The Absolute Continuity of the Spectrum of Maxwell Operator in a Periodic Media

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Abstract

We prove that the spectrum of Maxwell operator, $(B, E) \mapsto (-rot(E), a(x)rot(b(x)B)$, with non negative and periodic functions $a(x)$ and $b(x)$, is absolutely continuous.

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1 Introduction and main result

The propagation of waves in photonic crystals is subject to many interests in physics, applied mathematics and in mathematical physics, see for example the introduction and the bibliography of [1] and [2]. These waves are governed by the Maxwell equations in a periodic media, (see [1]), which can be written as follows

$$\frac{\partial}{\partial t}B(t,x) = -rot(E(t,x)), \qquad \frac{\partial}{\partial t}E(t,x) = a(x)rot(b(x)B(t,x)), \tag{1.1}$$

$$div(B(t,x)) = 0, \qquad div(\frac{1}{a(x)}E(t,x)) = 0.$$

if $rot(u(x)) = (\frac{\partial}{\partial x_2}u_3 - \frac{\partial}{\partial x_3}u_2, \frac{\partial}{\partial x_3}u_1 - \frac{\partial}{\partial x_1}u_3, \frac{\partial}{\partial x_1}u_2 - \frac{\partial}{\partial x_2}u_1)$ is the rotational of the vector field u of $\mathbf{R}^3$, and $div(u) = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}u_j$ its divergent.

a and b are non negative functions which are Γ-periodic,

$$a(x-c) = a(x),\ b(x-c) = b(x)\ \forall\ c\ \in \Gamma,\ \forall\ x\ \in \mathbf{R}^3; \tag{1.2}$$

$$a(x) > 0 \text{ and } b(x) > 0,$$

Γ is a lattice in $\mathbf{R}^3$, $\Gamma = \{c = \sum_{j=1}^{3} n_j e_j;\ n = (n_1, n_2, n_3) \in \mathbf{Z}^3\}$
with $\{e_1, e_2, e_3\}$ a basis of $\mathbf{R}^3$.

E denotes the electrical field, B the magnetic field, $1/a$ the dielectric constant and $1/b$ the magnetic permeability.

The Maxwell equations can be written in the following form

$$\frac{\partial}{\partial t}\Psi = iM(x,D)\Psi, \quad D.\Psi^+ = D.(\frac{1}{a(x)}\Psi^-) = 0, \tag{1.3}$$

if $\Psi = (\Psi^+, \Psi^-) = (B, E)$, (we denote $D.u = -idiv(u)$ for $u \in L^2(\mathbf{R}^3; \mathbf{C}^3)$). $M(x,D)$ is the Maxwell operator, which is a 6×6 system of differential operators of order one,

$$M(x,D) = \begin{pmatrix} 0 & iR_{ot}(D) \\ -ia(x)R_{ot}(D)b(x) & 0 \end{pmatrix} \tag{1.4}$$

$$\text{with } R_{ot}(\xi) = i\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}; \quad D = -i(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$$

When $a(x),\ b(x)\ \in C^1(\mathbf{R}^3)$, it is easy to see that the Maxwell operator $M(x,D)$ is essentially self-adjoint on

$$\mathbf{H} = L^2_b(\mathbf{R}^3; \mathbf{C}^3) \oplus L^2_{1/a}(\mathbf{R}^3; \mathbf{C}^3). \tag{1.5}$$

($L^2_w(\mathbf{R}^3) = L^2(\mathbf{R}^3; w(x)dx)$, if $w(x)$ a density, $w > 0$ and dx is the usual the Lebesgue measure).

If we denote by $M_1(x,D)$ the unique self-adjoint operator on $\mathbf{H}$ which is an extension of $M(x,D)$ from the regular vector fields with compact support $C_0^\infty(\mathbf{R}^3; \mathbf{C}^3) \oplus C_0^\infty(\mathbf{R}^3; \mathbf{C}^3)$, then $M_1(x,D)$ has zero as an eigenvalue of infinite multiplicity, (this comes from the non-ellipticity of the $M(x,D)$ as a differential operator).

We deal with the Maxwell operator, we still denote by $M(x,D)$, which is the restriction of $M_1(x,D)$ on the closure of its range $\mathbf{H}_1 = \overline{R(M_1(x,D))}$. $\mathbf{H}_1$ is a closed subspace of $\mathbf{H}$ and its orthogonal subspace is the kernel of $M_1(x,D)$, $\mathbf{H}_2 = Ker(M_1(x,D))$:

$$\mathbf{H} = L^2_b(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3) = \mathbf{H}_1 \oplus \mathbf{H}_2,$$

$$\mathbf{H}_1 = \{(u,v) \in L^2_b(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3);\ D.u = D.(\frac{1}{a}v) = 0\} \qquad (1.6)$$

$$\mathbf{H}_2 = \{(u,v) \in L^2_b(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3);\ R_{ot}(D)(bu) = R_{ot}(D)(v) = 0\}.$$

The domain of $M(x,D)$ is the intersection of $\mathbf{H}_1$ and the Sobolev space of order one

$$\mathcal{D}(M(x,D)) = \mathbf{H}_1 \bigcap (W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3;\mathbf{C}^3)) \qquad (1.7)$$

$$= \{(u,v) \in W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3;\mathbf{C}^3);\ D.u = D.(\frac{1}{a}v) = 0\},$$

and $M(x,D)$ is a self-adjoint operator on $\mathbf{H}_1$. More over, if a an b are in $C^\infty(\mathbf{R}^3)$, then $M(x,D)$ can be seen as a 4×4 elliptic pseudo-differential system, in the meaning of [3].

