Exponential mixing and $|\ln \hbar|$ time scales in quantized hyperbolic maps on the torus

Francesco Bonechi
INFN, Sezione di Firenze
Dipartimento di Fisica, Università di Firenze
Largo E.Fermi 2, 50125 Firenze, Italy
e-mail: bonechi@fi.infn.it
and
Stephan De Bièvre
UFR de Mathématiques et UMR AGAT
Université des Sciences et Technologies de Lille
59655 Villeneuve d'Ascq Cedex France
e-mail: debievre@gat.univ-lille1.fr

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Abstract

We study the behaviour, in the simultaneous limits $\hbar \to 0, t \to \infty$, of the Husimi and Wigner distributions of initial coherent states and position eigenstates, evolved under the quantized hyperbolic toral automorphisms and the quantized baker map. We show how the exponential mixing of the underlying dynamics manifests itself in those quantities on time scales logarithmic in $\hbar$. The phase space distributions of the coherent states, evolved under either of those dynamics, are shown to equidistribute on the torus in the limit $\hbar \to 0$, for times t between $\frac{1}{2}\frac{|\ln \hbar|}{\gamma}$ and $\frac{|\ln \hbar|}{\gamma}$, where γ is the Lyapounov exponent of the classical system. For times shorter than $\frac{1}{2}\frac{|\ln \hbar|}{\gamma}$, they remain concentrated on the classical trajectory of the center of the coherent state. The behaviour of the phase space distributions of evolved position eigenstates, on the other hand, is not the same for the quantized automorphisms as for the baker map. In the first case, they equidistribute provided $t \to \infty$ as $\hbar \to 0$, and as long as t is shorter than $\frac{|\ln \hbar|}{\gamma}$. In the second case, they remain localized on the evolved initial position at all such times.

1 Introduction

It has been known for a long time and proven in a large number of situations that the eigenfunctions of a quantum system that has an ergodic classical limit equidistribute on the relevant energy surface as $\hbar \to 0$ [Bo, BDB1, BDB2, CdV, DBDE, GL, HMR, Sc, Z1]. Similarly, if the underlying dynamics is mixing, this has an effect on the off-diagonal matrix elements of observables between eigenstates [Bo, CR1, Z2, Z3]. In other words, signatures of ergodicity or mixing on *spectral* properties of quantum systems have been extensively studied. In this paper we exhibit a phenomenon in the *time* domain that is a signature of the *exponential* mixing of the underlying dynamics: the equidistribution on the relevant energy surface of the Husimi and Wigner distributions of an evolved coherent state in the limit where simultaneously $\hbar \to 0$ and $\frac{1}{2}\frac{|\ln \hbar|}{\gamma} << t << \frac{|\ln \hbar|}{\gamma}$. We show this phenomenon occurs in the quantized cat and Baker maps. We expect it to hold also in other exponentially mixing systems, such as perturbed cat maps, chaotic billards, or manifolds of constant negative curvature. Rigorous proofs of these latter statements are still lacking for reasons that will be explained below.

More precisely, we give a rigorous proof of the occurrence of two distinct time scales, both logarithmic in $\hbar$, in the behaviour of the Wigner and Husimi phase space distributions of initial coherent states evolved under quantized hyperbolic toral automorphisms or the quantized baker map. (Theorems 1.1 and 1.2). Denoting by γ the Lyapounov exponent of the classical map, we show in particular a qualitative difference between the behaviour of the above quantities for times shorter than $\frac{|\ln \hbar|}{2\gamma}$ and for times between $\frac{|\ln \hbar|}{2\gamma}$ and $\frac{|\ln \hbar|}{\gamma}$. When $t << \frac{|\ln \hbar|}{2\gamma}$ the system is in what could be called, following [CC], the "orbital instability regime": the phase space distribution functions behave as a delta function situated on the classically evolved center of the initial coherent state. For times $\frac{|\ln \hbar|}{2\gamma} << t << \frac{|\ln \hbar|}{\gamma}$ the system is in the "mixing" or "statistical relaxation" regime: the phase space density spreads throughout the phase space and approaches the steady state given here by the uniform distribution. Some numerical evidence for the existence of such a transition at times of order $|\ln \hbar|$ was given in [TI] for the cat map and in [BV] for the Baker transformation without however pinning down its precise location.

The occurence of the transition at $\frac{|\ln \hbar|}{2\gamma}$ is linked to the support properties of the phase space distributions of coherent states and the situation is different for position eigenstates. When evolved under a quantized toral automorphism, the latter reach the mixing regime "immediately"; by this we mean in any limit, however slow, where $t \to \infty$ as $\hbar \to 0$, and for all times up to $\frac{|\ln \hbar|}{\gamma}$. When evolved under the quantized baker map, on the contrary, they remain in the "orbital instability regime" for all such times. For precise statements, see (1.6) and Proposition 4.2 (i).

We will argue below that the above phenomena can be understood in terms of the interaction between the exponential mixing properties of the classical dynamics and the support properties of the initial phase space distributions. This suggests that the results should remain valid – mutatis mutandis – for other exponentially unstable chaotic systems. To prove the existence of a transition in the behaviour of the coherent state matrix elements, however, one would need good control on the semi-classical approximation of the full quantum evolution of such systems for times longer than $|\ln \hbar|/2\gamma$, where γ is the classical mixing rate of the system. This is known to be a delicate matter: see [BR, BGP, CR2, HJ] for results in this direction.

The study of time-dependent quantities has attracted less attention in quantum chaos than that of spectral properties. Nevertheless, its importance has been stressed by several authors very early on [B] [BB] [BBTV]. There are two reasons for this. The first is that resolving fine spectral details is equivalent, via the Fourier transform, to understanding the (very) long time behaviour of the system. The other one, perhaps a bit more philosophical, is that chaotic properties of classical dynamical systems (ergodicity, mixing and mixing rates, Lyapounov exponents, ...) manifest themselves asymptotically in time ($t \to \infty$). It is well kown that bound quantum mechanical systems, having discrete spectrum, can not be chaotic in the classical sense. This poses the question of how classical chaos "emerges" from quantum mechanics as $\hbar \to 0$ and hence of the non-commutativity of the $\hbar \to 0$ and $t \to \infty$ limits. The following view on this problem has been expressed more or less explicitly by several authors [BV][OT][CC]. We know that for times that are not too long, but that could still go to infinity as $\hbar$ goes to 0, classical and quantum mechanics are close. It is reasonable to expect that if, within this time scale, the classical system exhibits certain chaotic properties, then those will also manifest themselves in the quantum system. Our results give a precise quantitative content to this idea by showing that, for the quantized cat and Baker maps, the classical exponential mixing manifests itself already in the quantum system for times shorter than $\frac{|\ln \hbar|}{\gamma}$.

Let us now give a precise statement of our results. For more details on the objects introduced, we refer to section 2 for the cat maps and to section 4.1 for the baker map. Let $\mathcal{H}_N(\kappa)(\kappa \in [0, 2\pi[\times[0, 2\pi[)$ be the quantum Hilbert space of states associated to the torus $\mathbf{T}^2$, where $N = 1/(2\pi\hbar) \in \mathbf{N}$, and where $Op^W_\kappa(f)$ is the Weyl quantization of the classical observable $f \in C^\infty(\mathbf{T}^2)$. We also introduce, for each $x = (q, p) \in \mathbf{T}^2$, the coherent state $|x, z, \kappa\rangle \in \mathcal{H}_N(\kappa)$ which is obtained by periodizing the Gaussian centered at x, with shape determined by $z \in \mathbf{C}$ ($Im\, z > 0$). To each hyperbolic toral automorphism $A \in \mathrm{SL}_2(\mathbf{Z})$ ($|\mathrm{Tr}A| > 2$) the metaplectic representation associates a unitary map $M_\kappa(A)$ on $\mathcal{H}_N(\kappa)$, the so called quantum map. To each (bounded) function f on the torus we associate the following (t, N) dependent function ($x_0 \in \mathbf{T}^2$):

$$O_f^{AW,\kappa}[t, N](x_0) = \langle x_0, z, \kappa|\mathrm{M}_\kappa(\mathrm{A})^{-t} Op^{AW}_{\kappa,z}(f)\mathrm{M}_\kappa(\mathrm{A})^t|x_0, z, \kappa\rangle\ . \tag{1.1}$$

Here $Op^{AW}_{\kappa,z}(f)$ stands for the anti-Wick quantization of f, so that

$$O_f^{AW,\kappa}[t, N](x_0) = \int_{\mathbf{T}^2} f(x)|\langle x, z, \kappa|\mathrm{M}_\kappa(\mathrm{A})^t|x_0, z, \kappa\rangle|^2\ \frac{dx}{2\pi\hbar}\ . \tag{1.2}$$

Our principal result for the quantized hyperbolic toral automorphisms is then:

Theorem 1.1 *For all $f \in C^\infty(\mathbf{T}^2)$, for all $\epsilon > 0$*

$$\lim_{\substack{N\to\infty \\ t<(1-\epsilon)\frac{\ln N}{2\gamma}}} \left(O_f^{AW,\kappa}[t,N](x_0) - f(A^t x_0)\right) = 0, \tag{1.3}$$

and

$$\lim_{\substack{N\to\infty \\ (1+\epsilon)\frac{\ln N}{2\gamma}<t<(1-\epsilon)\frac{\ln N}{\gamma}}} O_f^{AW,\kappa}[t,N](x_0) = \int_{\mathbf{T}^2} f(x)\, dx\ , \tag{1.4}$$

uniformly in $x_0 \in \mathbf{T}^2$ and in t in the indicated region. Here γ denotes the Lyapounov exponent of the classical map.

The first part of Theorem 1.1 asserts that if N goes to infinity with t "much" smaller than $\frac{\ln N}{2\gamma}$, then $O_f^{AW,\kappa}[t,N](x_0)$ tends to $f(A^t x_0)$. In other words, on this time scale, the behaviour of $O_f^{AW,\kappa}$ is determined by the motion of the center of the coherent state, which is erratic since the classical map is chaotic and which therefore depends strongly on the initial $x_0 \in \mathbf{T}^2$ and on A: the system is in the orbital instability regime. Note that, in view of (1.2), equation (1.3) means that the Husimi distribution of the evolved coherent states converges (weakly) to the delta function at $A^t x_0$ as $\hbar$ tends to 0. Changing from anti-Wick quantization to Weyl quantization, it is easy to see that the same statement holds for the Wigner distribution of the evolved coherent state as well.

Equation (1.4) shows that, at times beyond $\ln N/2\gamma$, the situation changes. In the asymptotic region where $\ln N/2\gamma << t << \ln N/\gamma$, $O_f^{AW,\kappa}[t,N](x_0)$ becomes a constant, independent of the coherent state center x_0 and of the dynamics A. This result is a consequence of the exponential mixing property of the classical map as we explain below.

Equation (1.3) is of course easily understood intuitively using the following simple argument about the evolution of coherent states. Recall first that (see also section 2) $M_\kappa(A)|x_0, z, \kappa\rangle = (\text{phase})|Ax_0, A\cdot z, \kappa\rangle$, where the change from z to $A\cdot z$ represents the stretching and squeezing of the initial coherent state under the classical evolution by a factor $\exp\gamma$. Hence, if $t \ll \frac{\ln N}{2\gamma}$ (i.e. $t \leq (\frac{1}{2}-\epsilon)\frac{\ln N}{\gamma}$), the region in which the coherent state lives is of maximal linear size $e^{\gamma t}/\sqrt{N} \approx N^{-\epsilon}$, which shrinks with N, even though t may go to infinity. As a result, the only contribution to the integral in calculating (1.2) comes from the point $A^t x_0$.

To understand (1.4), let us first recall that for the classical system one has (see [DB] for a simple proof of this well known fact):

$$\left|\int_{\mathbf{T}^2} f(x)\rho(A^{-t}x,t)\ dx - \int_{\mathbf{T}^2} f(x)\ dx\right| \leq C_f \left(\int_{\mathbf{T}^2} |\nabla\rho|(x,t)\ dx\right) \exp -\gamma t, \tag{1.5}$$

for any phase space distribution $\rho(x,t) \geq 0$, $\int_{\mathbf{T}^2} \rho(x,t)\ dx = 1$, which may depend explicitly on t, a remark we will use below. This means $\rho(x,t)$ converges (weakly) to 1 and does so exponentially fast provided $\sup_t \int_{\mathbf{T}^2} |\nabla\rho|(x,t)\ dx < \infty$: this is the property called exponential mixing, a consequence of the exponential instability of the dynamics. Note however that the convergence depends also on $\int_{\mathbf{T}^2} |\nabla\rho|(x,t)\, dx$, a quantity that should be thought of as measuring the scale on which $\rho(x,t)$ fluctuates: the faster the fluctuations in ρ, the bigger this quantity is and the slower the convergence in (1.5). Consider now

$$O_f^{AW,\kappa}[t,N](x_0) = \int_{\mathbf{T}^2} f(x) \frac{|\langle x_0, z, \kappa | A^{-t}x, A^{-t}\cdot z, \kappa\rangle|^2}{2\pi\hbar}\, dx.$$

This is precisely of the form $\int_{\mathbf{T}^2} f(x)\rho(A^{-t}x,t)\, dx$ with

$$\rho(x,t) = \frac{|\langle x_0, z, \kappa | x, A^{-t}\cdot z, \kappa\rangle|^2}{2\pi\hbar}.$$

Hence $\rho(x,t)$ is the Husimi distribution of the initial coherent state $|x_0, z, \kappa\rangle$ with respect to the "squeezed" coherent states $|x, z' = A^{-t}\cdot z, \kappa\rangle$: as a result, it is not unreasonable to expect that $\int_{\mathbf{T}^2} |\nabla\rho|(x,t)\, dx$ is of order $1/\sqrt{\hbar}$. If this were true, the exponential mixing property of the classical dynamics would immediately imply that

$$|O_f^{AW,\kappa}[t,N](x_0) - \int_{\mathbf{T}^2} f(x)\, dx| \leq C_f \frac{\exp -\gamma t}{\sqrt{\hbar}},$$

yielding (1.4). We will give two proofs of (1.4). The proof of Proposition 3.3 (ii) will follow essentially the above strategy which clearly brings out the role of the exponential mixing in the result, and its interaction with the natural length scale $\sqrt{\hbar}$ associated to the Husimi distribution of $|x_0, z, \kappa\rangle$. The proof of Proposition 3.1-3.2 is simpler, but perhaps less telling. Let us also point out that (1.4) remains true, even if x_0 is a periodic point of the underlying dynamics.

