Géométrie algébrique/Algebraic geometry

A Complex Hyperbolic Structure for Moduli of Cubic Surfaces

Daniel Allcock, James A. Carlson, and Domingo Toledo

Department of Mathematics
University of Utah
Salt Lake City, Utah, USA.
E-mail: allcock, carlson and toledo at math.utah.edu

Abstract. We show that the moduli space M of marked cubic surfaces is biholomorphic to $(B^4 - \mathcal{H})/\Gamma_0$ where B^4 is complex hyperbolic four-space, where Γ_0 is a specific group generated by complex reflections, and where $\mathcal{H}$ is the union of reflection hyperplanes for Γ_0. Thus M has complex hyperbolic structure, i.e., an (incomplete) metric of constant holomorphic sectional curvature.

Une structure hyperbolique complexe pour les modules des surfaces cubiques

Résumé. Nous montrons que l'éspace des modules M des surfaces cubiques marquées est biholomorphe à $(B^4 - \mathcal{H})/\Gamma_0$ oú B^4 est l'éspace complexe hyperbolique de dimenson quatre, oú Γ_0 est un groupe spécifique generé par des reflections complexes, et oú $\mathcal{H}$ est l'union de l'ensemble d'hyperplans de reflections de Γ_0. Donc M admet une structure hyperbolique complexe, c'est à dire une métrique (incomplète) de courbure holomorphe sectionelle constante.

Version française abrégée

A une surface cubique (marquée) correspond une variété cubique de dimension trois (marquée), à savoir le revêtement de $\mathbb{P}^3$ ramifié le long la surface. L'application des periodes f pour ces variétés de dimension trois est définie sur l'éspace des modules M des cubiques marquées, et cette application f prend ses valeurs dans une quotient de la boule unitaire dans $\mathbb{C}^4$ par l'action du groupe de monodromie projective. Ce groupe Γ_0 est generé par des reflections complexes dans un ensemble d'hyperplans dont la réunion nous notons par $\mathcal{H}$. Alors nous avons le resultat suivant:

Théorème. *L'application des periodes définit une biholomorphisme*

$$f : M \longrightarrow \left(B^4 - \mathcal{H}\right)/\Gamma_0.$$

De ce théorème on obtient des resultats sur la structure métrique de M et sur son groupe fondamentale:

First author partially supported by an NSF postdoctoral fellowship. Second and third authors partially supported by NSF grant DMS 9625463

Corollaires. *(1) L'éspace M des modules des surfaces cubiques marquées admet une structure hyperbolique complexe: une métrique (incomplète) de courbure holomorphe sectionelle constante. (2) Le groupe fondamentale de M contient un sousgroupe normale qui n'est pas de géneration finie. (3) Le groupe fondamentale de M n'est pas un réseau dans un groupe de Lie semisimple.*

Rémarque. Nos méthodes montrons aussi que la completée métrique de $(B^4 - \mathcal{H})/\Gamma_0$ est l'orbifolde B^4/Γ_0, isomorphe à l'éspace des modules des surfaces cubiques marquées stables.

Afin de préciser la notion de surface cubique lisse marquée, fixons un réseau L, une $\mathbb{Z}$-module libre avec une base e_0 , . . . , e_6 qui est muni de la forme quadratique telle que la base soit orthogonale et telle que $(\ e_0,\ e_0\ ) = 1$, $(\ e_k,\ e_k\ ) = -1$ pour $k > 0$. Soit $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Alors une *surface cubique marquée* est composée d'une surface cubique lisse S et d'une isométrie $\psi : L \longrightarrow H^2(S, \mathbb{Z})$ qui envoie η sur la classe d'un hyperplan. L'ensemble M de classes d'isomorphisme des surfaces cubiques marquées porte la structure d'une variété et de plus est une éspace de modules fines. Une construction de cette éspace a eté donée dans [9], oú on trouve aussi une compactification lisse C de M telle que les points de $C - M$ forme un diviseur à croisements normales.

