Fractal dimensions and the Phenomenon of Intermittency in Quantum Dynamics

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Abstract

We exhibit an intermittency phenomenon in quantum dynamics. More precisely we derive new lower bounds for the moments of order p associated to the state $\psi(t) = \mathrm{e}^{-itH}\psi$ and averaged in time between 0 and T. These lower bounds are expressed in terms of generalized fractal dimensions $D^{\pm}_{\mu_\psi}(1/(1+p/d))$ of the measure μ_ψ (where d is the space dimension). This improves notably previous results, obtained in terms of Hausdorff and Packing dimension.

1 Introduction

A by now wide number of articles deal with the links between the quantum dynamics of wave packets solution of the Schrödinger equation, and the spectral properties of the associated hamiltonian H. Actually, during the last decade, an analysis originated by Guarneri in [13] and refined by others [5, 7, 22, 14, 19] established that the fractal properties of the spectral measures were relevant for the study of the spreading of wave packets. Consider a separable Hilbert space $\mathcal{H}$, an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, and a self-adjoint operator H on $\mathcal{H}$. Let $\psi_t = \mathrm{e}^{-itH}\psi$ be the solution of the Schrödinger equation

$$\begin{cases} i\frac{\partial\psi_t}{\partial t} & = & H\psi_t \\ \psi_{t=0} & = & \psi \end{cases}$$

For $X = \sum_n \langle ., e_n \rangle e_n$ being the "position operator", we define the time averaged moments of order p for ψ as

$$\langle\langle |X|^p \rangle\rangle_{\psi,T} \equiv \frac{1}{T} \int_0^T \left\| |X|^{p/2} \mathrm{e}^{-itH} \psi \right\|_{\mathcal{H}}^2 \mathrm{d}t = \frac{1}{T} \int_0^T \sum_{n\in\mathbb{N}} |n|^p |\langle \psi_t, e_n \rangle|^2 \mathrm{d}t. \tag{1}$$

In the specific case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, more relevant from a physics point of view, $\{e_n\}$ is the canonical basis $\{\delta_n\}_{n\in\mathbb{Z}^d}$ and $\langle\langle |X|^p \rangle\rangle_{\psi,T} = \frac{1}{T} \int_0^T \sum_{n\in\mathbb{Z}^d} |n|^p |\psi_t(n)|^2 \mathrm{d}t$. It is now well known from a series of results [22, 7, 1, 13, 6], in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ (extendable to $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$) that if $d_H(\mu_\psi)$ is the Hausdorff dimension of the spectral measure μ_ψ, then

$$\alpha^-(\psi, p, d) \equiv \liminf_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq d_H(\mu_\psi)\, p/d\ . \tag{2}$$

More recently, it has been proven in [14] a lower bound for the upper oscillations of $\langle\langle |X|^p \rangle\rangle_{\psi,T}$, namely

$$\alpha^+(\psi, p, d) \equiv \limsup_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq d_P(\mu_\psi)\, p/d\ , \tag{3}$$

where $d_P(\mu_\psi)$ is the "packing" dimension of μ_ψ.

Those results are certainly not optimal. Some one-dimensional quantum systems with pure point spectrum can give rise to an almost ballistic motion [10], that is $\alpha^+(\psi, 2, 1) = 2$, meanwhile $d_P(\mu_\psi) = d_H(\mu_\psi) = 0$ for pure point measures. A similar phenomenon has been argued to hold for the random dimer model [11] [9].

An intermittency phenomenon in some quantum models for quasicrystals has been suggested by recent computations [23, 25, 29]. Namely, the growth exponents $\alpha^\pm(\psi, p, d)$ should grow faster than the one linear in p/d proposed in (2) and (3); one should observe a more complex law for the behaviour of $\alpha^\pm(\psi, p, d)$ in the variable p/d: $\alpha^\pm(\psi, p, d) = \beta^\pm(\psi, p/d) p/d$, where $\beta^\pm(\psi, p/d)$ are nondecreasing functions of p/d (and of course non smaller than d_H and d_P).

These recent numerical investigations emphasized, in this phenomenon of intermittency, the role of more refined fractal quantities, the so-called "q-th generalized fractal dimensions" $D_{\mu_\psi}(q)$. For a very specific one-dimensional model (d=1), the Julia matrices, it is proven in [4] that

$$\alpha^+(\psi, p, 1) \leq D_{\mu_\psi}(1-p)\, p.$$

Our aim in this paper is to establish rigorously, for a very large class of Schrödinger operators H, how and which generalized fractal dimensions enter the game. Our result (Theorem 2.1) is obtained in the setting of any separable Hilbert space $\mathcal{H}$. If we focus on the particular case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, our main result reads as follows

$$\alpha^-(\psi, p, d) \geq D^-_{\mu_\psi}\left(\frac{1}{1+p/d}\right) p/d$$

and

$$\alpha^+(\psi, p, d) \geq D^+_{\mu_\psi}\left(\frac{1}{1+p/d}\right) p/d.$$

where $D^-_{\mu_\psi}(q)$ and $D^+_{\mu_\psi}(q)$ are the lower and upper "q-th generalized Rényi dimensions" (see Definition 2.2), and X the usual "position" operator. To achieve this we first derive a p/d-dependent lower bound $L_\psi(T)$ for $\langle\langle |X|\rangle\rangle_{\psi,T}$ (Theorem 3.1), and then establish *via* Theorem 4.2 the connection between $L_\psi(T)$ and the generalized Rényi dimensions.

Moreover, we obtain a kind of optimality, in the sense that the quantity $L_\psi(T)$ that minors $\langle\langle |X|^p\rangle\rangle_{\psi,T}$ has its growth exponents exactly equal to $D^\pm_{\mu_\psi}(1/(1+p/d))\, p/d$.

We point out that $D^\pm_{\mu_\psi}(1/(1+p/d))$ are increasing function of p/d, and are respectively larger than $d_H(\mu_\psi)$ and $d_P(\mu_\psi)$ (for all $p/d > 0$). Therefore, our result improves notably the above (2) and (3) bounds.

These new bounds are a consequence of a double improvement of Guarneri-Combes-Last (G-C-L) [7, 22] and Barbaroux-Tcheremchantsev (BT) [5] approaches. The first improvement is due, after a decompostion $\psi = \varphi + \chi$, $\varphi \perp \chi$, to a better control of the key quantity

$$\begin{aligned} B_\psi(T,N) &\equiv \frac{1}{T}\int_0^\infty \sum_{n\le N} |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T)\mathrm{d}t \\ &\le B_\chi(T,N) + \frac{2}{T}\,\mathrm{Re}\int_0^\infty \sum_{n\le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle \overline{\langle \mathrm{e}^{-itH}\psi, e_n\rangle} h(t/T)\mathrm{d}t, \end{aligned} \tag{4}$$

and more particulary of the last term (the crossed term). Here $h(z)$ is some positive function in $C_0^\infty([0,1])$ such that $\int_0^1 h(z)\mathrm{d}z = 1$. Note that the consideration of this function h seems to be a necessary trick if one wants in whole generality to get rid of the negative values of the time t. Taking h as the usual gaussian $e^{-z^2/4}$ will only yield informations about the moments $|X|^p_\psi(t)$ averaged over $[-T,T]$, and not over $[0,T]$. This should be compared to the recent [15] where related results have been simultaneously obtained on $\mathcal{H} = \ell^2(\mathbb{N})$.

Afterwards, and this is Theorem 3.1, one is able to obtain a constant $C(\psi,p,h) > 0$, such that for all $T > 0$,

$$\langle\langle |X|^p\rangle\rangle_{\psi,T} \ge C(\psi,p,h) L_\psi(T), \quad \text{with} \quad L_\psi(T) = \sup_{\varphi\in\mathcal{H}_\psi} \left\{ \frac{|\langle\varphi,\psi\rangle|^{2+2\beta}}{\|\varphi\|^2\, (U_{\varphi,\psi}(T))^\beta} \right\},$$

where $\mathcal{H}_\psi$ is the cyclic subspace spanned by ψ and H,

$$U_{\varphi,\psi}(T) = \int_{\mathbb{R}^2} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\varphi(y) R(T(x-y)),$$

and R is some bounded fast decaying function defined by (11) in Section 3 (one should think to R as to the gaussian $e^{-w^2/4}$). Afterwards, as in [5, 14], one can "choose", for each T, a T-dependent vector φ in the decomposition $\psi = \varphi + \chi$, that contains enough spectral informations to approximate the supremum in $L_\psi(T)$. The second improvement then consists in the way one chooses this particular vector φ. We show (Theorem 4.1) that a judicious choice enables one to connect, up to a logarithmic factor, the quantity $L_\psi(T)$ to the integral

$$I_{\mu_\psi}\left(\frac{1}{1+p/d}, T^{-1}\right) = \int_{\mathbb{R}} \mathrm{d}\mu_\psi(x)\; \mu_\psi\left([x-T^{-1}, x+T^{-1}]\right)^{-\frac{p/d}{1+p/d}}$$

that defines the generalized fractal exponents $D^\pm_{\mu_\psi}(1/(1+p/d))$.

Our method also applies to the previous approaches (G-C-L) and (BT), where the crossed term in (4) wasn't well treated. It respectively yields, as stated in Appendix B, the fractal dimensions $D^{\pm}_{\mu_\psi}((1+\beta)/(1+2\beta))$ and $D^{\pm}_{\mu_\psi}((1+2\beta)/(1+3\beta))$. Since the functions $D^{\pm}_{\mu_\psi}(q)$ are nonincreasing functions of q, our Theorem 2.1 gives a better lower bound. It means that a better estimating of the crossed term in (4) does provide an improvement.

Eventually, with extra assumptions on the decay of the generalized eigenfunctions $u(n,x)$ of H (in the spirit of [18] [20]), it is possible to improve the above bounds. In particular, suppose that there exists a constant C such that for μ_ψ a.e.x, $\sum_{n<N}|u(n,x)|^2 \leq CN^d$ holds, then $\alpha^{\pm}(\psi,p,d) \geq D^{\pm}_{\mu_\psi}(1-p/d)$ (Theorem 4.3).

A related but weaker result has been independently proven to hold by Guarneri and Schulz-Baldes [15]. The method we use here is quite different. In particular we get rid of the Large Deviation Theorem needed in [15]. This enable us to obtain a more general result. We also refer the reader to Remarks 2.3 ii) and iii), 3.1, 4.3 concerning technical differences.

The results of the present article together with a flavor of the proof of Theorem 4.1 can be found in the short note [2]. And further results will be given in a forthcoming article [3].

The paper is organized as follows: in Section 2 we define the generalized fractal dimensions $D^{\pm}_{\mu}(q)$ and state our main result, *i.e.* Theorem 2.1. The next two sections are devoted to its proof. In Section 3 we derive an abstract lower bound for $\langle\langle |X|^p\rangle\rangle_{\psi,T}$, that is $L_\psi(T)$ (Theorem 3.1), and in Section 4 we relate this quantity $L_\psi(T)$ to the generalized fractal dimensions we defined (Theorems 4.1 and 4.2). We also prove that $D^{\pm}_{\mu_\psi}(q) \leq 1$ if μ_ψ has a bounded support.
In Appendix A we give the proof of statements ii) and iii) of Proposition 2.1, which show that our results do improve the existing ones.
In Appendix B we provide the analog of Theorems 4.1 and 4.2, for two others lower bounds, and discuss their relations to $L_\psi(T)$.
In Appendix C we extend our results to some measures with nonbounded support.
Finally, in Appendix D we give a simple example of pure point probability measure μ on $[0,1]$ which has strictly positive fractal dimensions $D^{\pm}_{\mu}(q)$ for some values of $q \in (0,1)$.