The toral automorphisms are obtained by folding a linear dynamics back over the torus and as a result they enjoy, both classically and quantum-mechanically, very special arithmetic properties [DEGI, KR, Ke]. This makes them rather particular and it is of interest to see if a result similar to Theorem 1.1 holds for more "generic" chaotic systems, such as the perturbed cat maps or the baker map. This would ensure that the phenomena observed are not due to the special nature of the toral automorphisms. A partial answer to this question is given for the baker map in Theorem 1.2 below. Our principal tool in the proof of this result is an Egorov theorem (Proposition 4.3) proven for this system in [DBDE], which allows us to control the dynamics for times up to $\log_2 N$. Recall that the Lyapounov exponent of the Baker map is $\ln 2$. As mentioned previously, such control is not available for perturbed toral automorphisms, nor for any other hyperbolic Hamiltonian system. Writing B for the baker map and V_B for its quantization on $\mathcal{H}_N(0)$, we have (for unexplained notations, see section 4):

Theorem 1.2 $\forall x_0 \in \mathbf{T}^2, \forall 0 < \epsilon < 1$, *there exists a sequence* $x_N = (q_N, p_N) \in \mathbf{T}^2, ||x_N - x_0|| \leq N^{-\epsilon/5}$, *so that, for all* $f \in C^\infty(\mathbf{T})$,

$$\lim_{\substack{N\to\infty \\ 0\leq t<(1-\epsilon)\frac{1}{2}\log_2 N}} \langle x_N, z, 0 | V_B^{-t} Op_{0,z}^{AW}(f) V_B^t | x_N, z, 0\rangle - f(2^t q_N) = 0$$

and

$$\lim_{\substack{N\to\infty \\ (1+\epsilon)\frac{1}{2}\log_2 N<t<(1-\epsilon)\log_2 N}} \langle x_N, z, 0|V_B^{-t} Op_{0,z}^{AW}(f) V_B^t|x_N, z, 0\rangle = \int_0^1 f(q)\, dq\ ,$$

The limits are uniform for t in the indicated region.

Comparing to Theorem 1.1, this result has two obvious shortcomings: the restriction to functions f of the q variable alone and the use of the sequences x_N. Their origin will be explained in section 4.2.

The occurence of a transition at $\frac{|\ln \hbar|}{2\gamma}$ in the qualitative behaviour of the matrix elements above is directly related to the choice of a coherent state as an initial state. To illustrate this, we also study the behaviour of $\langle e_j^\kappa, M_\kappa(A)^{-t} Op_{\kappa,z}^{AW}(f) M_\kappa(A)^t e_j^\kappa\rangle$ (section 3.3), and of $\langle e_j^\kappa, V_B^{-t} Op_{\kappa,z}^{AW}(f) V_B^t e_j^\kappa\rangle$ (section 4), where the e_j^κ are the "position eigenstates" in $\mathcal{H}_\hbar(\kappa)$ (see section 2). In the case of the cat maps, if one applied the same heuristic reasoning as above (based on (1.5)), one would conclude that the mixing regime should set in *no later than* for times of the order $\frac{\ln N}{2\gamma}$, since the Husimi distribution of the position eigenstates still has a spread of the order of $\sqrt{\hbar}$ in the q-direction. In fact, we will prove in section 3.3 that for all sequences $j_N (1 \leq j_N \leq N)$

$$\lim_{\substack{N\to\infty, t\to\infty \\ t<(1-\epsilon)\frac{\ln N}{\gamma}}} \langle e_{j_N}^\kappa, M_\kappa(A)^{-t} Op_{\kappa,z}^{AW}(f) M_\kappa(A)^t e_{j_N}^\kappa\rangle = \int_{\mathbf{T}^2} f(x) dx, \tag{1.6}$$

the limit being again uniform with respect to j_N: in other words, no orbital instability regime is observed here, mixing sets in as soon as $t \to \infty$. This results, intuitively at least, from a suitably adapted version of (1.5), proven in section 4, together with the observation that the phase space distributions of the position eigenstates are completely delocalized in the p-direction.

In the case of the baker map, the opposite phenomenon occurs: in this case, no mixing regime is observed for $\langle e_j^\kappa, V_B^{-t} Op_{\kappa,z}^{AW}(f) V_B^t e_j^\kappa\rangle$ at times shorter than $|\log_2 \hbar|$. For a detailed statement we refer to Proposition 4.2 (i). This result can again be understood in terms of the interaction of the mixing properties of the classical map with the support properties of the initial position state, the support of which is now exactly lined up with the stable manifold of the classical baker map.

As mentioned above, manifestations of classical chaos in the corresponding quantum system have been searched mainly in the energy domain. Nevertheless, several papers [B, BB, BBTV, BV, CC, OT, OTH, TH, TI] have analysed the behaviour of the Wigner function of semi-classically evolved initial states with initial support either in a point, such as coherent states, or on a Lagrangian submanifold, such as position eigenstates, and compared them numerically to the Wigner function of the quantum mechanically evolved states. The general picture that emerges from these studies is as follows: for times that are not too long the semi-classical approximation should be valid and is itself determined essentially by the behaviour of the support of the Wigner function of the initial state under the classical flow. The results of the present paper, as summarized above, corroborate this picture by proving some rigorous statements in this direction on a few simple systems, for times up to $|\ln \hbar|/\gamma$. As an example, Theorem 1.1 shows that the Husimi distribution of the *quantum mechanically evolved* coherent state (*i.e.* $\frac{|\langle x_0,z,\kappa|A^{-t}x, A^{-t}\cdot z,\kappa\rangle|^2}{2\pi\hbar}$) can not be distinguished, as $\hbar$ goes to 0, from the *classically evolved* Husimi distribution of the initial coherent state (*i.e.* , $\frac{|\langle x_0,z,\kappa|A^{-t}x,z,\kappa\rangle|^2}{2\pi\hbar}$) for times up to $\frac{\ln N}{\gamma}$. The change in behaviour of either of these quantities, at times of the orderof $\frac{\ln N}{2\gamma}$ is entirely due to the interaction of the exponential mixing of the classical dynamical system with the uncertainty principle which forces the initial Husimi distribution to have a spread of size $\sqrt{\hbar}$.

In addition, we exhibit one extra phenomenon: a slightly longer time scale on which the above picture breaks down, in the sense that the semi-classical or quantum evolution does no longer stay close to the underlying purely classical evolution. Indeed, we show in section 3.2 that, in quantized cat maps, the classical-quantum (and hence the classical-semi-classical) agreement breaks down "a little later than at $|\ln \hbar|/\gamma$", that is to say at times of order $\frac{3}{2}\frac{\ln N}{\gamma}$. More details, as well as an intuitive explanation of the phenomenon can be found in section 3.2.

It is important to add a few words of caution: we want to stress that our results have nothing to say about the problem of the existence or not of a so-called $|\ln \hbar|$-barrier, *i.e.* a time scale of the form $\alpha\frac{|\ln \hbar|}{\gamma}$ ($\alpha > 0$) beyond which semi-classical approximations to the quantum evolution may break down [B, BB, BBTV, BV, OT, OTH, TH]. What we have shown here is that the exponential mixing of the classical dynamics manifests itself in the quantum system already on time scales short enough to be accessible to semi-classical approximations. In other words, the phenomena we exhibit are taking place before such a breakdown – if any – is expected to occur. At any rate, the cat maps being linear, for them semi-classical approximations are exact at all times, and their study can not shed light on the above problem. For the baker map, we show that the semi-classical evolution of the Husimi or Wigner distributions of a position eigenstate (Proposition 4.2 (ii)) and of a coherent state (Proposition 4.1 (ii)) agrees indeed with the quantum mechanical one for times up to $\log_2 \hbar$, as expected.

At the risk of belabouring the point, we repeat one more time that we have not been able to exhibit a time scale on which the agreement between the semi-classical and the full quantum evolution breaks down.

2 Kinematical estimates on the torus

The kinematic framework of quantum mechanics on the two-torus $\mathbf{T}^2$ (with coordinates $(q,p) \in [0,1[\times[0,1[$) as well the quantization of the toral automorphisms is well known. We briefly recall the essential ingredients, following [Bo, BDB2], where proofs omitted here, as well as references to the original literature can be found.

First, recall that the quantum Hilbert space of a particle on the line is $L^2(\mathbf{R})$. Moreover, to each classical observable $f(q,p)$ ($f \in C^2(\mathbf{R}^2)$), Weyl quantization associates a quantum observable $Op^W(f)$ defined by

$$Op^W(f) = \int\int \tilde{f}(\mathbf{a}) U(\mathbf{a}) \frac{d\mathbf{a}}{2\pi\hbar}, \text{where} f(q,p) = \int \tilde{f}(\mathbf{a}) e^{-\frac{i}{\hbar}(a_1 p - a_2 q)} \frac{d\mathbf{a}}{2\pi\hbar}, \quad (2.1)$$

and where $U(a_1, a_2) = e^{-\frac{i}{\hbar}(a_1 P - a_2 Q)}$ are the phase space translation operators in the usual notations (see [BDB2] for further technical details). In particular, if f is periodic of period 1 in both q and p, one readily concludes that

$$Op^W(f) = \sum_{n\in\mathbf{Z}^2} f_n U(2\pi\hbar n_1, 2\pi\hbar n_2), \text{ where } f = \sum_n f_n e^{2\pi i(n_2 q - n_1 p)}. \quad (2.2)$$

We will always suppose $\sum_n |f_n| < \infty$ and shall write

$$f \in \check{\ell}_{1,k} \text{ iff } \sum_n |f_n| \|n\|^k < \infty. \quad (2.3)$$

For $\xi, \eta \in \mathbf{R}^2$, we introduce the notation $\langle \xi, \eta \rangle = \xi_1\eta_2 - \xi_2\eta_1$ and $\chi_\xi(x) = \exp 2\pi i \langle x, \xi \rangle$.

One constructs the quantum Hilbert space of states by searching for states ψ having the symmetry of the torus, i.e. states periodic of period 1 both in the position and the momentum variable:

$$U(1,0)\psi = e^{-i\kappa_1}\psi \qquad U(0,1)\psi = e^{i\kappa_2}\psi, \ (\kappa_1, \kappa_2) \in [0, 2\pi) \times [0, 2\pi), \quad (2.4)$$

where one allows for the usual phase change. The space of solutions $\mathcal{H}_\hbar(\kappa)$ to (2.4) is a space of tempered distributions of which one proves readily that either it is zero-dimensional, or there exists a positive integer N so that

$$2\pi\hbar N = 1, \quad (2.5)$$

in which case it is N-dimensional. Equation (2.5) will always be assumed to hold in the rest of this section.

Condition (2.5) is equivalent to $[U(0,1), U(1,0)] = 0$, so that (2.4) is nothing but the problem of simultaneously diagonalizing $U(0,1)$ and $U(1,0)$. Since their spectra are continuous, this leads to a direct integral decomposition of $L^2(\mathbf{R})$:

$$L^2(\mathbf{R}) \cong \int_0^{2\pi}\int_0^{2\pi} \frac{d\kappa}{(2\pi)^2} \mathcal{H}_\hbar(\kappa) \ , \quad \psi \cong \int_0^{2\pi}\int_0^{2\pi} \frac{d\kappa}{(2\pi)^2} \psi(\kappa). \quad (2.6)$$

This equips each "term" $\mathcal{H}_\hbar(\kappa)$ with a natural inner product. For each admissible $\hbar$ (i.e. $2\pi\hbar N = 1$), the spaces $\mathcal{H}_\hbar(\kappa)$, indexed by κ, are the quantum Hilbert spaces of states for systems having the torus as phase space. An orthonormal basis for $\mathcal{H}_\hbar(\kappa)$ is given by the position eigenstates $\{e_j^\kappa\}_{j=0}^{N-1}$, where

$$e_j^\kappa(y) = \sqrt{\frac{1}{N}} \sum_n e^{in\kappa_1} \delta(y - \frac{\kappa_2}{2\pi N} - \frac{j}{N} - n) \quad y \in \mathbf{R}\,. \quad (2.7)$$

The following unitary representation of the discrete phase space translations

$$U_\kappa\left(\frac{n_1}{N},\frac{n_2}{N}\right)\ e_j^\kappa = e^{i\pi\frac{n_1n_2}{N}}e^{i(\kappa_2+2\pi j)\frac{n_2}{N}}e^{\kappa}_{j+n_1} \tag{2.8}$$

is obtained by restriction of $U(\frac{n_1}{N},\frac{n_2}{N})$ to $\mathcal{H}_\hbar(\kappa)$. For each $\psi\in\mathcal{S}(\mathbf{R})$, we have that

$$\langle e_j^\kappa,\psi(\kappa)\rangle = \frac{1}{\sqrt{N}}\sum_{n\in\mathbf{Z}} e^{i\kappa_1 n}\psi(\frac{\kappa_2}{2\pi N}+\frac{j}{N}-n)\ . \tag{2.9}$$

For later use, we introduce the vectors $\epsilon_j\in L^2(\mathbf{R})$ that are the direct sum (see (2.6)) of the position eigenstates $e_j^\kappa\in\mathcal{H}_\hbar(\kappa)$.