Pour definir le groupe Γ_0, soit $\mathcal{E}$ l'anneau des entiers d'Eisenstein $\mathbb{Z}[\omega]$ oú $\omega = (-1 + \sqrt{-3})/2$, et considérons le produit Cartesian $\mathcal{E}^5$ muni d'une forme hermitien $h(v, w) = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Alors $(\mathcal{E}^5, h)$ est l'unique réseau autoduale sur les entiers d'Eisenstein qui est de signature $(4, 1)$. Donc $Aut(\mathcal{E}^5, h)$ est un sousgroupe discrète du groupe unitaire $U(h)$, qui agit sur $B^4 = \{\ \ell \in \mathbb{P}^4 \mid h|\ell < 0\ \}$. Notons que $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ est un corps de trois eléments et notons aussi qu'il y a une homomorphisme naturelle $Aut(\mathcal{E}^5, h) \longrightarrow Aut(\mathbb{F}_3^5, q)$ oú q est la forme quadratique obtenue par reduction de h modulo $\sqrt{-3}$. Notons par "P" la projectivization, et définissons un groupe Γ_0 d'automorphismes de B^4 par la suite exacte

$$1 \longrightarrow \Gamma_0 \longrightarrow PAut(\mathcal{E}^5, h) \longrightarrow PAut(\mathbb{F}_3^5, q) \longrightarrow 1.$$

Ce groupe est le groupe discrète du théorème principale. Les hyperplans de $\mathcal{H}$ sont définis par les equations $h(x, v) = 0$ pour des vecteurs v dans $\mathcal{E}^5$ avec $h(v) = 1$. Notons aussi que $PAut(\mathbb{F}_3^5, q)$ est isomorphe au groupe de Weyl du réseau E_6.

1. Main results

To a (marked) cubic surface corresponds a (marked) cubic threefold defined as the triple cover of $\mathbb{P}^3$ ramified along the surface. The period map f for these threefolds is defined on the moduli space M of marked cubic surfaces and takes its values in the quotient of the unit ball in $\mathbb{C}^4$ by the action of the projective monodromy group. This group Γ_0 is generated by complex reflections in a set of hyperplanes whose union we denote by $\mathcal{H}$. Then we have the following result:

Theorem. *The period map defines a biholomorphism*

$$f : M \longrightarrow \left(B^4 - \mathcal{H}\right)/\Gamma_0.$$

From this identification we obtain results on the metric structure and the fundamental group:

Corollaries. *(1) The moduli space of marked cubic surfaces carries a complex hyperbolic structure: an (incomplete) metric of constant holomorphic sectional curvature. (2) The fundamental group of the space of marked cubic surfaces contains a normal subgroup which is not finitely generated. (3) The fundamental group of the space of marked cubic surfaces is not a lattice in a semisimple Lie group.*

Remark. Our methods also show that the metric completion of $(B^4 - \mathcal{H})/\Gamma_0$ is the complex hyperbolic orbifold B^4/Γ_0, which is isomorphic to the moduli space of marked stable cubic surfaces.

To make precise the notion of smooth marked cubic surface, fix the lattice L to be the free $\mathbb{Z}$-module with basis $e_0, \dots, e_6$ endowed with the quadratic form for which the given basis is orthogonal and such that $(e_0, e_0) = 1$, $(e_k, e_k) = -1$ for $k > 0$. Let $\eta = 3e_0 - (e_1 + \cdots + e_6)$. Then a *marked cubic surface* consists of a smooth cubic surface S and an isometry $\psi : L \longrightarrow H^2(S, \mathbb{Z})$ which carries η to the hyperplane class. The set M of isomorphism classes of marked cubic surfaces has the structure of a variety and is a fine moduli space. A construction of it is described in [9], and a smooth compactification C is given for which the points of $C - M$ constitute a normal crossing divisor.

