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DIAMAGNETIC BEHAVIOR OF SUMS OF DIRICHLET EIGENVALUES

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Abstract.
The Li-Yau semiclassical lower bound for the sum of the first N eigenvalues of the Dirichlet–Laplacian is extended to Dirichlet– Laplacians with constant magnetic fields . Our method involves a new diamagnetic inequality for constant magnetic fields.

1. Introduction

When studying Schrödinger operators with magnetic fields, the diamagnetic inequality is the only general tool available for apriori estimates and comparisons with the free Laplacian. In its simplest form it says that

$$\Big| (-i\nabla + A(x))\, \psi(x) \Big|^2 \geq \Big| \nabla |\psi(x)| \Big|^2 , \tag{1.1}$$

but it also appears in the form of the following estimates on the heat kernel and the Green's function

$$\Big| e^{-t(-i\nabla+A)^2}(x,y) \Big| \leq e^{\Delta t}(x,y) \qquad t > 0 , \tag{1.2}$$

and

$$\left| \frac{1}{(-i\nabla + A)^2 + E}(x,y) \right| \leq \frac{1}{-\Delta + E}(x,y) , \qquad E \geq 0. \tag{1.3}$$

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(see [K 72], [S 77,79], [HSU 77], also [AHS 78] and [CFKS 87]). Here $A(x)$ is a one-form on $\mathbb{R}^n$ and the magnetic field is a two-form $B(x)$ given by

$$B(x) = \mathrm{d}A(x) \ . \tag{1.4}$$

The magnetic field determines the vector potential only up to an exact one-form $\mathrm{d}\phi$. In particular, in one dimension the vector potential can be gauged away, i.e., it can be removed by a suitable choice of ϕ. Thus we restrict our attention to the case $n \geq 2$.

One of the numerous applications of these estimates is the magnetic Lieb–Thirring bound, i.e. a bound on the moments of negative eigenvalues for the Schrödinger operator of the form

$$(-i\nabla + A(x))^2 + V(x) \tag{1.5}$$

acting on $L^2(\mathbb{R}^n)$, where V is an external potential (see [LT 75], [LT 76], [L 80], [AHS 78]). These estimates do not depend on B, in particular they do no improve as the magnetic field is increased, despite that negative eigenvalues typically disappear in strong fields. Moreover, for most cases, the constants in these bounds are not sharp. The notable exceptions are the recent bounds of Laptev and Weidl for higher Riesz means of the eigenvalues of (1.5) ([LW 99]). Denote by $\lambda_j = \lambda_j(B,V)$, $j = 1, 2, \ldots$ the negative eigenvalues of the operator (1.5), which depend only on the magnetic field B by gauge invariance. Laptev and Weidl showed that for $\gamma \geq 3/2$

$$\sum_j (-\lambda_j)^\gamma \leq L_{cl} \int_{\mathbb{R}^n} \max(-V(x), 0)^{n/2+\gamma} \mathrm{d}x \tag{1.6}$$

where

$$L_{cl} := \frac{\Gamma(\gamma+1)}{2^n \pi^{n/2} \Gamma(\gamma + \frac{n}{2} + 1)} \tag{1.7}$$

is the classical constant that appears in Weyl's asymptotic formula.

Related to the Lieb–Thirring bounds, the following inequalities of Li and Yau ([LY 83]) are known for the sum of eigenvalues of the Dirichlet Laplacian on a domain $U \subset \mathbb{R}^n$ with volume $|U|$.

$$\sum_{j=1}^{N} \lambda_j \geq C_n \frac{n}{n+2} N^{\frac{n+2}{n}} |U|^{-\frac{2}{n}} \tag{1.8}$$

where $C_n := (2\pi)^2 |B_n|^{-2/n}$, B_n is the unit ball in $\mathbb{R}^n$ and $|B_n|$ is its volume. Again, it follows from Weyl's asymptotic formula that the constant C_n is the best possible. Note that the Li–Yau result does not follow from the Laptev–Weidl result.

In this paper we prove a modest extension of the Li–Yau result to the magnetic Dirichlet Laplacian with a constant magnetic field. More specifically, for any domain $U \subset \mathbb{R}^n$ of finite volume we consider the operator

$$H = \Big(- i\nabla + A(x) \Big)^2$$

on $L^2(U)$ given by the closure of the form

$$(\psi, H\psi) := \int_U \left| (-i\nabla + A(x))\, \psi(x) \right|^2 \mathrm{d}x \tag{1.9}$$

on $C_c^\infty(U)$. The one-form $A(x)$ satisfies $\mathrm{d}A = B$, where B is a constant two-form. Our main result is the following:

Theorem 1 . *Let H be given by (1.9) where A generates a constant magnetic field. Then for any N orthonormal functions $\{\phi_j\}_{j=1}^N$ in the form domain of H we have the inequality*

$$\sum_{j=1}^{N} (\phi_j, H\phi_j) \geq \frac{n}{n+2} C_n N^{\frac{n+2}{n}} |U|^{-\frac{2}{n}} \ , \tag{1.10}$$

with C_n as before. Again the constant C_n is the best possible.

