. *Then* $f(\hat{H})\mathcal{D}_I \subset \mathcal{D}_I$, $f(\hat{H})\mathcal{D}(p_0) \subset \mathcal{D}(p_0)$ *and* $f(\hat{H})$ *is furthermore bounded as an operator on these domains equiped with their respective graph-norms.*

Proof. Since p_0 is a conjugate operator to $\hat{H}$ we know from abstract theory, see for example [Mø, Lemma 5.1 ii)], applied with $n = 0$, that the stated result holds for p_0. As for τ_I it is easy to see that $\exp(-is\tau_I)\mathcal{D}_0 \subset \mathcal{D}_0$ and

$$\|\hat{H}\exp(-is\tau_I)\psi\| \leq C(\|\hat{H}\psi\| + \|\psi\|),$$

uniformly in $|s| \leq 1$, for $\psi \in \mathcal{D}_0$ and hence also for $\psi \in \mathcal{D}(\hat{H})$ by Proposition 3.5. Furthermore $\mathcal{D}_I \cap \mathcal{D}(\hat{H})$ contains $\mathcal{D}_0$ and is therefore dense in $\mathcal{D}(\hat{H})$. We compute

$$i[\tau_I, \hat{H}] = V_t',$$

as a form on $\mathcal{D}_0$ and hence on $\mathcal{D}(\tau_I) \cap \mathcal{D}(\hat{H})$ by [Mo, Proposition II.1]. The desired result now follows from [Mø] again. □

In the following we abreviate $x_0 = \omega_0 \cdot x$ and assume for simplicity that the ρ appearing in Corollary 6.4 is smaller than $\frac{1}{2}$.

Proposition 6.6. *There exist* $\theta > 0$ *such that*

$$F(\frac{|x_0|}{\langle s \rangle^2} < \theta) \exp(-is\hat{H}) f(\hat{H}) \langle p_0 \rangle^{-1} \langle \tau_I \rangle^{-1} = O(\langle s \rangle^{-\rho}) \quad as \quad s \to \pm\infty,$$

for any $f \in C_0^\infty(\mathbb{R})$.

Proof. This proof is based on ideas used to solve an analogous problem in [HMS2, Appendix A] combined with the regularization procedure of Section 5.

Let $\varphi \in \hat{\mathcal{H}}$, $f \in C_0^\infty(\mathbb{R})$, $\psi = f(\hat{H})\langle p_0\rangle^{-1}\langle \tau_I\rangle^{-1}\varphi$ and $\psi_R = B_R\psi$, where

$$B_R = i(\frac{p^2+\langle x\rangle}{R}+i)^{-1}, \quad \text{for} \quad R > 0.$$

By Lemma 6.5 we find that

$$\psi \in \mathcal{D}_I \quad \text{and} \quad \psi_R \in \mathcal{D}_0.$$

We abbreviate $\chi(s) = \chi(\frac{|x_0|}{\langle s\rangle^2} < \theta)\exp(-is\hat{H})\psi \in \mathcal{D}(p_0) \cap \mathcal{D}(\langle x\rangle)$ and $\chi_R(s) = \chi(\frac{|x_0|}{\langle s\rangle^2} < \theta)\exp(-is\hat{H})\psi_R \in \mathcal{D}_0$. (Note that we verified in the proof of Proposition 3.5 that $\exp(-is\hat{H})\mathcal{D}_0 \subset \mathcal{D}_0$ and it is easy to check that $\exp(-is\hat{H})\mathcal{D}(p_0) \subset \mathcal{D}(p_0)$.)

We can now compute (with the convention that all constants $C > 0$ are independent of $R > 0$) writing $o(1)$ for errors converging to 0 as $R \to \infty$

$$\begin{aligned}\|p\chi_R(s)\|^2 &= \langle(\hat{H} + E_0\cdot x - V - \tau_I)\chi_R(s), \chi_R(s)\rangle \\ &\leq \theta C_1\langle s\rangle^2\|\chi_R(s)\|^2 + \|\tau_I\chi_R(s)\|\|\chi_R(s)\| + C_2\|\varphi\|^2 + o(1),\end{aligned} \tag{6.1}$$

where there error term comes from the commutators $i[\hat{H}, B_R]$ and $i[p_0, B_R]$ (see (5.1)). Since $\|\tau_I\chi_R(s)\| \leq C\langle s\rangle\|\varphi\|$ we get

$$\|p\chi_R(s)\|^2 \leq \theta C_3\langle s\rangle^2\|\chi_R(s)\|^2 + C_4\langle s\rangle\|\varphi\|^2 + o(1). \tag{6.2}$$