To define the group Γ_0, let $\mathcal{E}$ denote the ring of Eisenstein integers $\mathbb{Z}[\omega]$ where $\omega = (-1 + \sqrt{-3})/2$ is a cube root of unity, and consider the Cartesian product $\mathcal{E}^5$ endowed with the hermitian inner product $h(v, w) = -v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 + v_5\bar{w}_5$. Then $(\mathcal{E}^5, h)$ is the unique self-dual lattice over the Eisenstein integers with signature $(4, 1)$. Thus $Aut(\mathcal{E}^5, h)$ is a discrete subgroup of the unitary group $U(h)$, which acts on $B^4 = \{ \ell \in \mathbb{P}^4 \mid h|\ell < 0 \}$. Observe that $\mathcal{E}/\sqrt{-3}\mathcal{E} \cong \mathbb{F}_3$ is a field of three elements and that there is a natural homomorphism $Aut(\mathcal{E}^5, h) \longrightarrow Aut(\mathbb{F}_3^5, q)$ where q is the quadratic form obtained by reduction of h modulo $\sqrt{-3}$. Let "P" denote projectivization, and define a group Γ_0 of automorphisms of B^4 by

$$1 \longrightarrow \Gamma_0 \longrightarrow PAut(\mathcal{E}^5, h) \longrightarrow PAut(\mathbb{F}_3^5, q) \longrightarrow 1$$

This is the discrete group of the main theorem. The hyperplanes of $\mathcal{H}$ are defined by the equations $h(x, v) = 0$ for vectors v in $\mathcal{E}^5$ with $h(v) = 1$. Note that $PAut(\mathbb{F}_3^5, q)$ is isomorphic to the Weyl group of the E_6 lattice.

2. Construction of a period mapping

To construct the period mapping, we examine in detail the Hodge structures for the cubic threefolds. The underlying lattice $H^3(T, \mathbb{Z})$ is ten-dimensional, carries unimodular symplectic form Ω, and admits a Hodge decomposition of the form $H^3(T, \mathbb{C}) = H^{2,1} \oplus H^{1,2}$. Choose a generator σ for the group of automorphisms of T over $\mathbb{P}^3$, and note that it operates without fixed points on $H^3(T, \mathbb{Z})$. This action gives $H^3(T, \mathbb{Z})$ the structure of a five-dimensional module over the Eisenstein integers. It carries a hermitian form

$$h(x, y) = \frac{1}{2}(\Omega((\sigma - \sigma^{-1})x, y) + (\omega - \omega^{-1})\Omega(x, y))$$

which is unimodular and of signature $(4,1)$.

Now consider the quotient module $H^3(T,\mathbb{Z})/(1-\omega)H^3(T,\mathbb{Z})$ and observe that it can be identified isometrically with $(\mathbb{F}_3^5, q)$. We define a marking of T to be choice of such an isometry, and we claim that a marking of a cubic surface determines a marking of the corresponding threefold. Indeed, if γ is a primitive two-dimensional homology class on S the it is the boundary of a three-chain on T. Write this as $\partial\Gamma = \gamma$ and observe that $\sigma\Gamma$ has the same boundary. Thus the three-chain $c(\gamma) = (1-\sigma)\Gamma$ is a cycle. However, it is well-defined only up to addition of elements $(1-\sigma)\Delta$ where Δ is a three-cycle on T. Thus a homomorphism

$$c : H_2^{prim}(S,\mathbb{Z}) \longrightarrow H_3(T,\mathbb{Z})/(1-\sigma)$$

is defined. Since a marking of S can be viewed as a basis of $H_2^{prim}(S,\mathbb{Z})$, application of c to the basis elements defines a basis of $H_3(T,\mathbb{Z})/(1-\sigma)$, and this gives the required marking of the threefold.

The action of σ decomposes $H^3(T,\mathbb{C})$ into eigenspaces H^3_λ where λ varies over the primitive cube roots of unity. Because σ is holomorphic, the decomposition is compatible with the Hodge decomposition and one has

$$H^3_\omega = H^{2,1}_\omega \oplus H^{1,2}_\omega \qquad H^3_{\bar\omega} = H^{2,1}_{\bar\omega} \oplus H^{1,2}_{\bar\omega}.$$

The dimensions of the Hodge components can be found with the help of Griffiths' Poincaré residue calculus [5]. Details for this case are found in [3], section 5. One finds that

$$\dim H^{2,1}_\omega = 4, \quad \dim H^{1,2}_\omega = 1; \qquad \dim H^{2,1}_{\bar\omega} = 1, \quad \dim H^{1,2}_{\bar\omega} = 4$$

and from the Hodge-Riemann bilinear relations one finds that h has signature $(4,1)$.