The solution of the Maxwell equations is given by

$$\Psi_t = (B_t, E_t) = e^{itM(x,D)}\Psi_0,$$

if $\Psi_0 \in \mathbf{H}_1$.

Our main result is

Theorem 1.1 *Under the assumptions (1.2) and if*

$$a,\ b\ \in C^\infty(\mathbf{R}^3), \qquad (1.8)$$

then the Maxwell operator $M(x,D)$ with domain (1.7) is a self-adjoint operator on the Hilbert space $\mathbf{H}_1$ defined in (1.6), and its spectrum is purely absolutely continuous.

To prove Theorem (1.1) we need to consider the Maxwell operator as a component of an elliptic differential operator of Dirac type.

Let us consider the Hilbert space

$$\mathcal{H} = (L^2_b(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_b(\mathbf{R}^3)) \oplus (L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_a(\mathbf{R}^3)) \tag{1.9}$$

and the isometrical injection

$$J: \ \mathbf{H} \longmapsto \mathcal{H}, \quad J(u,v) = (u,0,v,0), \tag{1.10}$$

$\forall\ (u,v) \in \mathbf{H} = L^2_b(\mathbf{R}^3;\mathbf{C}^3) \oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3)$.

Let us define the 8×8 systems of differential operators

$$P_1(x,D) = J^\star M(x,D) J^\star, \qquad P_2(x,D)(u,f,v,g) = \tag{1.11}$$

$$-i(\frac{1}{b(x)} \bigtriangledown (a(x)b(x)g), a(x)div(\frac{1}{a(x)}v), \bigtriangledown(a(x)b(x)f), b(x)div(u)),$$

$\forall\ (u,f,v,g) \in\ W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3) \oplus W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3)$,
and $P(x,D) = P_1(x,D) + P_2(x,D)$, then

$$P(x,D)(u,f,v,g) = \tag{1.12}$$

$$-i(-rot(v)+\frac{1}{b(x)}\bigtriangledown(a(x)b(x)g), a(x)div(\frac{1}{a(x)}v), a(x)rot(b(x)u)+\bigtriangledown(a(x)b(x)f), b(x)div(u)),$$

$\forall(u,f,v,g)\ \in\ W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3) \oplus W^1(\mathbf{R}^3;\mathbf{C}^3) \oplus W^1(\mathbf{R}^3)$.

$P(x,D)$ is an elliptic differential 8×8 system, which is self-adjoint on $\mathcal{H}$, (with domain the Sobolev space $W^1(\mathbf{R}^3;\mathbf{C}^8)$). $P(x,D)$ is of Dirac type:

$$P^2(x,D)(u,f,v,g) = (Q_1(x,D)u, q_1(x,D)f, Q_2(x,D)v, q_2(x,D)g), \tag{1.13}$$

$\forall(u,f,v,g) \in\ W^2(\mathbf{R}^3;\mathbf{C}^3) \oplus W^2(\mathbf{R}^3) \oplus W^2(\mathbf{R}^3;\mathbf{C}^3) \oplus W^2(\mathbf{R}^3)$.

$Q_1(x,D)$ and $Q_2(x,D)$ are 3×3 elliptic differential systems of order 2, which are self-adjoint operators, on repectively $L^2_b(\mathbf{R}^3;\mathbf{C}^3)$ and $L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3)$, (with domain $W^2(\mathbf{R}^3;\mathbf{C}^3)$) :

$$Q_1(x,D)u = R_{ot}(D)[a(x)R_{ot}(D)b(x)u] + \frac{1}{b(x)}D[a(x)b^2(x)D.u], \tag{1.14}$$

$$Q_2(x,D)u = a(x)R_{ot}[b(x)R_{ot}u] + D[a^2(x)b(x)D.(\frac{1}{a(x)}u)],$$

$\forall\ u \in W^1(\mathbf{R}^3;\mathbf{C}^3)$; $\quad (D = -i\bigtriangledown)$.

$q_1(x,D)$ and $q_2(x,D)$ are elliptic differential operators of order 2, and they are self-adjoint on repectively, $L^2_b(\mathbf{R}^3)$ and $L^2_a(\mathbf{R}^3)$, (with domain $W^2(\mathbf{R}^3)$) :

$$q_1(x,D)f = a(x)D.[\frac{1}{a(x)}D(a(x)b(x)f)], \quad \text{and} \tag{1.15}$$

$$q_2(x,D)f = b(x)D.[\frac{1}{b(x)}D(a(x)b(x)f)], \quad \forall\ f \in W^2(\mathbf{R}^3).$$

It comes from the the isometry of the injection J that $P_1(x,D)$ is a self-adjoint operator on $\mathcal{H}_1$, if its domain is

$$\mathcal{D}(P_1(x,D)) = \tag{1.16}$$

$$\{(u,0,v,0) \in W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3)\oplus W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3);\ D.u = D.(\frac{1}{a(x)}v) = 0\},$$

