

Braid pictures for Artin groups

DANIEL ALLCOCK

Department of Mathematics, Harvard University, Cambridge, MA 02138

allcock@math.harvard.edu

<http://www.math.harvard.edu/~allcock>

Abstract

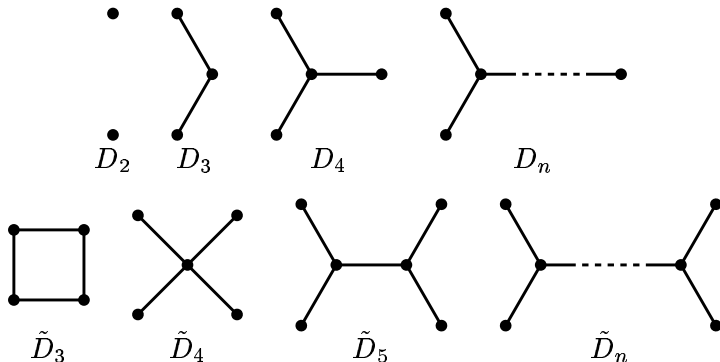
We define the braid groups of a two-dimensional orbifold and introduce conventions for drawing braid pictures. We use these to realize the Artin groups associated to the spherical Coxeter diagrams A_n , $B_n = C_n$ and D_n and the affine diagrams \tilde{A}_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n as subgroups of the braid groups of various simple orbifolds. The cases D_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n are new. In each case the Artin group is a normal subgroup with abelian quotient; in all cases except \tilde{A}_n the quotient is finite. We also illustrate the value of our braid calculus by giving a picture-proof of the basic properties of the Garside element of an Artin group of type D_n .

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1 Introduction

The purpose of this paper is to establish and explain a very close connection between two quite different-seeming generalizations of the classical braid group. One generalization is the class of Artin groups. This generalization is very natural when one studies the braid group from the point of view of singularity theory, or Lie theory, or the theory of reflection groups. We will be concerned mainly with the Artin groups $\mathcal{A}(D_n)$ and $\mathcal{A}(\tilde{D}_n)$ associated to the spherical and affine Coxeter diagrams



but we will also consider the diagrams associated to certain other Coxeter diagrams. These are the classical diagrams A_n and $B_n = C_n$ and the affine diagrams \tilde{A}_n , \tilde{B}_n and \tilde{C}_n . (Each diagram X_n has n nodes and each diagram \tilde{X}_n has $n + 1$. The remaining diagrams are given in section 4. The words ‘spherical’ and ‘affine’ indicate whether the corresponding reflection group acts naturally on the sphere or on Euclidean space.) The most direct way to define $\mathcal{A}(D_n)$ and $\mathcal{A}(\tilde{D}_n)$ is to give generators and relations: one takes one generator for each node of the diagram and imposes the relations that two of them commute (resp. braid) if the corresponding nodes are unjoined (resp. joined). When we say that two group elements x and y braid, we mean that they satisfy $xyx = yxy$. One can obtain the classical braid group on n strands by applying this construction to the Coxeter diagram A_{n-1} given below.

$$\bullet \text{---} \bullet \text{-----} \bullet \tag{1.1}$$

It turns out that this procedure of assigning a group to a Coxeter diagram is not just a random construction—the generators and relations arise from natural geometric considerations and are closely related to the associated Coxeter group. However, despite being natural objects, the Artin groups are still somewhat mysterious. The aim of this paper is to provide a very concrete way to understand some of the most important examples.

The other generalization of the classical braid group is the braid group of a two-dimensional orbifold. (An orbifold is a space locally modelled on open

diagram	orbifold features	quotient group	condition
A_{n-1}	none	1	$n > 1$
B_n, C_n	1 puncture	1	$n > 1$
D_n	1 cone point	$\mathbb{Z}/2$	$n > 1$
\tilde{A}_{n-1}	1 puncture	\mathbb{Z}	$n > 2$
\tilde{B}_n	1 puncture, 1 cone point	$\mathbb{Z}/2$	$n > 2$
\tilde{C}_n	2 punctures	1	$n > 1$
\tilde{D}_n	2 cone points	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$n > 2$

Table 1.1. Summary of results; entries should be interpreted as follows. For each row and each number n satisfying the condition in the last column, the Artin group associated to the given diagram is a normal subgroup of the n -strand braid group of the orbifold which is the plane equipped with the given features. All of the cone points indicated have order 2. The quotient of the braid group by the Artin group is given in the third column, and in each case the group extension splits.

sets in Euclidean space taken modulo finite groups. See chapter 13 of [18], section 2 of [17] or appendix A of [14] for more information.) The braid groups of certain two-orbifolds turn out to contain the Artin groups $\mathcal{A}(D_n)$ and $\mathcal{A}(\tilde{D}_n)$ as subgroups of very small index. The relevant orbifolds are among the simplest possible ones: one is the plane with a single cone point of order 2 and the other is the plane with two such cone points. A cone point of order n is the vertex of a cone \mathbb{R}^2/g where g is a rotation of order n ; orbifolds were introduced to make this notion precise. In the language of orbifolds, \mathbb{R}^2 is the universal cover of \mathbb{R}^2/g , with covering group \mathbb{Z}/n ; the orbifold fundamental group of \mathbb{R}^2/g coincides with the covering group and so it is also \mathbb{Z}/n . In sections 2 and 3 we treat the groups $\mathcal{A}(D_n)$ and $\mathcal{A}(\tilde{D}_n)$ in detail. There are similar results, some already known, for the Artin groups listed in table 1.1; we will discuss them in section 4. The cases D_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n are new, the case A_{n-1} is classical, B_n is treated in [11] and [3], and the result for \tilde{A}_{n-1} appears in [20]. Our interpretation of these Artin groups as braid groups leads to several curious coincidences, which we will discuss in section 4.

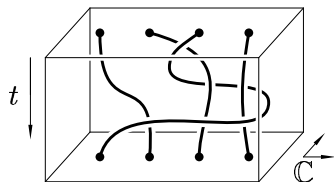
To define the braid groups, let L be any two-dimensional orbifold. Then its (n -strand) pure braid space is $L^n - \Delta_n$, where

$$\Delta_n = \{ (x_1, \dots, x_n) \in L^n \mid x_i = x_j \text{ for some } i \neq j \}.$$

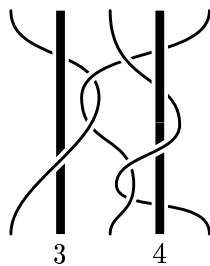
The symmetric group S_n acts on L^n in the obvious way; the action is free on $L^n - \Delta_n$ and defines a covering map to $X_n = (L^n - \Delta_n)/S_n$, which we call the (n -strand) braid space of L . We fix a basepoint $b = (b_1, \dots, b_n)$ for X_n , chosen so that it does not lie on the orbifold locus. Then the n -strand braid group $Z_n = Z_n(L)$ is defined to be the orbifold fundamental group of X_n .

We are interested in the case where the only orbifold singularities of L are cone points; in this case we may understand elements of $\pi_1(X_n)$ as follows. Each element of Z_n may be represented by a set of paths $\gamma_1, \dots, \gamma_n : [0, 1] \rightarrow L$ that miss the cone points of L and have the following properties. First, for some permutation π of $\{1, \dots, n\}$, each γ_i begins at b_i and ends at $b_{\pi(i)}$. Second, for each $t \in [0, 1]$, the points $\gamma_i(t)$ are all distinct. We call any n -tuple of paths satisfying these conditions a braid. Then $\gamma : t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ is a path in $L^n - \Delta_n$. If π is nontrivial then γ will not be a closed path, but its image in X_n will be, and hence define an element of Z_n .