Remark 1: That C_n is best possible follows again from the Weyl asymptotic formula, noting that the magnetic field does not contribute to the eigenvalue sum to leading order as $N \to \infty$.

Remark 2: Let $\lambda_j(B)$ be the eigenvalues of (1.9). Diamagnetism for eigenvalue sums in the strongest sense would mean that

$$\sum_{j=1}^{N} \lambda_j(B) \geq \sum_{j=1}^{N} \lambda_j(B=0) \ . \tag{1.11}$$

The diamagnetic inequality (1.1) shows that (1.11) is valid for $N = 1$, i.e. the lowest eigenvalue of the magnetic operator ((1.5) or (1.9)) increases as the magnetic field is turned on. But (1.11) fails in general even for $N = 2$. To see this, one can consider a planar domain where the second Dirichlet eigenvalue of $-\Delta$ is twofold degenerate. If we turn on a small constant magnetic field $B > 0$, the first eigenvalue increases quadratically with B, while the second one splits into two eigenvalues; one is raised and the other one is lowered proportionally to B. Thus the sum of the first two eigenvalues actually decreases for small B. A similar phenomenon can occur for the sum of the first N eigenvalues. Hence the eigenvalue sum $\sum_{j=1}^N \lambda_j(B)$ may decrease by turning on a nonzero magnetic field B; however our result says that it does not decrease so much as to violate the semiclassical bound (1.10). In this sense, Theorem 1 establishes a weak diamagnetic behavior for the eigenvalue sum.

This remark also applies to the result of Laptev and Weidl. The moment of negative eigenvalues in (1.6) may increase as B is turned on, but it never exceeds the classical value.

Let us say a few words about proofs. The strategy of [LY 83] does not work as smoothly in our problem as in the case without a magnetic field. The reason is that while the eigenfunctions of the problem on the whole space are explicitly known, the computation becomes fairly difficult in dimensions larger than two. Instead, we first reduce the problem to estimates on the integrated density of states (IDS) for the magnetic Hamiltonian defined on the whole space. Then we estimate the magnetic IDS (with a constant field) in terms of the IDS of the Laplacian without magnetic field. This estimate generalizes the known diamagnetic inequality in the following way.

Proposition 1 *(Generalized diamagnetic inequality) . Let B be a constant magnetic field in arbitrary dimension $n \geq 2$ and let $P = \chi_U$ be the characteristic function of an open set $U \subseteq \mathbb{R}^n$ with finite volume. Then,*

$$Tr\Big[Pf\big((-i\nabla + A)^2\big)P\Big] \leq Tr\big[Pf(-\Delta)P\big] \ , \tag{1.12}$$

or, in its pointwise form

$$f\big((-i\nabla + A)^2\big)(x,x) \leq f(-\Delta)(x,x) \ . \tag{1.13}$$

Here f is an arbitrary nonnegative convex function defined on $\mathbb{R}^+$ with $\lim_{\lambda\to\infty} f(\lambda) = 0$.

It is natural to ask whether our result in Theorem 1 holds for a general magnetic field. We do not know the answer to this question. However, we show that our new diamagnetic inequality does not hold generally for an inhomogeneous magnetic field.

In the following section we prove the two dimensional version of our theorem in two ways. We give then the proof for arbitrary dimension in the subsequent sections along with the proof of Proposition 1. We end the paper with a discussion of the results and the techniques.

2. A simple proof for the two dimensional case

Theorem 2 *. Let $U \subseteq \mathbb{R}^2$ be open, with finite volume. Assume that A is such that* $\operatorname{curl} A = B$ *with B constant, e.g. $A(x) = \frac{B}{2}(-x_2, x_1)$. Let $\{\phi_j\}_{j=1}^N$ be an orthonormal set of functions in $H_0^1(U)$. Then for any $N \geq 1$*

$$\sum_{j=1}^{N} \|(-i\nabla + A)\phi_j\|^2 \geq \frac{2\pi N^2}{|U|}. \tag{2.1}$$

We give two proofs of this theorem. The first one is a transcription of the proof given by Li–Yau for the free Laplace operator. The second one uses IDS of the infinite volume problem and we explain it in the next section.

