

A MONSTROUS PROPOSAL

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Dedicated to Domingo Toledo and to John McKay

ABSTRACT. We explain a conjecture relating the monster simple group to an algebraic variety that was discovered in a non-monstrous context.

The purpose of this note is to explain a conjecture I have circulated privately since 1997. On one level it is purely about group theory and complex hyperbolic geometry, but if it is true then the most natural explanation for it would be algebra-geometric: a certain ball quotient would be the moduli space for some sort of objects, the objects admitting some sort of marking related to the monster simple group. My original grounds for the conjecture were flimsy, but in his dissertation, Tathagata Basak discovered some very suggestive coincidences. The conjecture is still speculative, but now I am taking it seriously.

In brief, Conway described the bimonster $M \times M : 2$ as being generated by 16 involutions satisfying some braid and commutation relations, subject to the additional relation that a certain word w has order 10. On the other hand, I discovered a certain complex analytic orbifold X having a nice uniformization by complex hyperbolic 13-space B^{13} . There is nothing obviously monstrous about X , and I had no thought of the monster when constructing it. Then I noticed that its fundamental group $\pi_1^{\text{orb}}(X)$ contains 16 elements satisfying exactly the same braid and commutation relations, but having order 3 rather than 2. Later, Basak discovered that they generate all of $\pi_1^{\text{orb}}(X)$, that w has order 20, and that the generators can be extended to a set of 26 in exactly the same way as Conway's bimonster generators. This note formulates a conjecture that explains these coincidences by giving a uniform interpretation of both groups.

1. CONJECTURE

In [1], we considered a certain lattice L over the Eisenstein integers $\mathcal{E} = \mathbf{Z}[\omega = \sqrt[3]{1}]$, of signature $(1, 13)$; a lattice means a free module equipped

Date: June 30, 2007.

Partly supported by NSF grant DMS-0245120.

with an \mathcal{E} -valued Hermitian form $\langle | \rangle$. L is the most natural such lattice, in the sense that its underlying \mathbf{Z} -lattice is a scaled version of the unique even unimodular lattice of signature $(2, 26)$. We will describe L explicitly below, but for now it suffices to describe L as the unique \mathcal{E} -lattice of signature $(1, 13)$ satisfying $L = \theta L^*$, where $\theta = \omega - \bar{\omega} = \sqrt{-3}$ and the asterisk denotes the dual lattice. In particular, all inner products in L are divisible by θ . Because of this, if $r \in L$ has norm $r^2 = \langle r|r \rangle = -3$ (such an r is called a root of L), then the map

$$(1) \quad x \mapsto x - (1 - \omega) \frac{\langle x|r \rangle}{\langle r|r \rangle} r$$

is an isometry of L ; it sends r to ωr and fixes r^\perp pointwise. This is a complex reflection of order 3, also called a triflection, and r^\perp is called its mirror.

The importance of L in [1] was that it allowed the construction of a complex hyperbolic reflection group of record dimension, namely the subgroup Γ of $\text{Aut } L$ generated by the triflections in the roots of L . Now, $\text{Aut } L$ acts on the complex 13-ball B^{13} consisting of positive lines in $\mathbf{C}^{1,13} = L \otimes_{\mathcal{E}} \mathbf{C}$. I proved that Γ has finite-volume fundamental domain, and Basak has gone further [4], proving that Γ is all of $\text{Aut } L$.

We define $X = B^{13}/P\Gamma$, which is an algebraic variety and a complex analytic orbifold. We write \mathcal{H} for the union of the mirrors of the triflections, Δ for the image of \mathcal{H} in X , and G for the orbifold fundamental group $\pi_1^{\text{orb}}(X - \Delta)$. By a meridian we mean an element of G represented by a small loop in $X - \Delta$ that encircles Δ once positively at a generic point of Δ , or any conjugate of such a loop.

Conjecture. *The quotient of G by the normal subgroup generated by the squares of the meridians is the bimonster, i.e., the semidirect product $M \times M : 2$, where M is the monster simple group and $\mathbf{Z}/2$ acts by exchanging the factors.*

The conjecture gives a uniform description of the bimonster and $P\Gamma$, as the deck groups of the covering spaces of X which are universal among those having 2- and 3-fold branching along Δ . A geometric way to understand it is to consider the orbifold structure along $\Delta \subseteq X$. Because of the triflections, a generic point of Δ has local group $\mathbf{Z}/3$. The idea of the conjecture is to rub out this $\mathbf{Z}/3$ and replace it by $\mathbf{Z}/2$. This should change the orbifold fundamental group from $P\Gamma$ to the bimonster. The stated form of the conjecture expresses this idea without having to make sense of the new orbifold structure at non-generic points of Δ . (This can be done, but involves a digression; see remark (6) in section 3.)

