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The Finite Reflection Groups

We classify the finite reflection groups. Our treatment has several advantages over some other treatments—in particular, we avoid computing determinants and the use of the Perron-Frobenius Theorem. The ideas here can be found spread across several sections of Coxeter's *Regular Polytopes*. The only thing missing from our treatment is a construction of the finite groups.

By the norm v^2 of a vector v, we mean $v^2 = v \cdot v$; some people call this the squared norm of v.

1 Preliminaries

A reflection is an isometry of Euclidean space V that leaves a hyperplane (its mirror) fixed pointwise and exchanges the two components of its complement. A reflection group is a group generated by reflections. Suppose W is a finite reflection group. W stabilizes some point of Euclidean space (say, the centroid of the orbit of any point), which we will take to be the origin. W contains only finitely many reflections, and the complement in V of the union of the mirrors falls into finitely many components. We call the closure of any one of these components a Weyl chamber (or just a chamber). A mirror M is said to bound a chamber C if $C \cap M$ has the same dimension as M. The walls of C are the mirrors that bound C. A root of W is a vector r of norm 2 that is orthogonal to some mirror M of W; we sometimes refer to the reflection across M as the reflection in r. We fix one chamber and call it D. For each wall M of D we choose the root associated to M which has positive inner product with each element of the interior of D. We denote these vectors by $r_1, \ldots, r_n$ and call them the simple roots of W. We write R_i for the reflection in r_i, which negates r_i and fixes $r_i^\perp$ pointwise.

Lemma 1.1. *The R_i generate W, which acts transitively on its Weyl chambers.*

Proof: We say that 2 chambers C_1, C_2 are neighbors if they are both bounded by the same mirror M and $C_1 \cap M = C_2 \cap M$; in this case C_1 and C_2 are exchanged by the reflection across M. It is easy to see that any two chambers are equivalent under the equivalence relation generated by the relation of neighborliness. (Proof: choose points in the interiors of the 2 chambers in sufficiently general position that the segment joining them never meets an intersection of 2 mirrors. The sequence of chambers that this segment passes through provides a sequence of neighbors.)

If a subgroup G of W contains the reflections in the walls of a chamber C_1, and C_2 is a neighbor of C_1, then G also contains the reflections in the walls of C_2. Here's why: letting R be the reflection across the common wall of C_1 and C_2, we have $R \in G$ and we observe that the reflections in the walls of C_2 are the conjugates by R of those in the walls of C_1.

Letting G be the group generated by $R_1, \ldots, R_n$, we see that G contains the reflections in the walls of the neighbors of D, and of their neighbors, and so on. That is, G contains all the reflections of W, so equals W. Since any two neighboring chambers are equivalent under W, we also see that W acts transitively on chambers. □

Consider the subgroup H of W generated by the reflections in a pair of distinct simple roots r_i and r_j. In this paragraph we will restrict our attention to the span of r_i and r_j, since H acts trivially on $r_i^\perp \cap r_j^\perp$. Consider the chambers of H; these are even in number since each reflection of H permutes them freely. Furthermore, lemma 1.1 shows that they are all equivalent under H. Letting $2n_{ij}$ be the number of Weyl chambers, we deduce that the 2 mirrors bounding any chamber meet at an angle of π/n_{ij}. Because no mirror of H can cut the Weyl chamber D of W, the mirrors of R_i and R_j must bound the same chamber of H, so their interior angle is π/n_{ij}. Picture-drawing in the plane allows us to determine the angle between r_i and r_j, and we find

$$r_i \cdot r_j = -2\cos(\pi/n_{ij}). \tag{1.1}$$

We have already made the choice $r_i \cdot r_i = 2$, so we set $n_{ii} = 1$ to be consistent with (1.1). Note that the integers n_{ij} determine W: the mutual inner products of any set of vectors in Euclidean space determines them (up to isometry), so the n_{ij} determine the r_i, which determine the R_i, which by lemma 1.1 determine W.

A Coxeter diagram (sometimes just called a diagram) is a simplicial graph with each edge labeled by an integer > 2. The Coxeter diagram Δ_W of W is the diagram whose vertices are the r_i, with r_i and r_j joined by an edge marked with the integer n_{ij} when $n_{ij} > 2$. This definition depends on our choice D of Weyl chamber, but the transitivity of W on its chambers shows that a different choice of chamber leads to essentially the same diagram. We may recover the n_{ij} from Δ_W, so Δ_W determines W. For simplicity, when drawing a Coxeter diagram one omits the numeral 3 from edges that would be so marked.