Now let ϕ be a generator of the one-dimensional space $H^{2,1}_{\bar\omega}$ and let $\gamma_1, \ldots, \gamma_5$ be a standard basis of $H^3(T,\mathbb{Z})$ considered as an $\mathcal{E}$-module. By this we mean that the γ_k are orthogonal and that $h(\gamma_1,\gamma_1) = -1$ and $h(\gamma_k,\gamma_k) = 1$ for $k > 1$. Let $v(\phi,\gamma)$ be the vector in $\mathbb{C}^5$ with components

$$v_k = \int_{\gamma_k} \phi$$

One verifies that $h(v,v) < 0$ where now h is the hermitian form $-|z_1|^2 + |z_2|^2 + \cdots + |z_5|^2$. Thus the line generated by $v(\phi,\gamma)$ defines a point in $B^4 \subset \mathbb{P}^4$, and one checks that $v(\phi,\gamma) \notin \mathcal{H}$. By well-known constructions (the work of Griffiths), the period vector defines a holomorphic map from the universal cover of M to the ball which transforms according to the projectivized monodromy representation for marked cubic threefolds. The proof that Γ_0 is the projective monodromy group relies on the work of Libgober [6] and the first author [1]. Thus our construction yields a period map $f : M \longrightarrow (B^4 - \mathcal{H})/\Gamma_0$.

3. Properties of the period mapping

We must now show that f is bijective. For injectivity, consider once again the period vector $v(\phi,\gamma)$. The vectors γ_k can be decomposed into eigenvectors γ_k' and γ_k'' for σ, with eigenvectors ω and $\bar\omega$, respectively. Let $\hat\gamma_k'$ and $\hat\gamma_k''$ denote elements of the corresponding dual basis. Because ϕ is an eigenvector with eigenvalue $\bar{\omega}$, its integral over γ_k' vanishes, so that

$$\phi = \sum_k \hat{\gamma}_k'' \int_{\gamma_k''} \phi = \sum_k \hat{\gamma}_k'' \int_{\gamma_k} \phi$$

Thus the components of $v(\phi, \gamma)$ determine ϕ as an element of $H^3_{\bar{\omega}}$. Consequently the line $\mathbb{C}v(\phi, \gamma)$ determines the complex Hodge structure $H^3_{\bar{\omega}}$. Viewing the Hodge components of $H^3_{\bar{\omega}}$ as as subspaces of $H^3(T, \mathbb{C})$, we may take their conjugates to determine the complex Hodge structure H^3_{ω}. These two complex Hodge structures determine the Hodge structure on $H^3(T, \mathbb{Z})$. Thus, by the Torelli theorem of Clemens-Griffiths [4], the period vector $v(\phi, \gamma)$ determines the cubic threefold T up to isomorphism. It remains to show that T, which perforce is a cyclic cubic threefold, determines its ramification locus uniquely. This follows from the fact that the locus in question is a planar component of the Hessian surface.

To prove surjectivity we first consider a smooth compactification C of M by a normal crossing divisor D, e.g., the one given by Naruki [9], as well as the Satake compactification $\overline{B^4/\Gamma_0}$, obtained by adding fourty points, the "cusps," each corresponding to a null point of $P(\mathbb{F}_3^5, q)$. By well-known results [2] in complex variable theory, the period map has a holomorphic extension to a map $\bar{f}$ from C to the Satake compactification. Since C is compact, $\bar{f}$ is open, and $\overline{B^4/\Gamma_0}$ is connected, we conclude that $\bar{f}$ is surjective.

4. Boundary components

To pass from surjectivity of $\bar{f}$ to surjectivity of f, we must show that $\bar{f}$ maps the compactifying divisor D to the complement of $(B^4 - \mathcal{H})/\Gamma_0)$ in the Satake compactification. To this end write D as a sum of irreducible components, $D = \bigcup D_i' \cup \bigcup D_j''$, where D_i' parametrizes nodal cubic surfaces via the map to the geometric invariant theory compactification of the moduli space of smooth cubics, and where in the same way the D_j'' parametrize cubics with an A_2 singularity.