On the other hand we can estimate

$$\begin{aligned}\|p_0\chi_R(s)\|^2 &\geq \delta^2\langle s\rangle^2\|\chi(\frac{p_0^2}{\langle s\rangle^2} > \delta^2)\chi_R(s)\|^2 \\ &\geq \delta^2\langle s\rangle^2\|\chi_R(s)\|^2 - \delta^2\langle s\rangle^2\|\chi(\frac{p_0^2}{\langle s\rangle^2} < \delta^2)\chi_R(s)\|\|\chi_R(s)\| \\ &\geq \frac{1}{2}\delta^2\langle s\rangle^2\|\chi_R(s)\|^2 - C_5\langle s\rangle^2\|\chi(\frac{p_0^2}{\langle s\rangle^2} < \delta^2)\chi_R(s)\|^2.\end{aligned}$$

Combining this estimate with (6.2) and taking the limit $R \to \infty$ gives

$$(\frac{1}{2}\delta^2 - \theta C_3)\langle s\rangle^2\|\chi(s)\|^2 \leq C_5\langle s\rangle^2\|\chi(\frac{p_0^2}{\langle s\rangle^2} < \delta^2)\chi(s)\|^2 + C_4\langle s\rangle\|\varphi\|^2. \tag{6.3}$$

As in [HMS2, Appendix A] we have

$$[\chi(\frac{p_0^2}{\langle s\rangle^2} < \delta^2), \chi(\frac{|x|}{\langle s\rangle^2} < \theta)] = O(\langle s\rangle^{-3}).$$

By choosing $0 < \delta < \frac{\kappa}{4}$ we can use Corollary 6.4 to estimate (6.3) further

$$(\frac{1}{2}\delta^2 - \theta C_3)\langle s\rangle^2\|\chi(s)\|^2 \leq C_6\langle s\rangle^{2-2\rho}\|\varphi\|^2,$$

which yields the stated result by choosing $\theta < \frac{\delta^2}{2C_3}$. □

By the estimate (6.1) and Proposition 6.6 we get (since $p_0B_R \to p_0$ strongly on the domain of p_0 as $R \to \infty$)

Corollary 6.7. *Let $f \in C_0^\infty(\mathbb{R})$. We have the estimate*

$$p_0 \chi(\frac{|x_0|}{\langle s\rangle^2} < \theta) \exp(-is\hat{H}) f(\hat{H}) \langle p_0\rangle^{-1} \langle \tau_I\rangle^{-1} = O(\langle s\rangle^{1-\rho}).$$

Section 7: Asymptotic Completeness.

First we prove completeness for the Floquet Hamiltonians.

Proposition 7.1. *Assume V_t is short range and $E(t)$ satisfies Assumption 2.3. Then the wave operators*

$$\hat{W}^\pm = \mathrm{s} - \lim_{s\to\pm\infty} \exp(is\hat{H}) \exp(-is\hat{H}_0)$$

exist and are unitary.

Proof. By Proposition 4.5 it is sufficient to prove the result for constant non-zero fields. We will only prove existence of

$$\mathrm{s} - \lim_{s\to\pm\infty} \exp(is\hat{H}_0) \exp(-is\hat{H})$$

since existence of wave operators follows in the same way. By a standard argument this implies that the wave operators are unitary. Furthermore we restrict attention to the limit at $+\infty$.

Let $\varphi \in \hat{\mathcal{H}}$, $f \in C_0^\infty(\mathbb{R})$ and $\psi = f(\hat{H})\langle p_0\rangle^{-1}\langle\tau_I\rangle^{-1}\varphi$. For $\delta > 0$ we write $\chi_\delta(s)$ for $\chi(\frac{|x_0|}{\langle s\rangle^2} > \delta)$ and compute using Proposition 6.6

$$\begin{aligned}\exp(is\hat{H}_0)\exp(-is\hat{H})\psi &= \exp(is\hat{H}_0)\chi_{\frac{\theta}{2}}(s)\exp(-is\hat{H})\psi + o(1)\\ &= \int_0^s \exp(ir\hat{H}_0)\{i[\hat{H}_0, \chi_{\frac{\theta}{2}}(r)] + \chi_{\frac{\theta}{2}}(r)V + \partial_r \chi_{\frac{\theta}{2}}(r)\}\exp(-ir\hat{H})\psi dr.\end{aligned}$$