First proof. Without loss of generality we can assume that $B > 0$. Let Π_k^B be the projection onto the k-th Landau level of $(-i\nabla + A)^2$ defined on all of $L^2(\mathbb{R}^2)$,

$$(-i\nabla + A)^2 \Pi_k^B = (2k+1)B\ \Pi_k^B \ . \tag{2.2}$$

The projection Π_k^B has an explicit integral kernel whose value on the diagonal is given by

$$\Pi_k^B(x,x) = \frac{B}{2\pi} \ . \tag{2.3}$$

A simple calculation for functions $\phi_j \in C_c^\infty(U)$ shows that

$$\sum_{j=1}^N \|(-i\nabla + A)\phi_j\|^2 = \sum_{k=0}^\infty (2k+1)B \sum_{j=1}^N (\phi_j, \Pi_k^B \phi_j) \ , \tag{2.4}$$

which extends to $\phi_j \in H_0^1(U)$ by standard approximation. Set

$$a_k := \frac{2\pi}{B|U|} \sum_{j=1}^N (\phi_j, \Pi_k^B \phi_j) \ . \tag{2.5}$$

Assume that $\phi_1, \cdots, \phi_N$ is an orthonormal set, we extend it to an orthonormal basis of $L^2(U)$ and using (2.3) we see that

$$0 \le a_k \le \frac{2\pi}{B|U|} \sum_{j=1}^\infty (\phi_j, \Pi_k^B \phi_j) = \frac{2\pi}{B|U|} \mathrm{Tr}(\chi_U \Pi_k^B) = 1 \ , \tag{2.6}$$

where χ_U is the characteristic function of the set U. Since

$$\sum_{k=1}^\infty \Pi_k^B = \mathbb{I}$$

we get

$$\sum_{k=0}^\infty a_k = \frac{2\pi N}{B|U|} := \alpha \ . \tag{2.7}$$

Thus we have that

$$\sum_{j=1}^N \|(-i\nabla + A)\phi_j\|^2 = \frac{B|U|}{2\pi} \sum_{k=0}^\infty B(2k+1)a_k \ , \tag{2.8}$$

and we minimize the right side of (2.8) over all a_k satisfying (2.6) and (2.7) with a given α.

Applying the bathtub principle (see, e.g., [LL 97] p.28) we learn that the minimizer of this problem is given by

$$a_k = \begin{cases} 1 \ , & 0 \le k \le [\alpha] - 1 \\ \alpha - [\alpha] \ , & k = [\alpha] \\ 0 \ , & k > [\alpha] \end{cases} \tag{2.9}$$

Here [ ] brackets denote the integer part. An easy computation shows that the minimum of (2.8) is:

$$\frac{2\pi N^2}{\alpha^2|U|}\Big(\alpha-[\alpha]-(\alpha-[\alpha])^2+\alpha^2\Big)\ , \tag{2.10}$$

which is greater or equal than $\dfrac{2\pi N^2}{|U|}$. ■

3. Bathtub principle for the integrated density of states

First we give an abstract version of the bathtub principle used in Section 2 in terms of the IDS. Let H be a nonnegative selfadjoint operator on a Hilbert space $\mathcal{H}$ and let its spectral decomposition be $H=\int_0^\infty \lambda \mathrm{d}E_\lambda$, where E_λ is the spectral family associated with H. Recall the following properties of E_λ:

$$\lambda \to E_\lambda \ \text{ is continuous from the right} \tag{3.1}$$

and

$$E_\lambda \nearrow \mathbb{I} \ \text{ as } \ \lambda\to\infty\ . \tag{3.2}$$

Further, let h be a closed subspace of $\mathcal{H}$, and denote by P the projection from $\mathcal{H}$ onto h. Let ϕ_j, $j=1,\ldots N$ be an orthonormal set of functions in the intersection of h and the form domain of H.

Lemma 1 . *Let $f(\lambda):=Tr(PE_\lambda)$, then*

$$\sum_{j=1}^{N}(\phi_j,H\phi_j)\geq\int_0^\infty (N-f(\lambda))_+\mathrm{d}\lambda, \tag{3.3}$$

where $(N-f(\lambda))_+ := max\{N-f(\lambda),0\}$ is the positive part of $(N-f(\lambda))$.

Remark. In applications P will be the projection of $\mathcal{H}=L^2(\mathbb{R}^n)$ onto $h=L^2(U)$ with some $U\subset\mathbb{R}^n$. In this case $|U|^{-1}f(\lambda)$ is the integrated density of states, i.e. the number of states up to energy λ per unit volume.

Proof. Let

$$F_N(\lambda):=\sum_{j=1}^{N}(\phi_j,E_\lambda\phi_j)\ . \tag{3.4}$$

By (3.2) $F_N(\lambda)$ is increasing towards N as $\lambda\to\infty$ and for all λ

$$0\leq F_N(\lambda)\leq \mathrm{Tr}(PE_\lambda)=f(\lambda)\ . \tag{3.5}$$

By these properties, the function $F_N(\lambda)$ defines a measure $\mathrm{d}F_N(\lambda)$. Since

$$\sum_{j=1}^{N}(\phi_j,H\phi_j)=\int_0^\infty \lambda\,\mathrm{d}F_N(\lambda)=\int_0^\infty\Big(F_N(\infty)-F_N(\alpha)\Big)\mathrm{d}\alpha=\int_0^\infty (N-F_N(\alpha))\mathrm{d}\alpha\ , \tag{3.6}$$

the result follows from (3.5) and the fact that $F_N(\lambda) \leq N$ for all $\lambda \geq 0$. ■

The following elementary lemma is useful for computing explicit lower bounds.