$\mathcal{H}_1$ is the closed subspace of $\mathcal{H}$,

$$\mathcal{H}_1 = J(\mathbf{H}_1) = \tag{1.17}$$

$$\{(u,0,v,0) \in L^2_b(\mathbf{R}^3;\mathbf{C}^3)\oplus L^2_b(\mathbf{R}^3)\oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3)\oplus L^2_a(\mathbf{R}^3);\ D.u = D.(\frac{1}{a(x)}v) = 0\}.$$

In the same way, one can see easily that $P_2(x,D)$ is a self-adjoint operator on the orthogonal componet of $\mathcal{H}_1$, $\mathcal{H}_2$, with domain

$$\mathcal{D}(P_2(x,D)) = \tag{1.18}$$

$$\{(u,f,v,g) \in W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3)\oplus W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3);\ R_{ot}(D)(b(x)u) = R_{ot}(D)v = 0\}.$$

$\mathcal{H}_2$ can be written

$$\mathcal{H}_2 = \mathcal{H}_1^{\perp} = \tag{1.19}$$

$$\{(u,f,v,g) \in L^2_b(\mathbf{R}^3;\mathbf{C}^3)\oplus L^2_b(\mathbf{R}^3)\oplus L^2_{1/a}(\mathbf{R}^3;\mathbf{C}^3)\oplus L^2_a(\mathbf{R}^3);\ R_{ot}(D)(b(x)u) = R_{ot}(D)v = 0\}.$$

So $P(x,D) = P_1(x,D) \oplus P_2(x,D)$ and $P_1(x,D)$ is unitarily equivalent to the Maxwell operator $M(x,D)$. Then we have the proposition

Proposition 1.2 *Under the assumptions of Theorem (1.1), the spectrum of Maxwell operator $M(x,D)$ is included in the spectrum of the Dirac type operator $P(x,D)$ defined by (1.12). More precisely*

$$\sigma_{sc}(M(x,D)) \subset \sigma_{sc}(P(x,D)) \quad \text{and} \quad \sigma_p(M(x,D)) \subset \sigma_p(P(x,D)). \tag{1.20}$$

For an operator Q, $\sigma(Q)$, $\sigma_d(Q)$, $\sigma_{ac}(Q)$, $\sigma_{sc}(Q)$ and $\sigma_p(Q)$ denote its spectrum, discret spectrum, absolutely continuous spectrum, singular continuous spectrum and the point spectrum.

Theorem (1.1) comes from Proposition (1.2) and

Theorem 1.3 *Under the assumptions of Theorem (1.1), the spectrum of the Dirac type operator $P(x, D)$ defined by (1.12) is absolutely continuous.*

Consequently the operators $Q_1(x, D)$, $Q_2(x, D)$, $q_1(x, D)$ and $q_2(x, D)$ defined in (1.14) and (1.15) have only absolutely continuous spectrum.

Remark 1.4 *The assumption (1.8) can be weakened to $a,\ b\ \in C^3(\mathbf{R}^3)$, following our proof.*

Remark 1.5 *For any dimension n, if Γ is a lattice of $\mathbf{R}^n$ and if $a(x)$, $b(x)$ and $k(x)$ are Γ-periodic functions on $\mathbf{R}^n$, such that $a(x),\ b(x)\ \in C^\infty(\mathbf{R}^n;]0, +\infty[)$ and $k(x)$ is a real and bounded potential, $k(x) \in L^\infty(\mathbf{R}^n)$, then the Schrödinger operator $q(x, D)$ defined by $q(x, D)f = D.[a(x)D(b(x)f)] + k(x)f,\ \forall\ f \in W^2(\mathbf{R}^n)$, is self-adjoint on $L^2_b(\mathbf{R}^n)$.*

It comes easily from our proof that $q(x, D)$ has only absolutely continuous spectrum.

Remark 1.6 *In dimension two, we have just to take care of the modifications of the operators:*

$$R_{ot}(D)u = (\frac{\partial}{\partial x_2}u_3, -\frac{\partial}{\partial x_1}u_3, \frac{\partial}{\partial x_1}u_2 - \frac{\partial}{\partial x_2}u_1),$$

$$D.u = -i[\frac{\partial}{\partial x_1}u_1 + \frac{\partial}{\partial x_2}u_2],\ \ Df = -i(\frac{\partial}{\partial x_1}f, \frac{\partial}{\partial x_2}f, 0),\ \ \forall u \in W^1(\mathbf{R}^2; \mathbf{C}^3),\ \forall f \in W^1(\mathbf{R}^2).$$

The square of the equivalent of Maxwell operator, defined as in (1.4) (with $\xi_3 = 0$), becomes of diagonal type:

$$M^2(x, D)\Psi = (L_1(x, D)\tilde{u}, q_+(x, D)u_3, L_2(x, D)\tilde{v}, q_-(x, D)v_3),$$

$\forall\Psi = (u, v) \in W^1(\mathbf{R}^2; \mathbf{C}^3) \oplus W^1(\mathbf{R}^2; \mathbf{C}^3)$, *if* $\tilde{u} = (u_1, u_2)$ *and* $\tilde{v} = (v_1, v_2)$.