When L is the complex plane with some punctures and/or cone points then we can draw pictures of braids in essentially the same manner as in the classical case. One simply draws a picture of $L \times [0, 1]$ and plots the curves $t \mapsto (\gamma_i(t), t)$. One usually pictures the slices $L \times \{t\}$ as horizontal planes with points having large imaginary part appearing far from the viewer, and one regards t as increasing in the downward direction. If L is just \mathbb{C} with no cone points then we recover braid pictures in the usual sense, for example:



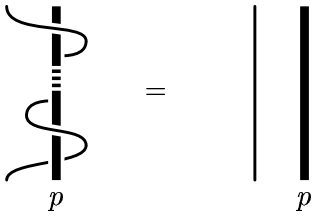
Of course we will typically draw less elaborate pictures. When L has cone points then we must indicate them in the picture of $L \times [0, 1]$. We do this by drawing each segment $\{c\} \times [0, 1]$, with c a cone point, as a thick vertical line with the order of the cone point indicated nearby. Here is an example of a braid when L has two cone points, of orders 3 and 4:



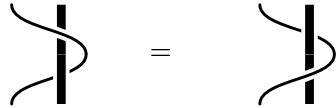
Finally, one multiplies braids in the obvious way: if B and B' are braids then we represent BB' by simply placing a picture of B' below a picture of B .

As usual, two braids represent the same element of Z_n if one may be deformed to the other through a continuum of braids. Also, when L has cone points then there is an additional ‘move’ that can be performed when a braid

strand links the ‘strand’ representing a cone point of order p exactly p times:

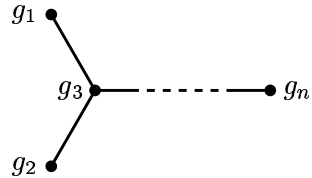
($2p$ crossings) 

These two braids represent the same element of $\pi_1(X_n)$ because a loop in L encircling an order p cone point exactly p times represents the trivial element of the orbifold fundamental group of L . When $p = 2$ the most useful formulation of this move is as

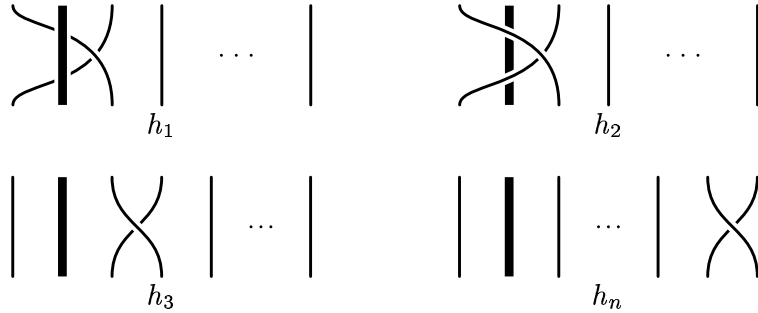

(1.2)

We have suppressed the numeral ‘2’ that would denote the order of the cone point. Since all heavy lines up until section 4 represent cone points of order 2, we will continue to do this.

We can now state the main results of the paper. We take $n \geq 2$ and label the standard generators of $\mathcal{A}(D_n)$ by

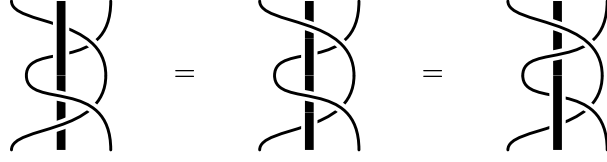

(1.3)

Let \mathcal{O} be the orbifold which is \mathbb{C} with a single cone point of order 2 and write h_1, \dots, h_n for the n -strand braids in \mathcal{O} given by the following diagrams:



It is easy to check that the map $g_i \mapsto h_i$ defines a homomorphism $\mathcal{A}(D_n) \rightarrow Z_n(\mathcal{O})$. The verification of each defining relation of $\mathcal{A}(D_n)$ except for the

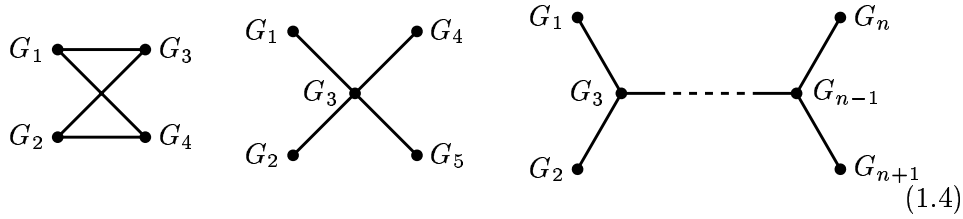
commutativity of h_1 and h_2 is easy and standard. The commutativity of h_1 and h_2 follows from a double application of the ‘orbifold move’:



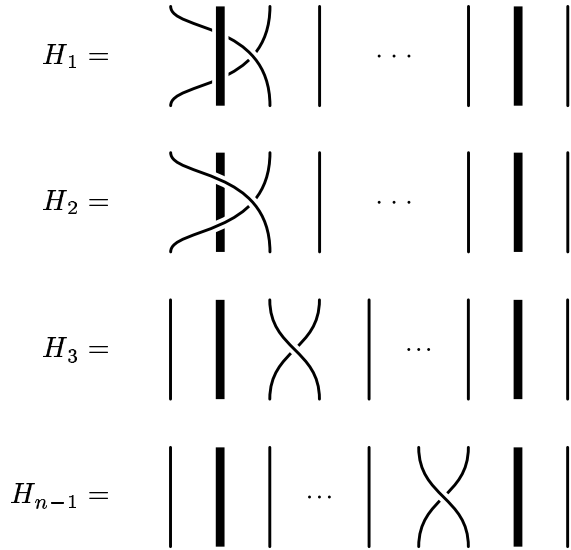
In section 2 we prove the following theorem.

Theorem 1.1. *For $n \geq 2$, the map $\mathcal{A}(D_n) \rightarrow Z_n(\mathcal{O})$ is an isomorphism onto its image, which has index two in $Z_n(\mathcal{O})$. There is a complementary subgroup $\mathbb{Z}/2$ whose nontrivial element acts on $\mathcal{A}(D_n)$ by exchanging g_1 with g_2 and fixing each of the remaining g_i .*

Something very similar happens for the affine Artin group $\mathcal{A}(\tilde{D}_n)$ for $n \geq 3$. We label the standard generators by



We take \mathcal{P} to be the orbifold which is \mathbb{C} with two cone points of order 2, and write H_1, \dots, H_{n+1} for the n -strand braids in \mathcal{P} given by the diagrams



$$\begin{array}{l}
H_n = \quad \left| \quad \left| \quad \left| \quad \cdots \quad \left| \quad \left(\begin{array}{c} \diagup \quad \left| \quad \diagdown \\ \diagdown \quad \left| \quad \diagup \end{array} \right. \right. \\
H_{n+1} = \quad \left| \quad \left| \quad \left| \quad \cdots \quad \left| \quad \left(\begin{array}{c} \diagup \quad \left| \quad \diagdown \\ \diagdown \quad \left| \quad \diagup \end{array} \right. \right.
\end{array}$$

The third and fourth pictures should be ignored when $n = 3$. By the same argument as in the spherical case, the map $G_i \mapsto H_i$ defines a homomorphism $\mathcal{A}(\tilde{D}_n) \rightarrow Z_n(\mathcal{P})$.

Theorem 1.2. *For $n \geq 3$, the map $\mathcal{A}(\tilde{D}_n) \rightarrow Z_n(\mathcal{P})$ is an isomorphism onto its image, which is normal in $Z_n(\mathcal{P})$ and has index 4. There is a complementary subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$; one element of this group acts on $\mathcal{A}(\tilde{D}_n)$ by exchanging H_1 with H_2 , fixing each of the remaining H_i , and another acts by exchanging H_n with H_{n+1} , fixing each of the remaining H_i .*

We treat the Artin groups of the other Coxeter diagrams of table 1.1 in section 4. The results and arguments are very similar to the D_n and \tilde{D}_n cases, so our presentation there is much more telegraphic.