Lemma 2 . *Let f and g be two nondecreasing functions on the positive line satisfying $\lim_{\lambda\to\infty} f(\lambda) = \lim_{\lambda\to\infty} g(\lambda) = +\infty$. Assume further that*

$$\int_0^E f(\lambda)\mathrm{d}\lambda \leq \int_0^E g(\lambda)\mathrm{d}\lambda \tag{3.7}$$

for all $E \geq 0$. Then

$$\int_0^\infty (N - f(\lambda))_+ \mathrm{d}\lambda \geq \int_0^\infty (N - g(\lambda))_+ \mathrm{d}\lambda \ , \tag{3.8}$$

for all $N \geq 0$.

Proof. With the definitions $\lambda_0 := \inf\{\lambda : f(\lambda) \geq N\}$ and $\mu_0 := \inf\{\mu : g(\mu) \geq N\}$ the problem is reduced to showing

$$\int_0^{\mu_0} g - \int_0^{\lambda_0} f \geq (\mu_0 - \lambda_0)N \ . \tag{3.9}$$

In the case $\mu_0 \geq \lambda_0$ we write the left side as

$$\int_0^{\mu_0} (g - f) + \int_{\lambda_0}^{\mu_0} f \ . \tag{3.10}$$

Since the first term in this sum is nonnegative and $f \geq N$ on (λ_0, μ_0) we obtain (3.9). In the case $\mu_0 \leq \lambda_0$ we write the left side of (3.9) as

$$\int_0^{\mu_0} (g - f) - \int_{\mu_0}^{\lambda_0} f \ . \tag{3.11}$$

Again the first term is nonnegative and on (μ_0, λ_0) we have $f \leq N$, which yields the result. ■

Remark. Armed with Lemma 2, one can use Berezin's trace inequality [B 72] to give an alternative proof of Lemma 1 without bathtub principle (we are grateful to Timo Weidl for pointing this out to us). Let $\widetilde{E}_\lambda$ be the spectral resolution of PHP. By the variational principle and the spectral theorem

$$\sum_{j=1}^N (\phi_j, H\phi_j) \geq \int_0^\infty \left(N - \mathrm{Tr}\ \widetilde{E}_\lambda\right)_+ \mathrm{d}\lambda \ .$$

Hence (3.3) would follow from Lemma 2 once we prove that for any $E \geq 0$

$$\int_0^E \mathrm{Tr}\ \widetilde{E}_\lambda\ \mathrm{d}\lambda \leq \int_0^E \mathrm{Tr}\big(PE_\lambda P\big)\ \mathrm{d}\lambda\ ,$$

but this is just Berezin's inequality Tr $\varphi(PHP) \leq$ Tr $P\varphi(H)P$ for the convex function $\varphi(u) := (E-u)_+$. Here we used that

$$\int_0^E E_\lambda \mathrm{d}\lambda = (E-H)_+\ . \tag{3.12}$$

and a similar relation for $\widetilde{E}_\lambda$.

With the help of these Lemmas we give now a second proof of Theorem 2 that can be easily generalized to higher dimensions.

4. Second proof of Theorem 2

Let $H = (-i\nabla + A)^2$ be the constant field operator, $\mathcal{H} = L^2(\mathbb{R}^2)$, $h = L^2(U)$. Let P be the orthogonal projection from $\mathcal{H}$ to h, in other words, the multiplication by the characteristic function χ_U. Then

$$\frac{1}{|U|}\mathrm{Tr}(PE_\lambda) = \frac{1}{|U|}\int_U E_\lambda^B(x,x)\mathrm{d}x = E_\lambda^B\ , \tag{4.1}$$

is the integrated density of states, where

$$E_\lambda^B(x,x) := \sum_{(2k+1)B\leq\lambda} \Pi_k^B(x,x) = \frac{B}{2\pi}\left[\frac{\lambda}{2B}+\frac{1}{2}\right] =: E_\lambda^B\ , \tag{4.2}$$

using (2.3). By translation invariance $E_\lambda^B(x,x)$ is clearly independent of x.