$q_+(x,D)f = D.[a(x)D(b(x)f)]$ *and* $q_-(x,D)f = a(x)D.[b(x)Df]$,
so by Remark (1.5) $q_+(x,D)$ *and* $q_-(x,D)$ *have only absolutely continuous spectrum, (as self-adjoint operator on* $L^2_b(\mathbf{R}^2)$ *and respectively on* $L^2_{1/a}(\mathbf{R}^2)$*).*

The operator $L_1(x,D) = (L_1^+(x,D), L_1^-(x,D))$ *is defined as follow:*

$L_1^+(x,D)w = -\frac{\partial}{\partial x_2}[a(x)(\frac{\partial}{\partial x_2}(b(x)w_1) - \frac{\partial}{\partial x_1}(b(x)w_2))]$,

$L_1^-(x,D)w = \frac{\partial}{\partial x_1}[a(x)(\frac{\partial}{\partial x_2}(b(x)w_1) - \frac{\partial}{\partial x_1}(b(x)w_2))])$,

$\forall\ w \in \mathcal{D}(L_1(x,D)) = W^2(\mathbf{R}^2;\mathbf{C}^2)\bigcap \mathbf{H}_1^+$,

$\mathbf{H}_1^+ = \{w \in L^2(\mathbf{R}^2;\mathbf{C}^2);\ div(w) = 0\}$.

In the same way $L_2(x,D) = (L_2^+(x,D), L_2^-(x,D))$, *if*

$L_2^+(x,D)w = -a(x)\frac{\partial}{\partial x_2}[b(x)(\frac{\partial}{\partial x_2}w_1 - \frac{\partial}{\partial x_1}w_2)]$

and $L_2^-(x,D)w = a(x)\frac{\partial}{\partial x_1}[b(x)(\frac{\partial}{\partial x_2}w_1 - \frac{\partial}{\partial x_1}w_2)])$,

$\forall\ w \in \mathcal{D}(L_2(x,D)) = W^2(\mathbf{R}^2;\mathbf{C}^2)\bigcap \mathbf{H}_1^-$,

$\mathbf{H}_1^- = \{w \in L^2(\mathbf{R}^2;\mathbf{C}^2);\ div(a^{-1}(x)w) = 0\}$.

The Theorem (1.1) is valid for $M(x,D)$, *so* $L_1(x,D)$ *and* $L_2(x,D)$ *are self-adjoint operators, on respectively the Hilbert space* $\mathbf{H}_1^+$ *and* $\mathbf{H}_1^-$, *and their spectrums are absolutely continuous.*

2 Proofs

<u>Proof of the essentially self-adjointness</u>

As $M(x,D)$ with domain $C_0^\infty(\mathbf{R}^3)\oplus C_0^\infty(\mathbf{R}^3)$ is a symetric operator on the Hilbert space $\mathbf{H}$, we have just to check that $M(x,D)\Psi = \pm i\Psi$ has no solution in $\mathbf{H}$.

If $\Psi = (u,v) \in \mathbf{H}$ is such that $M(x,D)\Psi = \pm i\Psi$,
then $div(u) = div(\frac{1}{a(x)}v) = 0$ in $L^2(\mathbf{R}^3)$.

But $v \in L^3(\mathbf{R}^3;\mathbf{C}^3)$, $rot(v) = \pm u\ \in L^2(\mathbf{R}^3;\mathbf{C}^3)$
and $div(v) = \frac{1}{a(x)}(\bigtriangledown a(x)).v\ \in L^2(\mathbf{R}^3)$, so $v \in W^1(\mathbf{R}^3;\mathbf{C}^3)$.

In the same way $u \in L^3(\mathbf{R}^3;\mathbf{C}^3)$, $rot(u) = \mp\frac{1}{a(x)b(x)}v - \frac{1}{b(x)}(\bigtriangledown b(x))\times u\ \in L^2(\mathbf{R}^3;\mathbf{C}^3)$ and $div(u) = 0\ \in L^2(\mathbf{R}^3)$, so $u \in W^1(\mathbf{R}^3;\mathbf{C}^3)$.

But $M(x,D)$ with domain $W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3;\mathbf{C}^3)$ is a symetric operator on $\mathbf{H}$, so $M(x,D)\Phi = \pm i\Phi$ has only $\Phi = 0$ as a solution in $W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3;\mathbf{C}^3)$, then $\Psi = (u,v) = (0,0)$ •

The previous proof shows easily that, if $\Psi = (u,v)\ \in \mathbf{H}$ is such $M(x,D)\Psi\ \in \mathbf{H}$ and $div(u) = div(\frac{1}{a(x)}v) = 0$,
then $\Psi \in W^1(\mathbf{R}^3;\mathbf{C}^3)\oplus W^1(\mathbf{R}^3;\mathbf{C}^3)$.

As for the proof of Maxwell operator, it is easy to show that $P_2(x,D)$ is essentially self-adjoint on $\mathcal{H}$ and,
if $\Phi = (u,f,v,g) \in \mathcal{H}$ is such that $P_2(x,D)\Phi \in \mathcal{H}$
and $rot(b(x)u) = rot(v) = 0$ then $\Phi \in W^1(\mathbf{R}^3;\mathbf{C}^8)$.

So $P_2(x,D)$ with domain given by (1.18) is a self-adjoint operator on $\mathcal{H}_2$.