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2 The spherical Artin group $\mathcal{A}(D_n)$

In this section we will prove theorem 1.1. Our earlier definition of the Artin group in terms of generators and relations conceals the geometric origin of the group. The group $\mathcal{A}(D_n)$ is important because it turns out to be the fundamental group of the space of conjugacy classes of regular semisimple elements of the Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$, and also of the discriminant complement in a miniversal deformation of the simple singularity D_n (see [1] and [2]). Both of these manifestations of $\mathcal{A}(D_n)$ are closely related to the appearance of $\mathcal{A}(D_n)$ in the much simpler context of finite reflection groups, to which we now turn.

Let $V^{\mathbb{R}} = \mathbb{R}^n$ be equipped with the standard Euclidean metric, and let V be the complexification of $V^{\mathbb{R}}$. The D_n root system is the set of vectors or ‘roots’ in $V^{\mathbb{R}}$ obtained from $(\pm 1, \pm 1, 0, \dots, 0)$ by arbitrary permutation of coordinates. The Weyl group $W = W(D_n)$ is the group generated by the reflections in these roots, that is, across the hyperplanes orthogonal to them. We write Σ for the set of reflections in W , which are precisely the reflections

in the roots. For each $s \in \Sigma$ we let $H_s^{\mathbb{R}}$ be the set of fixed points in $V^{\mathbb{R}}$ of s , which is just the mirror of the reflection. Then the complexification H_s of $H_s^{\mathbb{R}}$ is the set of fixed points of s in V . Because each H_s has real codimension two in V , the space $V_0 = V - \cup_{s \in \Sigma} H_s$ is connected. This is called the pure braid space for D_n and its fundamental group is called the pure Artin group of type D_n . It is known that W acts freely on V_0 , and the quotient manifold V_0/W is called the braid space for D_n .

It follows from work of Brieskorn [2] that $\pi_1(V_0/W) \cong \mathcal{A}(D_n)$. Indeed his work applies in greater generality, where one replaces $W(D_n)$ by any other finite reflection group W and constructs the corresponding space V_0/W ; then $\pi_1(V_0/W)$ is the Artin group corresponding to the chosen reflection group. For the constructions of Artin groups associated to even more general reflection groups, see [15], [21] or [22].

Proof of theorem 1.1: We take \mathcal{O} to be the orbifold $\mathbb{C}/(z \mapsto -z)$, and regard the map $z \mapsto z^2$ as an orbifold covering map from \mathbb{C} to \mathcal{O} . Recall that $X_n = (\mathcal{O}^n - \Delta_n)/S_n$. The idea of the proof is that there is a two-fold orbifold covering map $V_0/W \rightarrow X_n$. We begin by describing V_0 explicitly:

$$V_0 = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq \pm x_j \text{ if } i \neq j\}.$$

Using ATLAS [7] notation for group structures, the Weyl group W has structure $2^{n-1} : S_n$, where S_n permutes the coordinates and the group 2^{n-1} acts by changing the signs of any even number of them. The latter group is a subgroup of the larger group 2^n , also normalized by S_n , consisting of all sign-changes of coordinates.

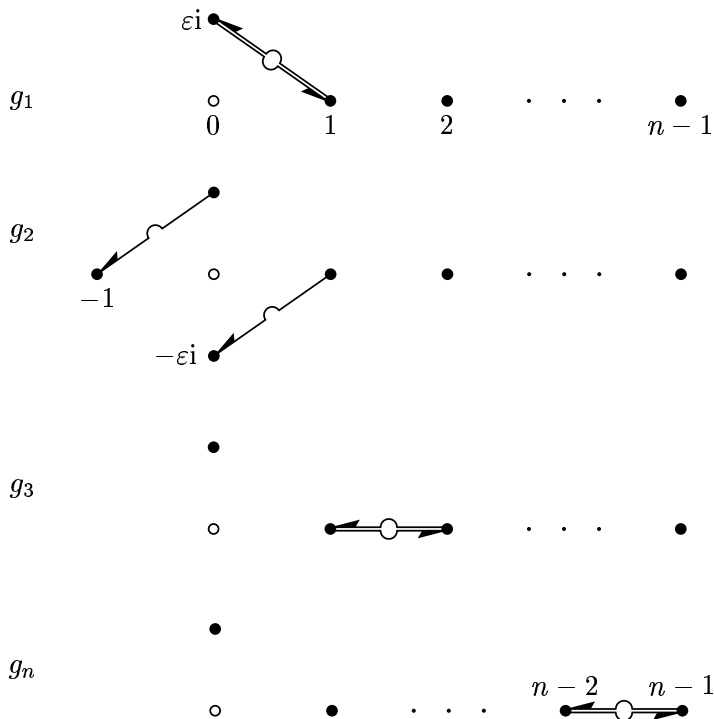
We define an orbifold covering map $\sigma : V \rightarrow V/2^n = \mathcal{O}^n$ by $(x_1, \dots, x_n) \mapsto (x_1^2, \dots, x_n^2)$. The image of V_0 in \mathcal{O}^n is exactly $\mathcal{O}^n - \Delta_n$. Furthermore, σ identifies the actions of S_n on its domain and range. Therefore the orbifolds $(V_0/2^n)/S_n$ and $(\mathcal{O}^n - \Delta_n)/S_n$ are isomorphic. The latter space is X_n and the former is $V_0/(2^n : S_n)$. Since $W = 2^{n-1} : S_n$ has index two in $2^n : S_n$, we see that V_0/W is a double cover of $V_0/(2^n : S_n) = X_n$. This proves that $\mathcal{A}(D_n)$ has index two in $Z_n(\mathcal{O})$.

Next we show that this 2-fold covering map induces the stated map on fundamental groups. For this we use Brieskorn's explicit description [2] of the standard Artin generators. We choose a set of simple roots

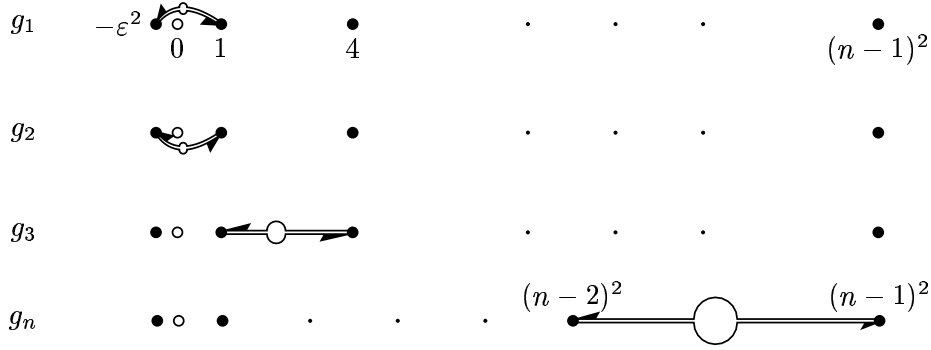
$$\begin{array}{c}
 (-1, 1, 0, \dots, 0) \\
 \bullet \\
 \diagdown \\
 (0, -1, 1, 0, \dots, 0) \bullet \text{-----} \bullet (0, \dots, 0, -1, 1) \\
 \diagup \\
 \bullet \\
 (1, 1, 0, \dots, 0)
 \end{array} \tag{2.1}$$

for D_n and let C be the (open) Weyl chamber they define. This is the set of all points in $V^{\mathbb{R}}$ with positive inner products with each simple root. We fix a basepoint p_0 in C , and for each simple root $s = 1, \dots, n$ with numberings as in (1.3), we let p_s be the image of p_0 under the reflection in that root. Then we let L_s be the complex line in V containing p_0 and p_s , and let $g_s : [0, 1] \rightarrow L_s$ be the path from p_0 to p_s that misses the mirrors and has the following properties. On $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$, g_s is linear in the real line through p_0 and p_s , and on $[\frac{1}{3}, \frac{2}{3}]$ it is a positively-oriented semicircle with center $(p_0 + p_s)/2$ and very small radius. Then g_s is not a loop but its image in V_0/W is. We have associated a loop in V_0/W with each node of the diagram, and Brieskorn's theorem says that we may take these to be the standard generators.