Thus, by Lemma 1

$$\int_0^\infty \left(N - |U|\frac{B}{2\pi}\left[\frac{\lambda}{2B}+\frac{1}{2}\right]\right)_+ \mathrm{d}\lambda \tag{4.3}$$

is a lower bound to the left side of (2.1). Finally, the bound

$$\int_0^\infty \left(N - |U|\frac{B}{2\pi}\left[\frac{\lambda}{2B}+\frac{1}{2}\right]\right)_+ \mathrm{d}\lambda \geq \int_0^\infty \left(N - \frac{\lambda}{4\pi}|U|\right)_+ \mathrm{d}\lambda = \frac{2\pi N^2}{|U|}\ , \tag{4.4}$$

is a consequence of Lemma 2 and the elementary but important observation that

$$\int_0^E E_\lambda^B \mathrm{d}\lambda = \int_0^E \frac{B}{2\pi}\left[\frac{\lambda}{2B}+\frac{1}{2}\right]\mathrm{d}\lambda \leq \int_0^E \frac{\lambda}{4\pi}\mathrm{d}\lambda = \int_0^E E_\lambda^0 \mathrm{d}\lambda\ , \tag{4.5}$$

for all $E \geq 0$. Here $E^0_\lambda := \lambda/(4\pi)$ is the IDS of the free Laplacian. It is useful to view (4.5) as a comparison of the IDS of two Laplace operators in the whole space: one with and the other without magnetic field.

A nice way to see inequality (4.5) is to notice that the left side is the integral (up to E) of the Landau staircase function $\lambda \to E^B_\lambda$, which is a function that has jumps of height $B/(2\pi)$ at the points $(2k+1)B, k = 0, 1, \ldots$. By comparing this function with the IDS of the free Laplacian, which is a straight line of slope $1/(4\pi)$ going through the middle point of each of the stairs, (4.5) follows easily. Note that this inequality is saturated exactly at values $E = 2kB$, $k = 0, 1, \ldots$. ∎

Remark. Formula (3.12) implies that (4.5) is equivalent to

$$\mathrm{Tr}\Big[P(E - (-i\nabla + A)^2)_+\Big] \leq \mathrm{Tr}\Big[P(E + \Delta)_+\Big] , \quad P = \chi_U . \tag{4.6}$$

5. Higher dimensions

The following lemma is the main device for passing to higher dimensions.

Lemma 3 . *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let A_j, B_j be nonnegative selfadjoint operators on $\mathcal{H}_j$, $j = 1, 2$. By a slight abuse of notation we denote by $A_1 + A_2$ the operator that acts on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $B_1 + B_2$ acts in a similar way. Let $P = P_1 \otimes P_2$, where P_1 and P_2 are nonnegative selfadjoint operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume further that*

$$\mathrm{Tr}\Big[P_1(E - A_1)_+\Big] \leq \mathrm{Tr}\Big[P_1(E - B_1)_+\Big] \tag{5.1}$$

and

$$\mathrm{Tr}\Big[P_2(E - A_2)_+\Big] \leq \mathrm{Tr}\Big[P_2(E - B_2)_+\Big] \tag{5.2}$$

hold for all $E \geq 0$. Then

$$\mathrm{Tr}\Big[P(E - A_1 - A_2)_+\Big] \leq \mathrm{Tr}\Big[P(E - B_1 - B_2)_+\Big] \tag{5.3}$$

for all $E \geq 0$. (The traces are taken on the respective Hilbert spaces where the operators are defined.)

Proof. For all real numbers x and y the following identity holds

$$(E - x - y)_+ = \int_0^\infty \theta(E - \beta - y)\big(1 - \theta(x - \beta)\big)\mathrm{d}\beta , \tag{5.4}$$

where

$$\theta(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} . \tag{5.5}$$

Via spectral calculus, this formula yields

$$\mathrm{Tr}\Big[P(E - A_1 - A_2)_+\Big] = \int_0^\infty \mathrm{Tr}\Big[P_1\theta(E - \beta - A_1)\Big]\ \mathrm{Tr}\Big[P_2(1 - \theta(A_2 - \beta))\Big]\mathrm{d}\beta . \tag{5.6}$$

The function $f(\beta) = \mathrm{Tr}\Big[P_2(1-\theta(A_2-\beta))\Big]$ is obviously monotonically increasing. By the layer cake representation (see [LL 97], p. 26) it can be written as

$$f(\beta) = \int_0^\infty \chi_{\{\nu : f(\beta) > \nu\}} d\nu \ . \tag{5.7}$$

For any ν fixed, $\beta \to \chi_{\{\nu : f(\beta) > \nu\}}$ are characteristic functions of half–lines starting at

$$\beta_0(\nu) := \inf\{\beta : f(\beta) > \nu\} \ . \tag{5.8}$$

In the case when $\{\beta : f(\beta) > \nu\}$ is an empty set we assume $\beta_0(\nu) = +\infty$.