Lemma 2.1 *Let $\epsilon_j\in L^2(\mathbf{R})$, $j=0,\ldots,N-1$, be defined by $\epsilon_j(x)=\sqrt{N}$ if $j/N\le x<(j+1)/N$ and 0 otherwise. Then*

$$\epsilon_j=\int_\oplus\frac{d\kappa}{(2\pi)^2}\ e_j^\kappa\ .$$

Proof. A direct computation shows that for each $\psi=\int_\oplus\frac{d\kappa}{(2\pi)^2}\ \psi(\kappa)$ one has $\langle\epsilon_j,\psi\rangle = \int d\kappa/(2\pi)^2\ \langle e_j^\kappa,\psi(\kappa)\rangle$. ■

Turning now to the quantization of observables, since for each f on $\mathbf{T}^2$, the commutator $[Op^W(f),U(k,\ell)]=0$, it is clear that, for each κ, $Op^W(f)\ \mathcal{H}_\hbar(\kappa)\subset\mathcal{H}_\hbar(\kappa)$ and hence

$$Op^W(f)\cong\int_0^{2\pi}\int_0^{2\pi}\frac{d\kappa}{(2\pi)^2}\ Op^W_\kappa(f), \tag{2.10}$$

where $Op^W_\kappa(f)$ denotes the restriction of $Op^W(f)$ to $\mathcal{H}_\hbar(\kappa)$. We conclude from (2.2) and (2.5) that

$$Op^W_\kappa(f)=\sum_{n\in\mathbf{Z}^2} f_n\ U_\kappa\left(\frac{n_1}{N},\frac{n_2}{N}\right).$$

We will make extensive use of coherent states in our analysis. Recall first the definition of the standard Weyl-Heisenberg coherent states. For $z\in\mathbf{C}$, and $Imz>0$, they are defined, for each $x=(q,p)\in\mathbf{R}^2$, by

$$\eta_{0,z}(y)=\left(\frac{Imz}{\pi\hbar}\right)^{\frac14} e^{\frac{i}{2\hbar}zy^2},\quad \eta_{x,z}(y)=\left(U(q,p)\eta_{0,z}\right)(y)\equiv\langle y\mid x,z\rangle\ y\in\mathbf{R}, \tag{2.11}$$

in the standard manner. We will systematically use the bra-ket notation of Dirac, i.e. $\eta_{x,z}(y)=\langle y\mid x,z\rangle$. Furthermore we recall here the explicit formula for the scalar product between coherent states ($x=(q,p)$, $x'=(q',p')$)

$$\langle x,z|x',z'\rangle = F(z,z')\ e^{\frac{i}{2\hbar}(qp'-pq')}e^{-\frac{1}{2\hbar}(x-x',B(z,z')(x-x'))}, \tag{2.12}$$

where, $F(z,z') = (\mathrm{Im}z\ \mathrm{Im}z')^{1/4}/\sqrt{(z'-z^*)/(2i)}$, and

$$B(z,z') = \frac{1}{z'-z^*}\begin{pmatrix} iz^*z' & -\frac{i}{2}(z'+z^*) \\ -\frac{i}{2}(z'+z^*) & i \end{pmatrix} .$$

If $z = z'$ then $B(z) = B(z,z)$ is a positive definite matrix; we will denote by $\beta_+(z)$ ($\beta_-(z)$) its greatest (smallest) eigenvalue.

The following property that generalizes the resolution of the identity for coherent states is valid for each $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbf{R})$:

$$\int_{\mathbf{R}^2} \frac{dx}{2\pi\hbar} \langle \psi_1 | U(x)\phi_1\rangle \langle U(x)\phi_2|\psi_2\rangle = \langle\psi_1|\psi_2\rangle\langle\phi_2|\phi_1\rangle\ . \tag{2.13}$$

The coherent states $|x,z,\kappa\rangle$ are defined on the torus implicitly by

$$|\ x, z\rangle \cong \int_0^{2\pi}\int_0^{2\pi} \frac{d\kappa}{(2\pi)^2}\ |\ x,z,\kappa\rangle\ . \tag{2.14}$$

They still satisfy an analogue of the generalized resolution of the identity.

Lemma 2.2 *For $i = 1,2$, let $\psi_{i,\kappa_0} \in \mathcal{H}_N(\kappa_0)$, $\phi_i \in \mathcal{S}(\mathbf{R})$. If we denote by $\int dk/(2\pi)^2[U(x)\phi_i](\kappa)$ the integral decomposition of $U(x)\phi_i$ for $x \in \mathbf{T}^2$, we have*

$$\int_{\mathbf{T}^2} \frac{dx}{2\pi\hbar}\langle\psi_{1,\kappa_0}|[U(x)\phi_1](\kappa_0)\rangle\langle[U(x)\phi_2](\kappa_0)|\psi_{2,\kappa_0}\rangle = \langle\psi_{1,\kappa_0}|\psi_{2,\kappa_0}\rangle\langle\phi_2|\phi_1\rangle\ .$$

Proof. Recall from [BDB2] that the map

$$S(\kappa) = (\sum_m \exp -i\kappa_2 m\ U(0,m))(\sum_n \exp i\kappa_1 n\ U(n,0))$$

defines a surjection of the space of Schwartz functions $\mathcal{S}(\mathbf{R})$ onto $\mathcal{H}_\hbar(\kappa) \subset \mathcal{S}'(\mathbf{R})$. Let $\phi, \psi \in \mathcal{S}(\mathbf{R})$ and write, as in (2.6), $\phi \cong \int \frac{d\kappa}{(2\pi)^2}\ \phi(\kappa), \psi \cong \int \frac{d\kappa}{(2\pi)^2}\ \psi(\kappa)$. Then, as a simple calculation shows,

$$\langle\phi(\kappa),\psi(\kappa)\rangle = \langle\phi, S(\kappa)\psi\rangle = \frac{1}{N}\sum_{\ell\in\mathbf{Z}}\bar{\phi}(x_\ell)\psi^{(\kappa_1)}(x_\ell),$$

where $x_\ell = \frac{\kappa_2}{2\pi N} + \frac{\ell}{N}$ and

$$\psi^{(\kappa_1)}(y) = \sum_{n\in\mathbf{Z}} \exp i\kappa_1 n\ \psi(y-n).$$

As a simple inspection shows, the matrix element $\langle\phi(\kappa),\psi(\kappa)\rangle$ is a smooth period function of κ and, as a result, is pointwise equal to the sum of its Fourier series. These observations can be used to justify the following formal computation. Let $\psi_i \in \mathcal{S}(\mathbf{R})$ such that $S(\kappa_0)\psi_i \cong \psi_{i,\kappa_0}$. Then we have

$$\int_{\mathbf{T}^2}\frac{dx}{2\pi\hbar}\langle\psi_{1,\kappa_0}|[U(x)\phi_1](\kappa_0)\rangle\langle[U(x)\phi_2](\kappa_0)|\psi_{2,\kappa_0}\rangle =$$

$$
\begin{aligned}
&= \sum_{mn} e^{i\pi N(m_1 m_2 + n_1 n_2)} e^{i((m_1+n_1)\kappa_{01} - (m_2+n_2)\kappa_{02})} \\
&\qquad\qquad \int_{\mathbf{T}^2} \frac{dx}{2\pi\hbar} \langle \psi_1 | U(m) U(x) \phi_1 \rangle \langle U(x)\phi_2 | U(n)\psi_2 \rangle \\
&= \sum_{ms} e^{i\pi N s_1 s_2} e^{i(s_1\kappa_{01} - s_2\kappa_{02})} \int_{\mathbf{T}^2} \frac{dx}{2\pi\hbar} \langle \psi_1 | U(m+x)\phi_1 \rangle \langle U(x+m)\phi_2 | U(s)\psi_2 \rangle \\
&= \sum_{s} e^{i\pi N s_1 s_2} e^{i(s_1\kappa_{01} - s_2\kappa_{02})} \int_{\mathbf{R}^2} \frac{dx}{2\pi\hbar} \langle \psi_1 | U(x)\phi_1 \rangle \langle U(x)\phi_2 | U(s)\psi_2 \rangle \\
&= \sum_{s} e^{i\pi N s_1 s_2} e^{i(s_1\kappa_{01} - s_2\kappa_{02})} \langle \psi_1 | U(s)\psi_2 \rangle \langle \phi_2 | \phi_1 \rangle = \langle \psi_{1,\kappa_0} | \psi_{2,\kappa_0} \rangle \langle \phi_2 | \phi_1 \rangle \ . \ \blacksquare
\end{aligned}
$$

Given $f \in L^\infty(\mathbf{T}^2)$, the anti-Wick quantization $Op^{AW}_{\kappa,z}(f)$ of f is the operator on $\mathcal{H}_\hbar(\kappa)$ defined by

$$
Op^{AW}_{\kappa,z}(f) = \int_{\mathbf{T}^2} f(x)\ |x,z,\kappa\rangle\langle x,z,\kappa| \frac{dx}{2\pi\hbar}. \tag{2.15}
$$

One has [BDB2],

$$
\mathrm{Id}_{\mathcal{H}_\hbar(\kappa)} = \int_{\mathbf{T}^2}\ |x,z,\kappa\rangle\langle x,z,\kappa| \frac{dx}{2\pi\hbar};\ Op^{AW}_{z}(f) = \int^{\oplus} \frac{d\kappa}{(2\pi)^2}\ Op^{AW}_{\kappa,z} f. \tag{2.16}
$$

In the following lemma we collect a useful relation between Weyl and anti-Wick quantization.

Lemma 2.3 *Let $z \in \mathbf{C}$, with* $\mathrm{Im} z > 0$.

(i) [Plane] *For each $\varphi_i \in L^2(\mathbf{R}), i = 1, 2,\ \beta \in \mathbf{R}^2$ we have that*

$$
\langle \varphi_1 | Op^{AW}_{z}(\chi_\beta) | \varphi_2 \rangle = \langle 2\pi\hbar\beta, z | 0, z \rangle \langle \varphi_1 | Op^{W}(\chi_\beta) | \varphi_2 \rangle\ .
$$

(ii)[Torus] *For each $\varphi_{i,\kappa} \in \mathcal{H}_N(\kappa)$, $\beta \in \mathbf{Z}^2$ we have that*

$$
\langle \varphi_{1,\kappa} | Op^{AW}_{\kappa,z}(\chi_\beta) | \varphi_{2,\kappa} \rangle = \langle \frac{\beta}{N}, z | 0, z \rangle \langle \varphi_{1,\kappa} | Op^{W}_{\kappa}(\chi_\beta) | \varphi_{2,\kappa} \rangle\ .
$$

Proof. (i) The result comes from (2.13) with $\phi_1 = |0,z\rangle$, $\phi_2 = U(2\pi\hbar\beta)|0,z\rangle$, $\psi_1 = \varphi_1$, $\psi_2 = U(2\pi\hbar\beta)\varphi_2$. Point (ii) is proven exactly as in (i) using now Lemma (2.2) with $\phi_1 = |0,z\rangle$, $\phi_2 = U(\beta/N)|0,z\rangle$, $\psi_{1,\kappa} = \varphi_{1,\kappa}$ and $\psi_{2,\kappa} = U_\kappa(\beta/N)\varphi_{2,\kappa}$. $\blacksquare$

Let us introduce some notation. If $s \in \mathbf{R}$, then we denote by $p(s)$ the nearest integer to s; if $s = m + 1/2$ is a half-integer ($m \in \mathbf{Z}$) then $p(m+1/2) = m$. With some abuse of notation, we write $p(\xi) = (p(\xi_1), p(\xi_2)) \in \mathbf{Z}^2$ for $\xi \in \mathbf{R}^2$.

We end this section with two propositions collecting a few simple but crucial semiclassical estimates on the Weyl quantized observables and the Husimi distributions, respectively.