To understand the image of each of these loops in X_n , it suffices to compute their images under $\sigma : V_0 \rightarrow \mathcal{O}^n - \Delta_n$, since the image in $(\mathcal{O}^n - \Delta_n)/S_n$ is obtained simply by 'forgetting' in the usual way the ordering of the strands. First, we observe that the point $(0, 1, \dots, n-1)$ lies in C , and that Brieskorn's theorem obviously still holds if we choose p_0 to be a perturbation of this point, say $p_0 = (\varepsilon i, 1, 2, \dots, n-1)$ where ε is a small positive number and $i = \sqrt{-1}$. (Actually, there is no need to restrict ε to be small.) Then the paths g_1, \dots, g_n in V_0 are given in by the following figure. For each s we have shown the union of the coordinate projections of g_s .



The images of the g_s under σ are given below; they are obtained by applying the squaring map to the paths above.



Drawing these braids using the conventions of section 1 shows that the images of the g_i are the h_i . Here, the basepoint for $\mathcal{O}^n - \Delta_n$ is $(-\varepsilon^2, 1, 4, \dots, (n-1)^2)$.

Now we show that $Z_n(\mathcal{O})$ has the claimed semidirect product structure. First we observe that the braid

$$\tau = \left[\text{braid diagram} \right] \dots \left[\text{braid diagram} \right] \quad (2.2)$$

has order 2. Furthermore, it does not lie in the group $\mathcal{A}(D_n)$ generated by the h_i . This follows from the fact that V_0/W is aspherical ([3], [8]) and hence its fundamental group $\mathcal{A}(D_n)$ is torsion-free. A more elementary way to see that $\tau \notin \mathcal{A}(D_n)$ is to note that its lift to V_0 joins the basepoint p_0 to $(-\varepsilon i, 1, 2, \dots, n-1)$ and that these two points are inequivalent under W . It is easy to check that conjugation by τ induces the stated automorphism of $\mathcal{A}(D_n)$. \square

Theorem 1.1 justifies the use of our braid calculus to perform calculations in $\mathcal{A}(D_n)$. As an application of this calculus, we will discuss the Garside (or fundamental) element Δ defined in [4] and [5]. From our perspective this is the braid w^{n-1} , where $w = h_1 h_2 \dots h_n$ is given in pictures by

$$w = \left[\text{braid diagram} \right] \dots \left[\text{braid diagram} \right] = \left[\text{braid diagram} \right] \dots \left[\text{braid diagram} \right] \quad (2.3)$$

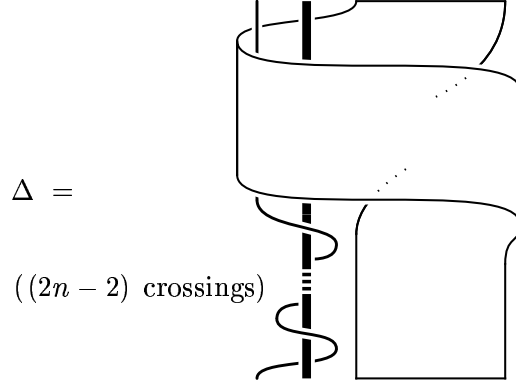


Figure 2.1. The fundamental element $\Delta = w^{n-1} = (h_1 \cdots h_n)^{n-1}$.

In the second picture we have used a ribbon to indicate some number of strands moving in parallel. This makes some pictures easier to understand. By stacking together $n - 1$ copies of w , one sees that Δ is represented by the braid of fig. 2.1.

Theorem 2.1. *Conjugation by Δ fixes each of h_3, \dots, h_n , and either swaps h_1 with h_2 (if n is odd) or fixes each of them (if n is even). Furthermore, Δ^2 (resp. Δ) is central in $Z_n(\mathcal{O})$ if n is odd (resp. even).*

Proof: It follows by inspection of fig. 2.1 that Δ commutes with h_i for all $i > 2$. Next we observe that $\Delta h_1 \Delta^{-1}$ is given in fig. 2.2. If n is odd then this braid is equivalent to that of fig. 2.3, where we have suppressed all but the first two braid strands. Similarly, if n is even then fig. 2.2 reduces to fig. 2.4. Therefore $\Delta h_1 \Delta^{-1} = h_1$ if n is even and $\Delta h_1 \Delta^{-1} = h_2$ if n is odd. A similar calculation shows that $\Delta h_2 \Delta^{-1} = h_2$ if n is even and $\Delta h_2 \Delta^{-1} = h_1$ if n is odd. Since $Z_n(\mathcal{O})$ is generated by τ and the h_i , to finish the proof it suffices to check that Δ and τ commute. This follows by inspection of fig. 2.1. Indeed w and τ also commute, by inspection of (2.3). □

This leads to another description of $Z_n(\mathcal{O})$ if n is odd:

Theorem 2.2. *If n is odd then a presentation for $Z_n(\mathcal{O})$ may be obtained from that of $\mathcal{A}(D_n)$ by adjoining a new central element z , subject to the relation that $z^2 = \Delta^2$.*

Proof: Take $z = \tau \Delta$. By theorems 1.1 and 2.1, the conjugation maps of Δ and $\tau = \tau^{-1}$ coincide, so z is central in $Z_n(\mathcal{O})$. Since Δ lies in $\mathcal{A}(D_n)$ but τ does not, $z \notin \mathcal{A}(D_n)$. Since $\mathcal{A}(D_n)$ has index two in $Z_n(\mathcal{O})$, the description of $Z_n(\mathcal{O})$ is completed by computing the square of z . In light of $\Delta \tau = \tau \Delta$, we have $z^2 = \tau \Delta \tau \Delta = \Delta^2$. □

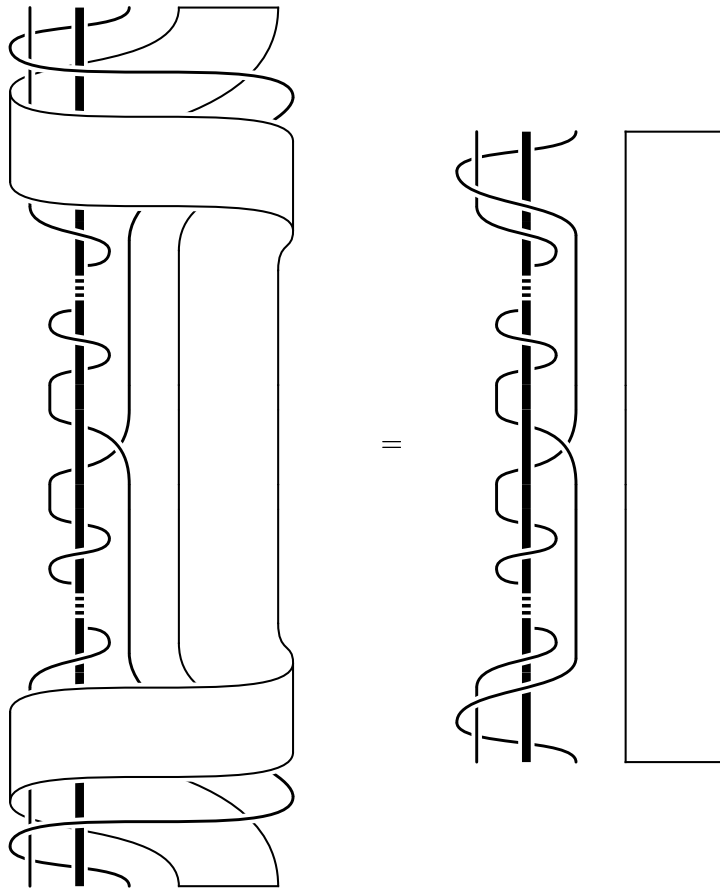


Figure 2.2. A picture of $\Delta h_1 \Delta^{-1}$.

