Edge currents without edges

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We investigate a charged two-dimensional particle in a homogeneous magnetic field interacting with a periodic array of point obstacles. We show that while Landau levels remain to be infinitely degenerate eigenvalues, between them the system has bands of absolutely continuous spectrum and exhibits thus a transport along the array. In distinction to the usual edge states, this is a purely quantum effect. We compute the band functions and the corresponding probability current.

The fact that the presence of boundaries can induce a transport in a system with a homogeneous magnetic field is known for long [1] and has numerous consequences in solid state physics. A fresh interest to the problem comes from the observation that the edge currents in a halfplane [2] survive a mild disorder so that away of the Landau levels the spectrum remains absolutely continous [3,4].

The wall producing the edge states need not be of a potential type. It is known, e.g., that a step of the magnetic field or another variation exhibiting a translational symmetry will smear again the Landau levels into a continuous spectrum [5,6]. Similarly to the usual edge states, this type of propagation allows for a classical explanation in terms of the cyclotronic radius changing with the magnetic field – see Ref. [7], Sec. 6.5.

The aim of the present paper is to present a simple model in which the magnetic transport is a purely quantum effect in the sense that a quantum particle propagates despite the fact that its classical counterpart moves on localized circular trajectories – apart of a zero-measure family of the initial conditions. Our model describes a charged quantum particle in the plane exposed to a homogeneous magnetic field of intensity B perpendicular to the plane and interacting with a periodic array of point obstacles situated at the x axis and described by δ potentials. Using the Landau gauge, we can write the Hamiltonian formally as [8]

$$H_{\alpha,\ell} = (-i\partial_x + By)^2 - \partial_y^2 + \sum_j \tilde\alpha\delta(x-x_0-j\ell)\,, \quad (1)$$

where $\ell>0$ is the array spacing. The interaction term, in particular the formal coupling constant $\tilde\alpha$, needs an explanation, since the two-dimensional δ potential is an involved object. We follow the usual definition [9] which determines the latter by means of the boundary conditions [10]

$$L_1(\psi,\vec a_j) + 2\pi\alpha L_0(\psi,\vec a_j) = 0\,, \quad j=0,\pm1,\pm2,\dots \quad (2)$$

with $\vec a_j := (x_0+j\ell,0)$, where L_k are the generalized boundary values

$$L_0(\psi,\vec a) := \lim_{|\vec x-\vec a|\to 0} \frac{\psi(\vec x)}{\ln|\vec x-\vec a|}\,,$$

$$L_1(\psi,\vec a) := \lim_{|\vec x-\vec a|\to 0} \Big[\psi(\vec x) - L_0(\psi,\vec a)\,\ln|\vec x-\vec a|\Big]\,,$$

and α is the (rescaled) coupling constant; the free (Landau) Hamiltonian corresponds to $\alpha=\infty$.

Using the periodicity, we can perform the Bloch decomposition in the x direction writing

$$H_{\alpha,\ell} = \frac{\ell}{2\pi}\int_{|\theta\ell|\le\pi}^{\oplus} H_{\alpha,\ell}(\theta)\,d\theta\,, \quad (3)$$

where the fiber operator $H_{\alpha,\ell}(\theta)$ is of the form (1) on the strip $0\le x\le\ell$ with the boundary conditions

$$\partial_x^i\psi(\ell-,y) = e^{i\theta\ell}\partial_x^i\psi(0+,y)\,, \quad i=0,1\,. \quad (4)$$

The Green's function of the operator $H_{\alpha,\ell}(\theta)$ is given by means of the Krein formula [9, App.A],

$$(H_{\alpha,\ell}(\theta)-z)^{-1}(\vec x,\vec x') = G_0(\vec x,\vec x';\theta,z) + (\alpha-\xi(\vec a_0;\theta,z))^{-1}G_0(\vec x,\vec a_0;\theta,z)G_0(\vec a_0,\vec x';\theta,z)\,, \quad (5)$$

where G_0 is the free Green's function and

$$\xi(\vec a;\theta,z) := \lim_{|\vec x-\vec a|\to 0}\left(G_0(\vec a,\vec x;\theta,z) - \frac{1}{2\pi}\ln|\vec x-\vec a|\right) \quad (6)$$

is its regularized value at the point $\vec a$. The Bloch conditions (4) determine eigenvalues and eigenfunctions of the transverse part of the free operator,