2 Controlling Δ.

Lemma 2.1. *Suppose $v \in V$ with $v = \sum_{i=1}^{n} v_i r_i$. If $v_i \geq 0$ and are not all 0, then $v^2 > 0$.*

Proof: Since each r_i has positive inner product with each element of the interior of C, so does v. Thus $v \neq 0$ and so $v^2 > 0$. □

A subdiagram of a Coxeter diagram Δ is a diagram whose vertex set is a subset of that of Δ, whose edge set consists of all edges of Δ joining pairs of these vertices, and whose edges are marked by the same numbers as in Δ. If Δ and Δ' are Coxeter diagrams with the same vertex set and with edge markings m_{ij} and n_{ij}, respectively, then we say that Δ' is an increasement of Δ if $n_{ij} \geq m_{ij}$ for all i and j. In terms of the diagrams, Δ' is a (strict) increasement of Δ if Δ' can be obtained from Δ by increasing edge labels or adding edges.

Lemma 2.2. *No diagram appearing in table 1 or table 2, nor any increasement of one, may appear as a subdiagram of Δ_W.*

Proof: Let Δ be a diagram from one of the tables, and Δ' an increasement of Δ that is a subdiagram of Δ_W. Identifying the vertices of Δ and Δ' with (some of) the simple roots r_i, we may construct the vector $v = \sum_i v_i r_i$, where v_i is the (positive) number adjacent to the vertex r_i on the table. One may compute the norm of v from knowledge of the edge labels n_{ij} of $\Delta' \subseteq \Delta_W$. If the edge labels of Δ are m_{ij} then

$$v^2 = \sum_{ij} -2v_i v_j \cos(\pi/n_{ij}) \leq \sum_{ij} -2v_i v_j \cos(\pi/m_{ij}), \tag{2.1}$$

the last inequality holding because Δ' is an increasement of Δ. In each case, computation reveals that the right hand side of (2.1) is at most 0, contradicting lemma 2.1. For reference, $-2\cos(\pi/n)$ equals 0, -1, $-\sqrt{2}$, $-\phi$ and $-\sqrt{3}$, for $n = 2$, 3, 4, 5 and 6, respectively, and $\phi = (1+\sqrt{5})/2 = 1.618\ldots$ is the golden mean.

The compuations are not even very tedious. For $\Delta = H_3$ or H_4 they are simplified by using the fact $\phi^2 = \phi + 1$. In all other cases (i.e., with Δ from table 1), the right hand side of (2.1) vanishes; to prove this one may compute inner products with the n_{ij} replaced by the m_{ij} and show that v is orthogonal to each r_i. Almost all cases are resolved by the following observation: if all the edges of Δ incident to r_i are marked 3 then $v \cdot r_i = 0$ just if twice the r_i label equals the sum of the labels of its neighbors. □