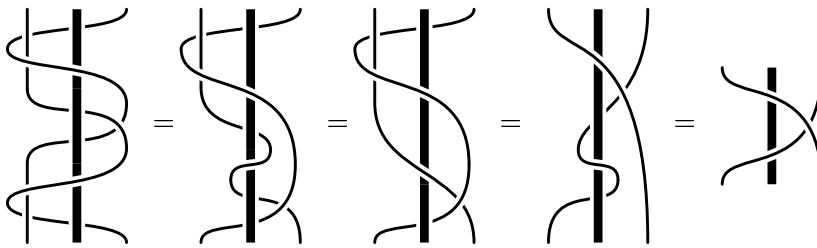


Figure 2.3. $\Delta h_1 \Delta^{-1} = h_2$ if n is odd.

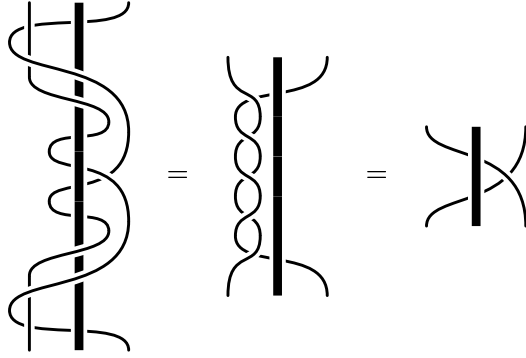


Figure 2.4. $\Delta h_1 \Delta^{-1} = h_1$ if n is even.

We now identify our fundamental element with those of Charney [5] and of Brieskorn and Saito [4]. It is easy to find the lift to V_0 of the path in $(\mathcal{O}^n - \Delta_n)/S_n$ represented by the braid of fig. 2.1. All we need is that the lift begins at $p_0 = (\varepsilon i, 1, 2, \dots, n-1)$ and ends at $p'_0 = ((-1)^{n-1} \varepsilon i, -1, -2, \dots, -(n-1))$. One can even read this fact directly from fig. 2.1: following the strands around gives the induced permutation of coordinates, with a sign-flip being present if the strand passes behind the ‘orbifold strand’ an odd number of times. Now, the real part of p'_0 lies in the Weyl chamber opposite to C , because it has negative inner product with each simple root. Therefore the standard map from $\mathcal{A}(D_n)$ to $W(D_n)$ carries Δ to the element α of W that exchanges C with $-C$. It is known that α is the unique longest element of W with respect to the standard generating set, with length $n(n-1)$. Since Δ is defined as a word of length $n(n-1)$ in the h_i , it is a minimal element of $\mathcal{A}(D_n)$ mapping to α . The description of the fundamental element in the proof of lemma 2.3 of [5] shows that our Δ coincides with Charney’s. (Our α is her g_0 .) Brieskorn and Saito [4, section 5.8] describe their fundamental element as a specific word of length $n(n-1)$ in the standard Artin generators that maps to α . This implies that their description coincides with Charney’s and hence with ours. It is interesting to note that their explicit expression for Δ differs from ours.

3 The affine Artin group $\mathcal{A}(\tilde{D}_n)$

In this section we will prove theorem 1.2. The ideas are similar to those of the previous section but the calculations are slightly more involved. The affine Weyl group $W = W(\tilde{D}_n)$ is a discrete cocompact group of isometries of n -dimensional real Euclidean space $V^{\mathbb{R}} = \mathbb{R}^n$. It is generated by $n+1$ Euclidean reflections, which we number as in (1.4). The first n reflections are across the hyperplanes orthogonal to the simple roots (2.1), and the last is the reflection

across the hyperplane whose elements have inner product -1 with the root $(0, \dots, 0, -1, -1)$. We take C to be the open region in $V^{\mathbb{R}}$ bounded by these hyperplanes, namely the set of points having inner product > -1 with the last root and positive inner product with each of the other simple roots. We write Σ for the set of reflections in W , and for each $s \in \Sigma$ we write $H_s^{\mathbb{R}}$ for the mirror of s . By the standard theory of Coxeter groups, the W -translates of C coincide with the components of $V_0^{\mathbb{R}} = V^{\mathbb{R}} - \cup_{s \in \Sigma} H_s^{\mathbb{R}}$.

Now we take V to be the complexification of $V^{\mathbb{R}}$ and for each $s \in \Sigma$ we take H_s to be the complexification of $H_s^{\mathbb{R}}$. By this we mean the unique complex hyperplane in V containing $H_s^{\mathbb{R}}$. (This definition is slightly different from the corresponding one in section 2, because here $H_s^{\mathbb{R}}$ and H_s might not contain the origin.) One can show that W acts freely on $V_0 = V - \cup_{s \in \Sigma} H_s$, and it turns out that $\pi_1(V_0/W) \cong \mathcal{A}(\tilde{D}_n)$. This is the content of the following theorem, which is due to Nguy en [15].

Theorem 3.1. *Let $p_0 \in V_0$ have real part lying in C . For each $s = 1, \dots, n+1$ let p_s be the image of p_0 under the s th generating reflection of W (i.e., across the s th wall of C), and let L_s be the complex line in V containing p_0 and p_s . Let G_s be the path $[0, 1] \rightarrow L_s$ from p_0 to p_s which misses the mirrors and is similar to the path g_s of section 2. That is, on $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ it is linear in the real line through p_0 and p_s and on $[\frac{1}{3}, \frac{2}{3}]$ it is a positively oriented semicircle with small radius and center at $(p_0 + p_s)/2$. Then $\pi_1(V_0/W)$ is isomorphic to $\mathcal{A}(\tilde{D}_n)$ and the loops in V_0/W represented by G_1, \dots, G_{n+1} may be taken as the standard Artin generators.*

Another description of W , more useful for some purposes, is the following. We denote by Λ the integral span of the roots of the D_n root system. This is called the D_n root lattice and consists of all vectors in \mathbb{Z}^n with even coordinate sum. It is known that W has structure $\Lambda : W_0$, where Λ acts by translations and W_0 is the finite Weyl group $W(D_n)$ studied in the previous section. Since W_0 has structure $2^{n-1} : S_n$ and the S_n normalizes Λ , we indicate the structure of W by $\Lambda : 2^{n-1} : S_n$. The mirrors of W are the Λ -translates of those passing through 0, which in turn are the mirrors of W_0 .

This allows us to describe the complex hyperplane arrangement associated to W . A point $x \in \mathbb{C}^n$ lies in the mirror of a reflection of W if and only if it has integral inner product with one of the roots of D_n . For if $x \cdot r = m \in \mathbb{Z}$ for a root r then upon choosing another root r' with $r \cdot r' = 1$, we have $(x - mr') \cdot r = 0$, so that x lies in a Λ -translate of r^\perp . On the other hand, if for some $\lambda \in \Lambda$, $x - \lambda$ lies in r^\perp for a root r , then $r \cdot x = r \cdot \lambda$, which is integral because all inner products of elements of Λ are integral. Therefore

$$V_0 = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \pm x_j \notin \mathbb{Z} \text{ if } i \neq j\}.$$

We can now prove theorem 1.2.

Proof of theorem 1.2: We recall that \mathcal{P} is the orbifold \mathbb{C} with two cone points of order two. We take the cone points to be at 0 and $1/2$, and we may regard the universal orbifold covering map $\mathbb{C} \rightarrow \mathcal{P}$ to be given by $x \mapsto \xi(x) = (1 - \cos(2\pi x))/4$. On the strip $\operatorname{Re} x \in [0, \frac{1}{2}]$ the map ξ may be visualized as follows. It fixes each of 0 and $1/2$, with branching of order 2 there, and each component of the boundary of the strip is folded upon itself (about either 0 or $1/2$) and carried to the real axis.