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2 Definition of fractal dimensions and main result

Definition 2.1. *The Hausdorff and Packing dimensions of a measure μ are respectively defined as (e.g. [14] and ref. therein)*

$$d_H(\mu) = \mu - \operatorname{ess\,sup}_{x\in\mathbb{R}} \underline{d}_\mu(x) \quad \textit{and} \quad d_P(\mu) = \mu - \operatorname{ess\,sup}_{x\in\mathbb{R}} \overline{d}_\mu(x) \tag{5}$$

where

$$\underline{d}_\mu(x) = \liminf_{\varepsilon\to 0} \frac{\log(\mu([x-\varepsilon,x+\varepsilon]))}{\log\varepsilon} \quad \textit{and} \quad \overline{d}_\mu(x) = \limsup_{\varepsilon\to 0} \frac{\log(\mu([x-\varepsilon,x+\varepsilon]))}{\log\varepsilon},$$

for $x \in \operatorname{supp}\mu$, and $\underline{d}_\mu(x) = \overline{d}_\mu(x) = +\infty$ for $x \notin \operatorname{supp}\mu$. The essential supremum is defined in the following way: take a set of full μ-measure and compute the supremum over this set, and then take the infimum over all these sets of full μ-measure.

Remark 2.1. Note that the above definitions we gave are indeed characterizations of these dimensions. The "usual" definitions, more intricate, can be found e.g. in [25] for the Hausdorff dimension and in [28] for the Packing dimension.

Definition 2.2. *Generalized Rényi dimensions of a measure [17, 8].*
Let μ be a (positive) Borelian probability measure. Let $q \in (-\infty, 1)$ and $\varepsilon \in (0,1)$. We consider the following function with values in $[1, \infty]$

$$I_\mu(q,\varepsilon) = \int_{\operatorname{supp}\mu} \mu([x-\varepsilon, x+\varepsilon])^{q-1} \mathrm{d}\mu(x).$$

The lower and upper generalized Rényi dimensions of μ are respectively defined as

$$D_\mu^-(q) = \frac{1}{1-q} \liminf_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log\varepsilon} \quad \text{and} \quad D_\mu^+(q) = \frac{1}{1-q} \limsup_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log\varepsilon}. \tag{6}$$

with the understanding that both are $+\infty$ if, for some $\varepsilon > 0$, I_μ takes the value $+\infty$.

Remark 2.2.
i) For our purpose, it is sufficient to discuss the case $q \in (-\infty, 1)$ (see e.g. [8] for the general case). We shall even only need in this paper $q \in (0,1)$.
ii) There exists actually a wide number of "generalized fractal dimensions". For example, they can be defined with the help of the so-called "singularity spectrum function" f_μ of the measure μ ([16], [24]), or as a solution of an implicit equation ([16, Formula 2.8], [24]).

The resulting dimensions coincide with each other in certain very specific cases, like e.g. Cookie Cutter measures in $\mathbb{R}$ [24].

In order to state our results we also define the following integrals that could be considered as approximations of the quantities $I_\mu(q,\varepsilon)$. The function R is a bounded even function with fast decay properties at $\pm\infty$ defined by (11).

$$K_\mu(q,\varepsilon) = \int_{\operatorname{supp}\mu} \mathrm{d}\mu(x) \left(\int_{\mathbb{R}} \mathrm{d}\mu(y) R((x-y)/\varepsilon) \right)^{q-1},$$

In Lemma 4.3 we shall prove that for any measure μ with a bounded support taking $K_\mu(q,\varepsilon)$ in (6) instead of $I_\mu(q,\varepsilon)$ leads to the same values of dimensions. Less restrictive assumptions, in order to get the same property, are studied in Appendix C.

We review below some properties of the fractal dimension numbers $D_\mu^\pm(q)$ that may be of interest for us.

Proposition 2.1. *Let μ be a Borel probability measure.*
i) $D_\mu^-(q)$ and $D_\mu^+(q)$ are nonincreasing functions of $q \in (-\infty, 1)$.
ii) For all $q \in (-\infty, 1)$, $D_\mu^-(q) \geq d_H(\mu)$.
iii) For all $q \in (-\infty, 1)$, $D_\mu^+(q) \geq d_P(\mu)$.
iv) If μ has a bounded support, then for all $q \in (0,1)$, $0 \leq D_\mu^-(q) \leq D_\mu^+(q) \leq 1$
v) If μ is "exact scaling", that is $\underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha$ for μ-a.e. x, then for all $q \in (-\infty, 1)$: $D_\mu^-(q) = D_\mu^+(q) = d_P(\mu) = d_H(\mu) = \alpha$.

Proof of Proposition 2.1.
Statement i) is already known (see *e.g.* [8]). This a straightforward consequence of concave and convex Jensen inequalities. For the proof of ii), iii) see Appendix A. The statement iv) is proved in Lemma 4.1. Note that iv) doesn't hold if one lets q vary in $(-\infty, 1)$ - See Appendix D. Finally, the proof of v) can be found e.g. in [25]. $\square$

From now on, each time we will refer to the exponent β it should be understood that $\beta = p$ if $\mathcal{H}$ is a general separable Hilbert space with basis $\{e_n\}_{n\in\mathbb{N}}$, and $\beta = p/d$ in the specific case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ equipped with its canonical basis $\{\delta_n\}_{n\in\mathbb{Z}^d}$.

Our main result is the following one:

Theorem 2.1. *Let H be a Schrödinger operator on a separable Hilbert space $\mathcal{H}$. Assume*

(H1) The spectral measure μ_ψ associated to $\psi \in \mathcal{H}$, $\|\psi\|_{\mathcal{H}} = 1$, has a bounded support.

Then for $\langle\langle |X|^p \rangle\rangle_{\psi,T}$ defined by (1) and with β as described just above, the following holds:

$$\liminf_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \beta D^-_{\mu_\psi}\left(\frac{1}{1+\beta}\right),$$

and

$$\limsup_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \beta D^+_{\mu_\psi}\left(\frac{1}{1+\beta}\right).$$

Remark 2.3.
i) As a consequence of i)-iii) of Proposition 2.1, this result does notably improve previous known bounds of [22, 1, 14].
ii) Since Hypothesis (H1) is particulary adapted to the discrete case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, we stated Theorem 2.1 with (H1). However Theorem 2.1 still holds under the more general assumptions (H2) and (H3) of Appendix C. See also [3] for further results. But at the moment, it is not clear to us whether Theorem 2.1 would remain true without any assumptions on μ_ψ. Concerning this technical point, we refer the reader to Remark 4.3.
iii) In [15], a similar but weaker version of Theorem 2.1 has been obtained independently and simultaneously. There, they assume in addition that for all $q \in (-\infty, +\infty)$ one has $D^+_\mu(q) = D^-_\mu(q)$. Furthermore, they consider the moments averaged over $[-T, T]$ instead of $[0, T]$.

Proof of Theorem 2.1.
The proof of Theorem 2.1 is the combination of Theorem 3.1 and Theorem 4.2 which are proved respectively in Sections 3 and 4. And so is the proof. $\square$

3 A general lower bound

This section can be regarded as the first part of the proof of our main result Theorem 2.1. Let H be a self-adjoint operator in Hilbert space $\mathcal{H}$, and $\{e_n\}$ an orthonormal basis in $\mathcal{H}$ labelled by $n \in \mathbb{N}$ or by $n \in \mathbb{Z}^d$. Let ψ be some vector in $\mathcal{H}$ such that $\|\psi\| = 1$. We are interested in lower bounds for the moments of the abstract position operator associated to the basis $\{e_n\}$, defined as

$$|X|_\psi^p(t) = \sum_n |n|^p |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2. \tag{7}$$

In particular, if $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, one can take the canonical basis $e_n(k) = \delta_{nk}, \quad n,k \in \mathbb{Z}^d$ to obtain the moments of the usual position operator. Our results can also be extended to the case $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ and

$$|X|_\psi^p(t) = \int_{\mathbb{R}^d} |x|^p |\psi(t,x)|^2 dx, \quad \psi(t,x) = (\mathrm{e}^{-itH}\psi)(x).$$

It can be done using the Theorem 3.2 in the same manner as in [5] (see Corollary 2.4 and Theorem 2.5 of [5], see also [22]). However, for the sake of simplicity, we shall consider in this paper only $|X|^p$ given by (7).

We derive a lower bound for the time averaged moments of position operator in term of an abstract quantity in the spirit of [5], namely

$$L_\psi(T) = \sup\left\{L_\psi(\varphi,T),\ \varphi \in \mathcal{H}_\psi,\ \ \langle\psi,\varphi\rangle \neq 0\right\}, \quad L_\psi(\varphi,T) = \frac{|\langle\varphi,\psi\rangle|^{2+2\beta}}{\|\varphi\|^2 U_{\varphi,\psi}(T)^\beta} \tag{8}$$

where $\mathcal{H}_\psi$ is the cyclic subspace spanned by ψ and H. The exponent $\beta = p$ if $n \in \mathbb{N}$ and $\beta = p/d$ if $n \in \mathbb{Z}^d$. In particular, notice that if $\varphi = \chi_\Omega(H)\psi$, where Ω is a borel set, then $L_\psi(\varphi,T)$ reads as

$$L_\psi(\varphi,T) = \frac{\|\varphi\|^{2+4\beta}}{U_{\varphi,\psi}(T)^\beta}. \tag{9}$$

The quantity $U_{\varphi,\psi}(T)$ is defined as follows.

$$U_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y) R((x-y)T), \tag{10}$$

where $R(w)$ is a bounded and fast decaying function defined in (11) below. In the sequel, we shall note (ε playing the role of $1/T$),

$$b^{(R)}(x,\varepsilon) \equiv \int_{\mathbb{R}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right), \quad \varepsilon = T^{-1}.$$

So that $U_{\varphi,\psi}(T) = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x) b^{(R)}(x, 1/T)$. The quantity $U_{\varphi,\psi}(T)$ is crucial since one may consider that it codes the determinating part of the spectral informations that are involved in the dynamical behaviour of the considered quantum system. The quantity in (10) should be compared to other quantities such as

$$\int_{|x-y|\leq 1/T} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\psi(y), \quad \text{or} \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\psi(y)\mathrm{e}^{-(x-y)^2T^2/4},$$

which already appear, in the limit $\varepsilon = T^{-1} \to 0$, as key quantities to discuss the nature of the spectrum [21, 22, 10, 5].

The main result of this section is the following.

Theorem 3.1. *Let H be a Schrödinger operator on $\mathcal{H}$ and let ψ be a vector in $\mathcal{H}$, $\|\psi\| = 1$. Let $\{e_n\}$ be some orthonormal basis in $\mathcal{H}$ and*

$$\langle\langle |X|^p \rangle\rangle_\psi(T) = \frac{1}{T}\int_0^T \sum_n |n|^p |\langle \mathrm{e}^{-itH}, e_n\rangle|^2 \mathrm{d}t.$$

Let $h \in C_0^\infty([0,1])$ be any positive function such that $\int_0^1 h(z)dz = 1$ and define

$$R(w) = \begin{cases} 1 & \text{if } |w| \leq 1 \\ |\hat{h}(w)|^2 & \text{if } |w| > 1, \end{cases} \tag{11}$$

where $\hat{h}$ stands for the Fourier transform of h. Then, for all $p > 0$ and with $L_\psi(T)$ defined Line (8), there exists a constant $C(\psi, p, h)$ such that for all $T > 0$

$$\langle\langle |X|^p \rangle\rangle_\psi(T) \geq C(\psi, p, h) L_\psi(T) \ .$$

Remark 3.1. As already pointed out in the introduction, h seems to be the necessary trick to take into account averaging on time between $[0, T]$ only (and not $[-T, T]$). This trick actually allows one to deal with the crossed term in (4) and to recover a function R with fast decay at $\pm\infty$. Replacing h by the usual gaussian $e^{-z^2/4}$ in (21) below and thus in (12) will lead to the same result but with $|X|_\psi^p(t)$ averaged over $[-T, T]$.