Here (as a form on $\mathcal{D}_0$)

$$i[\hat{H}_0, \chi_{\frac{\theta}{2}}(r)] = \langle r\rangle^{-2} b_1(r) p_0 \chi_\theta(r) + O(\langle r\rangle^{-4}) \tag{7.1}$$

and

$$\partial_r \chi_{\frac{\theta}{2}}(r) = \langle r\rangle^{-1} b_2(r) \chi_\theta(r).$$

where b_1 and b_2 denotes bounded functions from $\mathbb{R}_+$ into $\mathbb{R}$. By [Mø, Lema 2.2] applied with $H = p_0$, $A = \hat{H}_0$, $\tilde{H} = \chi_{\frac{\theta}{2}}(r)$ and $\mathcal{S} = \mathcal{D}_0$ we find that (7.1) holds as a form on $\mathcal{D}(\hat{H}) \cap \mathcal{D}(p_0)$ as well. Proposition 6.6 and Corollary 6.7 now yields existence of the integral. $\square$

Proof of Theorem 2.5. We restrict attention to the case $s = 0$ (since $W_\pm(s) = U(s,0)W_\pm(0)U_0(0,s)$). We know from the Avron-Herbst formula and a standard stationary phase argument that the physical wave operators

$$W_\pm(0) = \mathrm{s} - \lim_{t\to\pm\infty} U(0,t)U_0(t,0)$$

exist. One can compute as in [Ya1]

$$\hat{W}_\pm = UW_\pm(0)U_0^*,$$

which imply

$$\mathrm{Range}(\hat{W}_\pm) = UL^2([0,1]; \mathrm{Range}(W_\pm(0)))$$

and this shows by Theorem 7.1 that $W_\pm(0)$ are unitary. (See also [H2, Corollary 4.1].) The remaining statements follow from (3.1-2) and existence of wave operators. $\square$

Appendix A: Absolutely continuous vector-valued functions and the derivative.

In this Appendix we discuss absolutely continuous vector-valued functions as well as different realizations of the derivative on an interval.

We restrict attention to separable Hilbert-spaces since that is all we need in this paper and it makes the exposition simpler because the different notions of measurability coincide. See [RSI] and [T] for some material on measurability and integration. We just mention that integrals of vector-valued and operator-valued functions are weak and strong respectively.

In the case where $\mathcal{H}$ is finite-dimensional being absolutely continuous is equivalent to being an indefinit integral, see for example [R]. In general however this is not so (although being an indefinite integral implies absolute continuouity in the usual sense). In this section we work with indefinite integrals since these functions form natural domains for the derivative.

Definition A.1. *The space of absolutely continuous functions on the real line is defined by*

$$AC(\mathbb{R};\mathcal{H}) = \{f : \mathbb{R} \to \mathcal{H} : \exists g \in L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{H}) \ \text{and}\ \psi \in \mathcal{H}\ \text{s.t.}\ f(t) = \int_0^t g(s)ds + \psi\}$$

Note that $AC(\mathbb{R};\mathcal{H}) \subset C^0(\mathbb{R};\mathcal{H})$ the space of continuous $\mathcal{H}$-valued functions. It is easy to check that the map

$$(g,\psi) \to \int_0^t g(s)ds + \psi$$

from $L^1_{\text{loc}}(\mathbb{R};\mathcal{H}) \times \mathcal{H}$ onto $AC(\mathbb{R};\mathcal{H})$ is one to one and the following definition is therefore good

Definition A.2. *The derivative* $\partial : AC(\mathbb{R};\mathcal{H}) \to L^1_{\text{loc}}(\mathbb{R},\mathcal{H})$ *is given for* $g \in L^1_{\text{loc}}(\mathbb{R},\mathcal{H})$ *and* $\psi \in \mathcal{H}$ *by*

$$\partial(\int_0^t g(s)ds + \psi) = g.$$

We have as in the case $\mathcal{H} = \mathbb{C}$

Proposition A.3. *Let* $f \in AC(\mathbb{R};\mathcal{H})$*. Then the limit*

$$g(t) = \lim_{h\to 0} \frac{1}{h}(f(t+h) - f(t))$$

exists almost everywhere and $g = \partial f$*.*

Proof. Write $f(t) = \int_0^t \partial f(s)ds + f(0)$. Compute

$$\frac{1}{h}(f(t+h) - f(t)) - \partial f(t) = \frac{1}{h}\int_t^{t+h} \partial f(s) - \partial f(t)ds$$

and estimate

$$\|\frac{1}{h}(f(t+h) - f(t)) - \partial f(t)\| \leq \frac{1}{|h|}\int_t^{t+|h|} u_t(s)ds,$$

where $u_t(s) = \|\partial f(s) - \partial f(t)\|$ is in $L^1_{\text{loc}}(\mathbb{R})$ for almost every t. The limit on the right-hand side thus exists and equals $u_t(t) = 0$ almost everywhere. This concludes the proof. □