We have an inclusion of groups from $W = \Lambda : 2^{n-1} : S_n$ into $\overline{W} = \mathbb{Z}^n : 2^n : S_n$, where \mathbb{Z}^n acts on \mathbb{C}^n by translations, $2^{n-1} : S_n$ is W_0 , and the group 2^n acts by changing coordinates' signs, as in section 2. Because Λ has index two in \mathbb{Z}^n and 2^{n-1} has index two in 2^n , W has index four in \overline{W} . We will now show that $(\mathcal{P}^n - \Delta_n)/S_n \cong ((V_0/\mathbb{Z}^n)/2^n)/S_n$. The map $V \rightarrow V/\mathbb{Z}^n$ may be described by $(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$, where $y_k = \exp(2\pi i x_k)$. Writing \mathbb{C}^\times for $\mathbb{C} - \{0\}$, we see that the action of 2^n on $V/\mathbb{Z}^n = (\mathbb{C}^\times)^n$ induced by the action of $\mathbb{Z}^n : 2^n$ on V is by replacing some number of coordinates by their reciprocals. The map

$$y \mapsto \eta(y) = \frac{1}{4} \left(1 - \frac{1+y^2}{2y} \right)$$

identifies $y, y' \in \mathbb{C}^\times$ just if they are equal or reciprocal. Furthermore it carries the branch points (namely ± 1) of the map to 0 and $1/2$. Therefore we may regard it as an orbifold covering map (of degree 2) from \mathbb{C}^\times to \mathcal{P} . Therefore the map $V/\mathbb{Z}^n \rightarrow (V/\mathbb{Z}^n)/2^n = \mathcal{P}^n$ is given by $(y_1, \dots, y_n) \mapsto (z_1, \dots, z_n)$, where $z_k = \eta(y_k) = \xi(x_k)$. It is easy to compute the image of V_0 in these models of V/\mathbb{Z}^n and $V/(\mathbb{Z}^n : 2^n)$. We have

$$\begin{aligned} V_0/\mathbb{Z}^n &= \{ (y_1, \dots, y_n) \in (\mathbb{C}^\times)^n \mid y_i \neq y_j^{\pm 1} \text{ if } i \neq j \} \quad \text{and} \\ V_0/(\mathbb{Z}^n : 2^n) &= \{ (z_1, \dots, z_n) \in \mathcal{P}^n \mid z_i \neq z_j \text{ if } i \neq j \}. \end{aligned}$$

Finally, our map $V \mapsto \mathcal{P}^n$ identifies the actions of S_n on its domain and range. Therefore V_0/\overline{W} is the n -strand braid space of \mathcal{P} . This proves that $\mathcal{A}(\tilde{D}_n)$ has index 4 in $Z_n(\mathcal{P})$.

Next we compute the map on fundamental groups induced by this orbifold cover. We take our basepoint in V_0 to be

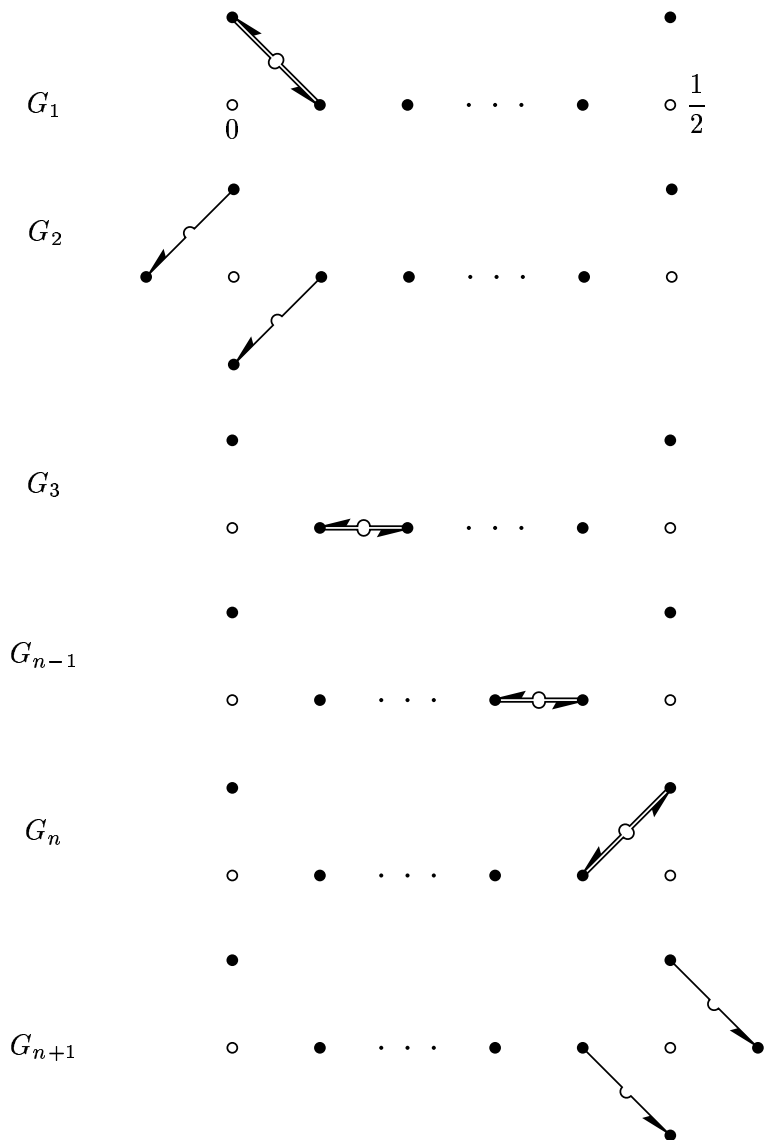
$$p_0 = \frac{1}{2n-2} (i, 1, 2, \dots, n-2, n-1+i).$$

It is easy to check that the real part of p_0 lies in C . We may take the standard generators for $\mathcal{A}(\tilde{D}_n)$ to be the paths G_s of theorem 3.1; most of the rest of the proof is a computation of their images in \mathcal{P}^n . For $s = 1, \dots, n$ the analysis is similar to that of the proof of theorem 1.1, because the reflections are the same. For $s = n+1$, the relevant reflection is across the hyperplane of \mathbb{C}^n

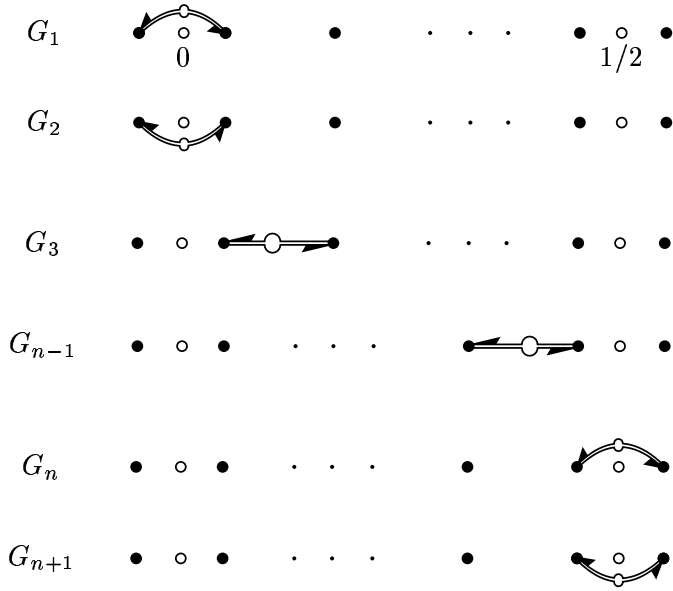
whose elements have inner product -1 with $r = (0, \dots, 0, -1, -1)$. The image of p_0 under this map is

$$p_{n+1} = \frac{1}{2n-2}(i, 1, \dots, (n-1) - i, n),$$

which may be verified by observing that $(p_0 + p_{n+1})/2$ lies on the mirror and that $p_0 - p_{n+1}$ is proportional to r . Here are the paths G_1, \dots, G_{n+1} , described as the g_s were in section 2, by simultaneously drawing all the coordinate projections of the path.



Their images in $\mathcal{P}^n = V/(\mathbb{Z}^n : 2^n)$ are given by applying the map ξ to the previous pictures. Our choice of the covering map $\xi : \mathbb{C} \rightarrow \mathcal{P}$ was made to make this step as easy as possible.



Drawing the braids associated to these paths and comparing them with the given braids H_i shows that our orbifold covering map $V_0/W \rightarrow X_n$ carries G_i to H_i .