As it is now well-known [13, 7, 22], a key point in the proof of lower bound for $|X|^p$ is a good control of the behaviour of the wave-packet inside a ball of radius N, namely

$$B_\psi(T, N) = \frac{1}{T}\int_0^\infty \sum_{|n|\leq N} |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T) \mathrm{d}t. \tag{12}$$

To that end, we shall need Theorem 3.2 below, which is a generalization of Theorem 2.1 in [5]. In order to state it we first recall well-known facts involving generalized eigenfunctions and the spectral measure μ_ψ (*see e.g.* [20] [27] and ref. therein).
Let W_ψ be the unitary map from the cyclic subspace $\mathcal{H}_\psi$ into the space $\mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Recall that $W_\psi(\psi) = 1$ and $W_\psi(\mathrm{e}^{-itH}\psi) = \mathrm{e}^{-itx}$ for all $x \in \mathbb{R}$. We shall denote by P_ψ the orthogonal projection on $\mathcal{H}_\psi$. The map W_ψ has a kernel $u(n, x)$ defined by $u(n) = W_\psi(P_\psi e_n)$, so that

$$\langle e^{-iHt}\psi, e_n\rangle = \int e^{-ixt}\overline{u(n,x)}\mathrm{d}\mu_\psi(x), \tag{13}$$

and more generally, for any vector $\xi \in \mathcal{H}$, one has

$$\langle P_\psi \xi, e_n\rangle = \int W_\psi\left(P_\psi \xi\right)(x)\overline{u(n,x)}\mathrm{d}\mu_\psi(x). \tag{14}$$

In the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and $e_n(k) = \delta_{nk}$, for each fixed $x \in \mathbb{R}$ the vector $(u(n,x)_{n\in\mathbb{Z}^d})$ is a generalized eigenfunction (*i.e.* in a distributional sense).

The expansions (13)-(14) are of course also possible with any vector $\varphi \in \mathcal{H}$ and corresponding generalized eigenfunctions $v(n, y) = W_\varphi(P_\varphi e_n)(\lambda)$ (actually if $\varphi = f(H)\psi$, with $f \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, then one checks that $v(n, y) = f(y)u(n, y)$).
We are now ready to formulate Theorem 3.2.

Theorem 3.2. *Let $h(z)$ be some function from $L^1(\mathbb{R})$, H a self-adjoint operator acting on $\mathcal{H}$ and A a Hilbert-Schmidt operator in $\mathcal{H}$. For any couple of vectors ψ, φ from $\mathcal{H}$ define the quantity*

$$D^{(h)}_{\varphi,\psi}(T) = \frac{1}{T}\int_{-\infty}^{+\infty} \langle A\mathrm{e}^{-itH}\varphi, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{dt}.$$

Let

$$U^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)|\hat{h}((x-y)T)|^2.$$

The following estimate holds:

$$|D^{(h)}_{\varphi,\psi}(T)| \le \|A\|_2 \left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}},$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of A.

In the special case $A = \sum_{|n|\le N}\langle\cdot, e_n\rangle e_n$, one has $\|A\|_2 \le CN^{d/2}$, and therefore

$$|D^{(h)}_{\varphi,\psi}(T,N)| = \left|\frac{1}{T}\int_{-\infty}^{+\infty}\sum_{|n|\le N}\langle \mathrm{e}^{-itH}\varphi, e_n\rangle\langle e_n, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{dt}\right| \le CN^{d/2}\left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}}.$$

Here $d = 1$ for abstract position operator associated with the base $\{e_n\}$ labelled by $n \in \mathbb{N}$, and $d \ge 1$ in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ equipped with the canonical basis $e_n = \delta_n$.

Proof of Theorem 3.2.
Since A is Hilbert-Schmidt, there exist two orthonormal bases $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a monotonely decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$, $E_n \ge 0$, such that $\sum_{n=1}^\infty E_n^2 = \|A\|_2^2 < +\infty$ and $A = \sum_{n=1}^\infty E_n\langle\cdot, f_n\rangle g_n$. Therefore,

$$D^{(h)}_{\varphi,\psi}(T) = \frac{1}{T}\int_{-\infty}^{+\infty}\sum_{n=1}^{\infty} E_n\langle \mathrm{e}^{-itH}\varphi, f_n\rangle\langle g_n, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{dt}. \tag{15}$$

Then (13) reads as

$$\langle \mathrm{e}^{-itH}\varphi, f_n\rangle = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{e}^{-itx}\overline{u(n,x)}, \tag{16}$$

where $u(n) = W_\varphi(P_\varphi f_n) \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\varphi)$. The similar formula holds for ψ with $v(n) = W_\psi(P_\psi g_n) \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. One obtains from (15) -(16) that

$$D^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)\hat{h}((x-y)T)S(x,y), \tag{17}$$

where

$$S(x,y) = \sum_{n=1}^{\infty} E_n\overline{u(n,x)}v(n,y).$$

The sum converges in $\mathrm{L}^2(\mathbb{R}^2, \mu_\varphi \times \mu_\psi)$. Applying the Cauchy-Schwarz inequality to (17), one gets

$$|D^{(h)}_{\varphi,\psi}(T)|^2 \le U^{(h)}_{\varphi,\psi}(T)\|S\|^2_{\mathrm{L}^2(\mathbb{R}^2, d\mu_\varphi\times d\mu_\psi)}. \tag{18}$$

One can easily see that

$$\|S\|^2_{\mathrm{L}^2(\mathbb{R}^2,d\mu_\varphi\times d\mu_\psi)} = \sum_{n,k=1}^{\infty} E_n E_k a_{nk}\overline{b_{nk}},$$

where

$$a_{nk} = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x) u(k,x)\overline{u(n,x)} = \langle W_\varphi(P_\varphi f_k), W_\varphi(P_\varphi f_n)\rangle_{\mathrm{L}^2(\mathbb{R},\mathrm{d}\mu_\varphi)} = \langle P_\varphi f_k, f_n\rangle_{\mathcal{H}}\ , \qquad (19)$$

where we used an analog of (14), and in the same manner

$$b_{nk} = \langle P_\psi g_k, P_\psi g_n\rangle_{\mathcal{H}} = \langle g_k, P_\psi g_n\rangle_{\mathcal{H}}.$$

We have also used the fact that both P_φ and P_ψ are orthogonal projections. By Parseval equality,

$$\sum_{n=1}^{\infty} |a_{nk}|^2 = \|P_\varphi f_k\|^2, \quad \sum_{k=1}^{\infty} |b_{nk}|^2 = \|P_\psi g_n\|^2.$$

Therefore, as $\|f_k\| = \|g_n\| = 1$ for all k, n,

$$\|S\|^4_{\mathrm{L}^2(\mathbb{R}^2,d\mu_\varphi\times d\mu_\psi)} \le \sum_{k=1}^{\infty} E_k^2 \|P_\varphi f_k\|^2 \sum_{n=1}^{\infty} E_n^2 \|P_\psi g_n\|^2 \le \left(\sum_{n=1}^{\infty} E_n^2\right)^2 = \|A\|_2^4. \qquad (20)$$

The first statement of the Theorem follows from (18) and (20). The proof of the second part is essentially the same. The only difference is that in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ the sums are taken over $n \in \mathbb{Z}^d:\ |n| \le N$. In particular, the estimate (20) reads as

$$\|S\|^4_{\mathrm{L}^2(\mathbb{R}^2,d\mu_\varphi\times d\mu_\psi)} \le \sum_{|k|\le N} \|P_\varphi e_k\|^2 \sum_{|n|\le N} \|P_\psi e_n\|^2 \le C\, N^{2d}.$$

The proof is completed. $\square$

One should stress that the proof we presented here is simpler than that of Theorem 2.1 in [5], because we do not use the product space $\mathcal{H}\otimes\mathcal{H}$. In the case $\psi = \varphi$ and $h(z) = \exp(-z^2/4)$ the result of Theorem 3.2 is equivalent to that of Theorem 2.1 in [5].

Proof of Theorem 3.1. Pick a positive function $h(z) \in C_0^\infty([0,1])$ such that $\int_0^1 h(z)\mathrm{d}z = 1$. The role of h is to supply a fast decaying function $|\hat{h}(w)|^2$. Note that one trivially has, for any $z \in [0,1]$, $h(z) \le \|h\|_\infty \chi_{[0,1]}(z)$. Then one verifies that

$$\langle\langle X^p\rangle\rangle_\psi(T) \ge \frac{1}{\|h\|_\infty}\int_0^\infty \sum_n |n|^p |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T)\frac{\mathrm{d}t}{T} \ge \frac{N^p}{\|h\|_\infty}\left(\|\psi\|^2 - B_\psi(T,N)\right), \quad (21)$$

with $B_\psi(T,N)$ defined line (12). As usual, one needs a control of this quantity $B_\psi(T,N)$ which represents the behaviour of the wave-packet in a ball of radius N. Decompose the vector ψ as $\varphi+\chi$, with $\langle\varphi,\chi\rangle = 0$ and $\varphi \ne 0$ (one should think to $\varphi = \chi_\Omega(H)\psi$). Thus

$$\begin{aligned} B_\psi(T,N) &= B_\varphi(T,N) + B_\chi(T,N) + \frac{2}{T}\,\mathrm{Re}\int_0^\infty \sum_{|n|\le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle\overline{\langle \mathrm{e}^{-itH}\chi, e_n\rangle} h(t/T)\mathrm{d}t, \\ &= -B_\varphi(T,N) + B_\chi(T,N) + \frac{2}{T}\,\mathrm{Re}\int_0^\infty \sum_{|n|\le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle\overline{\langle \mathrm{e}^{-itH}\psi, e_n\rangle} h(t/T)\mathrm{d}t. \end{aligned}$$

Then, taking into account that $1/T\int_0^\infty h(t/T)dt = 1$ and $h(z) \geq 0$, we have $B_\chi(T,N) \leq \|\chi\|^2 = \|\psi\|^2 - \|\varphi\|^2$. Let $A = \sum_{|n|\leq N}\langle\cdot, e_n\rangle e_n$. Then

$$B_\psi(T,N) \leq \|\psi\|^2 - \|\varphi\|^2 + 2\mathrm{Re}D^{(h)}_{\varphi,\psi}(T,N),$$

where $D^{(h)}_{\varphi,\psi}(T,N)$ was defined in the Theorem 3.2. The second statement of this Theorem gives immediately

$$B_\psi(T,N) \leq \|\psi\|^2 - \|\varphi\|^2 + CN^{\frac{d}{2}}\left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}}, \tag{22}$$

where, as in Theorem 3.2,

$$U^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)|\hat{h}((x-y)T)|^2.$$

As $|\hat{h}(w)| \leq 1$ for all w and by definition (11) of R, we clearly have $R(w) \geq |\hat{h}(w)|^2$ for all w. Therefore $U^{(h)}_{\varphi,\psi}(T) \leq U_{\varphi,\psi}(T)$, and the estimate (22) is valid with $U_{\varphi,\psi}(T)$ defined by (10), that is with the function R instead of $|\hat{h}|^2$.

We are now in position to finish the proof of Theorem 3.1. The basic strategy is standard: let N be the largest integer such that $CN^{d/2}U_{\varphi,\psi}(T)^{\frac{1}{2}} \leq \|\varphi\|^2/2$, it yields:

$$B_\psi(T,N) \leq \|\psi\|^2 - \frac{\|\varphi\|^2}{2}. \tag{23}$$

The inequalities (23) and (21) yield with some positive constant $C(\psi,p,h)$:

$$\langle\langle|X|^p\rangle\rangle_{\psi,T} \geq C(\psi,p,h)\frac{\|\varphi\|^{2+4\beta}}{U_{\varphi,\psi}(T)^\beta} = C(\psi,p,h)L_\psi(\varphi,T), \quad \beta = \frac{p}{d}. \tag{24}$$

One recovers $L_\psi(\varphi,T)$ as given Line (9). It is this latter lower bound that will be used in the proof of the lower bound in Theorem 4.1. However, in the more general case where φ is any function of $\mathcal{H}$ with $\langle\psi,\varphi\rangle \neq 0$, one gets the bound with

$$L_\psi(\varphi,T) \equiv \frac{|\langle\varphi,\psi\rangle|^{2+2\beta}}{\|\varphi\|^2 U_{\varphi,\psi}(T)^\beta}. \tag{25}$$

Indeed take such a φ, and then define as in [5] $\tilde{\varphi} = (\langle\varphi,\psi\rangle\|\varphi\|^{-2})\varphi$. So one checks that if $\chi = \psi - \tilde{\varphi}$, then $\langle\tilde{\varphi},\chi\rangle = 0$ and one is able to apply the result line (9) to ψ and $\tilde{\varphi}$. Taking into account that $\|\tilde{\varphi}\| = |\langle\varphi,\psi\rangle|\|\varphi\|^{-1}$ and that $U_{\tilde{\varphi},\psi}(T) = |\langle\varphi,\psi\rangle|^2\|\varphi\|^{-4}U_{\varphi,\psi}(T)$, one finds the announced expression (25). To optimise the lower bound, we should take the supremum of $L_\psi(\varphi,T)$ for a given T over all possible φ. One can show in the same manner as in [5], Lemma 3.1, that it is sufficient to take φ only from the cyclic subspace $\mathcal{H}_\psi$. This gives us $L_\psi(T)$ defined line (8).