We will now consider abolutely continuous functions from $\mathbb{R}$ into the bounded operators between separable Hilbert-spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We write $\|\cdot\|_1$ and $\|\cdot\|_2$ for their respective norms and $\|\cdot\|_{1,2}$ for the norm on $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$.

Definition A.4. *We say* $B \in AC(\mathbb{R};\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$ *if* $t \to (B\psi)(t) := B(t)\psi \in AC(\mathbb{R};\mathcal{H}_2)$ *for all* $\psi \in \mathcal{H}_1$ *and* $\sup_{\|\psi\|_1\leq 1} \|\partial(B\psi)(\cdot)\|_2 \in L^1_{\text{loc}}(\mathbb{R})$*.*

If $B \in AC(\mathbb{R};\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$ there exists a family $\partial B \in L^1_{\text{loc}}(\mathbb{R};\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$ such that $(\partial B\psi)(t) = (\partial B)(t)\psi$ almost everywhere. Compute for $t > s$

$$\|B(t) - B(s)\|_{1,2} \leq \int_s^t \sup_{\|\psi\|_1\leq 1} \|(\partial B)(s)\psi\|_2 ds.$$

By assumption the right-hand side converge to 0 as $t \to s$ and the map $t \to B(t)$ is therefore continuous.

Again one can verify that the map

$$(A, B_0) \to \int_0^t A(s)ds + B_0$$

from $L^1_{\text{loc}}(\mathbb{R}; \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)) \times \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)$ onto $AC(\mathbb{R}; \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2))$ is one to one, which justifies the following

Definition A.5. *The derivative $\partial : AC(\mathbb{R}; \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)) \to L^1_{\text{loc}}(\mathbb{R}, \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2))$ is given for $A \in L^1_{\text{loc}}(\mathbb{R}, \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2))$ and $B_0 \in \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)$ by*

$$\partial(\int_0^t A(s)ds + B_0) = A.$$

We similarly have, copying the proof of Proposition A.3,

Proposition A.6. *Let $B \in AC(\mathbb{R}; \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2))$. Then the limit*

$$A(t) = \lim_{h \to 0} \frac{1}{h}(B(t+h) - B(t))$$

exists almost everywhere and $A = \partial B$.

Let $f, g \in AC(\mathbb{R}; \mathcal{H})$. Since f, g are indefinite integrals one can verify that $t \to \langle f(t), g(t) \rangle$ is an absolutely continuous functions in the ordinary sense and hence itself an indefinite integral. This argument shows

$$f, g \in AC(\mathbb{R}; \mathcal{H}) \Rightarrow \langle f(\cdot), g(\cdot) \rangle \in AC(\mathbb{R}). \tag{A.1}$$

We turn to the analysis of the Hilbert-space $L^2([0,T]; \mathcal{H})$, $T > 0$, and consider the space of absolutely continuous functions with square integrable derivative

$$AC^2([0,T]; \mathcal{H}) = \{f \in AC([0,T]; \mathcal{H}) : \partial f \in L^2([0,T]; \mathcal{H})\}$$

and the operators $\tau_0 \subset \tau_V \subset \tau_*$, $V \in \mathcal{U}(\mathcal{H})$ (the unitary operators on $\mathcal{H}$), which are different realizations of $-i\partial$ on the interval, namely with the respective domains

$$\mathcal{D}_* = AC^2([0,T]; \mathcal{H}), \quad \mathcal{D}_0 = \{f \in \mathcal{D}_* : f(0) = f(T) = 0\}$$

and

$$\mathcal{D}_V = \{f \in \mathcal{D}_* : f(0) = Vf(T)\}.$$

The following result can be verified as in [RSI] (for the case $\mathcal{H} = \mathbb{C}$).