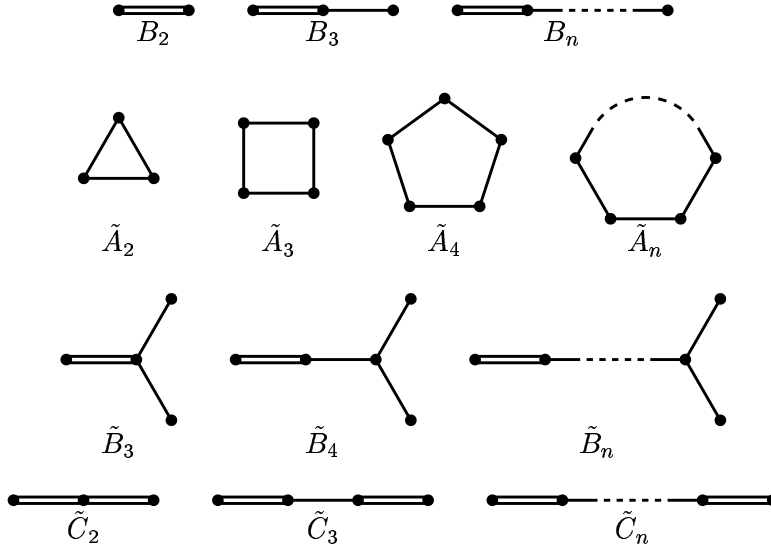
Now we establish the semidirect product structure. The following two elements of $Z_n(\mathcal{P})$ generate a group $(\mathbb{Z}/2)^2$.



The elements of this group are 1, τ_1 , τ_2 and $\tau_1\tau_2$. To see that of these only the trivial element lies in $\mathcal{A}(\tilde{D}_n)$ one can obtain the lifts to V_0 of the paths in $\mathcal{P}^n - \Delta_n$ represented by these braids and determine their final endpoints. These turn out to be $\frac{1}{2n-2}(\pm i, 1, \dots, n-2, n-1 \pm i)$ Since the real parts all lie in the open Weyl chamber C (indeed they coincide), they are inequivalent under W and hence only one of the paths represents a loop in V_0/W . Since $\mathcal{A}(\tilde{D}_n)$ has index 4 in $Z_n(\mathcal{P})$, we have found a complete set of coset representatives. It is easy to check that the conjugation maps of τ_1 and τ_2 have the properties stated. In particular, $\langle \tau_1, \tau_2 \rangle$ normalizes $\mathcal{A}(\tilde{D}_n)$, so that the semidirect product decomposition exists as claimed. \square

4 The remaining groups

In this section we sketch the arguments for the rest of the diagrams listed in table 1.1, namely the spherical diagrams A_{n-1} and $B_n = C_n$ and the affine diagrams \tilde{A}_{n-1} , \tilde{B}_n and \tilde{C}_n . The A_{n-1} diagram appears in (1.1). The remaining diagrams are given below.



Recall that a diagram X_n has n nodes and a diagram \tilde{X}_n has $n + 1$. The Artin groups are given by taking a generator for each node and declaring that two such generators x and y commute (resp. braid, or satisfy $xyxy = yxyx$) if the corresponding nodes are unjoined (resp. singly joined, or doubly joined). For each of the diagrams, the pure Artin group is the fundamental group of the complex hyperplane complement V_0 associated to the corresponding Weyl group. The Artin group itself is the fundamental group of the quotient of V_0 by the Weyl group. The standard Artin generators may be represented by paths in V_0 in the same manner as in sections 2 and 3. For background on this, see [2], [10] and [15]. Now we will show that each of these Artin groups may be realized as a normal subgroup of the n -strand braid group of a suitable 2-orbifold, as indicated in table 1.1. The results for the cases \tilde{B}_n and \tilde{C}_n are new. In each case the argument has much the same form as that of section 2 or 3, so our treatment is brief.

The diagram A_{n-1} : This approach to the classical braid group appears in [10]. Here,

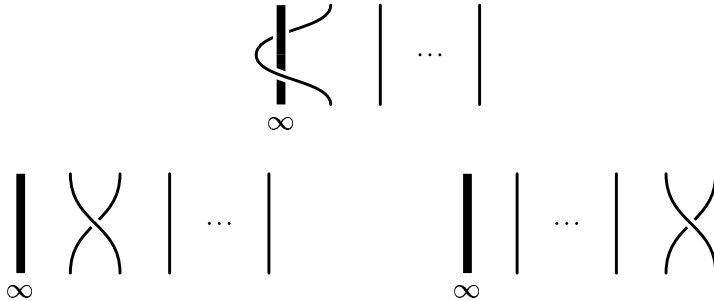
$$V_0 = \{ (x_1, \dots, x_n) \in \mathbb{C}^n \mid \sum x_i = 0, x_j \neq x_k \text{ if } j \neq k \}$$

and the Weyl group $W = W(A_{n-1}) = S_n$ acts by permuting the coordinates. The space V_0 is an S_n -equivariant deformation retract of

$$\{ (x_1, \dots, x_n) \in \mathbb{C}^n \mid x_j \neq x_k \text{ if } j \neq k \},$$

showing that V_0/W is homotopy-equivalent to the braid space of \mathbb{C} . It is easy to see that the standard Artin generators correspond to the standard braid generators. A picture of the fundamental element of $\mathcal{A}(A_{n-1})$ appears in [9, fig. 9.2].

The diagram $B_n = C_n$: The results here are implicit in [3]. Here $W = W(B_n)$ is the group $2^n : S_n$ used in section 2, so that W contains $W(D_n)$ as a subgroup of index 2. Each reflection negates a coordinate, or exchanges two coordinates, or exchanges two coordinates and negates both. The hyperplane complement is $V_0 = \{ (x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \mid x_j \neq \pm x_k \text{ if } j \neq k \}$. The map $V_0 \rightarrow V_0/2^n = (\mathbb{C}^\times)^n$ given by squaring each coordinate identifies $V_0/2^n$ with the pure braid space of \mathbb{C}^\times in an S_n -equivariant manner. This shows $\mathcal{A}(B_n) = Z_n(\mathbb{C}^\times)$. In terms of braids, the standard Artin generators (from left to right) are



The ‘ ∞ ’ below the heavy line indicates that the line represents the puncture. It is a pleasing exercise to verify directly the relation $xyxy = yxyx$ satisfied by the first two generators. The fundamental element of $\mathcal{A}(B_n)$ is given by the n th power of the product (from left to right) of the standard generators. It is easy to check that this element is central.

S. Lambropoulou [11] [12] [13] independently obtained our description of $\mathcal{A}(B_n)$, and she has established the relationship between braids in the punctured plane and knots in the solid torus and in some lens spaces, including an analogue of Markov’s theorem. For knots in such spaces she has also constructed analogues of the Jones polynomial by using the Hecke algebras of type B_n . T. tom Dieck ([19] [20]) has also obtained the isomorphism $Z_n(\mathbb{C}^\times) \cong \mathcal{A}(B_n)$ and studied the problem of generalizing to this case the machinery of tangle and skein theory associated to the classical braid group. We wonder whether our results for other Artin groups will lead to a whole series of skein theories in various orbifolds.