4 Towards the fractal dimensions

This section deals with the connection between the dynamic quantity $L_\psi(T)$ introduced in the previous section, and the fractal dimensions defined in Section 2, called the generalized Rényi dimensions. We prove here announced the intermittency phenomenon.

We shall prove

Theorem 4.1. *Let H be a Schrödinger operator on $\mathcal{H}$. Assume (H1), that is*

The spectral measure μ_ψ associated to $\psi \in \mathcal{H}$, $\|\psi\|_{\mathcal{H}} = 1$, has a bounded support.

Then, for all $\beta > 0$, there exists a constant $C_1 > 0$ such that, for all $\varepsilon > 0$:

$$\frac{C_1}{|\log \varepsilon|^{1+\beta}} K_{\mu_\psi}\left(\frac{1}{1+\beta}, \varepsilon\right)^{1+\beta} \le L_\psi(\varepsilon^{-1}) \le K_{\mu_\psi}\left(\frac{1}{1+\beta}, \varepsilon\right)^{1+\beta}, \tag{26}$$

where

$$K_{\mu_\psi}\left(\frac{1}{1+\beta}, \varepsilon\right) = \int_{\mathrm{supp}\mu_\psi} \mathrm{d}\mu_\psi(x) \left(\int_{\mathbb{R}} d\mu_\psi(y) R((x-y)/\varepsilon)\right)^{\frac{-\beta}{1+\beta}}.$$

This will actually imply

Theorem 4.2. *Under the same hypothesis (H1) as previously, one has*

$$\liminf_{T\to\infty} \frac{\log L_\psi(T)}{\log T} = \beta D^-_{\mu_\psi}\left(\frac{1}{1+\beta}\right), \quad \limsup_{T\to\infty} \frac{\log L_\psi(T)}{\log T} = \beta D^+_{\mu_\psi}\left(\frac{1}{1+\beta}\right).$$

Remark 4.1.
i) We point out that the left inequality in (26), that is the lower bound on $L_\psi(\varepsilon^{-1})$, is sufficient to prove our main result Theorem 2.1. However the right part of (26), that is the upper bound, is of interest too. In particular it says that once one derived the lower bound $L_\psi(T)$, one cannot hope a better result than the one we stated.
ii) For clarity, the above two theorems are stated under the very simple assumption (H1), namely the boundedness of the support of the spectral measure μ_ψ. Since the proof of Theorems 4.1 and 4.2 are based on the technical but central Lemma 4.1, it is a straightforward consequence of Appendix C that Theorem 4.1 and Theorem 4.2 remain true if one replaces assumption (H1) by the weaker assumption (H2) of Proposition 5.1 or (H3) of Proposition 5.2. See also Remark 2.3 Point ii).
iii) Note also that the right-hand-side inequality in (26), that is the upper bound, is true for any measures μ_ψ, not only those with bounded support.

We start by providing a proof for Theorem 4.1. Throughout this section, even if there is no mention, integrations must be understood on the support of the measure μ_ψ or μ_φ we consider.

Proof of the upper bound in Theorem 4.1.

The upper bound of (26) in Theorem 4.1 will be proved, if taking any function $f \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, one shows

$$L_\psi(f(H)\psi, \varepsilon^{-1}) \le K_{\mu_\psi}\left(\frac{1}{1+\beta}, \varepsilon\right)^{1+\beta} = \left(\int d\mu_\psi(x) b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}}\right)^{1+\beta}, \tag{27}$$

where $b^{(R)}(x,\varepsilon) = \int \mathrm{d}\mu_\psi(y)R((x-y)/\varepsilon)$. This result will follow from Cauchy-Schwarz and Hölder Inequalities. Pick $\varphi = f(H)\psi$, with $f \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Therefore

$$\mathrm{d}\mu_\varphi(x) = |f(x)|^2 \mathrm{d}\mu_\psi(x) \text{ and } \langle \varphi, \psi \rangle = \int \mathrm{d}\mu_\psi(x) f(x).$$

Remember also that $U_{\varphi,\psi}(T) = \int \mathrm{d}\mu_\varphi(x) b^{(R)}(x,\varepsilon)$. Then, rewriting $L_\psi((f(H)\psi, \varepsilon^{-1})$ as given in (8) one gets :

$$L_\psi(f(H)\psi, \varepsilon^{-1}) = \frac{\left| \int d\mu_\psi(x) f(x) \right|^{2+2\beta}}{\left(\int d\mu_\psi(x) |f(x)|^2 \right) \left(\int d\mu_\psi(x) |f(x)|^2 b^{(R)}(x,\varepsilon) \right)^\beta}. \tag{28}$$

One starts with a Cauchy-Schwarz inequality applied to the numerator, and to the functions $b^{(R)}(x,\varepsilon)^{\frac{-\beta}{2(1+\beta)}}$ and $f(x) b^{(R)}(x,\varepsilon)^{\frac{\beta}{2(1+\beta)}}$. It yields:

$$\begin{aligned} \left| \int d\mu_\psi(x) f(x) \right|^{2+2\beta} &\leq \left(\int d\mu_\psi(x) b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}} \right)^{1+\beta} \left(\int d\mu_\psi(x) b^{(R)}(x,\varepsilon)^{\frac{\beta}{1+\beta}} |f(x)|^2 \right)^{1+\beta} \quad (29) \\ &= K_{\mu_\psi}\left(\frac{1}{1+\beta}, \varepsilon \right)^{1+\beta} \left(\int d\mu_\psi(x) b^{(R)}(x,\varepsilon)^{\frac{\beta}{1+\beta}} |f(x)|^2 \right)^{1+\beta}. \end{aligned}$$

An Hölder inequality applied to the last term, and with the coefficients $p = 1+\beta$ and $p' = (1+\beta)/\beta$, leads to:

$$\begin{aligned} &\left(\int d\mu_\psi(x) b^{(R)}(x,\varepsilon)^{\frac{\beta}{1+\beta}} |f(x)|^2 \right)^{1+\beta} \\ &= \left(\int d\mu_\psi(x) (|f(x)|^2)^{\frac{1}{1+\beta}} \left(|f(x)|^2 b^{(R)}(x,\varepsilon) \right)^{\frac{\beta}{1+\beta}} \right)^{1+\beta} \\ &\leq \left(\int d\mu_\psi(x) |f(x)|^2 \right) \left(\int d\mu_\psi(x) |f(x)|^2 b^{(R)}(x,\varepsilon) \right)^\beta. \quad (30) \end{aligned}$$

One thus recovers exactly the denominator term, and (27) holds. □

Remark 4.2.
i) We stress out the very natural link that comes out between $L_\psi(\varepsilon^{-1})$ and the integral $K_{\mu_\psi}(q,\varepsilon)$ with the particular value $q = 1/(1+\beta)$. The same deep link appears to hold with the other lower bounds $L_1(T)$ and $L_2(T)$ defined in Appendix B and that come from the former approaches (G-C-L and BT). This shows how relevant are the fractal dimension numbers $D_\mu^\pm(q)$ with regards to the time behaviour of the quantities that have been studied for many years in quantum dynamics.
ii) We also point out that this is the first time that such upper bounds are obtained, supplying thereby the theoretic best result one can hope while studying the quantity $L_\psi(T)$ (which is actually a more refined lower bound than what one used to get [2]). It is remarquable that one happens to prove that these theoretic limits are actually reached.

We now turn to the second and main part of Theorem 4.1, that is the lower bound. Our basic strategy to get the lower bound in (26) is to estimate the quantity $L_\psi(\varphi, T)$ in (25), with a vector $\varphi = \chi_{\Omega(r)}(H)\psi$. And $\Omega(r)$ will be a "thin" set of the form $\Omega(r) = \{x \in \mathrm{supp}\mu_\psi,\ \varepsilon^{r+A/N} < b^{(R)}(x,\varepsilon) \leq \varepsilon^r\}$, but which supports, roughly speaking, a significant part of the mass of the integral $K_{\mu_\psi}(1/(1+\beta),\varepsilon)$. The constant A is fixed by Lemma 4.1 below, and has the value $A = 2(1+\beta)$. The integer N will stand for the integer part of $\log T$. Before going on with the proof of Theorem 4.1, we need the two following lemmas:

Lemma 4.1. *Let $q \in (0,1)$, and suppose that μ_ψ has a bounded support with length D (i.e. that (H1) holds for μ_ψ). Define $b(x,\varepsilon) = \int \mathrm{d}\mu_\psi(x) g((x-y)\varepsilon^{-1})$, with $g(w) = \chi_{[-1,1]}(w)$ or $R(w)$. Then there exists A (actually $A = 2/q$) such that*

$$\int_{b(x,\varepsilon)\leq\varepsilon^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \leq 2D\varepsilon. \tag{31}$$

Moreover, $D^{\pm}_{\mu_\psi} \in [0,1]$ for any $q \in (0,1)$.

Lemma 4.2. *Let q and μ_ψ be as in the previous lemma and $A = 2/q$. Let $N > 0$ be an integer. Then, for $b(x,\varepsilon)$ defined as in Lemma 4.1, there exists an r_0 and a set $\Omega(r_0) = \{x \in \mathrm{supp}\mu_\psi,\ \varepsilon^{r_0+A/N} < b(x,\varepsilon) \leq \varepsilon^{r_0}\}$ such that, for all ε small enough*

$$\int_{\Omega(r_0)} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \geq \frac{1}{2N}\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x).$$

Lemma 4.2 is the key lemma to get Theorem 4.1, since it is this lemma that supplies the set $\Omega(r)$, and so the vector $\varphi = \chi_{\Omega(r)}(H)\psi$, that is needed to prove Theorem 4.1. Lemma 4.1 will not enter explicitely in the proof of Theorem 4.1 but is an important ingredient that we shall use twice: once while proving Lemma 4.2 and then in Lemma 4.3.

We stress out that Lemma 4.1 is the only, but crucial, technical point where we shall take advantage of the boundedness assumption of the support of the measure μ_ψ (H1). Propositions 5.1 and 5.2 in Appendix C are analog of Lemma 4.1, but under the weaker assumptions (H2), resp. (H3). A consequence is that, as already mentionned in the Remark 4.1 above, the conclusion of Lemma 4.2, as well as the one of the forthcoming Lemma 4.3, remain valid under Hypotheses (H2) and (H3).

Proof of the lower bound in Theorem 4.1.

Let N be the integer part of $-\log\varepsilon$. For a sake of simplicity we shall use $N = -\log\varepsilon$ (rather than the integer part). Take A as in Lemma 4.2. Here $q = 1/(1+\beta)$, and thus $q-1 = -\beta/(1+\beta)$. For fixed ε, we choose $\varphi = \chi_{\Omega(r_0)}(H)\psi$ with r_0 given by Lemma 4.2. Thus, from the definition of $\Omega(r_0)$ we obtain

$$U_{\varphi,\psi}(\varepsilon^{-1}) = \int_{\Omega(r_0)} b^{(R)}(x,\varepsilon)\mathrm{d}\mu_\psi(x) \leq \varepsilon^{r_0}\mu_\psi(\Omega(r_0))\ ,$$

Therefore, using the expression (28) of $L_\psi(f(H)\psi,\varepsilon^{-1})$ with $f = \chi_{\Omega(r_0)}$, one has

$$\begin{aligned} L_\psi(\varepsilon^{-1}) &\geq \frac{\mu_\psi(\Omega(r_0))^{1+2\beta}}{\varepsilon^{r_0\beta}\mu_\psi(\Omega(r_0))^\beta} \geq \varepsilon^{\frac{A\beta}{N}}\left(\int_{\Omega(r_0)} b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}}\mathrm{d}\mu_\psi(x)\right)^{1+\beta} \\ &\geq \frac{e^{-A\beta}}{(-2\log\varepsilon)^{1+\beta}}\left(\int b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}}\mathrm{d}\mu_\psi(x)\right)^{1+\beta} = \frac{C_1}{|\log\varepsilon|^{1+\beta}}K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta}, \end{aligned} \tag{32}$$

where in the last inequality we used Lemma 4.2 and $N = -\log\varepsilon$. This holds for ε small enough. $\square$

Remark that the fact that r_0 disappears in the relation (32) is crucial. It is of course related to the particular value $q = 1/(1+\beta)$ of the fractal dimension that enters into account. This is the place where the deep link between $L_\psi(T)$ and $D^{\pm}_{\mu_\psi}(1/(1+\beta))$ comes out.