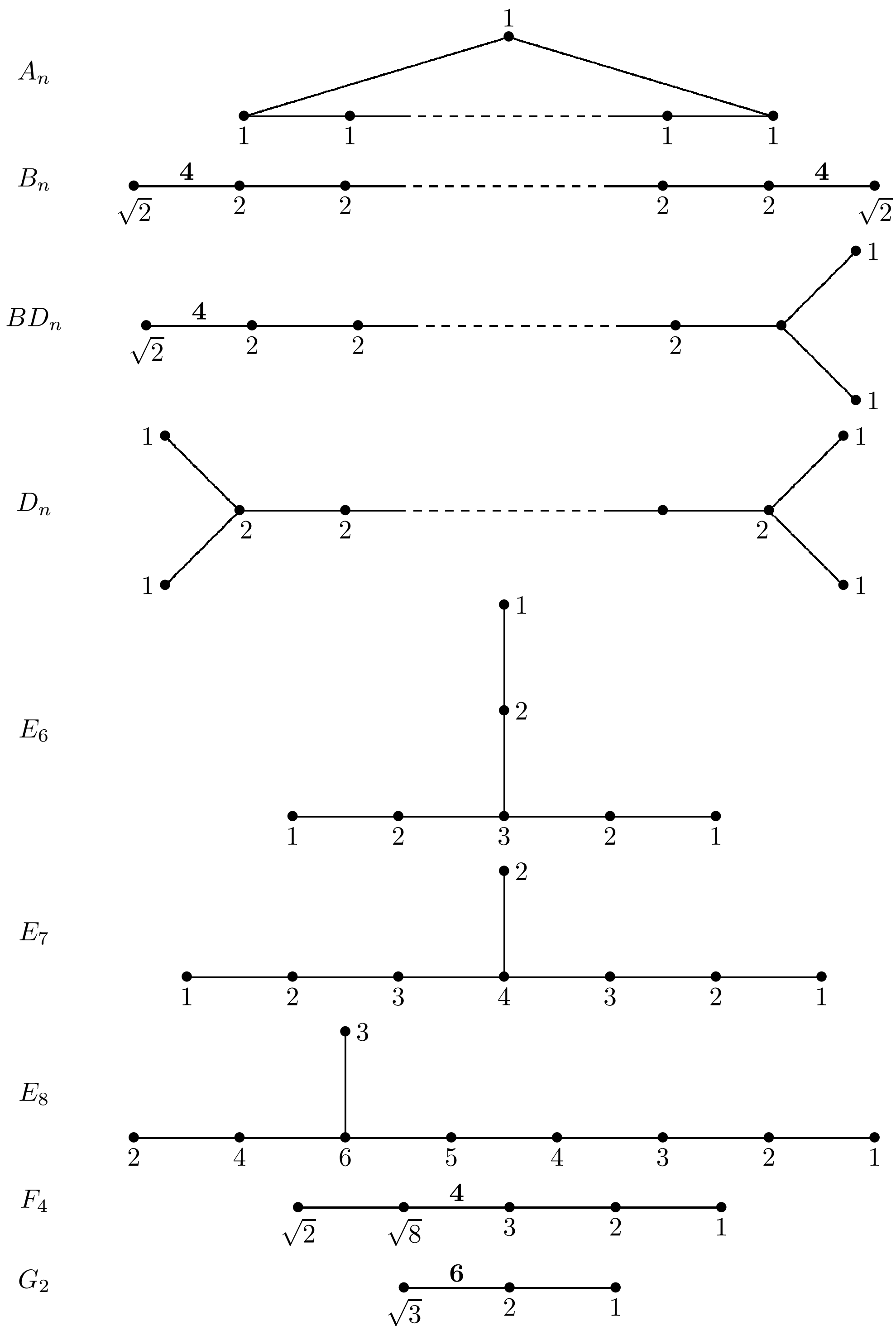

Table 1. A list of “affine” Coxeter diagrams. The numbers next to the vertices are used in the proof of lemma 2.2. A diagram X_n has $n+1$ vertices.

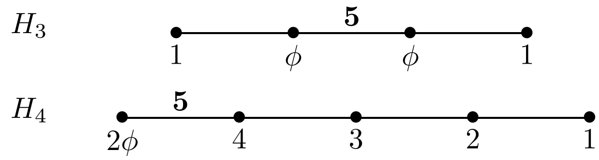

Table 2. Two examples of "hyperbolic" Coxeter diagrams. The numbers next to the vertices are used in the proof of lemma 2.2; $\phi = (1+\sqrt{5})/2$ is the golden mean.