The diagram \tilde{A}_n : Most of the results here are stated in [20]. The Weyl group W is generated by the reflections in the roots $(1, -1, 0, \dots, 0), \dots, (0, \dots, 0, 1, -1)$ in \mathbb{C}^n , together with the reflection across the hyperplane of elements having inner product -1 with $(-1, 0, \dots, 0, 1)$. Alternatively, $W = \Lambda : S_n$ where Λ is the integral span of the roots, which is the set of elements of \mathbb{Z}^n with vanishing coordinate sum. We have

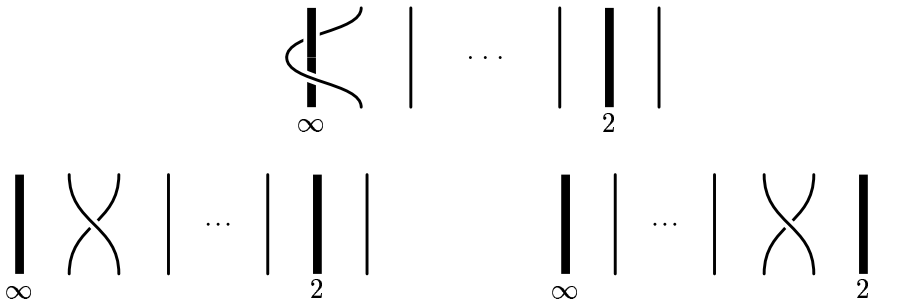
$$V_0 = \{ (x_1, \dots, x_n) \in \mathbb{C}^n \mid \sum x_j = 0, x_j - x_k \notin \mathbb{Z} \text{ if } j \neq k \} \quad .$$

The easiest way to compute the quotient of V_0 by Λ is to take the quotient of \mathbb{C}^n by \mathbb{Z}^n by applying the map $x \mapsto \exp(2\pi i x)$ for each coordinate. Then the n -strand pure braid space for \mathbb{C}^\times is $\{ (y_1, \dots, y_n) \in (\mathbb{C}^\times)^n \mid y_j \neq y_k \text{ if } j \neq k \}$, the map from this space to \mathbb{C}^\times given by $(y_1, \dots, y_n) \mapsto y_1 \cdots y_n$ is a locally trivial fibration, and V_0/Λ is the fiber over 1. The long exact homotopy sequence shows that $\mathcal{A}(\tilde{A}_{n-1})$ is normal in $Z_n(\mathbb{C}^\times)$ with quotient \mathbb{Z} . A good choice for a set of basepoints for n -strand braids in \mathbb{C}^\times is the set of n th roots of unity. Then the standard Artin generators correspond to the standard braid generators; each of these exchanges two adjacent basepoints in the simplest possible way.

The diagram \tilde{B}_n : The first n generators for $W = W(\tilde{B}_n)$ (all except for the top right node) may be taken to be the reflections in the roots $(-1, 0, \dots, 0), (1, -1, 0, \dots, 0), \dots, (0, \dots, 0, 1, -1)$ in \mathbb{C}^n , and the last may be taken to be across the hyperplane of points having inner product -1 with $(0, \dots, 0, 1, 1)$. From this one can obtain a description of W as $\Lambda : W(B_n)$ where Λ is the D_n root lattice considered in section 3. We observe that $W(D_n)$ has index 2 in W . Also,

$$V_0 = \{ (x_1, \dots, x_n) \in \mathbb{C}^n \mid x_j \pm x_k \notin \mathbb{Z} \text{ for } j \neq k \text{ and } x_j \notin \mathbb{Z} \text{ for all } j \} \quad .$$

Since Λ has index 2 in \mathbb{Z}^n , W has index 2 in $\mathbb{Z}^n : 2^n : S_n$. Closely following the analysis of section 3, with p_0 replaced by $\frac{1}{2n}(1, 2, \dots, n-1, n+i)$, shows that $\mathcal{A}(\tilde{B}_n)$ has index 2 in $Z_n(L)$ where L is \mathbb{C}^\times with an orbifold point of order two at $1/2$. One can show that the standard Artin generators correspond in the order given to the n -strand braids



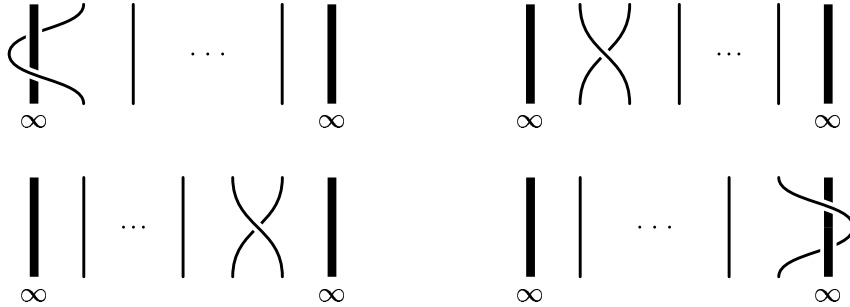


A subgroup $\mathbb{Z}/2$ complementary to $\mathcal{A}(\tilde{B}_n)$ may be obtained by taking the obvious analogue of the element τ of eq. (2.2).

The diagram \tilde{C}_n : The first n generators of $W = W(\tilde{C}_n)$ act on \mathbb{C}^n by the reflections in the roots $(1, 0, \dots, 1)$, $(-1, 1, 0, \dots, 0), \dots, (0, \dots, 0, -1, 1)$, and the last acts by the reflection across the hyperplane of vectors having inner product $-1/2$ with $(0, \dots, 0, -1)$. Alternatively, W is the group $\mathbb{Z}^n : 2^n : S_n$, so that $W(\tilde{D}_n)$ has index 4 and $W(\tilde{B}_n)$ has index 2. We have

$$V_0 = \{ (x_1, \dots, x_n) \in \mathbb{C}^n \mid x_j \pm x_k \notin \mathbb{Z} \text{ for } j \neq k \text{ and } x_j \notin \frac{1}{2}\mathbb{Z} \text{ for all } j \} .$$

The map $V_0 \rightarrow V_0 / (\mathbb{Z}^n : 2^n)$ may be described as in section 3, and this identifies $V_0 / (\mathbb{Z}^n : 2^n)$ with the pure braid space of $\mathbb{C} - \{0, \frac{1}{2}\}$. A suitable basepoint for the computation of the braids associated to the Artin generators is $p_0 = \frac{1}{2n+1}(1, \dots, n)$. The resulting braids are pictured below, in order.



We close by observing a few curiosities that our orbifold approach makes visible. We recall that \mathcal{O} is the orbifold $\mathbb{C}/(z \mapsto -z)$. First and most curious is the fact that $Z_4(\mathbb{C})$ has index 2 in $Z_3(\mathcal{O})$ because of the coincidence of diagrams $A_3 = D_3$. There is a similar coincidence $\tilde{A}_3 = \tilde{D}_3$, so that this affine Artin group has index 4 in $Z_3(\mathcal{P})$ and also index ∞ in $Z_4(\mathbb{C}^\times)$. I do not know of any elementary explanation for these coincidences.

The other curiosity arises from the natural map $Z_n(\mathbb{C}^\times) \rightarrow Z_n(\mathcal{O})$ given by filling in the puncture with a cone point of order 2. One can check that it carries $\mathcal{A}(\tilde{A}_{n-1}) \subseteq Z_n(\mathbb{C}^\times)$ onto $\mathcal{A}(D_n) \subseteq Z_n(\mathcal{O})$. It is not at all clear from the Coxeter diagrams that there is any surjection $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{A}(D_n)$ at all. One can also show that adjoining the relation that the leftmost generator of $\mathcal{A}(B_n) = Z_n(\mathbb{C}^\times)$ have trivial square reduces $Z_n(\mathbb{C}^\times)$ to $Z_n(\mathcal{O})$.

There is also a natural map $Z_n(\mathbb{C}^\times) \rightarrow Z_n(\mathbb{C})$ given by filling in the puncture with an ordinary point. This map induces a retraction $\mathcal{A}(\tilde{A}_{n-1}) \rightarrow \mathcal{A}(A_{n-1})$, where we regard $\mathcal{A}(A_{n-1})$ as embedded in $\mathcal{A}(\tilde{A}_{n-1})$ by any embedding of the Coxeter diagrams. This retraction has the pleasant property that it descends to the natural retraction $W(\tilde{A}_{n-1}) \rightarrow W(A_{n-1})$ given by considering the natural action of $W(\tilde{A}_{n-1})$ on the sphere at infinity of Euclidean space. We remark that the corresponding natural map $\mathcal{A}(\tilde{D}_n) \rightarrow \mathcal{A}(D_n)$ obtained by replacing a cone point of \mathcal{P} by an ordinary point does *not* have this property. I do not know if there is any retraction $\mathcal{A}(\tilde{D}_n) \rightarrow \mathcal{A}(D_n)$ which does descend to the natural retraction of Weyl groups.

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