Proposition 2.4 *Let $z \in \mathbf{C}$, with* $\mathrm{Im} z > 0$. *(i) For each function $f \in \check{\ell}_{1,0}$, for all $x = (q,p), x' = (q',p') \in \mathbf{T}^2$, we have that*

$$\begin{aligned}\langle x, z, \kappa | Op_\kappa^W(f) | x', z, \kappa\rangle &= \sum_{m \in \mathbf{Z}^2} (-)^{N m_1 m_2} e^{i\pi N \langle m, x'\rangle} e^{-i(m_1\kappa_1 - m_2\kappa_2)} \\ &\qquad \langle x, z | Op^W(f) | x' - m, z\rangle \,.\end{aligned}$$

(ii) For all n in $\mathbf{Z}^2$, for all $x \in \mathbf{T}^2$, we have that

$$\langle x, z, \kappa | U_\kappa(\frac{n}{N}) | x, z, \kappa\rangle = \sum_{m \in \mathbf{Z}^2} c_n^m(x) e^{-i(m_1\kappa_1 - m_2\kappa_2)} \,,$$

where

$$c_n^m(x) = \chi_{n - Nm}(x)\, (-)^{N m_1 m_2} e^{-i\pi <m,n>} e^{-\pi N (m - \frac{n}{N}, B(z)(m - \frac{n}{N}))} \,.$$

(iii) There exists $C > 0$ such that for each $n \in \mathbf{Z}^2$ and for each $x \in \mathbf{T}^2$ the following inequality holds

$$\left| \langle x, z, \kappa | U_\kappa(\frac{n}{N}) | x, z, \kappa\rangle - c_n^{p(\frac{n}{N})} e^{-i(p(\frac{n_1}{N})\kappa_1 - p(\frac{n_2}{N})\kappa_2)} \right| \le C \exp - \frac{\pi N \beta_-(z)}{4}.$$

Proof. Part (i) is easily obtained by calculating the Fourier coefficients of the function $\langle x, z, \kappa | Op_\kappa^W(f) | x', z, \kappa\rangle$ in κ. Indeed, for $m \in \mathbf{Z}^2$ let

$$c^m(f) = \int_{T_\kappa^2} \frac{d\kappa}{(2\pi)^2} \langle x, z, \kappa | Op_\kappa^W(f) | x', z, \kappa\rangle e^{i(m_1\kappa_1 - m_2\kappa_2)} \,.$$

Then, by using the direct sum decomposition of $L^2(\mathbf{R})$, of the coherent states and of $Op^W(f)$ we have

$$\begin{aligned} c^m(f) &= \int_{T_\kappa^2} \frac{d\kappa}{(2\pi)^2} \langle x, z, \kappa | Op_\kappa^W(f) U(-m_1, 0) U(0, -m_2) | x', z, \kappa\rangle \\ &= (-)^{N m_1 m_2} e^{i\pi N <m,x'>} \int_{T_\kappa^2} \frac{d\kappa}{(2\pi)^2} \langle x, z, \kappa | Op_\kappa^W(f) | x' - m, z, \kappa\rangle \\ &= (-)^{N m_1 m_2} e^{iN\pi <m,x'>} \langle x, z | Op^W(f) | x' - m, z\rangle \,. \end{aligned}$$

(ii) This is obtained from (i) using the explicit formula of the scalar product between two coherent states on the plane (2.12).

(iii) According to the definition given in (2.12) we have that

$$\begin{aligned} |\langle x, z, \kappa | U_\kappa(\frac{n}{N}) | x, z, \kappa\rangle - c_n^{p(\frac{n}{N})} e^{-i(p(\frac{n_1}{N})\kappa_1 - p(\frac{n_2}{N})\kappa_2)}| &\le \sum_{m \neq p(\frac{n}{N})} e^{-\pi N (m - \frac{n}{N}, B(z)(m - \frac{n}{N}))} \\ &\le \sum_{m \neq p(\frac{n}{N})} e^{-\pi N \beta_-(z) \left\| m - \frac{n}{N} \right\|^2} \,. \end{aligned}$$

The result then comes from the following inequality: let $a > 0$ and $x_o = p(x_o) + f_o \in \mathbf{R}$, with $0 \le |f_o| \le 1/2$; then we have

$$\begin{aligned} \sum_{n \neq p(x_o)} e^{-a(n - x_o)^2} &= \sum_{n \neq 0} e^{-a(n - f_o)^2} \le \sum_{n \neq 0} e^{-\frac{a}{2}|n - f_o|} \\ &= \frac{e^{-\frac{a}{2}}}{1 - e^{-\frac{a}{2}}} (e^{-\frac{a}{2} f_o} + e^{\frac{a}{2} f_o}) \le \frac{2\, e^{-\frac{a}{4}}}{1 - e^{-\frac{a}{2}}} \,. \ \blacksquare \end{aligned}$$

Notice in the above proof that it is convenient to estimate various κ-dependent quantities – which is an a priori difficult task – by estimating the terms of the corresponding κ-Fourier series expansion, written in terms of quantities defined on the plane which are often easy to calculate. This technique will be extensively used in the rest of the paper.

The following estimate makes precise the statement given in the introduction about quantum fluctuations of the Husimi distribution of a state ψ, both on the plane and on the torus.

Proposition 2.5 *Let* $z \in \mathbf{C}$, *Im* $z > 0$.

(i)[Plane] *For each* $\psi \in L^2(\mathbf{R})$, *we have that*

$$\begin{aligned}\int_{\mathbf{R}^2} |\partial_q |\langle\psi|x,z\rangle|^2| \frac{dx}{2\pi\hbar} &\leq |z| \sqrt{\frac{2}{\hbar \ \mathrm{Im} z}} \, \|\psi\|^2 \\ \int_{\mathbf{R}^2} |\partial_p |\langle\psi|x,z\rangle|^2| \frac{dx}{2\pi\hbar} &\leq \sqrt{\frac{2}{\hbar \ \mathrm{Im} z}} \, \|\psi\|^2\end{aligned}$$

(ii)[Torus] *For each* $\kappa_0 \in [0,2\pi) \times [0,2\pi)$, $\psi_{\kappa_0} \in \mathcal{H}_N(\kappa_0)$ *we have that*

$$\begin{aligned}\int_{\mathbf{T}^2} |\partial_q |\langle\psi_{\kappa_0}|x,z,\kappa_0\rangle|^2| \frac{dx}{2\pi\hbar} &\leq 2|z| \sqrt{\frac{\pi N}{\mathrm{Im} z}} \, \|\psi_{\kappa_0}\|^2_{\mathcal{H}_N} \\ \int_{\mathbf{T}^2} |\partial_p |\langle\psi_{\kappa_0}|x,z,\kappa_0\rangle|^2| \frac{dx}{2\pi\hbar} &\leq 2 \sqrt{\frac{\pi N}{\mathrm{Im} z}} \, \|\psi_{\kappa_0}\|^2_{\mathcal{H}_N}\end{aligned}$$

Proof. (i) By direct computation we obtain that ($x = (q,p)$)

$$\partial_q \eta_{x,z} = -\frac{i}{2\hbar} p \eta_{x,z} - \frac{iz}{\hbar} U(x) Q \ \eta_{0,z} \ , \quad \partial_p \eta_{z,x} = \frac{i}{2\hbar} q \eta_{z,x} + \frac{i}{\hbar} U(x) Q \ \eta_{z,0} \ ,$$

so that

$$\begin{aligned}\partial_q |\langle\psi|x,z\rangle|^2 &= \frac{2}{\hbar} \ \mathrm{Im} \left[z \langle x,z|\psi\rangle \langle\psi|U(x)Q|0,z\rangle \right] \\ \partial_p |\langle\psi|x,z\rangle|^2 &= -\frac{2}{\hbar} \ \mathrm{Im} \left[\langle x,z|\psi\rangle \langle\psi|U(x)Q|0,z\rangle \right] \ .\end{aligned} \tag{2.17}$$

Then we have

$$\begin{aligned}&\int_{\mathbf{R}^2} |\partial_q |\langle\psi|x,z\rangle|^2| \frac{dx}{2\pi\hbar} \\ &\leq \frac{2}{\hbar} \ |z| \int_{\mathbf{R}^2} |\langle\psi|x,z\rangle \langle\psi|U(x)Q|0,z\rangle| \frac{dx}{2\pi\hbar} \\ &\leq \frac{2}{\hbar} \ |z| \left[\int_{\mathbf{R}^2} |\langle\psi|x,z\rangle|^2 \frac{dx}{2\pi\hbar} \right]^{1/2} \left[\int_{\mathbf{R}^2} |\langle\psi|U(x)Q|0,z\rangle|^2 \frac{dx}{2\pi\hbar} \right]^{1/2} \\ &= \frac{2}{\hbar} |z| \|\psi\|^2 \|Q\eta_{0,z}\| \ ,\end{aligned}$$

where we used (2.13). The result then follows from $\|Q\eta_{0,z}\| = \sqrt{\frac{\hbar}{2\mathrm{Im}z}}$.

(ii) By using the Fourier transform in κ it is easy to show that the analogues of equations (2.17) are still valid in $\mathcal{H}_\hbar(\kappa_0)$, *i.e.*

$$\begin{aligned}
\partial_q |\langle \psi_{\kappa_0} | x, z, \kappa_0 \rangle|^2 &= 4\pi N \ \mathrm{Im}\,[z\langle x, z, \kappa_0 | \psi_{\kappa_0}\rangle \langle \psi_{\kappa_0} | [U(x) Q\eta_{0,z}](\kappa_0)\rangle] \\
\partial_p |\langle \psi_{\kappa_0} | x, z, \kappa_0 \rangle|^2 &= -4\pi N \ \mathrm{Im}\,[\langle x, z, \kappa_0 | \psi_{\kappa_0}\rangle \langle \psi_{\kappa_0} | [U(x) Q\eta_{0,z}](\kappa_0)\rangle] \ ,
\end{aligned}$$

where $[U(x)Q\eta_{z,0}](\kappa) \cong S(\kappa)U(x)Q\eta_{z,0}$. The result is then obtained exactly as in (i), by the use of Lemma 2.2. ■

3 The quantized toral automorphisms

Recalling the quantization of quadratic hamiltonians on $\mathbf{R}^2$ it is easy to be convinced that to any linear map specified by $A \in \mathrm{SL}_2(\mathbf{R})$ Weyl quantization associates a unitary propagator in $L^2(\mathbf{R})$

$$(M(A)\psi)(y) = \left(\frac{i}{2\pi\hbar a_{12}}\right)^{1/2} \int d\eta \ e^{\frac{i}{\hbar} S_A(y,\eta)} \psi(\eta) \ , \tag{3.1}$$

where S_A is the classical action associated to A (see [F] for details). A fundamental property of Weyl quantization with respect to this dynamics is the absence of an error term in the so called *Egorov theorem*, meaning that "quantization and evolution commute", i.e.

$$\mathrm{M(A)}^{-t} Op^W(f) \mathrm{M(A)}^t = Op^W(f \circ A^t) \ \forall t \in \mathbf{Z} \ . \tag{3.2}$$

The quantization of $A \in \mathrm{SL}_2(\mathbf{Z})$ on the torus is straightforward. It turns out that for all N there exists κ such that $M(A)\mathcal{H}_N(\kappa) \subset \mathcal{H}_N(\kappa)$ (see [BDB2] for the equation that κ must satisfy). We shall write $M_\kappa(A)$ for the restriction of $M(A)$ to $\mathcal{H}_N(\kappa)$. From this construction it follows that (3.2) holds for $Op^W_\kappa(f)$ as well. We finally recall that $M(A) \mid x, z\rangle = e^{i\phi(A)} \mid Ax, A\cdot z\rangle$ where $A \cdot z = (za_{22} + a_{21})/(za_{12} + a_{11})$ (closely related to the usual homographic action), and where $\phi(A)$ is a phase we don't specify. As a result,

$$M_\kappa(A) \mid x, z, \kappa\rangle = e^{i\phi(A)} \mid Ax, A\cdot z, \kappa\rangle. \tag{3.3}$$

3.1 The proof of Theorem 1.1

Our strategy for the first proof of (1.3) and (1.4) will be as follows. We will first consider

$$O_f^{W,\kappa}[t,N](x_0) = \langle x_0, z, \kappa | \mathrm{M}_\kappa(\mathrm{A})^{-t} Op^W_\kappa(f) \mathrm{M}_\kappa(\mathrm{A})^t | x_0, z, \kappa\rangle \tag{3.4}$$

rather than $O_f^{AW,\kappa}[t,N](x_0)$ (see (1.1)) and prove the results for those objects (Propositions 3.1-3.2). Equations (1.3) and (1.4) then follow immediately from the observation [BDB2] that

$$||Op^{AW}_{\kappa,z}(f) - Op^{W}_{\kappa}(f)|| \le C_f \frac{1}{N}.$$

To study the limiting behaviour of $O_f^{W,\kappa}[t,N]$, we proceed in two steps: first, we show the limiting behaviour of $O_f^{W,\kappa}[t,N](x_0)$ is identical to that of

$$O_f^{W}[t,N](x_0) = \langle x_0, z|\mathrm{M(A)}^{-t} Op^W(f) \mathrm{M(A)}^t |x_0, z\rangle \tag{3.5}$$

for times up to $\frac{\ln N}{\gamma}$ (Proposition 3.2); here M(A) is defined in (3.1) and the coherent states $|x_0, z\rangle$ in (2.11). Hence the problem is reduced to the analysis of the limiting behaviour of $O_f^W[N,t]$, which is essentially trivial since $M(A)$ is just the quantization of the ordinary linear dynamics on the full phase plane, rather than on the torus (Proposition 3.1).

Let $A \in \mathrm{SL}_2(\mathbf{R})$ with $|\mathrm{Tr}A| > 2$. We write $Av_\pm = \lambda^{\pm 1} v_\pm, \lambda = \exp\gamma > 1, v_\pm = (\cos\theta_\pm, \sin\theta_\pm)$ for the eigenvectors and the eigenvalues of A. We will always write $N = 1/(2\pi\hbar)$, although N will be assumed to be an integer only when we deal with the quantum map on $\mathcal{H}_N(\kappa)$ (in which case automatically $A \in \mathrm{SL}_2(\mathbf{Z})$), in Proposition 3.2.