<u>Proof of Proposition</u> (1.2)

The ellipticity of $P(x,D)$ ensure that $P(x,D)$ with domain $W^1(\mathbf{R}^3;\mathbf{C}^8)$ is self-adjoint on $\mathcal{H}$, which is the closure of $P(x,D)$ with initial domain $C_0^\infty(\mathbf{R}^3;\mathbf{C}^8)$. It is easy to see that zero is not an eigenvalue of $P(x,D)$, so $\mathcal{H} = \overline{R(P(x,D))}$. More precisely, for every $\Phi \in \mathcal{H}$, there exists a sequence (Φ_j) in $C_0^\infty(\mathbf{R}^3;\mathbf{C}^8)$, such that $(P(x,D)\Phi_j)$ converges in $\mathcal{H}$ to Φ. But
$P_1(x,D)\Phi_j \in \mathcal{H}_1$, $P_2(x,D)\Phi_j \in \mathcal{H}_2$ and
$P_1(x,D)P_2(x,D)\Psi = P_2(x,D)P_1(x,D)\Psi = 0, \ \forall \Psi \in W^2(\mathbf{R}^3;\mathbf{C}^8)$.

So $P_1(x,D)\Phi_j$ and $P_2(x,D)\Phi_j$ are orthogonal in $\mathcal{H}$ for every integer j, then $\Phi = \Phi^+ + \Phi^-$, with Φ^+ the limit in $\mathcal{H}$ of $(P_1(x,D)\Phi_j)$ and Φ^- the one of $(P_2(x,D)\Phi_j)$. It is easy to see that the decomposition

$$\Phi = \Phi^+ + \Phi^-, \ \Phi^+ \in \mathcal{H}_1 = \overline{P_1(x,D)(W^1(\mathbf{R}^3;\mathbf{C}^8))}, \tag{2.1}$$

$$\Phi^- \in \mathcal{H}_2 = \overline{P_2(x,D)(W^1(\mathbf{R}^3;\mathbf{C}^8))},$$

is unique, so the sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is orthogonal.

More over, if $\Phi \in W^1(\mathbf{R}^3;\mathbf{C}^8)$, then $\Phi^+, \ \Phi^- \in W^1(\mathbf{R}^3;\mathbf{C}^8)$.
To be convinced, consider the equivalent norm on $W^1(\mathbf{R}^3;\mathbf{C}^8)$,
$\|\Psi\|^2 = \|P(x,D)\Psi\|_{\mathcal{H}}^2 + \|\Psi\|_{\mathcal{H}}^2$, for any $\Psi \in W^1(\mathbf{R}^3;\mathbf{C}^8)$,
and use that $P^3(x,D)(C_0^\infty(\mathbf{R}^3;\mathbf{C}^8))$ is dense in $\mathcal{H}$ to see that $P(x,D)(C_0^\infty(\mathbf{R}^3;\mathbf{C}^8))$ is dense in $W^1(\mathbf{R}^3;\mathbf{C}^8)$.

We have proved that

$$P(x,D) = P_1(x,D) \oplus P_2(x,D) : \tag{2.2}$$

$$[W^1(\mathbf{R}^3;\mathbf{C}^8)\bigcap \mathcal{H}_1] \oplus [W^1(\mathbf{R}^3;\mathbf{C}^8)\bigcap \mathcal{H}_2] \ \longmapsto \ \mathcal{H}_1 \oplus \mathcal{H}_2,$$

the last sum is orthogonal, the $P_k(x,D)$ are self-adjoint on $\mathcal{H}_k$, so Proposition (1.2) follows from (2.2) and the unitary equivalence between $M(x,D)$ and $P_1(x,D)$.

Proof of Theorem (1.3)

For any differential $n \times n$ system $S(x,D) = \sum_{|\alpha| \le m} A_\alpha(x) D^\alpha$ on $\mathbf{R}^3$, with Γ-periodic $n \times n$ matrices $A_\alpha(x)$, and for any $\Theta \in \mathbf{C}^3$, we will consider the differential $n \times n$ system on the torus $\mathbf{T}^3 = \mathbf{R}^3/\Gamma$,
$S(x, D-\Theta) = \sum_{|\alpha| \le m} A_\alpha(x)(D-\Theta)^\alpha$.

Then the ellipticity of $P(x, D-\Theta)$, ensure that, with domain $W^1(\mathbf{T}^3; \mathbf{C}^8)$, $P(x, D-\Theta)$ is a closed operator on the Hilbert space $\mathcal{H}(\mathbf{T}^3)$,

$$\mathcal{H}(\mathbf{T}^3) = L^2_b(\mathbf{T}^3; \mathbf{C}^3) \oplus L^2_b(\mathbf{T}^3) \oplus L^2_{1/a}(\mathbf{T}^3; \mathbf{C}^3) \oplus L^2_a(\mathbf{T}^3). \tag{2.3}$$

Its adjoint is $P(x, D-\overline{\Theta})$, so $P(x, D-\theta)$ is self-adjoint for any $\theta \in \mathbf{R}^3$.

But $P(x, D-\Theta)$ is of compact resolvent, so its spectrum is discrete, (for any $\Theta \in \mathbf{C}^3$).

The Floquet theory, (see [6] or [4]), enables to see that the spectrum of $P(x,D)$ is composed of the bands formed by the eigenvalues of $P(x, D-\theta)$ when θ varies in $\mathbf{R}^3$.

For any fixed $\theta \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$, ($\mathbf{S}^2$ denotes the two-dimensional unit sphere), for z varying in $\mathbf{C}$, the eigenvalues of $P(x, D-\theta-z\omega)$ can be labelled such that they become analytical in z in a neighbourhood of the real line, (see [6] for the particular case of Laplace operator).