Hence we obtain

$$\begin{aligned} \mathrm{Tr}\Big[P(E - A_1 - A_2)_+\Big] &= \int_0^\infty \mathrm{d}\nu \int_0^\infty \mathrm{d}\beta \ \mathrm{Tr}\Big[P_1\theta(E-\beta-A_1)\Big] \ \chi_{\{\nu : f(\beta) > \nu\}} = \\ &= \int_0^\infty \mathrm{d}\nu \ \mathrm{Tr}\Big[P_1(E-\beta_0(\nu)-A_1)_+\Big] \leq \\ &\leq \int_0^\infty \mathrm{d}\nu \ \mathrm{Tr}\Big[P_1(E-\beta_0(\nu)-B_1)_+\Big] = \mathrm{Tr}\Big[P(E-B_1-A_2)_+\Big] \ . \end{aligned} \tag{5.9}$$

Here we used (5.1). By the same reasoning we have

$$\mathrm{Tr}\Big[P(E - B_1 - A_2)_+\Big] \leq \mathrm{Tr}\Big[P(E - B_1 - B_2)_+\Big] \ . \tag{5.10}$$

■

Now we are ready to prove Theorem 1.

Proof of Theorem 1. If the dimension n is even, then the operator $(-i\nabla + A)^2$ acting on all $\mathbb{R}^n$ is unitarily equivalent to a sum of $\frac{n}{2}$ two dimensional magnetic Schrödinger operators that act on $L^2(\mathbb{R}^2)$. This follows from the fact that the magnetic field, being a two form with constant coefficients, can be transformed into the form

$$\sum_{k=1}^{\frac{n}{2}} B_k \ \mathrm{d}x_{2k-1} \wedge \mathrm{d}x_{2k} \ , \tag{5.11}$$

by an orthogonal change of coordinates. If the dimension n is odd the magnetic field looks as above (except the summation is up to $\frac{n-1}{2}$) but the operator is of the form

$$\sum_{k=1}^{\frac{n-1}{2}} H(B_k) + H_0 \ . \tag{5.12}$$

Here $H(B_k)$ is a two dimensional magnetic Schrödinger operator with constant field B_k acting on the (x_{2k-1}, x_{2k}) coordinate plane and $H_0 := -\partial^2_{x_n}$.

By Lemma 1 we know that

$$\sum_{j=1}^{N}(\phi_j, H\phi_j) \geq \int_0^\infty (N - f(\lambda))_+ \mathrm{d}\lambda \ , \tag{5.13}$$

where

$$f(\lambda) = \mathrm{Tr}(PE_\lambda) \tag{5.14}$$

is determined by the integrated density of states of the operator $(-i\nabla + A)^2$ in the whole space and $P = \chi_U$. We make an induction argument over dimensions. ¿From Theorem 2 we know that

$$\mathrm{Tr}\Big[P\big(E - (-i\nabla + A)^2\big)_+\Big] \leq \mathrm{Tr}\Big[P(E + \Delta)_+\Big] \ , \tag{5.15}$$

is true in $\mathbb{R}^2$, where $P = \chi_U$.

Suppose (5.15) holds in $\mathbb{R}^n$, where n is even. Then it is valid in $\mathbb{R}^{n+1}$ by means of Lemma 3. We choose A_1 to be the n–dimensional magnetic Schrödinger operator, B_1 is the n–dimensional minus Laplacian both acting on the L^2 space of the first n coordinates. Finally A_2 and B_2 are both equal to $-\partial^2_{x_{n+1}}$ as in (5.12).

To prove (5.15) in $\mathbb{R}^{n+2}$ we choose A_1 and B_1 as above; and we let A_2 and B_2 be two dimensional Laplacians on the (x_{n+1}, x_{n+2}) coordinate plane with and without magnetic field, respectively.

This induction argument works not only for domains of the form $U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}$ and $U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^2$ suggested by Lemma 3 but also for any finite volume domain. The reason is that all the operators considered are actually on the full Euclidean space, hence they are translation invariant up to a gauge and their kernels at the (x, x) diagonal are independent of x. Therefore $\mathrm{Tr}(PE_\lambda)$ equals to the integrated density of states multiplied by $|U|$. So actually we proved the pointwise form of (5.15)

$$\Big(E - (-i\nabla + A)^2\Big)_+(x, x) \leq \big(E + \Delta\big)(x, x) \ . \tag{5.16}$$

By applying (5.13)-(5.15) and Lemma 2 (via the identity (3.12) the inequality (5.15) plays the role of (3.7) in Lemma 2), we obtain the lower bound

$$\sum_{j=1}^{N}(\phi_j, H\phi_j) \geq \int_0^\infty \Big(N - \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n}|B_n||U|\Big)_+ \mathrm{d}\lambda \ , \tag{5.17}$$

where $\dfrac{\lambda^{\frac{n}{2}}}{(2\pi)^n}|B_n|$ is the integrated density of states of $-\Delta$ acting on $L^2(\mathbb{R}^n)$. An easy computation shows that the right side of (5.17) is equal to

$$\frac{n}{n+2} C_n N^{\frac{n+2}{n}} |U|^{-\frac{2}{n}} \ , \tag{5.18}$$

with

$$C_n = (2\pi)^2 |B_n|^{-\frac{2}{n}} \ ,$$

where B_n is the unit ball in $\mathbb{R}^n$. This completes the proof of our main Theorem. ■

Proof of Proposition 1:

Using the spectral calculus and the identity $f(\lambda) = \int_0^\infty (E-\lambda)_+ f''(E)\mathrm{d}E$ the result follows from (5.15) and (5.16). ■

Remark: Due to the close connection between the magnetic operator with a constant field and the harmonic oscillator it is natural to ask whether the analogue of (1.13) is true for the n–dimensional harmonic oscillator $H_\omega = -\Delta + \omega^2 x^2$. While it is true for the two dimensional harmonic oscillator and hence for even dimensional ones, it definitely fails in one dimension. This can be seen by straightforward calculations.