Proposition A.7. *We have*

i) τ_* *is closed.*
ii) τ_0 *is closed and symmetric.*
iii) τ_V *is self-adjoint for any* $V \in \mathcal{U}(\mathcal{H})$.
iv) $\sigma(\tau_0) = \emptyset$ *and* $\sigma_{\mathrm{pp}}(\tau_*) = \mathbb{C}$.
v) *The adjoint of* τ_0 *equals* τ_*.
vi) *The spectrum of the* τ_V*'s are periodic with period* $\frac{2\pi}{T}$.

Furthermore, the resolvent of τ_V *is given pointwisely by*

$$((\tau_V - \lambda)^{-1} f)(t) = i \int_0^t e^{i\lambda(s-t)} f(s) ds + i(V^* - e^{i\lambda T})^{-1} \int_0^T e^{i\lambda(s-T)} f(s) ds \quad \text{(A.4)}$$

for $\lambda \in \mathbb{C}$, *operatornameIm*$\lambda \neq 0$.

Let $B \in \mathcal{B}(\mathcal{H})$. We lift B to a bounded operator on $L^2([0,T];\mathcal{H})$ by $(Bf)(t) = Bf(t)$ for $f \in C^0([0,T];\mathcal{H})$ and extend it by continuity. The following identity is easily verified.

$$S^* \tau_V S = \tau_{S^*VS} \quad \text{for} \quad S \in \mathcal{U}(H). \quad \text{(A.3)}$$

Here we have used the same symbol for the operator itself and its lifting. If there is cause for confusion we will write $\oint B$ for the lifted operator. We have for example

$$\sigma(\oint B) = \sigma(B), \quad \text{(A.4)}$$

which follows since resolvents of $\oint B$ maps the subspace of constant functions into itself. In fact $(\oint B - z)^{-1} = \oint (B-z)^{-1}$.

We now determine the flow generated by τ_V. Let $f \in C^0([0,T];\mathcal{H})$ and define

$$(U_V(s)f)(t) = V^{-\frac{[t-s]}{T}} f(t - s - [t-s]),$$

where $[s]$ is the largest multiple of T smaller than s. This is clearly a one-parameter strongly continuous unitary group.

Proposition A.8. *Let* $V \in \mathcal{U}(H)$ *and* $s \in \mathbb{R}$. *Then* $\exp(-is\tau_V) = U_V(s)$.

Proof. It is sufficient to prove that the generator of U_V coincide with τ_V on $f \in \mathcal{D}_V \cap C^\infty([0,T];\mathcal{H})$ which by (A.2) is a core for τ_V. We compute

$$\begin{aligned} \|i\frac{1}{s}(U_V(s)f - f) - \tau_V f\|^2 = & \int_0^s \|\frac{1}{s}(Vf(T+t-s) - f(t)) + f'(t)\|^2 dt \\ & + \int_s^T \|\frac{1}{s}(f(t-s) - f(t)) + f'(t)\|^2 dt. \end{aligned}$$

By the choice of f and the Lebesgue Theorem on dominated convergence we see that the right-hand side converge to zero as s tends to zero, which proves the result. □

We end the appendix with a structure result

Proposition A.9. *Let $V \in \mathcal{U}(\mathcal{H})$. Then*

$$\begin{aligned}
\lambda \in \sigma_{\mathrm{pp}}(\tau_V) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{pp}}(V), \quad \mathcal{H}_{\mathrm{pp}}(\tau_V) = L^2([0,T]; \mathcal{H}_{\mathrm{pp}}(V)),\\
\lambda \in \sigma_{\mathrm{ac}}(\tau_V) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{ac}}(V), \quad \mathcal{H}_{\mathrm{ac}}(\tau_V) = L^2([0,T]; \mathcal{H}_{\mathrm{ac}}(V)),\\
\lambda \in \sigma_{\mathrm{sc}}(\tau_V) &\iff e^{-i\lambda T} \in \sigma_{\mathrm{sc}}(V), \quad \mathcal{H}_{\mathrm{sc}}(\tau_V) = L^2([0,T]; \mathcal{H}_{\mathrm{sc}}(V)).
\end{aligned}$$

Proof. By Proposition A.8 we find that $\exp(-iT\tau_V) = \oint V$. This shows the equivalence of pure point spectrums and it implies the result for the pure point subspace. It also shows that for a Borel set $\Omega \subset S^1$ (the unit circle) and $f \in L^2([0,T]; \mathcal{H})$

$$\langle f, P_\Omega(\exp(-iT\tau_V))f\rangle = \int_0^T \langle f(t), P_\Omega(V)f(t)\rangle,$$

where P_Ω denotes the characteristic function for the set Ω. This identity shows the stated identity for the absolutely continuous subspace and since the spectral subspaces decomposes the Hilbert space we find the identity for the singular continuous subspaces as well. The equivalence of the last two spectra now follows from this discussion and (A.4). $\square$

Appendix B: The time-dependent Schrödinger equation.