Now consider a one-parameter family of cubic surfaces with smooth total space acquiring a node. Its local equation near the node has the form $x^2 + y^2 + z^2 = t$ and the corresponding family of cyclic cubic threefolds has the form $x^2 + y^2 + z^2 + w^3 = t$. The local monodromy of the latter has order six, its eigenvalues are primitive sixth roots of unity, and the space of vanishing cycles is two-dimensional. (These facts are well-known adn the relevant literature and arguments are summarized in [3], section 6). From [7] we conclude that coefficients of the period vector on vanishing cycles are of the form $A(t)t^{1/6} + B(t)t^{5/6}$ where A and B are holomorphic. Now the space of vanishing cycles is invariant under the action of σ and so constitutes a rank one $\mathcal{E}$-submodule. One can choose a generator δ for it so that $h(\delta, \delta) = 1$, and then one has

$$\lim_{t \to 0} \int_\delta \phi = 0$$

Thus the limiting value of the period vector lies in the orthogonal complement of δ. In other words, $\bar{f}(D_i')$ lies in $\mathcal{H}/\Gamma_0$, as required.

Consider next a one-parameter family of cubic surfaces with smooth total space whose central fiber acquires an A_2 singularity. Its local equation is $x^2 + y^2 + z^3 = t$ and the corresponding family of cyclic cubic threefolds has local equation $x^2 + y^2 + z^3 + w^3 = t$. In this case the local monodromy is of infinite order. After replacing t by t^3 one finds an expansion of the form $\phi(t) = A(t)(\log t)\,\hat{\gamma} +$ terms bounded in t, where $A(0) \neq 0$ and where $\hat{\gamma}$ is an integer cohomology class which is isotropic for h. Consequently the line $\mathbb{C}\phi(t)$ converges to the isotropic line $\mathbb{C}\hat{\gamma}$ as t converges to zero, hence converges to a cusp in the Satake compactification.

6. The corollaries

Finally, we comment on the collaries. Part (a) is immediate. For part (b) let K denote the kernel of the map $\pi_1(M) \longrightarrow \Gamma_0$. Then K is isomorphic to the fundamental group of $B^4 - \mathcal{H}$ and it is easy to see that its abelianization is not finitely generated. We remark that K is not free: there are many sets of commuting elements corresponding to normal crossings of $\mathcal{H}$. For (c) we note first that for lattices in semisimple Lie group of real rank greater than one, the results of Margulis [8] imply finite generation of all normal subgroups. The rank one case can be treated separately, as was shown to us by Michael Kapovich.

References

[1] D. Allcock, New complex and quaternion-hyperbolic reflection groups, submitted, http://www.math.utah.edu/~allcock

[2] A. Borel, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays. J. Differential Geometry 6 (1972), 543–560.

[3] J. Carlson and D. Toledo, Discriminant complements and kernels of monodromy representations, 22 pp, submitted, http://www.math.utah.edu/~carlson/eprints.html

[4] C. H. Clemens and P. A. Griffiths, the intermediate Jacobian of the cubic threefold, Ann. of Math. 95 (1972) 281–356.

[5] P.A. Griffiths, On the periods of certain rational integrals: I and II, Ann. of Math. **90** (1969) 460-541

[6] A. Libgober, On the fundamental group of the space of cubic surfaces, Math. Zeit. **162** (1978) 63–67.

[8] B. Malgrange, Intégrales asymptotiques et monodromie, Ann. Sci. Ecole Norm. Sup., ser. 4 1974, tome 7, 405–430.

[8] G. A. Margulis, Quotient groups of discrete subgroups and measure theory, Funct. Anal. Appl. 12 (1978), 295–305.

[9] I. Naruki, Cross ratio variety as a moduli space of cubic surfaces, Proc. London Math. Soc. (3) 45 (1982) 1-30.