6. The generalized diamagnetic inequality for inhomogeneous large fields

Now we wish to consider whether our technique of comparing IDS works for general magnetic fields. We can show that our generalized diamagnetic inequality (5.16) [hence (1.13)] remains valid for inhomogeneous magnetic fields in the large field limit. However, by constructing a counterexample in the next section, we also show that it fails in general.

To consider the large field limit case, we work in $n = 2$ dimensions for simplicity.

Proposition 2. *Let the function $\psi : \mathbb{R}^2 \to \mathbb{R}^+$ be nonnegative, smooth and compactly supported. Let A be smooth, generating a magnetic field satisfying $B(x) \geq c > 0$ for all $x \in \mathbb{R}^2$. Assume that $|\nabla B(x)| \geq c > 0$ on an open set U, where c is any fixed positive constant. Then for any fixed $E > 0$*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} Tr\Big[\psi\left(\lambda E - (-i\nabla + \lambda A)^2\right)_+\Big] \leq \lim_{\lambda\to\infty} \frac{1}{\lambda^2} Tr\Big[\psi\left(\lambda E + \Delta\right)_+\Big] \ , \tag{6.1}$$

where ψ acts as a multiplication operator. The right side of (6.1) is independent of λ even before the limit and equals to $\dfrac{E^2}{8\pi}\displaystyle\int \psi$. The inequality is sharp if supp $\psi \cap U \neq \emptyset$.

The heart of the proof is the following semiclassical statement.

Proposition 3. *Under the stated conditions,*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} Tr\Big[\psi\left(\lambda E - (-i\nabla + \lambda A)^2\right)_+\Big] = \int_{\mathbb{R}^2} \psi(x) \frac{B(x)}{2\pi} \sum_{k=0}^{\infty} \Big(E - (2k+1)B(x)\Big)_+ \mathrm{d}x \tag{6.2}$$

¿From this result Proposition 2 follows easily since

$$\frac{B(x)}{2\pi} \sum_{k=0}^{\infty} \Big(E - (2k+1)B(x)\Big)_+ \leq \frac{E^2}{8\pi} \tag{6.3}$$

for each individual x, by the staircase argument. Integrating (6.3) against $\psi(x)$ we obtain (6.1). Since $B(x)$ is continuous but not constant on the support of ψ, we see that the staircase inequality (6.3) is strict on an open set of x inside the support of ψ.

The proof of Proposition 3 is a microlocal result (see Theorem 6.4.13, statement (6.4.57) in [Iv 98]). We rescale the problem so that the strong field limit becomes the standard semiclassical strong field limit. Clearly

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\mathrm{Tr}\Big[\psi\left(\lambda E-(-i\nabla+\lambda A)^2\right)_+\Big]=\lim_{h\to 0}h^2\mathrm{Tr}\Big[\psi\big(E-(-ih\nabla+\mu A)^2)_+\big)\Big] \tag{6.4}$$

with $\mu := h^{-1}$ and $h := \lambda^{-1/2}$. Moreover, $\mathrm{Tr}\Big[\psi\left(E-(-ih\nabla+\mu A)^2\right)_+\Big]$ is exactly the expression

$$\int \mathrm{d}x\int (E-\tau)_+\psi(x)\mathrm{d}_\tau e(x,x,-\infty,\tau') \tag{6.5}$$

in (6.4.50) [Iv 98] (changing τ' to τ). Here

$$e(x,x,-\infty,\tau) := \Pi_{(-\infty,\tau)}(H)(x,x) \tag{6.6}$$

is the kernel of the spectral projection on the diagonal with

$$H = H(h,\mu) = (-ih\nabla+\mu A)^2\ . \tag{6.7}$$

The other term in (6.4.50) [Iv 98] is

$$\begin{aligned} h^{-2}\int\mathrm{d}x\int(E-\tau)_+\psi(x)\frac{B(x)}{2\pi}\mathrm{d}_\tau\Big(\sum_{k=0}^{\infty}\theta(\tau-(2k+1)B(x))\Big) &= \\ = h^{-2}\int\psi(x)\frac{B(x)}{2\pi}\sum_{k=0}^{\infty}\Big(E-(2k+1)B(x)\Big)_+\mathrm{d}x\ , \end{aligned} \tag{6.8}$$

where $\theta(t)$ is given in (5.5). Hence (6.4.57) [Iv 98] says that with some constant C, depending on the smoothness of B

$$\left|h^2\mathrm{Tr}\Big[\psi\big(E-(-ih\nabla+\mu A)^2\big)_+\Big]-\int\psi(x)\frac{B(x)}{2\pi}\sum_{k=0}^{\infty}\Big(E-(2k+1)B(x)\Big)_+\mathrm{d}x\right|\le Ch^2\ , \tag{6.9}$$

which clearly goes to zero as $h\to 0$.