$$\mu_m(\theta) = \left(\frac{2\pi m}{\ell}+\theta\right)^2,\ \eta_m^\theta(x) = \frac{1}{\sqrt{\ell}}\,\mathrm{e}^{i(2\pi m+\theta\ell)x/\ell}\,, \quad (7)$$

where m runs through integers. Then we have

$$G_0(\vec x,\vec x';\theta,z) = -\sum_{m=-\infty}^{\infty}\frac{u_m^\theta(y_<)v_m^\theta(y_>)}{W(u_m^\theta,v_m^\theta)}\,\eta_m^\theta(x)\overline{\eta_m^\theta}(x')\,, \quad (8)$$

where $y_<, y_>$ is the smaller and larger value, respectively, of y, y', and u_m^θ, v_m^θ are solutions to the equation

$$-u''(y) + \left(By + \frac{2\pi m}{\ell} + \theta\right)^2 u(y) = zu(y) \qquad (9)$$

such that u_m^θ is L^2 at $-\infty$ and v_m^θ is L^2 at $+\infty$; in the denominator we have their Wronskian. By the argument shift we get

$$u_m^\theta(y) = u\left(y + \frac{2\pi m + \theta\ell}{B\ell}\right) \qquad (10)$$

and a similar relation for v_m^θ, where u, v are the corresponding oscillator solutions. Of course, we have $W(u_m^\theta, v_m^\theta) = W(u, v)$. The functions u, v express in terms of the confluent hypergeometric functions [11, Chap. 13]:

$$v(y) = e^{-By^2/2} U\left(\frac{B-z}{4B}, \frac{1}{2}; By^2\right) \qquad (11)$$

away from zero, and u is obtained by analytical continuation in the y^2 variable; together we have

$$\left\{ (Font) LaTeX Font Warning : Command invalid in math mode 1011.5pt c \begin{matrix} u \\ v \end{matrix} \right\}(y) = \sqrt{\pi}\, e^{-By^2/2} \left[\frac{M\left(\frac{B-z}{4B}, \frac{1}{2}; By^2\right)}{\Gamma\left(\frac{3B-z}{4B}\right)} \pm 2\sqrt{B}y \frac{M\left(\frac{3B-z}{4B}, \frac{3}{2}; By^2\right)}{\Gamma\left(\frac{B-z}{4B}\right)} \right]. \quad (12)$$

From here and Ref. [11, Chap. 6] we compute the Wronskian; in combination with (7) we get

$$G_0(\vec{x}, \vec{x}'; \theta, z) = -\frac{2^{(z/2B)-(3/2)}}{\sqrt{\pi B}\ell} \Gamma\left(\frac{B-z}{2B}\right) \mathrm{e}^{i\theta(x-x')}$$
$$\times \sum_{m=-\infty}^{\infty} u\left(y_< + \frac{2\pi m + \theta\ell}{B\ell}\right) v\left(y_> + \frac{2\pi m + \theta\ell}{B\ell}\right)$$
$$\times\, \mathrm{e}^{2\pi i m(x-x')/\ell}. \qquad (13)$$

As expected the function has singularities which are independent of θ and coincide with the Landau levels, i.e., $z_n = B(2n+1)$, $n = 0, 1, 2, \ldots$. Let us observe first that each z_n remains to be infinitely degenerate eigenvalue of the "full" fiber operator $H_{\alpha,\ell}(\theta)$. To this end, one has to adapt the argument of Ref. [12] to the set of functions $w^k \sin\left(\frac{\pi w}{\ell}\right) e^{-B|w|^2/4}$, $k = 0, 1, \ldots$, with $w := x+iy$ which vanish at the points of the array so the conditions (2) are satisfied for them automatically.

On the other hand, $H_{\alpha,\ell}(\theta)$ has also eigenvalues away of z_n which we denote as $\epsilon_n(\theta) \equiv \epsilon_n^{(\alpha,\ell)}(\theta)$. In view of (5) they are given by the implicit equation

$$\alpha = \xi(\vec{a}_0; \theta, \epsilon) \qquad (14)$$

and the corresponding eigenfunctions are

$$\psi_n^{(\alpha,\ell)}(\vec{x}; \theta) = G_0(\vec{x}, \vec{a}_0; \theta, \epsilon_n(\theta)). \qquad (15)$$