Proposition 3.1 *Let* $z \in \mathbf{C}$*, with* $\mathrm{Im} z > 0$*.*

(i) Let $A \in \mathrm{SL}_2(\mathbf{R})$ *with* $|\mathrm{Tr}A| > 2$*. Then, for each* $f \in \check{\ell}_{1,2}$*, there exists* $C_f > 0$ *such that* $\forall\, x_0 \in \mathbf{T}^2$

$$\left| O_f^W[t,N](x_0) - (f \circ A^t)(x_0)\right| \le C_f \beta_+(z) \frac{\lambda^{2t}}{N}.$$

(ii) Let $A \in \mathrm{SL}_2(\mathbf{Z})$ *with* $|\mathrm{Tr}A| > 2$*. If* $f \in \check{\ell}_{1,k}$*,* $(k \in \mathbf{N}^*)$ *then there exist* $C_{f,1}, C_{f,2} > 0$ *such that* $\forall\, x_0 \in \mathbf{T}^2$ *and* $\forall\,(t,N)$

$$\left| O_f^W[t,N](x_0) - \int_{\mathbf{T}^2} dx\ f(x)\right| \le \frac{C_{f,1}}{\beta_-(z)^k}\left(\frac{N}{\lambda^{2t}}\right)^{\frac{k}{2}} + C_{f,2}\, e^{-\pi \frac{\lambda^{2t}}{N} c_+^2 \beta_-(z)},$$

where $c_+ = \mathrm{Max}\{|\cos\theta_+|, |\sin\theta_+|\}$*.*

Proof. (i) Since $\chi_\xi(x) = \exp 2\pi i \langle x, \xi\rangle$, we have $Op^W(\chi_\xi) = \mathrm{U}(\xi_1/N, \xi_2/N)$ and hence we obtain, using (2.12),

$$O_{\chi_\xi}^W[0,N](x_0) = e^{2i\pi\langle x_0, \xi\rangle} e^{-\frac{\pi}{N}(\xi, B(z)\xi)}. \tag{3.6}$$

Using (3.2), it then follows that $O_{\chi_\xi}^W[t,N] = O_{\chi_{\xi_t}}^W[0,N]$ where $\xi_t = A^{-t}\xi$. As a result, if $f = \sum_n f_n \chi_n$, then

$$O_f^W[t,N](x_0) = \sum_{n\in\mathbf{Z}^2} f_n e^{-\frac{\pi}{N}(n_t, B(z) n_t)} \chi_{n_t}(x_0). \tag{3.7}$$

Using (3.7) we have that

$$\begin{aligned}
\left|O_f^W[t,N](x_0) - f\circ A^t(x_0)\right| &\leq \sum_{n\in\mathbf{Z}^2} |f_n|(1-e^{-\frac{\pi}{N}(n_t, B(z)n_t)}) \\
&\leq \frac{\pi}{N}\sum_{n\in\mathbf{Z}^2} |f_n|(n_t, B(z)n_t)\ , \\
&\leq \frac{\pi\beta_+(z)}{N}\sum_{n\in\mathbf{Z}^2} |f_n|\|n_t\|^2\ ,
\end{aligned}$$

where we used the inequality $1-e^{-y}\leq y$, for each $y\geq 0$. The result then follows from the properties of $\|A^{-t}\|$ and from the regularity of f.

(ii) We recall that $s_+ = \cot\theta_+$ is a quadratic irrational. The basic properties of quadratic irrationals that we will use can be found in [Kh]. Using the Schwarz inequality we have that

$$\left\|A^{-t}n\right\| \geq \left|\langle v_+, A^{-t}n\rangle\right| = \left|\langle A^t v_+, n\rangle\right| = \lambda^t|\sin\theta_+||n_1 - n_2 s_+|\ ,$$

and from (3.7) we obtain the following estimate

$$\left|O_f^W[t,N](x_0) - \int_{\mathbf{T}^2} dx\, f(x)\right| \leq \sum_{n\neq\mathbf{0}} |f_n|\, e^{-\pi\beta_-(z)\frac{\lambda^{2t}}{N}\sin^2\theta_+|n_1-n_2s_+|^2}\ .$$

Without loss of generality suppose that $s_+>0$. It is convenient to divide the sum in two parts, I_1 and I_2. In I_1 we sum over $n_1\cdot n_2\leq 0$, from which we easily obtain the exponential term in the estimate. To discuss I_2, we recall that since s_+ is a quadratic irrational, there exists a constant $C>0$ such that $\forall\, n_1, n_2>0$

$$|n_1 - s_+n_2| > \frac{1}{Cn_2}\ .$$

Then,

$$I_2 \leq \sum_{n_1,n_2>0} (|f_{n_1,n_2}| + |f_{-n_1,-n_2}|)\, e^{-\pi\beta_-(z)\frac{\lambda^{2t}}{N}\frac{\sin^2\theta_+}{C^2n_2^2}}\ .$$

We then obtain the final estimate using the inequality $\exp(-x) < C_k'/x^k$, valid for some $C_k'>0$ and for each $x>0$. ∎

From now on, $N = 1/(2\pi\hbar)$ is an integer and let $\mathcal{H}_\hbar(\kappa)$ be the Hilbert space defined in section 2. The key observation allowing a simple analysis of the behaviour of $O_f^{W,\kappa}[t,N]$ in the (t,N)-plane is the following proposition, which shows that for times $t << \frac{\ln N}{\gamma}$, $O_f^{W,\kappa}[t,N]$ is close to $O_f^W[t,N]$, so that its behaviour can be read off from Proposition 3.1 above.

Proposition 3.2 *Let $z\in\mathbf{C}$, with $\mathrm{Im}z>0$. If $f\in\check{\ell}_{1,k}$, then there exist $C_{f,1}, C_{f,2} > 0$ such that for each $x_0\in\mathbf{T}^2$ and all t, N*

$$\left|O_f^{W,\kappa}[t,N](x_0) - O_f^W[t,N](x_0)\right| < C_{f,2}\left(\frac{\lambda^t}{N}\right)^k + C_{f,1}\exp -\frac{\pi N}{4}\beta_-(z)\ .$$

Proof. Using the inequality of Proposition 2.4(iii) and recalling that $c_n^0 = O^W_{\chi_n}[0,N]$, it is easy to find $C_{f,1}$ such that

$$\left|O_f^{W,\kappa}[t,N](x_0) - O_f^W[t,N](x_0)\right| \le C_{f,1} e^{-\frac{\pi N}{4}\beta_-(z)} +$$
$$\sum_n |f_n| \left| c_{n_t}^{p(\frac{n_t}{N})}(x_0) e^{-i(p(\frac{n_{t1}}{N})\kappa_1 - p(\frac{n_{t2}}{N})\kappa_2)} - c_{n_t}^0(x_0)\right| .$$

Because $p(x) \neq 0$ only if $|x| > 1/2$ the sum is limited to n such that $|n_{t1}/N|$ or $|n_{t2}/N| > 1/2$. Then, we have

$$\begin{aligned}\left|O_f^{W,\kappa}[t,N](x_0) - O_f^W[t,N](x_0)\right| &\le C_{f,1} e^{-\frac{\pi N}{4}\beta_-(z)} + 2 \sum_{\|A^{-t}n\| > \frac{N}{2}} |f_n| \\ &\le C_{f,1} e^{-\frac{\pi N}{4}\beta_-(z)} + 2 \sum_n |f_n| \left(\frac{2\|A^{-t}n\|}{N}\right)^k \\ &\le C_{f,1} e^{-\frac{\pi N}{4}\beta_-(z)} + 2 \left(\frac{2\|A^{-t}\|}{N}\right)^k \sum_n |f_n| \|n\|^k . \ \blacksquare\end{aligned}$$

It is now clear that the results of Propositions 3.1 and 3.2 imply Theorem 1.1.

We now give a more direct proof of the mixing regime of Theorem 1.1 for the Husimi distribution, based on the intuitive picture presented in the introduction and in particular on (1.5).

Proposition 3.3 *Let* $z \in \mathbf{C}$, $(\mathrm{Im} z > 0)$, *and* $A \in \mathrm{SL}_2(\mathbf{Z})$, *with* $\mathrm{Tr} A > 2$.

(i) Let $f \in \check{\ell}_{1,1}$. *There exists* $C_{f,A} > 0$ *such that* $\forall x_0 \in \mathbf{R}^2$, $\hbar, t > 0$,

$$|\langle x_0, z| M(A)^{-t} Op_z^{AW}(f) M(A)^t |x_0, z\rangle - \int_{\mathbf{T}^2} dx\ f(x)| \le C_{f,A} \frac{|z|+1}{\sqrt{\mathrm{Im} z}} \frac{1}{\lambda^t \sqrt{\hbar}} .$$

(ii) Let $f \in \check{\ell}_{1,k}$, *with* $k \ge 1$. *There exist* $C_{f,A}, C_{f_1}, C_{f_2} > 0$ *such that* $\forall x_0 \in \mathbf{T}^2$, $N, t > 0$

$$\begin{aligned}&|\langle x_0, z, \kappa| M_\kappa(A)^{-t} Op_{\kappa,z}^{AW}(f) M_\kappa(A)^t |x_0, z, \kappa\rangle - \int_{\mathbf{T}^2} dx\ f(x)| \\ &\le C_{f,A} \frac{\sqrt{N}}{\lambda^t} + C_{f_2} \left(\frac{\lambda^t}{N}\right)^k + C_{f_1} e^{-\pi N \beta_-(z)/4} .\end{aligned}$$

Proof.(i) Let $f_1 = \sum_n f_{1,n}\chi_n \in \check{\ell}_{1,1}$ (with $\int_{\mathbf{T}^2} dx\ f_1(x) = 0$) and $f_2 \in C^\infty(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2} dy\ [|\partial_q f_2| + |\partial_p f_2|] < \infty$. Adapting the proof given in Theorem 4 of [DB] it can be shown that

$$|\int_{\mathbf{R}^2} dy\ (f_1 \circ A^t)(y) f_2(y)| \le C_A \lambda^{-t} (\sum_n |f_{1,n}| \|n\|) \int_{\mathbf{R}^2} dy\ [|\partial_q f_2| + |\partial_p f_2|] ,$$

for some $C_A > 0$. Then, if $\int_{\mathbf{T}^2} f = 0$

$$|\langle x_0, z| M(A)^{-t} Op_z^{AW}(f) M(A)^t |x_0, z\rangle| = |\int_{\mathbf{R}^2} dy\ (f \circ A^t)(y) \frac{|\langle y, A^{-t} \cdot z | x_0, z\rangle|^2}{2\pi\hbar}|$$

$$
\begin{aligned}
&= |\int_{\mathbf{R}^2} dy\ (f\circ A^t)(y)\frac{|\langle 0, A^{-t}\cdot z|x_0-y,z\rangle|^2}{2\pi\hbar}| \\
&= |\int_{\mathbf{R}^2} d\xi\ (f\circ A^t)(x_0-\xi)\frac{|\langle 0, A^{-t}\cdot z|\xi,z\rangle|^2}{2\pi\hbar}| \\
&\leq C_A(\sum_n |f_n|\|n\|)\lambda^{-t}\int_{\mathbf{R}^2} dy\ [|\partial_q h| + |\partial_p h|],
\end{aligned}
$$

where $h(\xi) = |\langle 0, A^{-t}\cdot z|\xi, z\rangle|^2/(2\pi\hbar)$ is the Husimi distribution of $|\psi\rangle = |0, A^{-t}z\rangle$. The result then follows from Proposition 2.5(i).

(ii) The only difficulty to repeat on the torus the estimate given under (i) comes from the fact that $|\langle y, z_1, \kappa|x, z_2, \kappa\rangle|^2 \neq |\langle 0, z_1, \kappa|x-y, z_2, \kappa\rangle|^2$. Indeed,

$$
\langle x_0, z, \kappa|M(A)^{-t}Op^{AW}_{\kappa,z}(f)M(A)^t|x_0, z, \kappa\rangle =
$$

$$
\int_{\mathbf{T}^2} dy\ (f\circ A^t)(x_0-y)\frac{|\langle 0, A^{-t}\cdot z;\kappa|y,z;\kappa\rangle|^2}{2\pi\hbar} + \int_{\mathbf{T}^2} dy\ G(x_0, z; y, A^{-t}\cdot z)(f\circ A^t)(y)\ ,
$$

where $G(x, z_1; y, z_2) = N(|\langle y, z_2, \kappa|x, z_1, \kappa\rangle|^2 - |\langle 0, z_2, \kappa|x-y, z_1, \kappa\rangle|^2)$. The first term can be evaluated as in part (i), using this time Proposition 2.5 (ii). Let $G(x, z_1; y, z_2) = \sum_\beta G_\beta(x, z_1; z_2)\exp i2\pi\langle\beta, y\rangle$. By using the definition of anti-Wick quantization we easily rewrite

$$
G_\beta(x, z_1; z_2) = \langle x, z_1, \kappa|Op^{AW}_{\kappa,z_2}(\chi_\beta)|x, z_1, \kappa\rangle - \chi_\beta(x)\langle 0, z_2, \kappa|Op^{AW}_{\kappa,z_1}(\chi^*_\beta)|0, z_2, \kappa\rangle\ .
$$

By subtracting the same quantity calculated on the plane, which is zero, and by using Lemma 2.3 we finally have

$$
\begin{aligned}
G_\beta(x, z_1; z_2) &= \langle\frac{\beta}{N}, z_2|0, z_2\rangle\left[\langle x, z_1, \kappa|Op^W_\kappa(\chi_\beta)|x, z_1, \kappa\rangle - \langle x, z_1|Op^W(\chi_\beta)|x, z_1\rangle\right] \\
&- \chi_\beta(x)\langle 0, z_1|\frac{\beta}{N}, z_1\rangle\left[\langle 0, z_2, \kappa|Op^W_\kappa(\chi^*_\beta)|0, z_2, \kappa\rangle - \langle 0, z_2|Op^W(\chi^*_\beta)|0, z_2\rangle\right]\ .
\end{aligned}
$$

After some simple calculation we can finally write ($\beta_t = A^{-t}\beta$)

$$
\begin{aligned}
&\int_{\mathbf{T}^2} dy\ G(x_0, z; y, A^{-t}\cdot z)(f\circ A^t)(y) = \\
&\quad\sum_\beta f_\beta\langle\frac{\beta}{N}, z|0, z\rangle\left[\langle x_0, z, \kappa|Op^W_\kappa(\chi_{\beta_t})|x_0, z, \kappa\rangle - \langle x_0, z|Op^W(\chi_{\beta_t})|x_0, z\rangle\right] - \\
&\quad\sum_\beta f_\beta\chi_{\beta_t}(x_0)\langle 0, z|\frac{\beta_t}{N}, z\rangle\left[\langle 0, z, \kappa|Op^W_\kappa(\chi^*_\beta)|0, z, \kappa\rangle - \langle 0, z|Op^W(\chi^*_\beta)|0, z\rangle\right]\ .
\end{aligned}
$$

The result then follows by making use of Proposition 3.2. ∎

3.2 The Period of the Quantum Map

It is natural to wonder if (1.4) holds beyond $\frac{\ln N}{\gamma}$ and in particular if it holds possibly for times that are polynomial in N. We shall show (see (3.11)-(3.12)) with an example that the mixing regime may break down on a time scale which is only logarithmic in N, thereby showing that on this scale the agreement between the classical and quantum or semi-classical evolutions breaks down.