It is easy to check that the required conditions in Theorem 6.4.13 of [Iv 98] are satisfied in our situation. ∎

7. Counterexample to the generalized diamagnetic inequality

The following counterexample shows that (5.16) is not true for arbitrary magnetic field.

Let us consider the magnetic field in $\mathbb{R}^3$ which is given by a vector–potential $\lambda A(|x|)$ that satisfies the following conditions:

1) A is smooth, compactly supported, $\mathrm{supp}A = \Big\{x \in \mathbb{R}^3 \,:\, r_1 \le |x| \le r_2\Big\}$;

2) $x \cdot A(|x|) = 0$ for all $x \in \mathrm{supp}A$;

3) $\mathrm{div}A = 0$,

and λ is a small coupling constant. By the spectral resolution

$$\Big(E - (-i\nabla + \lambda A)^2\Big)_+(x,x) = \int_{\mathbb{R}^3} (E - k^2)_+ |\varphi_k(x)|^2 \mathrm{d}k \ , \tag{7.1}$$

where $\varphi_k(x)$ satisfies the Lippmann–Schwinger equation ([RS 79]):

$$\varphi_k(x) = \frac{e^{ik\cdot x}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V_\lambda \varphi_k)(y)\mathrm{d}y \tag{7.2}$$

with

$$V_\lambda = -2i\lambda A\cdot\nabla + \lambda^2 A^2 \ . \tag{7.3}$$

We solve (7.2) for $\varphi_k(x)$ perturbatively and use them in (7.1) keeping only the terms of the order up to λ^2. Integration by parts and the properties of A imply that at $x = 0$ only the $\lambda^2 A^2$ term contributes with respect to the nonmagnetic problem, namely

$$|\varphi_k(0)|^2 = \frac{1}{(2\pi)^3}\Big(1 - \frac{\lambda^2}{|k|}\int_{r_1}^{r_2} A^2(r)\sin(2|k|r)\mathrm{d}r\Big) + o(\lambda^2) \tag{7.4}$$

and, after some calculation

$$\begin{aligned}&\Big(E - (-i\nabla + \lambda A)^2\Big)_+(0,0) - (E+\Delta)_+(0,0)\\ &= -\frac{\lambda^2}{(4\pi)^2}\int_{r_1}^{r_2}\frac{A^2(r)}{r^4}\Big(- \xi^2\sin\xi - 3\xi\cos\xi + 3\sin\xi\Big)\mathrm{d}r + o(\lambda^2) \ ,\end{aligned} \tag{7.5}$$

where $\xi := 2r\sqrt{E}$.

Clearly r_1, r_2 and E can be chosen such that

$$-\xi^2\sin\xi - 3\xi\cos\xi + 3\sin\xi < 0 \tag{7.6}$$

for all relevant values of ξ. Hence, for sufficiently small values of λ the right side of (7.5) can be made positive which contradicts to the comparison (5.16).

Since $|x|^{-1} \in L^{3-\epsilon}(\mathbb{R}^3) + L^{3+\epsilon}(\mathbb{R}^3)$, $|x|^{-2} \in L^{\frac{3}{2}-\epsilon}(\mathbb{R}^3) + L^{\frac{3}{2}+\epsilon}(\mathbb{R}^3)$, and $|A|$, $A^2 \in L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$, we have that

$$\int_{\mathbb{R}^3}\frac{|A(|y|)|}{|x-y|}\mathrm{d}y \ , \int_{\mathbb{R}^3}\frac{A^2(|y|)}{|x-y|}\mathrm{d}y \ \text{ and } \ \int_{\mathbb{R}^3}\frac{|A(|y|)|}{|x-y|^2}\mathrm{d}y$$

are finite by Young's inequality. This provides the finiteness of the $L^\infty \to L^\infty$ norm of the integral operator

$$(Q_\lambda \varphi)(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V_\lambda \varphi)(y) \mathrm{d}y$$

in the Lippmann–Schwinger equation (7.2). Moreover, for λ sufficiently small Q_λ clearly becomes a contraction which yields the convergence of the Neumann series for φ_k. This justifies the applicability of the perturbation argument. ■

This counterexample shows that the approach used to prove Theorem 1 cannot be directly generalized to the case of a general magnetic field. However, the generalization of Theorem 1 to arbitrary magnetic fields remains open. Another open question: Which is the most general class of functions f for which the diamagnetic inequality (1.13) is true?

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