Let $\{H(t)\}_{t\in\mathbb{R}}$ be a family of self-adjoint operators on a separable Hilbert-space $\mathcal{H}$ which satisfies that there exists a dense subspace $\mathcal{S}$ with

$$\mathcal{S} \subset \cap_{t\in\mathbb{R}} \mathcal{D}(H(t)).$$

We wish to discuss the time-dependent Schrödinger equation corresponding to the family $H(t)$, in particular what is a solution supposed to satisfy. The following suggestion is a natural one: A (two-parameter) family of unitary operators $\{U(t,s)\}_{t,s\in\mathbb{R}}$ is a solution to the time-dependent Schrödinger equation if

(1) $U(t,s)\mathcal{S} \subset \mathcal{S}$ for all $t, s \in \mathbb{R}$.

(2) For any $\varphi \in \mathcal{S}$ the map $t \to U(t,s)\varphi$, admits a pointwise derivative almost everywhere and its derivative satisfies the vector-valued differential equation

$$i\frac{d}{dt}U(t,s)\varphi = H(t)U(t,s)\varphi, \quad U(s) = I, \tag{B.1}$$

almost everywhere.

Solutions to this equation are not unique and we therefore need to discuss which one we will consider (if any exist).

Let $\mathcal{S}$ be as above. A natural class of unitary families would be those for which $U\varphi \in AC(\mathbb{R};\mathcal{H})$ for all $\varphi \in \mathcal{S}$. It is however not clear why this family should be stable under composition, which makes it difficult to work with.

Instead we have a slightly weaker result which we choose to supply since it covers what is needed in the present paper.

We consider the case where $\mathcal{H}_1 \subset \mathcal{H}_2$ is a dense sub Hilbert-space equipped with a stronger norm.

$$\|\psi\|_2 \leq C\|\psi\|_1, \quad \psi \in \mathcal{H}_1,$$

for some $C > 0$. (Compared to above $\mathcal{S} = \mathcal{H}_1$ and $\mathcal{H} = \mathcal{H}_2$.) We write $\tilde{\mathcal{S}}_{1,2}$ for the space of symmetric operators with domain containing $\mathcal{H}_1$ and we say $H_1 \sim H_2$ if $H_{1|\mathcal{H}_1} = H_{2|\mathcal{H}_1}$ and define $\mathcal{S}_{1,2} = \tilde{\mathcal{S}}_{1,2}/\sim$. Since a symmetric operator $H \in \mathcal{S}_{1,2}$ is closable we find that H as an operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ is closed and hence bounded by the Closed Graph Theorem. There is thus a canonical inclusion of $\mathcal{S}_{1,2}$ into $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$. In the norm $\sup_{\|\psi\|_1\leq 1}\|H\psi\|_2$ the space $\mathcal{S}_{1,2}$ is complete and the inclusion into $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ is an isometry.

We will consider operator families H from the space $L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{S}_{1,2})$ which we identify with a subspace of $L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$. We are interested in solutions $U(\cdot,s)$ to the evolution equation

$$i\partial U = HU, \quad U(s) = I. \tag{B.2}$$

Solutions will be sought in the set $\mathcal{U}_{1,2}$.

Definition B.1. *The set $\mathcal{U}_{1,2}$ consists of $U : \mathbb{R} \to \mathcal{U}(\mathcal{H}_2)$ which are measurable and satisfy that $U_{|\mathcal{H}_1} \in AC(\mathbb{R};\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$ and $U\psi, U^*\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ for all $\psi \in \mathcal{H}_1$.*

Note that being a solution to (B.2) is consistent with being an element of $\mathcal{U}_{1,2}$. All operators are identified with operators on the large Hilbert-space $\mathcal{H}_2$ and adjoints are taken with respect to the inner product $\langle\cdot,\cdot\rangle_2$.

Note that families in $\mathcal{U}_{1,2}$ are strongly continuous and by the Uniform Boundedness Principle we have

$$\sup_{|t|\leq T}\left(\|U(t)\|_{1,1} + \|U^*(t)\|_{1,1}\right) < \infty, \tag{B.3}$$

for any $T > 0$.