The intuitive "proof" of (1.4) given in the introduction, which is based on (1.5), does not predict any breakdown of (1.4) at long times. It is possible to get an intuitive understanding of the breakdown of the mixing regime using the uncertainty principle as follows. First remark that, given any $\psi \in \mathcal{H}_N(\kappa)$, the support of its Wigner distribution has, due to the uncertainty principle, necessarily a linear size of the order of at least $\hbar$ in all directions: indeed, from $\Delta X \Delta P \geq \hbar$ and $\Delta X, \Delta P \leq 1$, one concludes $\Delta X, \Delta P \geq \hbar$. Consider now $\mathrm{M(A)}^t|x, z, \kappa\rangle$; we expect its Wigner distribution to have a spread of $\sqrt{\hbar}\exp\gamma t$ along the unstable direction and therefore to wrap around the torus as soon as $\sqrt{\hbar}\exp\gamma t >> 1$. The transversal distance between the successive windings so obtained can be estimated by $\sqrt{\hbar}^{-1}\exp -\gamma t$. When this distance becomes less than $\hbar$, the support of the Wigner distribution of $\mathrm{M(A)}^t|x, z, \kappa\rangle$ can no longer separate the separate windings, because of the previous remark, and as a result one expects the classical evolution picture may break down at such times, given by $\sqrt{\hbar}^{-1}\exp -\gamma t \sim \hbar$ or $t \sim \frac{3}{2}\frac{|\ln\hbar|}{\gamma}$. We shall now exhibit precisely such a phenomenon for the quantized cat maps.

For that purpose we consider A such that either $a_{12} = 1$ or $a_{21} = 1$ and $[a_{11}]_2 = [a_{22}]_2 = 0$ (we use the notation $[x]_n = x \bmod n$, both for numbers and matrices). We know that $\kappa = 0$ gives an admissible quantization; furthermore it was shown in [HB] that the corresponding quantum map $M_o(A)$ is periodic with period $n(N) = \mathrm{Min}\left\{t \mid M_o(A)^t = e^{i\phi_N}\ ,\ \phi_N \in \mathbf{R}\ \right\}$. These periods have been studied in [Ke], were it is argued that they behave "on average" linearly in N, but with great fluctuations about this average. It is of course clear that the "mixing regime" must break down before the period. In the following we will show that there exists a sequence N_{2k+1} of values of N for which the period is extremely short: $n(N_{2k+1}) \sim 2\frac{\ln N_{2k+1}}{\gamma}$, leading to the announced result.

We will need the following simple formulas for A. If we call λ the biggest eigenvalue of A, we know that for each $t \in \mathbf{N}^+$

$$A^t = p_t A - p_{t-1}\ , \qquad p_{t+1} = \mathrm{Tr}(A)p_t - p_{t-1}\ , \text{where } p_t = \frac{\lambda^t - \lambda^{-t}}{\lambda - \lambda^{-1}}\ . \tag{3.8}$$

We also introduce $T_N = \mathrm{Min}\ \{t \mid [A^t]_N = 1\,\}$. We denote by A_N the matrix with integer entries such that $A^{T_N} = 1 + NA_N$. Then, following [HB], $n(N) = T_N$ if N is odd or if N is even and $[(A_N)_{12}]_2 = [(A_N)_{21}]_2 = 0$; otherwise $n(N) = 2T_N$. We finally define for $k \in \mathbf{N}^+$

$$N_k \equiv \mathrm{Max}\ \left\{N \mid [A^k]_N = 1\ \right\}\ , \tag{3.9}$$

and we prove the following result.

Proposition 3.4 *For each $k \in \mathbf{N}^+$ we have that $T_{N_k} = k$ and*

$$N_{2k} = 2p_k, \qquad N_{2k+1} = p_k + p_{k+1}\ . \tag{3.10}$$

Proof. Using (3.8), we see that N_k is the greatest integer such that

$$\begin{array}{ll} [p_k a_{11} - p_{k-1} - 1]_{N_k} = 0 & [p_k a_{12}]_{N_k} = 0 \\ [p_k a_{22} - p_{k-1} - 1]_{N_k} = 0 & [p_k a_{21}]_{N_k} = 0 \,, \end{array}$$

or, because of the hypothesis about the off-diagonal terms of A, $[p_k]_{N_k} = 0$, $[p_{k-1}]_{N_k} = -1$. This means that N_k is the greatest common divisor of p_k and $p_{k-1} + 1$, i.e. $N_k = (p_k, p_{k-1} + 1)$. We are going to show by induction that, for each $s = 0, \ldots, k-1$,

$$N_k = (p_{k-s} + p_s, p_{k-(s+1)} + p_{s+1}) \,.$$

Since $p_0 = 0$ and $p_1 = 1$, this is clearly true for $s = 0$. Supposing it is true for s, we have

$$\begin{aligned} N_k &= (\mathrm{Tr}A\, p_{k-s-1} - p_{k-s-2} + p_s,\ p_{k-s-1} + p_{s+1}) \\ &= (\mathrm{Tr}A\,(p_{k-s-1} + p_{s+1}) - \mathrm{Tr}A\, p_{s+1} - p_{k-s-2} + p_s,\ p_{k-s-1} + p_{s+1}) \\ &= (p_{s+2} + p_{k-(s+2)},\ p_{k-(s+1)} + p_{s+1}) \,, \end{aligned}$$

so that it is true for $s+1$. In the third line we used the identity $(a,\ ca - b) = (a,\ b)$, valid for all a, b, c, and formula (3.8).

If $k = 2\ell$, then, setting $s = \ell$ in the above formula, we have

$$N_{2\ell} = (2p_\ell,\ p_{\ell-1} + p_{\ell+1}) = (2p_\ell,\ \mathrm{Tr}A\, p_\ell) = 2p_\ell; \,,$$

because $\mathrm{Tr}A$ is pair by hypothesis. If $k = 2\ell + 1$, then $N_{2\ell+1} = p_\ell + p_{\ell+1}$. For each k we have that $[A^k]_{N_k} = [A^{T_{N_k}}]_{N_k} = [A^{T_{N_k}}]_{N_{T_{N_k}}} = 1$. From the definition of the period we have that $T_{N_k} \leq k$ and from the definition of N_k it follows that $N_k \leq N_{T_{N_k}}$. Since the sequence $\{N_k\}$ is increasing (see (3.8)-(3.10)) we conclude that $T_{N_k} \geq k$ and hence that $T_{N_k} = k$. ∎

Since $[\mathrm{Tr}A]_2 = 0$, we have that $[p_{2k}]_2 = 0$ and $[p_{2k+1}]_2 = 1$, so that $[N_{2k+1}]_2 = 1$. Using the results of Proposition 3.4, it then follows that $2k + 1$ is the quantum period for $N_{2k+1} = p_k + p_{k+1}$, i.e. $n(N_{2k+1}) \approx 2\frac{\ln N_{2k+1}}{\gamma}$. Keeping in mind that $M(A)^{-1} = M(A^{-1})$, so that $M(A)^t = M(A)^{t-n(N_{2k+1})} = M(A^{-1})^{n(N_{2k+1})-t}$, $(0 \leq t \leq n(N_{2k+1}))$, we can apply (1.3) and (1.4) to A^{-1} to conclude that if we perform the limits running only over N_{2k+1}, we have

$$\lim_{\substack{k\to\infty \\ \frac{\ln N_{2k+1}}{2\gamma} \ll t \ll \frac{3}{2}\frac{\ln N_{2k+1}}{\gamma}}} O_f^{AW,0}[t, N_{2k+1}](x_0) = \int_{\mathbf{T}^2} f(x')\, dx' \tag{3.11}$$

$$\lim_{\substack{k\to\infty \\ \frac{3}{2}\frac{\ln N_{2k+1}}{\gamma} \ll t \ll 2\frac{\ln N_{2k+1}}{\gamma}}} |O_f^{AW,0}[t, N_{2k+1}](x_0) - \left(f \circ A^{t-n(N_{2k+1})}\right)(x_0)| = 0 \tag{3.12}$$

Equation (3.12) clearly shows the breakdown of the mixing regime for times beyond $\frac{3}{2}\frac{\ln N}{\gamma}$, in the cases considered here. It is of course still possible that, "generically", it remains valid for much longer times, as the classical intuition would predict.

3.3 Position eigenstates

In this subsection we study the Wigner and Husimi distributions of evolved position eigenstates by studying the matrix elements $\langle e_j^\kappa, \mathrm{M(A)}^{-t} Op_\kappa^W(f) \mathrm{M(A)}^t e_j^\kappa \rangle$ and $\langle e_j^\kappa, \mathrm{M(A)}^{-t} Op_{\kappa,z}^{AW}(f) \mathrm{M(A)}^t e_j^\kappa \rangle$ in the limits $t \to \infty$ and $N \to \infty$. If we applied the heuristic argument of the introduction to this case, we would conclude that the mixing regime should set in *no later than* at times of order $\frac{\ln N}{2\gamma}$. As a matter of fact, it sets in much sooner, as Proposition 3.6 shows. We need the following preparatory result.

Lemma 3.5 *For each $f \in \check{\ell}_{1,k}$, there exist $C_f > 0$ such that for each $N, t \geq 0$, $j = 0, \ldots N-1$ and $|J| < N/2$ we have that*

$$\begin{aligned} |\langle e_{j+J}^\kappa, \mathrm{M(A)}^{-t} Op_\kappa^W(f) \mathrm{M(A)}^t e_j^\kappa \rangle \quad &- \quad \int_0^1 dp\, (f \circ A^t)(\frac{\kappa_2/\pi + 2j + J}{2N}, p) e^{i2\pi J p}| \\ &\leq \quad C_f \left(\frac{\|A^{-t}\|}{N} \right)^k . \end{aligned} \tag{3.13}$$

Proof. Using the Egorov theorem and (2.8) we find that

$$\begin{aligned} \langle e_{j+J}^\kappa, \mathrm{M(A)}^{-t} Op_\kappa^W(f) \mathrm{M(A)}^t e_j^\kappa \rangle &= \sum_n f_{A^t n} e^{i\pi \frac{n_1 n_2}{N}} e^{i(\kappa_2 + 2\pi j)\frac{n_2}{N}} \langle e_{j+J}^\kappa, e_{j+n_1}^\kappa \rangle \\ &= \sum_n f_{A^t(J+n_1 N, n_2)} e^{-ik_1 n_1} e^{i\frac{\pi}{N} n_2 (J+Nn_1)} e^{i(\kappa_2+2\pi j)\frac{n_2}{N}} . \end{aligned}$$

It is clear that the integral in (3.13) is obtained posing $n_1 = 0$. The remaining terms are easily estimated:

$$\begin{aligned} \sum_{\substack{n \\ n_1 \neq 0}} |f_{A^t(J+Nn_1, n_2)}| &\leq \sum_{\|n\| \geq \frac{N}{2}} |f_{A^t n}| \\ &\leq \sum_n \left(\frac{2\|n\|}{N} \right)^k |f_{A^t n}| \leq \left(\frac{2\|A^{-t}\|}{N} \right)^k \sum_n \|n\|^k |f_n| . \blacksquare \end{aligned}$$

Proposition 3.6 *For each $f \in \check{\ell}_{1,k}$, there exist $C_{f,1}, C_{f,2}$ such that, for all t, N, $j = 0, \ldots, N-1$, one has*

$$|\langle e_j^\kappa, \mathrm{M(A)}^{-t} Op_\kappa^W(f) \mathrm{M(A)}^t e_j^\kappa \rangle - \int_{\mathbf{T}^2} dx\, f| \leq C_{f,1} \left(\frac{\|A^{-t}\|}{N} \right)^k + \frac{C_{f,2}}{\lambda^{tk}} .$$

Proof. From (3.13) it is clear that it will be enough to show that it exists $C_{f,2}$ such that, for all $0 \leq q \leq 1$,

$$|\int_0^1 (f \circ A^t)(q, p) dp - \int_{\mathbf{T}^2} f(x) dx| \leq \frac{C_{f,2}}{\lambda^{tk}}.$$

Indeed,

$$|\int_0^1 (f \circ A^t)(q,p) dp - \int_{\mathbf{T}^2} f(x) dx| \quad \leq \quad \sum_{n \neq 0} |f_{A^t(0,n)}| \leq \sum_{\|m\| \geq \lambda^t |\cos\theta_-|} |f_m|$$

$$\leq \frac{1}{\lambda^{tk}|\cos\theta_-|^k}\sum_m |f_m|\|m\|^k \ ,$$

where we used the inequality $\|A^t(0,n)\| \geq |\langle v_-, A^t(0,n)\rangle| \geq \lambda^t|\cos\theta_-|$, valid for each $n \neq 0$. ∎

It is clear that we actually have, for any sequence $\theta_N \to \infty$,

$$\lim_{\substack{N\to\infty \\ \theta_N < t < (1-\epsilon)\frac{\ln N}{\gamma}}} \langle e_j^\kappa, \mathrm{M(A)}^{-t} Op_\kappa^W(f)\mathrm{M(A)}^t e_j^\kappa\rangle = \int_{\mathbf{T}^2} f(x)\ dx \ .$$

Therefore there is no "orbital instability regime" for initial position eigenstates, but this is easily understood in terms of the classical dynamics applied to the support of the initial Wigner distribution.