We are left with the proofs of Lemmas 4.1 and 4.2.

Proof of Lemma 4.1.

Let us first cover suppμ_ψ with disjoint intervals $I_j = [x_j - \varepsilon/2, x_j + \varepsilon/2[$. If D is the length of the support of μ_ψ, then note that one needs at most $2D/\varepsilon$ such intervals I_j to cover suppμ_ψ. Then define $B_A = \{x \in \mathrm{supp}\mu_\psi,\ b(x,\varepsilon) \leq \varepsilon^A\}$ and for any j such that $\mu_\psi(I_j \cap B_A) \neq 0$,

$$\underline{b}_j = \inf\{b(x,\varepsilon),\ x \in I_j \cap B_A\}.$$

Remark that $\underline{b}_j \leq \varepsilon^A$. We first prove that

$$\mu_\psi(I_j \cap B_A) \leq \underline{b}_j. \tag{33}$$

To that end we show that $\mu_\psi(I_j) \leq b(x,\varepsilon)$, for all $x \in I_j$; it will then clearly implies the claimed estimate (33). Pick $x \in I_j$. Since $I_j = [x_j - \varepsilon/2, x_j + \varepsilon/2[\subset [x-\varepsilon, x+\varepsilon]$, one has

$$\mu_\psi(I_j) \leq \mu_\psi([x-\varepsilon, x+\varepsilon]) = \int \mathrm{d}\mu_\psi(y)\chi_{[-1,1]}\left(\frac{x-y}{\varepsilon}\right) \leq \int \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right).$$

The latter means that $\mu_\psi(I_j) \leq b(x,\varepsilon)$, whenever $g(w)$ is defined as $\chi_{[-1,1]}(w)$ or $R(w)$. Thus (33) holds. As $q \in (0,1)$, one has, taking into account only the intervals I_j such that $\mu_\psi(I_j \cap B_A) \neq 0$ (so that $\underline{b}_j > 0$):

$$\int_{B_A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \leq \sum_j \underline{b}_j^{q-1}\mu_\psi(I_j \cap B_A) \leq \sum_j \underline{b}_j^q \leq \frac{2D}{\varepsilon}\varepsilon^{qA}, \tag{34}$$

where we used (33). Now we are in position to finish the proof. First, taking $A = 2/q$, we get (31). Next, obviously

$$\int_{b(x,\varepsilon)>\varepsilon} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \leq \varepsilon^{q-1}. \tag{35}$$

This bound together with (34) for $A = 1$ gives

$$\int_{\mathrm{supp}\mu_\psi} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \leq (2D+1)\varepsilon^{q-1},$$

so $D^{\pm}_{\mu_\psi}(q) \leq 1$. This ends the proof of Lemma 4.1. $\square$

Proof of Lemma 4.2.

To alleviate the notations, denote as in the previous lemma $B_A = \{x \in \mathrm{supp}\mu_\psi,\ b(x,\varepsilon) \leq \varepsilon^A\}$ and also $B^A = \{x \in \mathrm{supp}\mu_\psi,\ b(x,\varepsilon) > \varepsilon^A\}$. Then, using the bound of Lemma 4.1, one has

$$\begin{aligned}\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) &= \int_{B^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) + \int_{B_A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \\ &\leq \int_{B^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) + 2D\varepsilon.\end{aligned}$$

Remark that one always has $b(x,\varepsilon) \le 1$ (since $R(w) \le 1$). Thus one can divide the remaining set B^A into N parts: $\Omega(kA/N) = \{x \in \mathrm{supp}\mu_\psi, \varepsilon^{(k+1)A/N} < b(x,\varepsilon) \le \varepsilon^{kA/N}\}$, with $k = 0, 1, \cdots, N-1$. At least one of these N sets $\Omega(kA/N)$ gives rise to an integral bigger than $1/N$ times the integral over the whole set B^A. And so are picked k_0, $r_0 = Ak_0/N$, and thus the set $\Omega(r_0)$ of the Lemma.

To end the proof, remark that $\int b(x,\varepsilon)^{q-1} \mathrm{d}\mu_\psi(x) \ge 1$, since $q < 1$. Therefore, for $\varepsilon \le 1/(4D)$, one gets

$$\begin{aligned}\int_{\Omega(r_0)} b(x,\varepsilon)^{q-1} \mathrm{d}\mu_\psi(x) &\ge \frac{1}{N}\left(\int b(x,\varepsilon)^{q-1} \mathrm{d}\mu_\psi(x) - 2D\varepsilon\right) \\ &\ge \frac{1}{2N}\int b(x,\varepsilon)^{q-1} \mathrm{d}\mu_\psi(x).\end{aligned}$$

□

We now turn to the proof of Theorem 4.2, which is actually a consequence of Theorem 4.1 and of the following Lemma 4.3 that relates the integrals $K_{\mu_\psi}(q,\varepsilon)$ to the integrals $I_{\mu_\psi}(q,\varepsilon)$ that enter into account in the definition of the generalized Rényi dimension $D^\pm_{\mu_\psi}(q)$. More precisely, Lemma 4.3 says that both $K_{\mu_\psi}(q,\varepsilon)$ and $I_{\mu_\psi}(q,\varepsilon)$ give rise to the same dimension numbers. This is of course, as already said, because of the fast decay properties of the function R we have chosen (*via* the choice of $h \in C_0^\infty([0,1])$).

Lemma 4.3. *Let $q \in (0,1)$. Suppose that μ_ψ has a bounded support (H1). Then for all $\nu \in (0,1)$*

$$\frac{1}{2^{2-q}} I_{\mu_\psi}(q, \varepsilon^{1-\nu}) \le K_{\mu_\psi}(q,\varepsilon) \le I_{\mu_\psi}(q,\varepsilon), \tag{36}$$

where the left inequality holds for ε small enough ($\varepsilon \le \varepsilon(\nu)$). As a consequence

$$\frac{1}{1-q}\liminf_{\varepsilon\to 0}\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} = D^-_{\mu_\psi}(q) \quad \text{and} \quad \frac{1}{1-q}\limsup_{\varepsilon\to 0}\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} = D^+_{\mu_\psi}(q). \tag{37}$$

Remark 4.3. This lemma remains true if we replace the boundedness assumption on the support of μ_ψ (H1) by assumption (H2) or (H3) given in Appendix B. In [3] we show that $\mu_\psi \in P_1(\mathbb{R})$ (see [24] for a definition) is also a sufficient condition for (37) to be true. But it is not clear for us whether or not Lemma 4.3 (and more precisely (37)) is true in whole generality as it is stated in the appendix of [15]. However, it comes out from Hypotheses (H1), (H2), (H3) and also from [3] that the integrals $K_{\mu_\psi}(q,\varepsilon)$ and $I_{\mu_\psi}(q,\varepsilon)$ do lead to the same dimension numbers $D^\pm_{\mu_\psi}(q)$ for a very large class of measure. See Remark 5.1.

Proof of Lemma 4.3.

Throughout this proof we shall denote $B(x,\varepsilon) = [x-\varepsilon, x+\varepsilon]$. One has

$$I_{\mu_\psi}(q,\varepsilon) = \int \mu_\psi(B(x,\varepsilon))^{q-1}\mathrm{d}\mu_\psi(x) = \int \mathrm{d}\mu_\psi(x)\left(\int \chi_{[-1,1]}\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}\mu_\psi(x)\right)^{q-1}.$$

Since, by (11), $\chi_{[-1,1]}(w) \leq R(w)$, and $q-1<0$, one has $K_{\mu_\psi}(q,\varepsilon) \leq I_{\mu_\psi}(q,\varepsilon)$. So we need to get the lower bound of (36). First notice that

$$\begin{aligned} b^{(R)}(x,\varepsilon) &= \int \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) \\ &= \int_{\left|\frac{x-y}{\varepsilon}\right|<\varepsilon^{-\nu}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) + \int_{\left|\frac{x-y}{\varepsilon}\right|\geq\varepsilon^{-\nu}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) \\ &\leq \mu_\psi(B(x,\varepsilon^{1-\nu})) + \sup_{|w|\geq\varepsilon^{-\nu}} R(w), \end{aligned}$$

where we used $R(w) \leq 1$. Let $\overline{R}(z) = \sup_{|w|\geq z} R(w)$. Define also

$$A_\nu^{(R)}(\varepsilon) = \{x \in \mathrm{supp}\mu_\psi | \mu_\psi(B(x,\varepsilon^{1-\nu})) \geq \overline{R}(\varepsilon^{-\nu})\} \quad \text{and} \quad \tilde{A}_\nu^{(R)}(\varepsilon) = \left(A_\nu^{(R)}(\varepsilon)\right)^c \cap \mathrm{supp}\mu_\psi.$$

Thus, with these notations, one has

$$\begin{aligned} K_{\mu_\psi}(q,\varepsilon) &= \int \mathrm{d}\mu_\psi(x) b^{(R)}(x,\varepsilon)^{q-1} \geq \int_{A_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x) b^{(R)}(x,\varepsilon)^{q-1} \\ &\geq \int_{A_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x) \Big(2\mu_\psi(B(x,\varepsilon^{1-\nu}))\Big)^{q-1} \\ &= \frac{1}{2^{1-q}} \left(I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - \int_{\tilde{A}_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x) \mu_\psi(B(x,\varepsilon^{1-\nu}))^{q-1} \right). \end{aligned}$$

Since $R(w)$ decays at $+\infty$ faster than any inverse power, for ε small enough, that is $\varepsilon \leq \varepsilon(\nu)$, one has $\overline{R}(\varepsilon^{-\nu}) \leq \varepsilon^A$. Hence, with $\tilde{A}_\nu(\varepsilon) = \{x \in \mathrm{supp}\mu_\psi | \mu_\psi(B(x,\varepsilon^{1-\nu})) \leq \varepsilon^A\}$,

$$\begin{aligned} 2^{1-q} K_{\mu_\psi}(q,\varepsilon) &\geq I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - \int_{\tilde{A}_\nu(\varepsilon)} \mathrm{d}\mu_\psi(x) \mu_\psi(B(x,\varepsilon^{1-\nu}))^{q-1} \quad (38) \\ &\geq I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - 2D\varepsilon^{1-\nu} \geq \frac{1}{2} I_{\mu_\psi}(q,\varepsilon^{1-\nu}), \quad (39) \end{aligned}$$

if in addition $\varepsilon^{1-\nu} \leq 1/(4D)$ (since $I_{\mu_\psi}(q,\varepsilon^{1-\nu}) \geq 1$). Line (38) we used Lemma 4.1, with the function $g(w) = \chi_{[-1,1]}(w)$. This ends the proof of the first part. The conclusion of the lemma then follows: (39) yields

$$\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} \geq \frac{\log\left(2^{q-2} I_{\mu_\psi}(q,\varepsilon^{1-\nu})\right)}{-\log\varepsilon^{1-\nu}}(1-\nu),$$

for ε small enough. Then take respectively the lim inf and lim sup and notice that the result is valid for all $\nu > 0$ as small as one wants. □

Proof of Theorem 4.2.
Theorem 4.2 is a direct consequence of Theorem 4.1 and of Lemma 4.3. □

As already mentioned at the end of Section 2, our main Theorem, namely Theorem 2.1, is a direct consequence of Theorem 3.1 (Section 3) and of Theorem 4.2.