Then, as for Theorem 2.1 of [5], the Floquet theory and an argument due to Thomas [7] and generalized by Reed and Simon [6], lead to

Lemma 2.1 *The singular continuous spectrum of* $P(x,D)$ *is empty:*

$$\sigma_{sc}(P(x,D)) = \emptyset.$$

$\mu \in \mathbf{R}$ *is an eigenvalue of* $P(x,D)$ *iff* μ *is an eigenvalue of* $P(x, D-\Theta)$, $\forall\, \Theta \in \mathbf{C}^3$.

It remains to prove that $P(x,D)$ has no eigenvalue, or equivalently that $P(x, D-\Theta)$ has no Θ-independent eigenvalue.

By [7] or [6], we know the existence of $\theta \in \mathbf{R}^3$, $\omega \in \mathbf{S}^2$ and a constant $C > 0$ such that

$$\|(D-\theta-i\lambda\omega)^2 f\|_{L^2(\mathbf{T}^3)} \ge \lambda \frac{1}{C} \|f\|_{L^2(\mathbf{T}^3)}, \quad \forall\ f \in W^1(\mathbf{T}^3),\ \lambda > C. \tag{2.4}$$

Suppose now μ is an eigenvalue of $P(x,D)$, then μ is an eigenvalue of $P(x,D-\theta-i\lambda\omega)$ for any real λ. Let Φ^λ be an associated eigenvector of unit norm:

$$P(x,D-\theta-i\lambda\omega)\Phi^\lambda = \mu\Phi^\lambda \text{ and } \|\Phi^\lambda\|_{\mathcal{H}(\mathbf{T}^3)} = 1. \tag{2.5}$$

The ellipticity of $P(x,D-\theta-i\lambda\omega)$ ensure that $\Phi^\lambda \in W^3(\mathbf{T}^3;\mathbf{C}^8)$, (in fact $\Phi^\lambda \in C^\infty(\mathbf{T}^3;\mathbf{C}^8)$ with the hypothesis (1.8), see for example [3]), so for $k=1$ or $k=2$, $P_k(x,D-\theta-i\lambda\omega)(P(x,D-\theta-i\lambda\omega)\Phi^\lambda) =$
$P_k^2(x,D-\theta-i\lambda\omega)\Phi^\lambda = \mu P_k(x,D-\theta-i\lambda\omega)\Phi^\lambda$.

Let us prove that either $P_1(x,D-\theta-i\lambda\omega)\Phi^\lambda \neq 0$ or $P_2(x,D-\theta-i\lambda\omega)\Phi^\lambda \neq 0$.

If $P_1(x,D-\theta-i\lambda\omega)\Phi^\lambda = P_2(x,D-\theta-i\lambda\omega)\Phi^\lambda = 0$, and if $\Phi^\lambda = (u^\lambda, f^\lambda, v^\lambda, g^\lambda)$, with $u^\lambda,\ v^\lambda \in W^3(\mathbf{T}^3;\mathbf{C}^3)$, $f^\lambda,\ g^\lambda \in W^3(\mathbf{T}^3;\mathbf{C})$, then

$$R_{ot}(D-\theta-i\lambda\omega)v^\lambda = R_{ot}(D-\theta-i\lambda\omega)(b(x)u^\lambda)$$
$$= (D-\theta-i\lambda\omega)(a(x)b(x)g^\lambda) = (D-\theta-i\lambda\omega)(a(x)b(x)f^\lambda) = 0$$

and $(D-\theta-i\lambda\omega).(\frac{1}{a(x)}v^\lambda) = (D-\theta-i\lambda\omega).(u^\lambda) = 0$;
so if (2.4) is satisfied $f^\lambda = g^\lambda = 0$.

Let $\mathbf{1}_3$ denotes the 3×3 identity matrix and let us remind the formulas, $\forall\ w \in W^2(\mathbf{T}^3;\mathbf{C}^3)$,

$$(D-\theta-i\lambda\omega)^2\mathbf{1}_3 w = R_{ot}^2(D-\theta-i\lambda\omega)w + (D-\theta-i\lambda\omega)[(D-\theta-i\lambda\omega).w], \tag{2.6}$$

$$(D-\theta-i\lambda\omega).R_{ot}(D-\theta-i\lambda\omega)w = 0 \text{ and } R_{ot}(D-\theta-i\lambda\omega)[(D-\theta-i\lambda\omega)w] = 0.$$

As $R_{ot}(D-\theta-i\lambda\omega)(b(x)u^\lambda) = 0$, if λ is large enough, then, by (2.4) and (2.6), we can write

$$b(x)u^\lambda = (D-\theta-i\lambda\omega)h^\lambda, \tag{2.7}$$

$h^\lambda = s(D-\theta-i\lambda\omega)[(D-\theta-i\lambda\omega).(b(x)u^\lambda)]$, if $s(D-\theta-i\lambda\omega)$ is the inverse of $(D-\theta-i\lambda\omega)^2$.

But $b(x)u^\lambda \in W^3(\mathbf{T}^3;\mathbf{C}^3)$, then $h^\lambda \in W^4(\mathbf{T}^3;\mathbf{C})$.