4 The Quantum Baker Map

4.1 Presentation of the model and of the results

The Baker map is a discontinuous map B on the torus, defined on $x = (q,p) \in \mathbf{T}^2$ by

$$Bx = \begin{cases} B_o x & q \in [0, 1/2[\\ B_o x + (-1, 1/2) & q \in]1/2, 1] \end{cases} \ , \tag{4.1}$$

where B_o is the linearized map given by

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} .$$

The following relation concerning the action of B on the characters χ_n, $n \in \mathbf{Z}^2$, will be crucial:

$$\chi_{2n_1,n_2} \circ B = \chi_{n_1,2n_2} \ . \tag{4.2}$$

With the usual assumption $[N]_2 = 0$ and posing $\kappa = 0$ (see [BV] or [DBDE] and further references therein for the general scheme) the quantization of B is given by the following unitary operator on $\mathcal{H}_N \equiv \mathcal{H}_N(0)$:

$$V_B = \mathcal{F}_N^{-1} \circ \begin{pmatrix} \mathcal{F}_{N/2} & 0 \\ 0 & \mathcal{F}_{N/2} \end{pmatrix}$$

where $(\mathcal{F}_N)_{k,\ell} = \dfrac{1}{\sqrt{N}} e^{-i2\pi\frac{k\ell}{N}}$ are the matrix elements of $\mathcal{F}$ in the position eigenstate basis $\{e_j \equiv e_j^0\}_{j=0}^{N-1}$.

In this section we will mean by semi-classical evolution of the Husimi function associated with $\psi \in \mathcal{H}_N$ the following function

$$g_\psi^{sc}(x,t) = \frac{|\langle B^{-t}x, B_o^{-t}\cdot z|\psi\rangle|^2}{2\pi\hbar} \ , \tag{4.3}$$

where $B_o^{-t} \cdot z = 4^t z$. We will write g_j^{sc}, respectively $g_{x_0,z}^{sc}$ when $\psi = e_j$, respectively $\psi = |x_0, z; 0\rangle$. We will compare this quantity to the Husimi distribution of the quantum mechanically evolved state.

Since throughout the rest of this section $\kappa = 0$, and in order to alleviate the notations, we will drop the κ-dependence of many symbols. In the following propositions we list our principal results about the quantum and semi-classical evolutions of coherent states and position eigenstates from which Theorem 1.2 will follow easily. The proofs are postponed to the next subsection. From now on, we will deal with observables depending only on the q-variable, for which we will write $f \in C^\infty(\mathbf{T})$.

We consider for each N a lattice of $M_N < N$ points $x_j^{(N)}$ on the torus, given by $x_j^{(N)} = (\frac{j_1}{\sqrt{M_N}}, \frac{j_2}{\sqrt{M_N}})$, $1 \leq j_1, j_2 \leq \sqrt{M_N}$. Proposition 2.4 (i) with $f = 1$ and $z = i$ then immediately implies

$$|\langle x_k^{(N)}, i | x_j^{(N)}, i\rangle - \delta_{kj}| \leq C \ \exp -\frac{\pi}{2}\frac{N}{M_N}, \tag{4.4}$$

where we wrote, with some abuse of notation $|x_j^{(N)}, z\rangle \equiv |x_j^{(N)}, z, 0\rangle \in \mathcal{H}_\hbar(0) \sim \mathcal{H}_\hbar$ (Do not confuse with (2.11)). Since we will need to apply Lemma 4.5 below to the $|x_j^{(N)}, i>$, we will always need the sequence M_N to be such that

$$8CM_N \exp -\frac{\pi}{2}\frac{N}{M_N} \leq 1. \tag{4.5}$$

Consequently, from now on, we will always take $\frac{\pi}{2}\frac{N}{\ln 8CN} - 1 \leq M_N \leq \frac{\pi}{2}\frac{N}{\ln 8CN}$. We can now state the main ingredient for the proof of Theorem 1.2:

Proposition 4.1 [Coherent states] *(i) Let $N > 0, 0 < \epsilon < 1$ and $0 < \alpha < \frac{\epsilon}{4}$ be given and let M_N be as above. Then there exists a subset $\mathcal{M}_N$ of the index set $1 \leq i_1, i_2 \leq \sqrt{M_N}$ with the following properties:*

(a) $0 \leq 1 - \frac{\sharp \mathcal{M}_N}{M_N} \leq \frac{1}{N^{2\alpha}}\frac{N}{M_N}$;

(b) *For all $f \in C^\infty(\mathbf{T})$, and for all $k \in \mathbf{N}$, there exists a constant $C_{f,k}$ so that, for all $0 \leq t < (1-\epsilon)\log_2 N$ and for all $j \in \mathcal{M}_N$,*

$$|\langle x_j^{(N)}, i | V_B^{-t} Op_0^W(f) V_B^t | x_j^{(N)}, i\rangle \ - \ \langle x_j^{(N)}, i | Op_0^W(f \circ B^t) | x_j^{(N)}, i\rangle| \leq C_{f,k}[\frac{1}{\sqrt{M_N}} + N^{\alpha - \frac{\epsilon}{4}} + N^{-k\frac{\epsilon}{2}}];$$

(ii) For each $f \in C^\infty(\mathbf{T})$ there exist C_{f_1}, C_{f_2}, C_{f_3} such that $\forall x_0 \in \mathbf{T}^2$, $N, t > 0$

$$|\int_{\mathbf{T}^2} dy \ g_{x_0,z}^{sc}(y,t) f(y) - \int_{\mathbf{T}^2} dy \ f(y)| \leq C_{f_1}\frac{\sqrt{N}}{2^t} + C_{f_2}\frac{2^t}{N} + C_{f_3} e^{-\pi N \beta_-(z)/4} \ .$$

The second part of this proposition says that the semi-classically evolved Husimi distribution equidistributes provided $\sqrt{N} << 2^t << N$. Comparing to Theorem 1.2, we conclude that its behaviour is identical to that of the quantum mechanically evolved distribution on that time scale. We unfortunately have no idea what happens at later times. Analogously, for the position eigenstates we have:

Proposition 4.2 [Position eigenstates] *Let $N > 0$, $0 < \epsilon < 1$ and $0 < \alpha < \epsilon/4$ be given. Let $f \in C^\infty(\mathbf{T})$. Then*

(i) There exists $\mathcal{M}_N \subset \{1, \ldots, N\}$ satisfying

$$0 \leq 1 - \frac{\#\mathcal{M}_N}{N} \leq \frac{1}{N^{2\alpha}} \ ;$$

and for all $f \in C^\infty(\mathbf{T})$, for all $k \in \mathbf{N}$, a constant $C_{f,k} > 0$ such that for all $0 \leq t < (1-\epsilon)\log_2 N$ and for each $j \in \mathcal{M}_N$

$$|\langle e_j, \ V_B^{-t} Op_0^W(f) V_B^t e_j\rangle - f(\frac{2^t j}{N})| \leq C_{f,k}\left(\frac{1}{N^{\epsilon/4-\alpha}} + \frac{1}{N^{\epsilon k/2}}\right);$$

(ii) There exists $C > 0$ s.t., for $0 \leq j \leq N-1$, $f \in C^\infty(\mathbf{T})$, $t, N > 0$ we have

$$|\int_{\mathbf{T}^2} g_j^{sc}(x,t) f(x) \ dx - \frac{1}{2}[f(\frac{2^t j}{N}) + f(\frac{2^t(j+1)}{N})]| \leq C\frac{\|f'\|_\infty}{\sqrt{N}} \ .$$

The first part of this result is again readily understood in terms of evolution of the support of the Wigner function of the initial state e_j, which is the vertical strip at j/N, of width $1/N$. The dynamics contracts this strip vertically, stretches it horizontally to size $2^t/N$, and centers it at $2^t j/N$. As a result, one does not expect mixing to set in before times of order $\log_2 N$, i.e. when $2^t/N \sim 1$; this is indeed confirmed by the above result. Comparing furthermore the first part of the proposition to the second, one concludes again that the semi-classical evolution can not be distinguished from the quantum-mechanical one up to times $2^t \ll N$.

4.2 Proof of Propositions 4.1 - 4.2 and of Theorem 1.2

To prove part (i) of Proposition 4.2 as well as of Proposition 4.1, we first of all need the following Egorov theorem, proven in [DBDE]. Let's define, for each $t > 0$ (compare this to (3.2)),

$$E_t(f) = V_B^{-t} Op_0^W(f) V_B^t - Op_0^W(f \circ B^t) \ ,$$

and $E_t(m) = E_t(\chi_m)$.

Proposition 4.3 *[DBDE] Let $\eta_N > 0$ and $t_N > 0$ such that $2^{t_N}\eta_N < N$. Then there exists a subspace $\mathcal{G}_{\eta_N}(t_N) \subset \mathcal{H}_N$ such that*

i) $\dim \mathcal{G}_{\eta_N}(t_N) \geq N - 2^{t_N}\eta_N$*;*

ii) for each $\psi \in \mathcal{G}_{\eta_N}(t_N)$, $|n| < \eta_N$ and $0 \leq t \leq t_N$

$$E_t(n,0)\ \psi = 0\ .$$

The Proposition asserts roughly that, provided one looks at times shorter than $\log_2 N$, and provided one restricts one's attention to a "good" subspace $\mathcal{G}_{\eta_N}$, there is no error in the Egorov theorem for trigonometric polynomials of degree at most η_N. The good subspace gets smaller as the time gets larger, but is non-trivial on the time-scale considered. Two obvious weaknesses of the above result are that it does not describe the good space explicitly and that it deals only with functions of q. These are at the origin of the limitations of Theorem 1.2 pointed out in the introduction.

The following corollary is an easy consequence of Proposition 4.3. We write $P_{\mathcal{B}_{\eta_N}}$ for the projector onto $\mathcal{B}_{\eta_N}$, the orthogonal complement of $\mathcal{G}_{\eta_N}$.

Corollary 4.4 *Let $\{\phi_j\}_{j=1}^{M_N}$ be a family of orthonormal vectors in $\mathcal{H}_N$. Let η_N, t_N be as in Proposition 4.3, $\delta_N > 0$ and*

$$\mathcal{M}_N = \{1 \leq j \leq M_N \mid \langle\phi_j|P_{\mathcal{B}_{\eta_N}}|\phi_j\rangle < \delta_N^2\}\ .$$

Then

i) $\#\mathcal{M}_N \geq M_N - 2^{t_N}\eta_N/\delta_N^2$*;*

ii) for each $f \in C^\infty(\mathbf{T})$ and $k \in \mathbf{N}$, there exists $C_{f,k} > 0$ such that, for each $0 \leq t \leq t_N$ and $j \in \mathcal{M}_N$,

$$\|E_t(f)\phi_j\| \leq C_{f,k}(\delta_N + \eta_N^{-k})\ .$$

We will always choose things in such a way that $2^{t_N}\eta_N/\delta_N^2/M_N \to 0, \delta_N \to 0, \eta_N \to \infty$, so that the Corollary asserts that the error is "small" on "many" ϕ_j.

Proof. (i) From Proposition 4.3 we have that $\dim \mathcal{B}_{\eta_N} \leq 2^{t_N}\eta_N$ and

$$\dim \mathcal{B}_{\eta_N} = \mathrm{Tr}P_{\mathcal{B}_{\eta_N}} \geq \sum_{j\notin\mathcal{M}_N} \langle\phi_j|P_{\mathcal{B}_{\eta_N}}|\phi_j\rangle \geq (M_N - \#\mathcal{M}_N)\delta_N^2\ ,$$

from which the statement follows.