We end this section with a few words concerning the case where, as in [18] and [20], one assumes some further properties on the spatial behaviour of the eigenfunctions of H. Suppose that for some constant C, independent of the energy x, and for $\gamma \leq d$ one knows that: $\sum_{|n|\leq N} |u(n,x)|^2 \leq CN^\gamma$, for μ_ψ *a.e* x (Hypothesis (HS)), then one can not only get a better result taking $\beta = p/\gamma$ instead of $\beta = p/d$, but it turns out that one can push one step further our approach to reach the dimension numbers $D^\pm_{\mu_\psi}(1-\beta)$. The gain takes place in Inequality (18), where a direct estimate is made possible thanks to Hypothesis (HS). This thereby enable us to avoid the Cauchy-Schwarz inequality we made in order to split the integral into a spatial part and a spectral part. This therefore leads to the lower bound $\langle\langle |X|^p\rangle\rangle_{\psi,T} \geq CL'_\psi(T,\varphi) \equiv C\frac{\|\varphi\|^{2+2\beta}}{U_{\varphi,\psi}(T)^\beta}$. Then the same technic as the one used to prove Theorem 4.1 supplies the following result.

Theorem 4.3. *Suppose in addition to the hypotheses of Theorem 2.1 that the spatial Hypothesis (HS) above holds for some γ and constant C. Then, if $\beta = p/\gamma < 1$, one has*

$$\liminf_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} \geq \liminf_{T\to\infty} \frac{\log L'_\psi(T)}{\log T} = \beta D^-_{\mu_\psi}(1-\beta),$$

and

$$\limsup_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} \geq \limsup_{T\to\infty} \frac{\log L'_\psi(T)}{\log T} = \beta D^+_{\mu_\psi}(1-\beta).$$

Further results concerning Theorem 4.3 will be found in the forthcoming article [3]. In particular we shall discuss how far Hypothesis (HS) is relevant, and also provide a version of this theorem that holds for all β (and not only $\beta < 1$).

5 Appendices

Appendix A: Complement of Section 2.

For the reader convenience, we provide the proof of statements ii) and iii) of Proposition 2.1.

Proof of ii) of Proposition 2.1. We first note that from convex Jensen Inequality, and since $q-1<0$, we have, for all set $A \subset \mathbb{R}$

$$\begin{aligned} I_\mu(q,\varepsilon/2)^{\frac{1}{q-1}} &= \left(\int_{\mathbb{R}} \mu(x-\varepsilon/2, x+\varepsilon/2)^{q-1}\mathrm{d}\mu(x)\right)^{\frac{1}{q-1}} \leq \left(\int_A \mu(x-\varepsilon, x+\varepsilon)^{q-1}\mathrm{d}\mu(x)\right)^{\frac{1}{q-1}} \\ &\leq \int_A \mu(x-\varepsilon, x+\varepsilon)\mathrm{d}\mu(x)\ . \end{aligned} \tag{40}$$

Consider now, for all $\nu \in (0,1)$, the set

$$A^{(H)}_\nu(\varepsilon) \equiv \{x\in\mathbb{R} \mid \mu(x-\varepsilon,x+\varepsilon) < \varepsilon^{d_H(\mu)-\nu}\}.$$

We have

$$A^{(H)}_\nu \equiv \liminf_{\varepsilon\to 0} A^{(H)}_\nu(\varepsilon) \supset \{x\in\mathbb{R} \mid \underline{d}_\mu(x) > d_H(\mu)-\nu\}.$$

By the definition of $d_H(\mu)$ given Line (5) we get $\mu(\{x \in \mathbb{R} \mid \underline{d}_\mu(x) > d_H(\mu) - \nu\}) > 0$. Furthermore, $\liminf_{\varepsilon\to 0} \mu(A_\nu^{(H)})(\varepsilon) \geq \mu(A_\nu^{(H)}) \geq \mu(\{x \in \mathbb{R} \mid \underline{d}_\mu(x) > d_H(\mu) - \nu\}) > 0$. Thus, letting $A = A_\nu^{(H)}(\varepsilon)$ in (40), we obtain

$$D_\mu^-(q) = \liminf_{\varepsilon\to 0} \frac{\log I_\mu(q, \varepsilon/2)^{\frac{1}{q-1}}}{\log(\varepsilon/2)} \geq \liminf_{\varepsilon\to 0} \frac{\log\left(\mu(A_\nu^{(H)}(\varepsilon))\varepsilon^{d_H(\mu)-\nu}\right)}{\log(\varepsilon/2)} = d_H(\mu) - \nu\ .$$

Since the above inequality is valid for all $\nu \in (0,1)$, ii) of Proposition 2.1 is proven.

Proof of iii) of Proposition 2.1. We define, for $\varepsilon_k = \mathrm{e}^{-k}$

$$A_\nu^{(P)}(\varepsilon_k) \equiv \{x \in \mathbb{R} \mid \mu(x - \varepsilon_k, x + \varepsilon_k) < \varepsilon_k^{d_P(\mu)-\nu}\}\ .$$

Since $\lim_{k\to\infty} \log\varepsilon_k / \log\varepsilon_{k+1} = 1$, we have $\limsup_{k\to\infty} \log\mu(x - \varepsilon_k, x + \varepsilon_k)/\log\varepsilon_k = \limsup_{\varepsilon\to 0} \log\mu(x - \varepsilon, x + \varepsilon)/\log\varepsilon$. Thus we get

$$\begin{aligned} A_\nu^{(P)} &\equiv \limsup_{k\to\infty} A_\nu^{(P)}(\varepsilon_k) \\ &\supset \{x \in \mathbb{R} \mid \limsup_{k\to\infty} \frac{\mu(x - \varepsilon_k, x + \varepsilon_k)}{\log\varepsilon_k} > d_P(\mu) - \nu\} \\ &= \{x \in \mathbb{R} \mid \overline{d}_\mu(x) > d_P(\mu) - \nu\}. \end{aligned}$$

Therefore, from the definition of $d_P(\mu)$ given Line (5) and the above inclusions, we get $\mu(A_\nu^{(P)}) > 0$. Using the Borel Cantelli lemma (as done in [14]) implies that $\sum_k \mu(A_\nu^{(P)}(\varepsilon_k)) = \infty$ and thus, there exists a subsequence $\varepsilon_{k(n)} \searrow 0$ of ε_k such that $\mu(A_\nu^{(P)}(\varepsilon_{k(n)})) \geq k(n)^{-2} = (\log\varepsilon_{k(n)})^{-2}$. Thus, letting $A = A_\nu^{(P)}(\varepsilon_{k(n)})$ in (40), we obtain

$$D_\mu^+(q) = \limsup_{\varepsilon\to 0} \frac{\log I_\mu(q, \varepsilon)^{\frac{1}{q-1}}}{\log\varepsilon} \geq \lim_{n\to\infty} \frac{\log\left(\mu(A_\nu^{(P)}(\varepsilon_{k(n)}))\varepsilon_{k(n)}^{d_P(\mu)-\nu}\right)}{\log\varepsilon_{k(n)}} = d_P(\mu) - \nu\ .$$

Again, since the result is true for all $\nu \in (0,1)$, statement iii) of Proposition 2.1 is proven. □

Appendix B: Relation to other lower bounds.

Getting a growth exponent in terms of fractal dimensions $D_{\mu_\psi}^\pm(q)$ is not specific to our lower bound $L_\psi(T)$ (8). It is also possible to get such relations from the lower bounds formerly derived, either directly (Barbaroux-Tcheremchantsev's lower bound $L_2(T)$ below), either improving them (Guarneri-Combes-Last's lower bound, improved by optimizing in φ for each single T, see $L_1(T)$ below). We shall briefly explain this point in this section, and thereby propose to the reader a link between this work and the former methods.

Since we shall focus on the relation between those lower bounds and the fractal dimensions $D_{\mu_\psi}^\pm(q)$, and for a sake of simplicity, we shall only consider vectors φ of the form $\chi_\Omega(H)\psi$ instead of general $\varphi \in \mathcal{H}_\psi$ as in (8).

We first consider the lower bound that appears in [5]. For some constant $C(\psi,p)>0$, it is shown that for all $T>0$,

$$\langle\langle |X|^p\rangle\rangle_{\psi,T}\geq C(\psi,p)L_2(T),\quad \text{with}\quad L_2(T)=\sup\left\{L_2(\varphi,T):=\frac{\|\varphi\|^{2+8\beta}}{S_\varphi(T)^\beta},\ \varphi=\chi_\Omega(H)\psi\right\},$$

where

$$S_\varphi(T)=\int_{\mathbb{R}^2}\mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y)R(T(x-y)).$$

Here $R(w)$ can be chosen as the usual gaussian $e^{-w^2/4}$. Then one can mimic the proof of Theorem 3.1, and relate the quantity $L_2(\varphi,T)\equiv\|\varphi\|^{2+8\beta}/S_\varphi(T)^\beta$ to the integral $K_{\mu_\psi}(\frac{1+2\beta}{1+3\beta},T^{-1})$. More precisely, the same kind of Hölder Inequalities as in (29)-(30) supplies an upper bound for $L_2(T)$ and using a set $\Omega(r)$ in the same spirit as previously yields the lower bound, again up to a logarithmic factor. The following theorem then holds:

Theorem 5.1. *Under the same hypotheses as in Theorem 2.1, one has*

$$\liminf_{T\to\infty}\frac{\log L_2(T)}{\log T}=\beta D^-_{\mu_\psi}\left(\frac{1+2\beta}{1+3\beta}\right),\quad \text{and}\quad \limsup_{T\to\infty}\frac{\log L_2(T)}{\log T}=\beta D^+_{\mu_\psi}\left(\frac{1+2\beta}{1+3\beta}\right).$$

The second lower bound we want to discuss in this section is the improved version, using [5], of the first kind of quantity that has been considered in order to bound from below $\langle\langle |X|^p\rangle\rangle_{\psi,T}$, and that comes from Guarneri-Combes-Last [13, 7, 22]. Roughly, the basic idea is to take into account the function $G_\varphi(T)\equiv\mu_\varphi-\operatorname{ess\,sup}\int\mathrm{d}\mu_\varphi(y)R(T(x-y))$ where $R(w)=\exp(-w^2/4)$ (notice that $G_\varphi(T)$ will be smaller than $CT^{-\alpha}$ if μ_φ is uniformly α-Hölder continuous, see (3.10) in [22]). One then uses $G_\varphi(T)$ in order to bound $B_\varphi(T,N)$ defined as in (12) but with $1/T\int_0^T$ rather than $1/T\int_0^{+\infty}h(t/T)$. For instance, using $2|v(n,x)v(n,y)|\leq|v(n,x)|^2+|v(n,y)|^2$,

$$\begin{aligned}B_\varphi(T,N)&\leq C\int_{\mathbb{R}}\int_{\mathbb{R}}\mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y)R((x-y)T)\sum_{|n|\leq N}|v(n,x)v(n,y)|\\&\leq CG_\varphi(T)\int_{\mathbb{R}}\mathrm{d}\mu_\varphi(x)\sum_{|n|\leq N}|v(n,x)|^2\leq C'G_\varphi(T)N^d\end{aligned}$$

Following [22, 5], this leads to the lower bound

$$\langle\langle |X|^p\rangle\rangle_{\psi,T}\geq C(\psi,p)L_1(T),\quad \text{with}\quad L_1(T)=\sup\left\{L_1(\varphi,T):=\frac{\|\varphi\|^{2+4\beta}}{G_\varphi(T)^\beta},\ \varphi=\chi_\Omega(H)\psi\right\}.$$

One should compare the obtained lower bound $L_1(T)$ to the expression (6.10) in [22].