Instead of (B.2) one could seek solutions $U(s,\cdot)$ from $\mathcal{U}_{1,2}$ to the equation

$$i\partial U = -UH, \quad U(s,s) = I \tag{B.4}$$

and in fact we have the following result

Lemma B.2. *We have*

i) *Suppose* $U(\cdot, s) \in \mathcal{U}_{1,2}$ *solves* (B.2). *Then* $U^*(\cdot, s) \in \mathcal{U}_{1,2}$ *and it solves* (B.4).
ii) *Suppose* $U(s, \cdot) \in \mathcal{U}_{1,2}$ *solves* (B.4). *Then* $U^*(s, \cdot) \in \mathcal{U}_{1,2}$ *and it solves* (B.2).

Proof. We only verify i) since ii) can be proved in similar fashion. Since $U(\cdot, s)$ is in $AC(\mathbb{R}; \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2))$ and solves (B.2) we find

$$iU(t, s) = I + \int_s^t H(r)U(r, s)dr.$$

Due to the construction of the integral (see Appendix A) we thus find, for $\psi, \varphi \in \mathcal{H}_1$,

$$\langle \psi, iU(t, s)\varphi\rangle = \langle \psi, \varphi\rangle + \int_s^t \langle U^*(r, s)H(r)\psi, \varphi\rangle dr,$$

which implies that

$$-iU^*(t, s) = I + \int_s^t U^*(r, s)H(r)dr.$$

Hence $U^*(\cdot, s) \in \mathcal{U}_{1,2}$ and it solves (B.4). □

The solution set is a group, more precisely

Proposition B.3. *Let* $U_1, U_2 \in \mathcal{U}_{1,2}$ *and* $\psi \in AC(\mathbb{R}; \mathcal{H}_2) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathcal{H}_1)$. *Then we have*

i) $U_1^* \in \mathcal{U}_{1,2}$, $\mathcal{H}_1 \subset \mathcal{D}((\partial U)^*)$ *and* $\partial U^* = (\partial U)^*_{|\mathcal{H}_1}$.
ii) $U_1\psi \in AC(\mathbb{R}; \mathcal{H}_2) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathcal{H}_1)$ *and* $\partial U_1\psi = (\partial U_1)\psi + U\partial\psi$.
iii) $U_2U_1 \in \mathcal{U}_{1,2}$ *and* $\partial(U_2U_1) = (\partial U_2)U_1 + U_2\partial U_1$.

Proof. Compute for $\psi, \varphi \in \mathcal{H}_1$ using (A.1) and Proposition A.6

$$0 = \partial\langle\psi, \varphi\rangle_2 = \partial\langle U_1(\cdot)\psi, U_1(\cdot)\varphi\rangle_2 = \langle(\partial U_1)(\cdot)\psi, U_1(\cdot)\varphi\rangle_2 + \langle U_1(\cdot)\psi, (\partial U_1)(\cdot)\varphi\rangle_2,$$

which shows that $U_1^*(i\partial U_1) \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathcal{S}_{1,2})$. For $\psi, \varphi \in \mathcal{H}_1$ we thus have

$$\langle\psi, (\partial U_1)(\cdot)\varphi\rangle_2 = \langle U_1^*(\cdot)\psi, U_1^*(\cdot)(\partial U_1)(\cdot)\varphi\rangle_2 = -\langle U_1^*(\cdot)(\partial U_1)(\cdot)U_1^*(\cdot)\psi, \varphi\rangle_2.$$

This implies that $\mathcal{H}_1 \subset \mathcal{D}((\partial U_1)^*(t))$ for almost all t and

$$(\partial U_1)^*_{|\mathcal{H}_1} = -U_1^*(\partial U_1)U^*_{1|\mathcal{H}_1}. \tag{B.5}$$

As in the proof of Lemma B.2, this shows that $U_1^* \in \mathcal{U}_{1,2}$.