(ii) Let $j \in \mathcal{M}_N$ and $f = \sum_n c_n\chi_{n0} \in C^\infty(\mathbf{T}^2)$. Then

$$\|E_t(f)\phi_j\| \leq \sum_{|n|<\eta_N} |f_n|\|E_t(n,0)\phi_j\| + \sum_{|n|\geq\eta_N} |f_n|\|E_t(n,0)\phi_j\|.$$

We then have, using Proposition 4.3,

$$\sum_{|n|<\eta_N} |f_n| \|E_t(n,0)\phi_j\| = \sum_{|n|<\eta_N} |f_n| \left\| E_t(n,0) P_{\mathcal{B}_{\eta_N}} \phi_j \right\| \le \delta_N 2 \sum_{|n|<\eta_N} |f_n| \ ,$$

and

$$\sum_{|n|\geq\eta_N} |f_n| \|E_t(n,0)\phi_j\| \le \frac{2}{\eta_N^k} \sum_{|n|>\eta_N} |f_n||n|^k \ .$$

The result then follows with $C_{f,k} = 2\sum_n |f_n||n|^k$. ■

The idea of the proofs of Propositions 4.1 (i) and 4.2 (i) is to apply the above corollary to the families $|x_i^{(N)}, i\rangle$ and $|e_j\rangle$, respectively. Since the former family is not orthogonal, we will furthermore use the following result from linear algebra, which is proven through an application of the Gram-Schmidt orthogonalization procedure.

Lemma 4.5 *Let $u_1, u_2, \dots u_M \in \mathcal{H}_N$ be such that, for some ϵ satisfying $8\epsilon M \le 1$,*

$$|\langle u_i, u_j\rangle - \delta_{ij}| \le \epsilon. \tag{4.6}$$

Then the u_i are linearly independent and there exists an orthonormal basis $w_1, w_2, \dots w_M$ of span$\{u_1, u_2, \dots u_M\}$ so that

$$||w_i - u_i|| \le 7\epsilon\sqrt{M} \le \frac{1}{\sqrt{M}}. \tag{4.7}$$

Proof. We will prove both statements at once using the Gram-Schmidt orthogonalization procedure as follows. First, let $v_i = \frac{u_i}{||u_i||}$, then

$$||v_i - u_i|| = |1 - ||u_i||\,| \le \epsilon. \tag{4.8}$$

Now define, for all $1 \le i \le M$, $v_i' = v_i - \chi_i$, where $\chi_1 = 0$ and χ_i is the orthogonal projection of v_i onto the span of $v_1, \dots v_{i-1}$. It is then clear that, for all $1 \le i \ne j \le M$, $\langle v_i', v_j'\rangle = 0$ and that the span of $u_1, \dots u_M$ equals the span of $v_1', \dots v_M'$. We now first show that, for all $1 \le i \le M$, $v_i' \ne 0$ by showing that

$$||\chi_i||_{\mathcal{H}_N} \le 3\epsilon\sqrt{M}. \tag{4.9}$$

For $i = 1$, (4.9) is trivial. Consider then a fixed value of i between 2 and M and write $\chi_i = \sum_{j=1}^{i-1} \lambda_i^j v_j$, where the λ_i^j are obtained by solving $(1 \le k < i)$

$$\langle v_k, \chi_i\rangle = \langle v_k, v_i\rangle = \sum_{j=1}^{i-1} \lambda_i^j \langle v_k, v_j\rangle.$$

Introducing the matrices $[V_i]^{kj} = \langle v_k, v_j\rangle$, $[\tau_i]^k = \langle v_k, v_i\rangle$ $(1 \le k, j < i)$, this can be rewritten $\tau_i = V_i\lambda_i$. Consequently, since $V_i^* = V_i$, we have

$$\langle \chi_i, \chi_i\rangle = \lambda_i^* V_i \lambda_i = \tau_i^* V_i^{-1} V_i V_i^{-1} \tau_i = \tau_i^* V_i^{-1} \tau_i,$$

and

$$||\chi_i||^2_{\mathcal{H}_N} \leq ||V_i^{-1}|| \left(\sum_{k=1}^{i-1} |\langle v_k, v_i\rangle|^2 \right).$$

To estimate $||V_i^{-1}||$, note that $V_i = I + \epsilon S_i$, where S_i is an off-diagonal self-adjoint matrix with matrix elements $(k \neq j)$

$$|S_i^{kj}| = \frac{1}{\epsilon}|\langle v_k, v_j\rangle| = \frac{|\langle u_k, u_j\rangle|}{||u_k||\ ||u_j||\ \epsilon} \leq \frac{2\epsilon}{\epsilon}.$$

Consequently,

$$||S_i|| = \sup_{||a||_{\mathbf{R}^{(i-1)}}=1} |\langle a, Sa\rangle_{\mathbf{R}^{(i-1)}}| \leq 2 \sup_{||a||_{\mathbf{R}^{(i-1)}}=1} \sum_{k,l=1}^{i-1} |a_k|\ |a_l| \leq 2M.$$

As a result

$$||V_i^{-1}|| \leq \sum_{r=0}^{\infty} \epsilon^r ||S_i||^r \leq \frac{1}{1-\epsilon||S_i||} \leq \frac{4}{3}$$

and consequently $||\chi_i||^2_{\mathcal{H}_N} \leq (4/3)4\epsilon^2 M \leq 6\epsilon^2 M$, from which (4.9) follows.

Since all $v_i' \neq 0$, we can define $w_i = v_i'/||v_i'||$ with $\langle w_i, w_j\rangle = \delta_{ij}$. Since furthermore

$$\mathrm{span}\{u_1, u_2, \ldots u_M\} = \mathrm{span}\{v_1', v_2', \ldots v_M'\} = \mathrm{span}\{w_1, w_2, \ldots w_M\}$$

it follows that the u_i are linearly independent. It remains to show (4.7). To that end, compute, using (4.8),

$$||w_i - u_i|| \leq ||w_i - v_i'|| + ||v_i' - v_i|| + ||v_i - u_i|| \leq |1 - ||v_i'||\ | + ||\chi_i|| + \epsilon.$$

Since $|1 - ||v_i'||\ | = |\ ||v_i|| - ||v_i'||\ | \leq ||v_i' - v_i|| = ||\chi_i||$, the result follows from (4.9).
■

Proof of Proposition 4.1. (i) We first use Lemma 4.5 and (4.4)-(4.5) to assert the existence of an orthonormal set $\phi_j, 1 \leq j_1, j_2 \leq \sqrt{M_N}$ such that

$$||\phi_j - |x_j^{(N)}, i\rangle|| \leq 7\sqrt{M_N} C e^{-\frac{\pi}{2}\frac{N}{M_N}} \leq \frac{1}{\sqrt{M_N}}.$$

We then apply Corollary 4.4 with $\eta_N = N^{\frac{\epsilon}{2}}, \delta_N = N^{\alpha-\frac{\epsilon}{4}}$ and $t_N = (1-\epsilon)\log_2 N$, so that

$$\frac{\sharp\mathcal{M}_N}{M_N} \geq 1 - \frac{N^{1-\epsilon}}{M_N} N^{\epsilon/2} N^{-2\alpha+\frac{\epsilon}{2}}$$

and for all $j \in \mathcal{M}_N,\ 0 \leq t \leq t_N$

$$||E_t(f)\phi_j|| \leq C_{f,k}(N^{\alpha-\frac{\epsilon}{4}} + N^{-k\frac{\epsilon}{2}}).$$

Consequently,

$$|\langle x_j^{(N)}, i|V_B^{-t} Op_0^W(f) V_B^t|x_j^{(N)}, i\rangle \quad - \quad \langle x_j^{(N)}, i|Op_0^W(f\circ B^t)|x_j^{(N)}, i\rangle|$$

$$
\begin{aligned}
&\leq \||x_j^{(N)}, i\rangle\| \|E_t(f)|x_j^{(N)}, i\rangle\| \\
&\leq C(\|E_t(f)(\phi_j - |x_j^{(N)}, i\rangle)\| + \|E_t(f)\phi_j\|) \\
&\leq C_{f,k}[\frac{1}{\sqrt{M_N}} + N^{\alpha-\frac{\epsilon}{4}} + N^{-k\frac{\epsilon}{2}}],
\end{aligned}
$$

which is the desired result.

(*ii*) Since the Baker map is linear on $f \in C^\infty(\mathbf{T})$, this is a direct consequence of the proof of Proposition 3.3. ∎

Proof of Theorem 1.2. With the notations of Proposition 4.1, let $x_0 \in \mathbf{T}^2$ and consider

$$d_N(x_0) = \min\{|x_j^{(N)} - x_0| \mid j \in \mathcal{M}_N\}.$$

Since $M_N - \sharp\mathcal{M}_N \leq \frac{N}{N^{2\alpha}}$, it is clear that

$$d_N(x_0) \leq \sqrt{\frac{N}{M_N}\frac{1}{N^\alpha}} \ .$$

Let's choose $\alpha > \epsilon/5$; then the sequence M_N defined after (4.5) is such that $N/M_N < N^{2(\alpha-\epsilon/5)}$. Consequently $\forall N, \exists j_N \in \mathcal{M}_N$ such that $||x_0 - x_{j_N}^{(N)}|| \leq N^{-\epsilon/5}$. We write $x_N = x_{j_N}^{(N)}$.

Let $t < \frac{1-\epsilon}{2}\log_2 N$ and consider the following inequality:

$$|\langle x_N, i|V_B^{-t}Op_{0,z}^{AW}(f)V_B^t|x_N, i\rangle - f(2^t q_N)| \leq C||Op_0^W(f) - Op_{0,z}^{AW}(f)||+$$

$$|\langle x_N, i|E_t(f)|x_N, i\rangle| + |\langle x_N, i|Op_0^W(f\circ B^t)|x_N, i\rangle - f(2^t q_N)| \ .$$

The result then comes from Proposition 4.1, the general estimate $||Op_0^W(f) - Op_z^{AW}(f)|| \leq C_f/N$, Proposition 3.1(i) with $A = B_o$ and a straightforward adaptation of Proposition 3.2 to the case $A = B_o$ and $f = f(q)$. In the same way we obtain the result for the mixing regime. ∎

Proof of Proposition 4.2. (*i*) With respect to Corollary 4.4 let $M_N = N$, $\phi_j = e_j$, $\eta_N = N^{\epsilon/2}$, $\delta_N = N^{\alpha-\epsilon/4}$ and $t_N \leq (1-\epsilon)\log_2 N$. Then, we have that $\#\mathcal{M}_N/N \geq 1 - 1/N^{2\alpha}$ and

$$|\langle e_j|E_t(f)|e_j\rangle| \leq C_{f,k}\left(N^{\alpha-\epsilon/4} + N^{-k\epsilon/2}\right) \ .$$

Finally, by a simple computation we find that, if $\partial_p f = 0$,

$$\langle e_j|Op_0^W(f\circ B^t)|e_j\rangle = f(\frac{2^t j}{N}) \ .$$

(*ii*) Following the method of Proposition 2.4, we write the Fourier series in κ and obtain for $\kappa = 0$

$$\int_{\mathbf{T}^2} g_j^{sc}(x,0)f(x)dx = \sum_{\ell\in\mathbf{Z}^2}\langle\epsilon_j|Op_z^{AW}(f)|U(\ell)\epsilon_j\rangle$$

$$= N^2 \sum_{\ell \in \mathbf{Z}^2} \int_{\mathbf{R}^2} dx f(x) \int_{\frac{j}{N}}^{\frac{j+1}{N}} ds\ \eta_{x,z}(s) \int_{\frac{j}{N}-\ell_1}^{\frac{j+1}{N}-\ell_1} dw\ \eta^*_{x,z}(w) e^{i2\pi N\ell_2 w}\ ,$$

where we used the definition of ϵ_j (Lemma 2.1) and the explicit action of $U(\ell)$. Here $\eta_{x,z}$ are the coherent states defined in (2.11). Introducing furthermore $\phi_{q,z}(s) = \exp(i\pi Npq)\exp(2i\pi Nps)\eta_{x,z}(s)$ and recalling that f depends only on q one obtains, as result of the integration in p, $\delta(s-w)$; after integrating over w this yields

$$\int_{\mathbf{T}^2} g^{sc}_j(x,0) f(x) dx = N \sum_{\ell_2} \int dq f(q) \int_{\frac{j}{N}}^{\frac{j+1}{N}} ds |\phi_{q,z}(s)|^2 e^{i2\pi N\ell_2 s}\ .$$

If we introduce the characteristic function χ_j of the interval $[j, j+1]$ we see that

$$\begin{aligned}
\int_{\mathbf{T}^2} g^{sc}_j(x,0) f(x) dx &= \int dq\ f(q) \sum_{\ell \in \mathbf{Z}} \int_{\mathbf{R}} |\phi_{q,z}(\frac{s}{N})|^2 \chi_j(s) e^{i2\pi \ell s}\ ds \\
&= \frac{1}{2} \int dq\ f(q) (\lim_{\epsilon \to 0^+} + \lim_{\epsilon \to 0^-}) \sum_{\ell} |\phi_{q,z}(\frac{\ell+\epsilon}{N})|^2 \chi_j(\ell+\epsilon) \\
&= \frac{1}{2} \int dq\ f(q) \left[|\phi_{q,z}(\frac{j}{N})|^2 + |\phi_{q,z}(\frac{j+1}{N})|^2 \right]\ ,
\end{aligned}$$

where we used in the second line the convergence properties of the Fourier series of $|\phi_{q,z}(s/N)|^2 \chi_j(s)$. If $z = i\omega$, using the explicit form of $\phi_{q,z}$ we easily obtain the following estimate with $C > 0$ an universal constant

$$|\int dx g^{sc}_j(x,0) f(x) - \frac{1}{2}(f(\frac{j}{N}) + f(\frac{j+1}{N}))| \leq C \frac{1}{\sqrt{\omega N}} \|\partial_q f\|_\infty\ .$$

To obtain the final result concerning the t-dependence, observe that it is enough to change $f \to f \circ B^t$ and $\omega \to 4^t \omega$. ∎

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