Then one can again mimic the proof of Theorem 4.1, and relate the quantity $L_1(T)$ to the integral $K_{\mu_\psi}(\frac{1+\beta}{1+2\beta},T^{-1})$. This leads to the following theorem:

Theorem 5.2. *Under the same hypotheses as in Theorem 2.1, one has*

$$\liminf_{T\to\infty}\frac{\log L_1(T)}{\log T}=\beta D^-_{\mu_\psi}\left(\frac{1+\beta}{1+2\beta}\right),\quad \text{and}\quad \limsup_{T\to\infty}\frac{\log L_1(T)}{\log T}=\beta D^+_{\mu_\psi}\left(\frac{1+\beta}{1+2\beta}\right).$$

One should also compare the expression of $L_1(T)$ and $L_2(T)$ to the expression of $L_\psi(T)$ given Line (9), that is where as above the supremum is taken over the set of vectors $\varphi = \chi_\Omega(H)\psi$, namely the supremum of $\|\varphi\|^{2+4\beta}/U_{\varphi,\psi}(T)^\beta$. One may have the right to wonder whether there are some links between these quantities, and also whether $L_\psi(T)$ is effectively a better lower bound than $L_1(T)$ and $L_2(T)$. Theorem 5.3 below answers to these questions. This theorem is actually a consequence of Theorems 4.1, 5.2 and 5.1, and of the nonincreasing property of the functions $D^\pm_{\mu_\psi}(q)$ (point i) of Proposition 2.1).

Theorem 5.3. *Under the same hypotheses as previously, one has*

$$\liminf_{T\to\infty} \frac{\log L_\psi(T)}{\log T} \geq \liminf_{T\to\infty} \frac{\log L_1(T)}{\log T} \geq \liminf_{T\to\infty} \frac{\log L_2(T)}{\log T},$$

and

$$\limsup_{T\to\infty} \frac{\log L_\psi(T)}{\log T} \geq \limsup_{T\to\infty} \frac{\log L_1(T)}{\log T} \geq \limsup_{T\to\infty} \frac{\log L_2(T)}{\log T}.$$

Remarks.
1) It is worth to point out that such a comparison is made possible thanks to the upper bounds obtained for $L_\psi(T)$, $L_1(T)$ and $L_2(T)$; upper bounds that are for the first time derived for such quantities.
2) Theorem 5.3 tells us that it is worth to deal with the crossed term in (4) while one developps $B_\psi(T,N)$ with $\psi = \varphi + \chi$. Dealing with the crossed term does lead to an improvement, with regards to the former approaches (G-C-L) and (BT) where this term wasn't well treated.
3) One can show that $L_\psi(\varphi,T) \geq c_1 L_1(\varphi,T)$ with some constant $c_1 > 0$ uniform in T,φ. The latter is not true as one compares $L_\psi(\varphi,T)$ and $L_2(\varphi,T)$ for some φ. We confess that the inequalities involving $L_2(T)$ in Theorem 5.3 are quite a surprise for us, since we were expecting the exponents of $L_2(T)$ to be bigger than those of $L_1(T)$.
4) We believe a stronger version of Theorem 5.3 to be true, namely: $L_\psi(T) \geq c_1 L_1(T) \geq c_2 L_2(T)$.

Appendix C: Alternative hypotheses concerning the measure.

Lemma 4.1 is the only technical lemma where we used the assumption (H1), that is the boundedness of the support of the measure μ_ψ. We shall below propose two versions of this Lemma extended to more general hypotheses, which in particular enable one to deal with non compactly supported spectrum. Note that since Lemma 4.2 and 4.3 are proven independently of the assumption on μ_ψ, provided Equation (31) is fulfilled, they remain valid if one supposes for μ_ψ (H2) of Proposition 5.1 or (H3) of Proposition 5.2 below instead of the assumption (H1) of Theorem 2.1. We also refer the reader to a forthcoming article [3] for further results.

Proposition 5.1 deals with values q in $(0,1)$, and Proposition 5.2 with values in $(-\infty,1)$. In the next two statements $b(x,\varepsilon)$ is either $\int \mathrm{d}\mu(y)\chi_{[-1,1]}((x-y)/\varepsilon)$ or $\int \mathrm{d}\mu(y)R((x-y)/\varepsilon)$.

Proposition 5.1. *Let $0 < q < 1$ and $b(x,\varepsilon)$ defined as in Lemma 4.1. Assume*

$$\textbf{(H2)} \quad \text{For some } \lambda > \frac{1-q}{q}, \text{ one has } \int_{\mathbf{R}} |x|^\lambda d\mu(x) < +\infty.$$

Then there exists $A = A(q, \lambda) > 0$ such that

$$\int_{b(x,\varepsilon)\leq\varepsilon^A} b(x,\varepsilon)^{q-1} d\mu(x) \leq C(q,\lambda)\varepsilon,$$

Moreover, $D^{\pm}_{\mu}(q) \leq 1$. As a consequence the conclusions of Lemmas 4.2 and 4.3 hold.

Proposition 5.2. *Let q be in $(-\infty, 1)$ and $b(x,\varepsilon)$ defined as in Lemma 4.1. Assume*

$$\textbf{(H3)} \quad \text{For some } q' < q, \text{ one has } \; D^{+}_{\mu}(q') < \infty.$$

Then there exists $A = A(q, q') > 0$ such that the following holds:

$$\int_{b(x,\varepsilon)\leq\varepsilon^A} b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) \leq \varepsilon. \tag{41}$$

As a consequence the conclusions of Lemmas 4.2 and 4.3 hold.

Remark 5.1.
In particular Proposition 5.2 says that if $q^* = \inf\{q, D^{+}_{\mu}(q) < +\infty\}$ (possibly $q^* = -\infty$), then for any $q \in]q^*, 1[$ the integrals $K_{\mu_\psi}(q,\varepsilon)$ and $I_{\mu_\psi}(q,\varepsilon)$ give rise to the same dimensions $D^{\pm}_{\mu}(q)$. In other words $K_{\mu_\psi}(q,\varepsilon)$ and $I_{\mu_\psi}(q,\varepsilon)$ can lead to different values of fractal dimensions only if $D^{+}_{\mu}(q) = +\infty$ or also at the critical point $q = q^*$.

Proof of Proposition 5.1.
One starts as in Lemma 4.1. Define

$$\Omega(\varepsilon) = \{x \in \mathrm{supp}\mu : b(x,\varepsilon) \leq \varepsilon^A\},$$

and $I_j = [(j-1/2)\varepsilon, (j+1/2)\varepsilon) = [x_j - \varepsilon/2, x_j + \varepsilon/2)$, where $x_j \equiv j\varepsilon$, $j \in \mathbb{Z}$. For any j such that $\mu(I_j \cap \Omega(\varepsilon)) \neq 0$ define $\underline{b}_j = \inf\{b(x,\varepsilon),\ x \in I_j \cap \Omega(\varepsilon)\}$. As in the proof of Lemma 4.1 note that $\mu(I_j \cap \Omega(\varepsilon)) \leq \underline{b}_j \leq \varepsilon^A$ and one has

$$\int_{\Omega(\varepsilon)} b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) \leq \sum_j \underline{b}_j^q.$$

But now one can rewrite this summation as follows:

$$\sum_j \underline{b}_j^q = \sum_{k=0}^{+\infty} S_k, \quad S_k \equiv \sum_{j\in J_k} \underline{b}_j^q, \tag{42}$$

where $J_k = \{j | I_j \subset [e^k, e^{k+1}) \cup [-e^{k+1}, -e^k)\}$ for $k > 0$ and $J_0 = \{j | I_j \subset [-e, e)\}$. To be rigourous Equality (42) holds if one replaces in the definition of the sets J_k the quantity e^k by $[e^k/\varepsilon]\varepsilon + \varepsilon/2$, where $[\cdot]$ stands for the integer part, but for a sake of simplicity we shall use the definition of J_k given above.

Let $\gamma \in (0,1)$. For $k \geq 0$, by a Hölder inequality, and using that $\mathrm{Card} J_k \leq 2(e^{k+1}-e^k)/\varepsilon \leq 4e^k/\varepsilon$,

$$\begin{aligned} S_k &= \sum_{j\in J_k} \underline{b}_j^q = \sum_{j\in J_k} \underline{b}_j^{\gamma q}\underline{b}_j^{(1-\gamma)q} \leq \left(\sum_{j\in J_k}\underline{b}_j\right)^{\gamma q}\left(\sum_{j\in J_k}\underline{b}_j^{\frac{(1-\gamma)q}{1-\gamma q}}\right)^{1-\gamma q} \\ &\leq \varepsilon^{A(1-\gamma)q}\left(\frac{4e^k}{\varepsilon}\right)^{1-\gamma q}\left(\sum_{j\in J_k}\underline{b}_j\right)^{\gamma q} = 4\varepsilon^{A(1-\gamma)q-(1-\gamma q)}e^{k(1-\gamma q)}\left(\sum_{j\in J_k}\underline{b}_j\right)^{\gamma q} \end{aligned}$$

Assume that

$$\sum_{j\in J_k}\underline{b}_j \leq C(\lambda)e^{-k\lambda}. \tag{43}$$

Then the proof can be completed as follows:

$$\sum_j \underline{b}_j^q \leq 4C(\lambda)\varepsilon^{A(1-\gamma)q-(1-\gamma q)}\sum_{k=0}^{+\infty} e^{-k(\lambda\gamma q-(1-\gamma q))} \leq C(q,\lambda)\varepsilon, \tag{44}$$

provided $\lambda > (1-q)/q$, γ close enough to 1, and finally $A = (2-\gamma q)/(1-\gamma)q$. Moreover, taking $A = 1$ in (44), one gets that $\int_{b(x,\varepsilon)\leq\varepsilon} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x) \leq C(q,\lambda)\varepsilon^{q-1}$. The latter implies (see the proof of Lemma 4.1) that $D_\mu^\pm(q) \leq 1$.

We are then left with the proof of Estimate (43). This will be a consequence of Hypothesis (H2) and of the Chebychev Inequality. Indeed from these two points one derives the following estimate:

$$\mu(|x| \geq D) \leq C(\lambda)D^{-\lambda}, \tag{45}$$

for some $C(\lambda) \leq +\infty$. Moreover, observe that $\underline{b}_j \leq b(x_j,\varepsilon)$ (where $x_j = j\varepsilon$). Indeed, if $x_j \in \Omega(\varepsilon)$, clearly,

$$\underline{b}_j = \inf\{b(x,\varepsilon),\ x \in I_j \cup \Omega(\varepsilon)\} \leq b(x_j,\varepsilon).$$

On the other hand, if $x_j \notin \Omega(\varepsilon)$, then $b(x_j,\varepsilon) > \varepsilon^A \geq \underline{b}_j$. Therefore, in any cases,

$$\sum_{j\in J_k}\underline{b}_j \leq \sum_{j\in J_k} b(x_j,\varepsilon) \tag{46}$$

First consider the case where $b(x,\varepsilon) = \mu(x-\varepsilon, x+\varepsilon)$. Since $\cup_{j\in J_k}[x_j-\varepsilon, x_j+\varepsilon)$ covers, roughly speaking, two times the set $\{e^{k+1} \geq |x| \geq e^k\}$, the r.h.s of (46) is then bounded from above by $2\mu(e^{k+1} \geq |x| \geq e^k) \leq 2\mu(|x| \geq e^k)$. Thus, combining this with (45) one gets the desired result.

Estimate (43) is a little bit more tricky in the second case, that is when $b(x,\varepsilon)$ is defined with the function R. In this case, the summation other the x_j cannot be interpreted as a summation of balls $[x_j-\varepsilon, x_j+\varepsilon)$, which led previously to $\mu(e^{k+1} \geq |x| \geq e^k)$. One has to handle with the remainder part of R outside $[-1,1]$.