In order to evaluate them, we have to assess the convergence of the series in (13). Using the asymptotic behavior

$$\left\{ (Font) LaTeX Font Warning : Command invalid in math mode 1011.5pt c \begin{matrix} u \\ v \end{matrix} \right\}(y) = e^{\mp\{\pm\}By^2/2} \left(\mp\sqrt{B}y\right)^{\frac{z-B}{2B}} \left(1 + \mathcal{O}(|y|^{-2})\right) \qquad (16)$$

for $y \to \mp\infty$, we find that the product

$$s_m := u\left(y_< + \frac{2\pi m + \theta\ell}{B\ell}\right) v\left(y_> + \frac{2\pi m + \theta\ell}{B\ell}\right)$$

is for $y \neq y'$ governed by the exponential term,

$$s_m \sim \exp\left\{\frac{B}{2}\left(y_<^2 - y_>^2\right) + \left(\theta - \frac{2\pi|m|}{\ell}\right)(y_> - y_<)\right\}$$
$$\times \left(|m|^{-1} + \mathcal{O}(|m|^{-2})\right)$$

as $|m| \to \infty$, while for $y = y'$ we have

$$s_m = -\frac{1}{4\pi}|m|^{-1} + \mathcal{O}(|m|^{-2}),$$

so the series (13) is not absolutely convergent. Summing now the contributions from $\pm m$ we see that in the limit $x' \to x$ it diverges at the same rate as the Taylor series of $-(1/2\pi)\ln\zeta$ does for $\zeta \to 0$. Hence we get

$$\xi(\vec{x}; \theta, z) = \sum_{m=-\infty}^{\infty} \left\{ \frac{1 - \delta_{m,0}}{4\pi|m|} \right.$$
$$\left. - \frac{2^{-2\zeta-1}}{\sqrt{\pi B}\ell} \Gamma(2\zeta)\, (uv)\left(y + \frac{2\pi m + \theta\ell}{B\ell}\right) \right\}, \qquad (17)$$

where $\zeta := \frac{B-z}{4B}$. The expression is independent of x, because the regularized resolvent does not change if the array is shifted in the x direction. We can write it by means of the first hypergeometric function alone, since

$$(uv)(\xi/\sqrt{B})$$
$$= \pi\, \mathrm{e}^{-\xi^2} \left[\frac{M(\zeta, \frac{1}{2}; \xi^2)^2}{\Gamma(\zeta + \frac{1}{2})^2} - 4\xi^2 \frac{M(\zeta + \frac{1}{2}, \frac{3}{2}; \xi^2)^2}{\Gamma(\zeta)^2} \right], \qquad (18)$$

where $\xi := \sqrt{B}\left(y + \frac{2\pi m + \theta\ell}{B\ell}\right)$.

Spectral bands of our model are given by the ranges of the functions $\epsilon_n(\cdot)$. Solutions of the condition (14) do not cross the Landau levels, because $\xi(\vec{a}_0; \theta, \cdot)$ is increasing in the intervals $(-\infty, B)$ and $(B(2n-1), B(2n+1))$ and diverges at the endpoints; this is a general feature [13]. The spectrum will be continuous away of z_n if the latter are nowhere constant. In view of the spectral condition (14) one has to check that $\xi(\vec{x}; \theta, z)$ is nowhere constant as a function of θ. Notice that each term in (17) is real-analytic for real z and the series has a convergent majorant independent of θ; hence $\xi(\vec{x}; \cdot, z)$ is real-analytic as well and one has to check that it is non-constant in the whole Brillouin zone $[-\pi/\ell, \pi/\ell)$.

Suppose that the opposite is true. Then the Fourier coefficients

$$c_k := \int_{-\pi/\ell}^{\pi/\ell} \xi(\vec{x};\theta,z)\, e^{ik\ell\theta}\, d\theta \tag{19}$$

should vanish for any non-zero integer k. Since the summand in (17) behaves as $\mathcal{O}(|m|^{-2})$ as $|m|\to\infty$, we may interchange the summation and integration. A simple change of variables then gives

$$\begin{aligned} c_k = &-\frac{2^{-2\zeta-1}}{\sqrt{\pi B}\ell}\,\Gamma(2\zeta) \\ &\times \lim_{M\to\infty}\int_{-\pi(2M+1)}^{\pi(2M+1)} (uv)\left(y+\frac{\vartheta}{B\ell}\right) e^{ik\vartheta}\, d\vartheta\,, \end{aligned} \tag{20}$$

so

$$\hat{F}_y(k) := \int_{-\infty}^{\infty} F_y(\vartheta)\, e^{ik\vartheta}\, d\vartheta = 0\,, \tag{21}$$