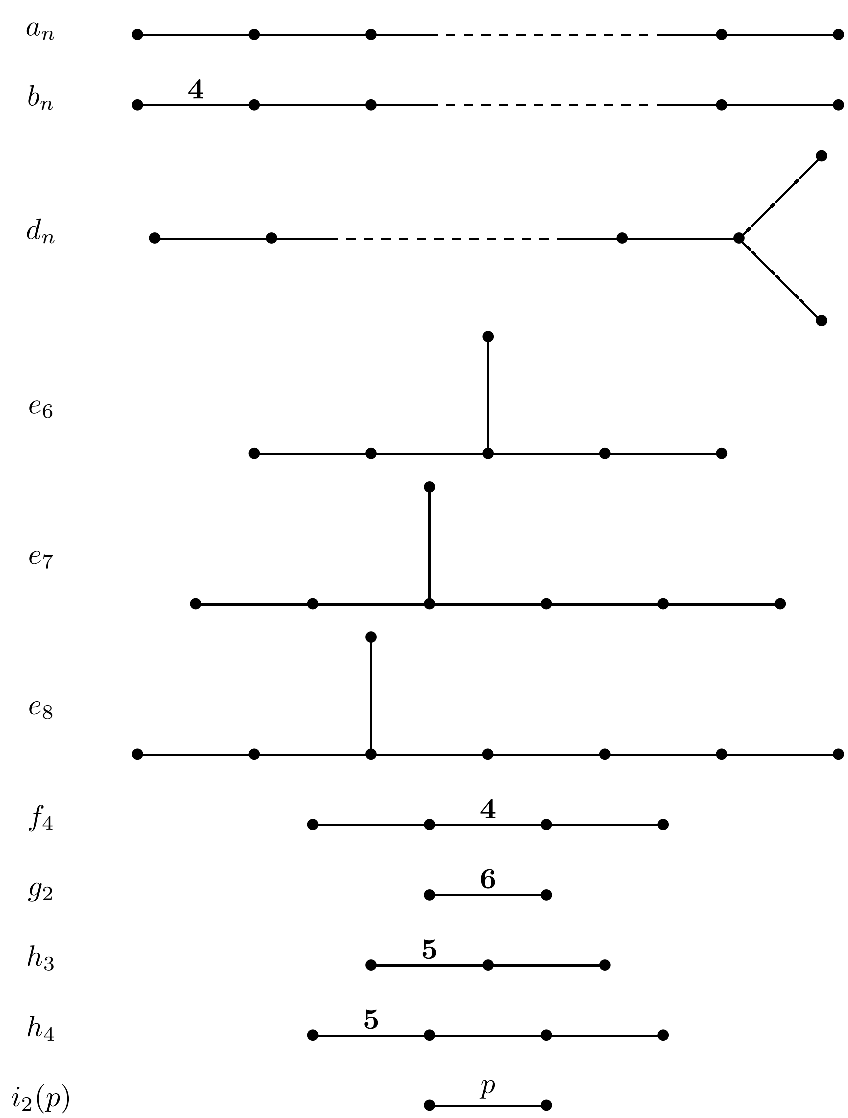

Table 3. A complete list of possible connected components of Coxeter diagrams of finite reflection groups. (See theorem 3.1.) A diagram x_n has n vertices.

3 The Classification

In light of the fact that Δ_W determines W, the following theorem classifies the finite reflection groups.

Theorem 3.1. *If W is a finite reflection group, then Δ_W is a disjoint union of copies of the Coxeter diagrams appearing in table 3.*

Proof: (This is the usual combinatorial argument.) Let Δ be a connected component of Δ_W. Δ can contain no cycles, else the subdiagram spanned by the vertices of a shortest cycle would be an increasement of A_n for some n. We will express this sort of reasoning by statements like "By A_n, Δ contains no cycles."

Suppose that an edge of Δ has marking $p \geq 4$. By B_n, Δ contains just one edge so marked. By BD_n, Δ has no branch points, so Δ is a simple chain of edges. By G_2, if $p > 5$ then the edge is the whole of Δ, so Δ is $i_2(p)$. If $p = 5$ then by H_3 the edge must be at an end of Δ, and then by H_4, Δ must have fewer than 4 edges. We deduce that if $p = 5$ then Δ is $i_2(5)$, h_3 or h_4. If $p = 4$ and the edge is not at an end of Δ then by F_4 we have $\Delta = f_4$. If $p = 4$ and the edge is at an end of Δ then $\Delta = b_n$ for some n.

It remains to consider the case in which all edge labels are 3. If Δ has no branch points then $\Delta = a_n$ for some n. By D_4, each branch point of Δ has valence 3, and by D_n (for $n > 4$), Δ has at most one branch point. Therefore it suffices to consider Δ with exactly one branch point, of valence 3. By a 'leg' of Δ we mean one of the 3 subgraphs of Δ consisting of the edges of the path in Δ joining the branch point to one of the 3 endpoints of Δ; the length of the leg is the number of these edges. Let ℓ_1, ℓ_2, ℓ_3 be the lengths of the legs, with $\ell_1 \leq \ell_2 \leq \ell_3$. By E_6, $\ell_1 = 1$. If we also have $\ell_2 = 1$ then $\Delta = d_n$ for some n. If $\ell_2 > 1$ then by E_7 we have $\ell_2 = 2$ and then by E_8 we have $\ell_3 < 5$, so Δ is one of e_6, e_7 and e_8. □