To that aim we shall consider annulars centered at points $x_j = j\varepsilon$, $j \in J_k$. As previously we shall denote $\overline{R}(x) = \sup_{|w|\geq x} R(w)$. Note that as the function R, $\overline{R}$ decays faster than any inverse power. With some abuse of notation, one has, using first (46),

$$\begin{aligned}\sum_{j\in J_k}\underline{b}_j &\leq \sum_{j\in J_k} b(x_j,\varepsilon)\leq \sum_{j\in J_k}\int \mathrm{d}\mu(y)R\left(\frac{x_j-y}{\varepsilon}\right)\\ &\leq \sum_{j\in J_k}\left(\sum_{n=0}^{(e^{k-2}/\varepsilon)-1}\int_{n\varepsilon\leq|x_j-y|<(n+1)\varepsilon}\mathrm{d}\mu(y)R\Big(\frac{x_j-y}{\varepsilon}\Big)+\int_{e^{k-2}\leq|x_j-y|}\mathrm{d}\mu(y)R\Big(\frac{x_j-y}{\varepsilon}\Big)\right)\\ &\leq \sum_{n=0}^{(e^{k-2}/\varepsilon)-1}\left(\overline{R}(n)\sum_{j\in J_k}\mu(n\varepsilon\leq|x_j-y|<(n+1)\varepsilon)\right)+\mathrm{Card}J_k\,\overline{R}(e^{k-2}/\varepsilon).\end{aligned}$$

Notice that for any fixed $n\leq(e^{k-2}/\varepsilon)-1$ the union of the annulars $n\varepsilon\leq|x_j-y|<(n+1)\varepsilon$, $j\in J_k$, is included in $\{|y|\geq e^{k-1}\}$, covering it at most twice. Therefore, and using again that $\mathrm{Card}J_k\leq 4e^k/\varepsilon$,

$$\begin{aligned}\sum_{j\in J_k}\underline{b}_j &\leq 2\sum_{n=0}^{(e^k/\varepsilon)-1}\overline{R}(n)\mu(|y|\geq e^{k-1})+\frac{4e^k}{\varepsilon}\overline{R}(e^{k-2}/\varepsilon)\\ &\leq C\mu(|x|\geq e^{k-1})+4e^2\frac{e^{k-2}}{\varepsilon}\overline{R}(e^{k-2}/\varepsilon)\leq C(q,\lambda)e^{-k\lambda}.\end{aligned}$$

The latter inequality comes from (45) and the fast decay property of $\overline{R}$. The proof is completed. □

Proof of Proposition 5.2. For $A>0$ to be fixed later on, and for all $\varepsilon\in(0,1)$ let

$$\Omega(\varepsilon):=\{x\mid b(x,\varepsilon)\leq\varepsilon^A\},\quad\text{and also}\quad\Omega_k(\varepsilon):=\left\{x\mid \frac{\varepsilon^A}{e^{k+1}}<b(x,\varepsilon)\leq\frac{\varepsilon^A}{e^k}\right\},$$

for any $k\in\mathbb{N}$. Note that $\Omega(\varepsilon)=\cup_{k\in\mathbb{N}}\Omega_k(\varepsilon)$ and that for all $k\in\mathbb{N}$ we have

$$\Omega_k(\varepsilon)\subset\left\{x\mid\frac{\varepsilon^A/e^k}{b(x,\varepsilon)}\geq 1\right\}.\tag{47}$$

Chebychev Inequality and (47), for any $q'<1$, imply

$$\mu(\Omega_k(\varepsilon))\leq\left(\frac{\varepsilon^A}{e^k}\right)^{1-q'}\int_{\mathrm{supp}\mu}b(x,\varepsilon)^{q'-1}\mathrm{d}\mu(x)\leq\left(\frac{\varepsilon^A}{e^k}\right)^{1-q'}I_\mu(q',\varepsilon),\tag{48}$$

and this whatever is the definition of $b(x,\varepsilon)$: with $\chi_{[-1,1]}(w)$ or $R(w)$ - because of the choice of R Line (11). Thus

$$\int_{b(x,\varepsilon)\leq\varepsilon^A}b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)\leq\sum_{k\in\mathbb{N}}\int_{\Omega_k(\varepsilon)}b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)\leq\sum_{k\in\mathbb{N}}\left(\frac{e^{k+1}}{\varepsilon^A}\right)^{1-q}\mu(\Omega_k(\varepsilon)).$$

Using now (48),

$$\int_{b(x,\varepsilon)\leq\varepsilon^A}b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)\leq e^{1-q}\sum_{k\in\mathbb{N}}\left(\frac{\varepsilon^A}{e^k}\right)^{q-q'}I_\mu(q',\varepsilon)=C(q,q')\varepsilon^{A(q-q')}I_\mu(q',\varepsilon),\tag{49}$$

for some constant $C(q,q')$ and since $q-q'>0$. Now, by hypothesis $D_\mu^+(q')<\infty$, and thus one has for ε small enough: $I_\mu(q',\varepsilon)\le(1/\varepsilon)^{(D_\mu^+(q')+1)(1-q')}$. For those ε, and with any $A\ge[(D_\mu^+(q')+1)(1-q')+1]/(q-q')$, one has

$$\int_{b(x,\varepsilon)\le\varepsilon^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)\le C\varepsilon.$$

□

Appendix D: An example of finite pure point measure with nontrivial generalized fractal dimensions.

For $\lambda>1$ and $\alpha>0$, let $a_n=a/n^\lambda$, $x_n=1/n^\alpha$, where $a>0$ is a normalization constant. We define the finite pure point probability measure $\mu=\sum_{n=1}^\infty a_n\delta_{x_n}$. As the measure μ has a bounded support, $0\le D_\mu^\pm(q)\le 1$ for any $q\in(0,1)$ - see Point iv) of Proposition 2.1. We denote by $B(x,\varepsilon)$ the closed ball of center x and radius ε. For any given $\varepsilon>0$, take N to be the integer part of $\varepsilon^{-1/(1+\alpha)}$. Thus, there exists a constant $c>0$ uniform in N (and ε) such that for all $n\le cN$, $\mu(B(x_n,\varepsilon))=\mu(\{x_n\})=a_n$. Let $q<1/\lambda$. Then we have

$$I_\mu(q,\varepsilon)=\sum_n a_n\mu(B(x_n,\varepsilon))^{q-1}\ge\sum_{n=1}^{cN}\mu(B(x_n,\varepsilon))^{q-1}a_n=\sum_{n=1}^{cN}a_n^q\sim C\,\varepsilon^{-\frac{1-q\lambda}{1+\alpha}}\ . \tag{50}$$

Therefore

$$D_\mu^\pm(q)\ge\frac{1-q\lambda}{(1-q)(1+\alpha)}.$$

This implies that for $0<q<1/\lambda$, one obtains strictly positive generalized Rényi dimensions $D_\mu^\pm(q)$ for the pure point measure μ. Moreover, by taking q and α small, one can render these dimensions as close to 1 as one wants.

The estimate (50) is actually also valid for $q<0$ (and one can show that the dimensions $D_\mu^\pm(q)$ are finite for any $q<0$). However, the behaviour of the fractal dimensions is rather strange: they can be greater than 1 and the bigger is λ, the bigger are $D_\mu^\pm(q)$ as $q\to-\infty$:

$$\liminf_{q\to-\infty}D_\mu^\pm(q)\ge\frac{\lambda}{1+\alpha}>1,$$

if α is small enough. Furthermore note that if μ is now defined with $a_n=ae^{-\lambda n}$, then $D_\mu^\pm(q)=+\infty$ whenever $q<0$. This shows that one should be very cautious when considering possible physical applications of $D_\mu^\pm(q)$ with $q<0$. Further discussions will be found in [3].

□

References

[1] Barbaroux J-M., Combes J-M., Montcho R., *Remarks on the relation between quantum dynamics and fractal spectra*, J. Math. Anal and Appl. **213** (1997), 698-722.

[2] Barbaroux J-M., Germinet F., Tcheremchantsev S., *Intermittency in Quantum Dynamics*, to appear in C.R. Acad. Sci, Paris, Série I (1999), or mp-arc 99-351.

[3] Barbaroux J-M., Germinet F., Tcheremchantsev S., *Fractal dimensions and the Phenomenon of Intermittency in Quantum Dynamics II*, In preparation.

[4] Barbaroux J-M., Schulz-Baldes H., *Anomalous quantum transport in presence of self-similar spectra*, Ann. Inst. Henri Poincaré, **Vol 71**, Numero 5 (1999) 1-21.

[5] Barbaroux J-M., Tcheremchantsev S., *Universal Lower Bounds for Quantum Diffusion*, to appear in J. Funct. Anal. (1999).

[6] Bellissard J., Schulz-Baldes H., *Anomalous Transport: A Mathematical Framework*, Rev. Math. Phys. **10** (1998), 1-46.

[7] Combes J-M., *Connection between quantum dynamics and spectral properties of time evolution operators* in "Differential Equations and Applications in Mathematical Physics", Eds. W.F. Ames, E.M. Harrel, J.V. Herod (Academic Press 1993), 59-69.

[8] Cutler C.D., *Some results on the behaviour and estimation of the fractal dimensions of distributions on attractors*, J. Stat. Phys. **62**, Nos 3/4 (1991), 651-708.

[9] De Bièvre S., Germinet F., *Dynamical Localization for the random dimer model*, Preprint (1999), or mp-arc 99-250.

[10] Del Rio R., Jitomirskaya S., Last Y., Simon B., *Operators with singular continuous spectrum IV: Hausdorff dimensions, rank one pertubations and localization*, J. d'Analyse Math. **69**, 153-200 (1996).

[11] Dunlap D.H., Wu H.-L., Phillips P., *Absence of localization in a random Dimer model*, Phys. Rev. Lett. **65**, 88 (1990).

[12] Falconer K. J., *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics, No 85, Cambridge Univ. Press, UK, 1985.

[13] Guarneri I., *Spectral properties of quantum diffusion on discrete lattices*, Europhys. Lett. 10 (1989), 95-100; *On an estimate concerning quantum diffusion in the presence of a fractal spectrum*, Europhys. Lett. **21** (1993), 729-733.

[14] Guarneri I., Schulz-Baldes H., *Lower bounds on wave packet propagation by packing dimensions of spectral measures*, Math. Phys. Elec. J. **5**, paper 1 (1999).

[15] Guarneri I., Schulz-Baldes H., *Intermittent lower bound on quantum diffusion*, Preprint (1999), or mp-arc 99-342.

[16] Halsey T.C., Jensen M., Kaddanov L., Procaccia I., Schraiman B., *Fractal Measures and their Singularities: the Characterization of Strange stes*, Phys. Rev. A **33** (1986), 1141-1151.

[17] Hentschel H.G.E., Procaccia I., *The Infinite Number of Generalized Dimensions of Fractals and Strange Attractors*, Phys. D **8** (1983), 435-444.

[18] Ketzmerick R., Kruze K., Kraut S., Geisel T., *What Determines the Spreading of a Wave Packet*, Phys. Rev. A **79** (1997), 1959-1963.

[19] Ketzmerick R., Petschel G., Geisel T, *Slow decay of temporal correlations in quantum systems with Cantor spectra*, Phys. Rev. Lett. **69** (1992), 695-698.

[20] Kiselev A., Last Y., *Solutions, spectrum, and dynamics for Schrödinger operators on infinte domains*, Preprint (1999), or mp-arc 99-206.

[21] Kunz H., Souillard B., *Sur le spectre des opérateurs aux différences finies aléatoires*, Commun. Math. Phys. **78**, 201-246 (1980).

[22] Last Y., *Quantum Dynamics and Decomposition of Singular Continuous Spectrum*, J. Funct. Anal **142** (1996), 406-445.

[23] Mantica G. *Quantum intermittency in almost periodic systems derived from their spectral properties*, Physica D **103** (1997), 576-589; *Wave Propagation in Almost-Periodic Structures*, Physica D **109** (1997), 113-127.

[24] Olsen L., *A multifractal Formalism*, Adv. Math. **116**, 82-195 (1995).

[25] Pesin Y.B., *Dimension Theory in Dynamicals Systems: Contemporary Views and Applications*, Univ. Chicago Press, (1996).

[26] Piechon F., *Anomalous Diffusion Properties of Wave Packets on Quasiperiodic Chains*, Phys. Rev. Lett. **76**, n^o23, 4372-4375 (1996).

[27] B. Simon, *Schrödinger semi-groups*, Bull. Amer. Math. Soc. **Vol.7**, n^o3, p.447-526 (1982).

[28] Taylor S. J., Tricot C., *Packing Measure, and its Evaluation for a Brownian Path*, Trans. Amer. Math. Soc. **288**, 679-699 (1985).

[29] Wilkinson M., Austin E.J., *Spectral dimension and dynamics for Harper's equation*, Phys. Rev. B **50** (1994), 1420-1429.