As $(D-\theta-i\lambda\omega).u^\lambda = 0$, then (2.7) entails

$$(D-\theta-i\lambda\omega)^2 h^\lambda + i\frac{1}{b(x)}(\bigtriangledown b(x)).[(D-\theta-i\lambda\omega)h^\lambda] =$$

$$\sqrt{b(x)}[(D-\theta-i\lambda\omega)^2(\frac{1}{\sqrt{b(x)}}h^\lambda)+(\Delta\frac{1}{\sqrt{b(x)}})h^\lambda]=0. \qquad (2.8)$$

If λ is large enough, (2.4) and (2.8) lead to $h^\lambda=0$ and then $u^\lambda=0$.

In the same way $R_{ot}(D-\theta-i\lambda\omega)v^\lambda=0$, leads to

$$v^\lambda=(D-\theta-i\lambda\omega)k^\lambda, \qquad (2.9)$$

$k^\lambda=s(D-\theta-i\lambda\omega)[(D-\theta-i\lambda\omega).v^\lambda]$, (if λ is large enough), and $(D-\theta-i\lambda\omega).(\frac{1}{a(x)}v^\lambda)=0$ entails

$$(D-\theta-i\lambda\omega)^2k^\lambda+i\frac{1}{a(x)}(\nabla a(x)).[(D-\theta-i\lambda\omega)k^\lambda]=$$

$$\sqrt{a(x)}[(D-\theta-i\lambda\omega)^2(\frac{1}{\sqrt{a(x)}}k^\lambda)+(\Delta\frac{1}{\sqrt{a(x)}})k^\lambda]=0, \qquad (2.10)$$

this last equation will give $k^\lambda=0$ and then $v^\lambda=0$, (if λ is large enough, thanks to (2.4)).

We have proved that, if λ is large enough, then either
$P_1(x,D-\theta-i\lambda\omega)\Phi^\lambda\neq 0$
or $P_2(x,D-\theta-i\lambda\omega)\Phi^\lambda\neq 0$.
So equivalently to (2.5), we can suppose that, either

$$P(x,D-\theta-i\lambda\omega)\Phi^\lambda=P_1(x,D-\theta-i\lambda\omega)\Phi^\lambda=\mu\Phi^\lambda,\quad \|\Phi^\lambda\|_{\mathcal{H}(\mathbf{T}^3)}=1 \quad (2.11)$$

$\Phi^\lambda=(u^\lambda,0,v^\lambda,0)$, $u^\lambda,\ v^\lambda\ \in W^2(\mathbf{T}^3;\mathbf{C}^3)$ and $(D-\theta-i\lambda\omega).u^\lambda=(D-\theta-i\lambda\omega).(\frac{1}{a(x)}v^\lambda)=0$

or

$$P(x,D-\theta-i\lambda\omega)\Phi^\lambda=P_2(x,D-\theta-i\lambda\omega)\Phi^\lambda=\mu\Phi^\lambda,\quad \|\Phi^\lambda\|_{\mathcal{H}(\mathbf{T}^3)}=1 \quad (2.12)$$

$$\Phi^\lambda=(u^\lambda,f^\lambda,v^\lambda,g^\lambda),\ u^\lambda,\ v^\lambda\in W^2(\mathbf{T}^3;\mathbf{C}^3),\ f^\lambda,\ g^\lambda\ \in W^2(\mathbf{T}^3)$$

and $R_{ot}(D-\theta-i\lambda\omega)(b(x)u^\lambda)=R_{ot}(D-\theta-i\lambda\omega)v^\lambda=0$.
(It comes from this proof that, if λ is large enough then $\mu\neq 0$).

1) Suppose (2.11).
For any $w\ \in W^2(\mathbf{T}^3;\mathbf{C}^3)$,

$$R_{ot}(D-\theta-i\lambda\omega)[a(x)R_{ot}(D-\theta-i\lambda\omega)b(x)w]=a(x)b(x)R^2_{ot}(D-\theta-i\lambda\omega)w+$$

$$(\bigtriangledown(a(x)b(x)))\times R_{ot}(D-\theta-i\lambda\omega)w+ia(x)[(D-\theta-i\lambda\omega).w](\bigtriangledown b(x))- \quad (2.13)$$
$$i[a(x)(\bigtriangledown b(x)).(D-\theta-i\lambda\omega)]w+V(x)w$$

with $V(x)$ the 3×3 matrix,
$(V_{ij}(x))_{1\leq i,j\leq 3} = -div(a(x)(\bigtriangledown b(x)))\mathbf{1}_3 + (\frac{\partial}{\partial x_j}(a(x)\frac{\partial}{\partial x_i}b(x)))_{1\leq i,j\leq 3}$.

When using (2.13), the equations (2.11) lead to

$$a(x)b(x)(D-\theta-i\lambda\omega)^2\mathbf{1}_3u^\lambda = a(x)b(x)R_{ot}^2(D-\theta-i\lambda\omega)u^\lambda = \quad (2.14)$$

$$\mu^2u^\lambda-(\bigtriangledown(a(x)b(x)))\times R_{ot}(D-\theta-i\lambda\omega)u^\lambda+i[a(x)(\bigtriangledown b(x)).(D-\theta-i\lambda\omega)]u^\lambda+V(x)u^\lambda$$

and then, using that
$\sqrt{b(x)}[(D-\Theta)^2f-i\frac{1}{b(x)}(\bigtriangledown b(x)).(D-\Theta)f] = (D-\Theta)^2(\sqrt{b(x)}f)+(\Delta(\sqrt{b(x)}))f$,
we get

$$(D-\theta-i\lambda\omega)^2\mathbf{1}_3(\sqrt{b(x)}u^\lambda) = -(\Delta(\sqrt{b(x)}))u^\lambda+ \quad (2.15)$$
$$\frac{1}{a(x)}\frac{1}{\sqrt{b(x)}}\{\mu^2u^\lambda-(\bigtriangledown(a(x)b(x)))\times R_{ot}(D-\theta-i\lambda\omega)u^\lambda+V(x)u^\lambda\}.$$