where $F_y(\vartheta) := (uv)\left(y+\frac{\vartheta}{B\ell}\right)$. The same reasoning applies to any finitely periodic extension of $\xi(\vec{x};\theta,z)$, hence (21) is valid for each non-zero rational k. However, the function decays $\mathcal{O}(|\vartheta|^{-1})$ and the integral makes sense only as the principal value. We shall use the above mentioned asymptotic behavior which implies, in particular,

$$F_y(\vartheta) = -\frac{1}{4\pi\sqrt{1+\vartheta^2}} + f_y(\vartheta)\,, \tag{22}$$

where $f_y(\vartheta) = \mathcal{O}(|\vartheta|^{-2})$ uniformly in $y\in[0,\ell]$. Thus

$$\hat{F}_y(k) = -\frac{1}{2\pi}\, K_0(k) + \hat{f}_y(k)\,, \tag{23}$$

see [14, 3.754.2]. Since $f_y \in L^1$, the second term at the r.h.s. is continuous w.r.t. k and the same is then true for $\hat{F}_y$; this means that the relation (21) is valid for any nonzero k. Furthermore, $\hat{f}_y$ is bounded and K_0 diverges logarithmically at $k=0$, hence $\int_{-N}^{N} F_y(\vartheta)\, e^{ik\vartheta}\, d\vartheta$ can be bounded by an integrable function independent of N. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{F}_y(k)\phi(k)\, dk &= \int_{-\infty}^{\infty} dk\, \phi(k) \lim_{N\to\infty}\int_{-N}^{N} F_y(\vartheta)\, e^{ik\vartheta}\, d\vartheta \\ &= \int_{-\infty}^{\infty} F_y(\vartheta)\, \hat{\phi}(\vartheta)\, d\vartheta \end{aligned} \tag{24}$$

holds for any $\phi\in\mathcal{S}(\mathbf{R})$, i.e., $\hat{F}_y(k)$ is the Fourier transform of $F_y(\vartheta)$ in the sense of tempered distributions. Since this is a one-to-one correspondence [15, Thm.IX.2], we arrive at the absurd conclusion that $F_y = 0$. We get thus the following result [16]:

Theorem. For any real α the spectrum of $H_{\alpha,\ell}$ consists of the Landau levels $B(2n+1)$, $n=0,1,2,\ldots$, and absolutely continuous spectral bands situated between adjacent Landau levels and below B.

This result says a little about the character of the transport. To get a better idea we solve the spectral condition (14) numerically for several values of the parameters. The results are plotted in Fig. 1 for the second and 21st spectral band. We see that the bands move downwards with decreasing α and their profile becomes more complicated with the band index n; a higher B tends to smear the structure.

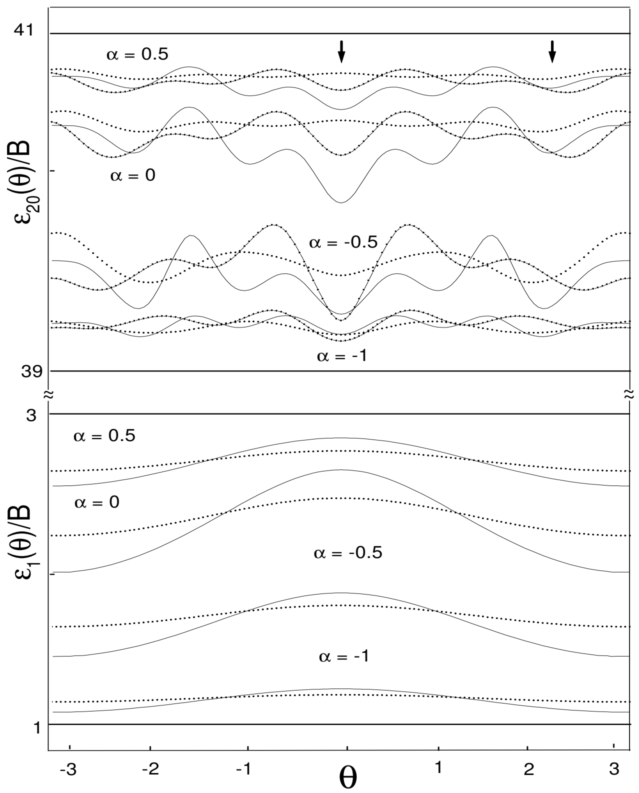

FIG. 1. The eigenvalues of $H_{\alpha,l}(\theta)$ in the units of B. (Top) $n=20$, $B=4$ (full line), $B=6$ (full dotted line) and $B=8$ (dotted line). (Bottom) $n=1$, $B=10$ (full line) and $B=15$ (dotted line). The thick lines represent the Landau levels.