Let U_1 and ψ be as in the statement of the Proposition. Compute for $\varphi \in \mathcal{H}_1$ using (A.1) and the above

$$\langle \varphi, U_1(t)\psi(t)\rangle = \int_0^t h(s)ds + \langle \varphi, U_1(0)\psi(0)\rangle,$$

where by Proposition A.6

$$\begin{aligned} h &= \partial\langle U_1^*(\cdot)\varphi, \psi(\cdot)\rangle \\ &= \langle (\partial U_1)^*(\cdot)\varphi, \psi(\cdot)\rangle + \langle U_1^*(\cdot)\varphi, (\partial\psi)(\cdot)\rangle \\ &= \langle \varphi, (\partial U_1)(\cdot)\psi(\cdot) + U_1(\cdot)(\partial\psi)(\cdot)\rangle. \end{aligned}$$

By construction of the integral and (B.3) this concludes the proof of ii) and iii) follows directly from ii). □

Notice that by (B.5) the maps

$$\Phi_L(U) = (i\partial U)U^* \quad \text{and} \quad \Phi_R(U) = U^*(i\partial U)$$

both map $\mathcal{U}_{1,2}$ into $L^1_{\mathrm{loc}}(\mathbb{R}; S_{1,2})$ and U is hence a solution to (B.2) with $H = \Phi_L(U)$ and to (B.4) with $H = \Phi_R(U)$.

Theorem B.4 (Uniqueness). *Let $s \in \mathbb{R}$ and $H \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathcal{S}_{1,2})$.*

i) *There can be at most one solution $U(\cdot, s) \in \mathcal{U}_{1,2}$ to* (B.2).
ii) *There can be at most one solution $U(s, \cdot) \in \mathcal{U}_{1,2}$ to* (B.4).

Proof. We restrict attention to i). Assume there exist $U_1(\cdot, s), U_2(\cdot, s) \in \mathcal{U}_{1,2}$ such that $\Phi_L(U_1(\cdot, s)) = \Phi_L(U_2(\cdot, s))$. Compute using Proposition B.3 i), iii) and (B.5)

$$\begin{aligned} \partial(U_2^*U_1) &= (\partial U_2^*)U_1 + U_2^*(\partial U_1) \\ &= -U_2^*(\partial U_2)U_2^*U_1 + U_2^*(\partial U_1)U_1^*U_1 \\ &= -iU_2^*\Phi_L(U_2)U_1 + iU_2^*\Phi_L(U_1)U_1 \\ &= 0. \end{aligned}$$

This shows that $U_2(\cdot, s)^*U_1(\cdot, s) = U_0(s)$, a unitary operator, and completes the proof since the left hand side equals the identity at $t = s$. □

We have the following easy consequences of Lemma B.2, Proposition B.3 and Theorem B.4.

Corollary B.5. *Let $U(t, s)$ be a family of unitary operators. The following three statements are equivalent*

i) $U(\cdot, s), U(s, \cdot) \in \mathcal{U}_{1,2}$ *and solves both* (B.2) *and* (B.4).
ii) $U(\cdot, s) \in \mathcal{U}_{1,2}$ *solves* (B.2) *and* $U^*(t, s) = U(s, t)$.
iii) $U(s, \cdot) \in \mathcal{U}_{1,2}$ *solves* (B.4) *and* $U^*(t, s) = U(s, t)$.

Corollary B.6 (Chapman-Kolmogorov). *Suppose the family $U(t,s)$ satisfies either one of the three conditions in Corollary B.5. Then*

$$U(t,s)U(s,r) = U(t,r) \quad \textit{for all} \quad t,s,r \in \mathbb{R}.$$

In the light of Theorem B.4 and Corollary B.5 we will in the present paper employ the following definition of what a solution to the time-dependent Schrödinger equation is

Definition B.7. *Let $H(t)$ be a family of Hamiltonians for which there exists a Hilbert-space $\mathcal{H}_1 \subset \mathcal{H}_2 = \mathcal{H}$ such that $H \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathcal{S}_{1,2})$. A family of unitary operators $U(t,s)$ is said to solve the time-dependent Schrödinger equation if*

i) $U(\cdot,s), U(s,\cdot) \in \mathcal{U}_{1,2}$.
ii) $U(\cdot,s)$ *solves* (B.2) *and* $U(s,\cdot)$ *solves* (B.4).

For other purposes one might want to assume the Hamiltonians $H(t)$ to be essentially self-adjoint on $\mathcal{H}_1$ for almost all t instead of just symmetric. This is however not nescessary in our situation. In construction procedures one often considers a corresponding integral equation and invariance of some domain is a typical byproduct, see [Ta] and [Ya1]. In [DG, Appendix B.3] a situation similar to ours is considered and it is shown how one can use control of certain commutators to obtain invariance of a domain under the flow. Their approach is independent of a construction procedure. These two observations indicate that Definition B.7 is not to restrictive.

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