Using (2.11) again we get

$$(D-\theta-i\lambda\omega)^2\mathbf{1}_3(\sqrt{b(x)}u^\lambda) = -(\Delta(\sqrt{b(x)}))u^\lambda+ \quad (2.16)$$

$$\frac{1}{a(x)}\frac{1}{\sqrt{b(x)}}\{\mu^2u^\lambda-(\bigtriangledown(a(x)b(x)))\times[i\mu\frac{1}{a(x)b(x)}v^\lambda-\frac{1}{b(x)}(\bigtriangledown b(x))\times u^\lambda]+V(x)u^\lambda\}.$$

The estimate (2.4), the equations (2.11) and the equality (2.16) entail the existence of a constant $C_1>0$ such that

$$\|u^\lambda\|_{L^2(\mathbf{T}^3)} \leq C_1\frac{1}{\lambda}, \quad \forall\ \lambda > C_1(1+|\mu|). \quad (2.17)$$

In the same way the equations from (2.11),
$a(x)R_{ot}(D-\theta-i\lambda\omega)(b(x)R_{ot}(D-\theta-i\lambda\omega)v^\lambda) = \mu^2v^\lambda$,
and $(D-\theta-i\lambda\omega).(\frac{1}{a(x)}v^\lambda) = 0$,
lead to the equivalent equation to (2.15)

$$(D-\theta-i\lambda\omega)^2\mathbf{1}_3(\frac{1}{\sqrt{a(x)}}v^\lambda) = -(\Delta(\sqrt{a(x)}))\frac{v^\lambda}{a(x)}+ \quad (2.18)$$

$$\frac{1}{\sqrt{a(x)}}\frac{1}{b(x)}\{\mu^2\frac{v^\lambda}{a(x)} - (\bigtriangledown(a(x)b(x))) \times R_{ot}(D-\theta-i\lambda\omega)\frac{v^\lambda}{a(x)} + W(x)\frac{v^\lambda}{a(x)}\},$$

$W(x)$ is defined as $V(x)$ interchanging $a(x)$ and $b(x)$.

(2.11) and (2.18) ensure the equivalent estimate to (2.17), for some constant $C_2 > 0$,

$$\|v^\lambda\|_{L^2(\mathbf{T}^3)} \leq C_2\frac{1}{\lambda}, \quad \forall\ \lambda > C_2(1+|\mu|). \tag{2.19}$$

But (2.11) gives the existence of a constant C_3 such that

$$\|u^\lambda\|^2_{L^2(\mathbf{T}^3)} + \|v^\lambda\|^2_{L^2(\mathbf{T}^3)} \geq \frac{1}{C_3}, \quad \forall\ \lambda > C_3, \tag{2.20}$$

then, if λ is large enough, (2.17) coupled with (2.19), and (2.20) are contradictory, so (2.11) is not accurate.

2) Suppose (2.12).

From (2.12) we get

$$a(x)(D-\theta-i\lambda\omega).[\frac{1}{a(x)}(D-\theta-i\lambda\omega)(a(x)b(x)f^\lambda)] = \mu^2 f^\lambda, \tag{2.21}$$

or equivalently, (using the formula $(D-\Theta)^2 f + i\frac{1}{a(x)}(\bigtriangledown a(x)).(D-\Theta)f = \sqrt{a(x)}[(D-\Theta)^2(\frac{1}{\sqrt{a(x)}}f) + (\Delta(\frac{1}{\sqrt{a(x)}}))f])$, that

$$(D-\theta-i\lambda\omega)^2(\sqrt{a(x)}b(x)f^\lambda) = \tag{2.22}$$

$$\frac{1}{\sqrt{a(x)}}\mu^2 f^\lambda - (\Delta(\frac{1}{\sqrt{a(x)}}))a(x)b(x)f^\lambda.$$

We have also

$$b(x)(D-\theta-i\lambda\omega).[\frac{1}{b(x)}(D-\theta-i\lambda\omega)(a(x)b(x)g^\lambda)] = \mu^2 g^\lambda \tag{2.23}$$

which is equivalent to

$$(D-\theta-i\lambda\omega)^2(a(x)\sqrt{b(x)}g^\lambda) = \tag{2.24}$$

$$\mu^2 \frac{1}{\sqrt{b(x)}} g^\lambda - (\Delta(\frac{1}{\sqrt{b(x)}})) a(x) b(x) g^\lambda].$$

Applying again (2.4), to the equalities (2.22) and (2.24), leads to
$f^\lambda = g^\lambda = 0$, and then by (2.12) $u^\lambda = v^\lambda = 0$, so $\Phi^\lambda = 0$.

We have proved that, for any fixed real μ, if (2.4) is satisfied, for any large λ, μ cannot be an eigenvalue of $P(x, D - \theta - i\lambda\omega)$, then by Lemma (2.1) μ is not an eigenvalue of $P(x, D)$,
so $\sigma_p(P(x, D)) = \emptyset$, using again the same Lemma, we get Theorem (1.3) $\bullet$

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