The Bloch eigenfunctions (15) are in general complex-valued and yield thus a nontrivial probability current, $\vec{\jmath}_n(\vec{x};\theta) = 2\,\mathrm{Im}\left(\bar{\psi}_n^{(\alpha,\ell)}(\vec{\nabla}-i\vec{A})\psi_n^{(\alpha,\ell)}\right)(\vec{x};\theta)$. The current pattern changes with θ oscillating between a symmetric "two-way" picture and the situations where one direction clearly prevails, these extremal behaviors occurring at the extrema of the corresponding band function. This is illustrated in Fig. 2.

In addition, while the pattern has predominantly "laminar" character, in some parts current vortices may form, mainly in low spectral bands as it is illustrated in Fig. 3.

In conclusion, we have analyzed the behavior of a quantum particle in the plane exposed to a homogeneous mag-

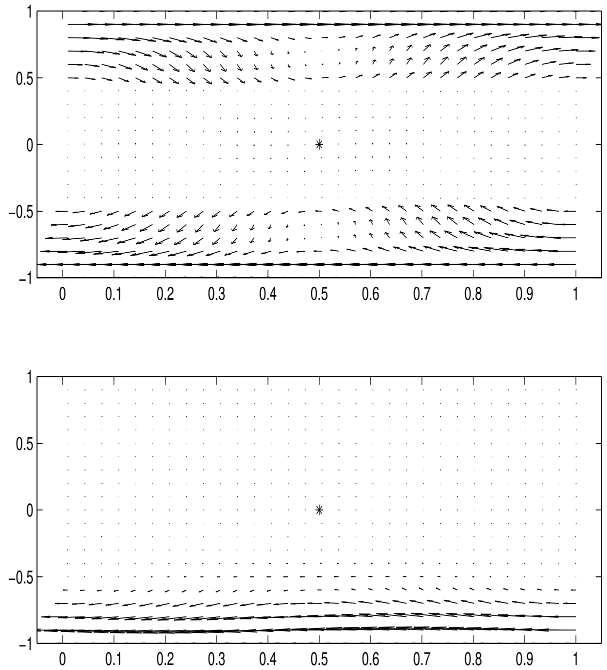

FIG. 2. Probability current for $n = 20$, $B = 4$, $\alpha = 0.5$ and two different values of θ corresponding to the extremal points of $\epsilon_{20}(\cdot)$: $\theta = 0$ (top), $\theta = 2.2$ (bottom), see the arrows in FIG.1. The star marks the point perturbation position.

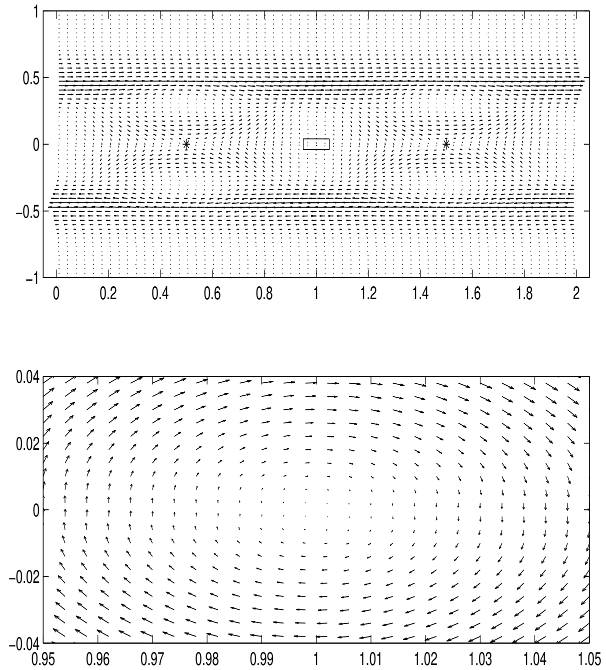

FIG. 3. Probability current for $n = 1$, $B = 20$, $\alpha = -1$ and $\theta = 0$. In the bottom graph the inset shows a vortex between the point perturbations.

netic field and interacting with a periodic array of point perturbations. We have shown that while the Landau levels survive, the spectrum develops an absolutely continuous part, i.e. a sequence of spectral bands. Depending on the quasimomentum, the particle is transported along the array with zero or nonzero mean longitudinal momentum, and the probability current pattern may exhibit vortices in some regions.

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