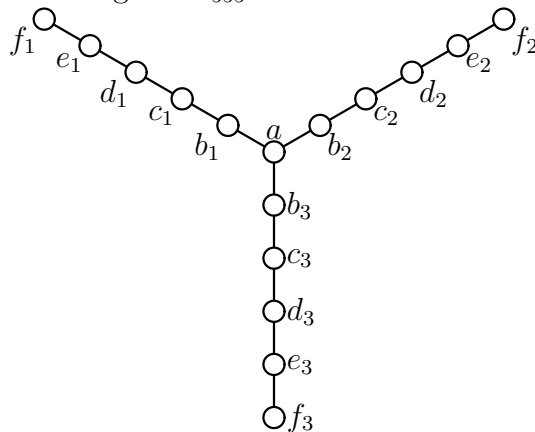
If the conjecture holds then we should expect some reason for it to be true. One possibility that would be very pretty is for X to have a moduli interpretation. If \mathcal{M} is a moduli space parameterizing some sort of algebra-geometric object, then a cover $\widetilde{\mathcal{M}}$ of \mathcal{M} , possibly ramified somewhere, can be thought of as a moduli space parameterizing suitably marked versions of the objects.

For example we may take \mathcal{M} to be the moduli space of unordered 12-tuples in $\mathbf{C}P^1$ (say, in the sense of geometric invariant theory) and $\widetilde{\mathcal{M}}$ to be the moduli space of ordered 12-tuples. In this case the marking is the ordering, the deck group is the symmetric group S_{12} , and $\widetilde{\mathcal{M}}$ is the universal cover of \mathcal{M} with 2-fold branching along the discriminant. This example is especially relevant because \mathcal{M} may also be described as a 9-ball quotient; indeed B^9 is the universal cover of $\mathcal{M} - (\text{one point})$ with 3-fold branching along the discriminant. So this situation is an exact analogue of that of the conjecture, with B^{13} replaced by B^9 and the bimonster by S_{12} . In fact, the group acting on B^9 is also the analogue of our $P\Gamma$. See [1], [10] and [22] for more information.

It would be very pretty if the same phenomenon happened for X ; then the objects it parameterized would admit a sort of marking, and varying the objects in a family would permute the markings by an action of the bimonster. The branched covering $B^{13} \rightarrow X$ would parameterize the same objects, but equipped with a different notion of marking.

2. EVIDENCE

The origin of the conjecture is a coincidence of diagrams. Conway conjectured that the bimonster can be presented as the quotient of the Coxeter group with diagram Y_{555}



by a single extra relation (see [7]). There are several ways to write the extra relation, one of which is $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$, called the spider relation. Ivanov [13] and Norton [18] proved his conjecture.

The diagram Y_{550} arose as figure 5.1 of my paper [1], in a manner suggesting its extension to Y_{555} . It describes an arrangement of 11 vectors in $\mathbf{C}^{1,9}$ of norm -3 such that their triflections braid ($aba = bab$) or commute ($ab = ba$) according to whether the corresponding nodes of the figure are joined or not. The obvious generalization to Y_{555} is the arrangement of vectors in $\mathbf{C}^{1,13}$

$$\begin{aligned} a &= (\quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad 1, \bar{\omega}) \\ b_1 &= (\quad , \quad , \quad , \quad \bar{\theta}; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad 0, 1) \\ c_1 &= (\quad , \quad 1, \quad 1, \quad 1; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad) \\ d_1 &= (\quad , \quad \theta, \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad) \\ e_1 &= (-1, -1, \quad 1, \quad ; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad) \\ f_1 &= (\quad \bar{\theta}, \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad , \quad , \quad ; \quad , \quad) \end{aligned}$$

with blanks indicating zeros and b_2, \dots, f_2 and b_3, \dots, f_3 got from b_1, \dots, f_1 by permuting the first three blocks of coordinates. Here we are referring to the Hermitian form with inner product matrix $\text{diag}[-1, \dots, -1] \oplus \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$. The triflections in these roots braid or commute according to the Y_{555} diagram, and the roots span a copy of L in the form $L = \Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H$. Here, $\Lambda_4^{\mathcal{E}}$ is the E_8 root lattice regarded as a 4-dimensional \mathcal{E} -lattice and H is the ‘‘hyperbolic cell’’ $\begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$. For each $i = 1, 2, 3$, the i th summand $\Lambda_4^{\mathcal{E}}$ is spanned by c_i, \dots, f_i .

The Artin group A of Y_{555} is the group with one generator for each node of the diagram, subject to the same commutation and braiding relations as in the Coxeter group. Forcing the generators to have order 2 yields the Coxeter group of Y_{555} , so Conway provides us with a surjection from A to the bimonster. On the other hand, forcing them to have order 3 gives a map from A into Γ by sending the Artin generators to the triflections in a and the b_i, \dots, f_i . Basak has proven that these 16 triflections generate $\Gamma = \text{Aut } L$, so both the bimonster and Γ are quotients of A , with generators of order 2 and 3 respectively. I expect that there is also a map $A \rightarrow G$, sending the Artin generators to meridians, so that $A \rightarrow P\Gamma$ is the composition $A \rightarrow G \rightarrow P\Gamma$.

This suggests that G is really the central object, and that it is ‘like’ A , subject to some extra relations. In the presence of these extra relations, forcing the meridians to have order 3 reduces the group to Γ . On the other hand, with luck, when the meridians are forced to have order 2, the extra relations become equivalent to the spider relation, giving the conjectured map $G \rightarrow M \times M : 2$. A natural sanity check is whether the spider relation is compatible with this picture. Basak considered

the word $S = ab_1c_1ab_2c_2ab_3c_3$ as an element of A and computed the order of its image in Γ , which turns out to be 20. This implies that S , regarded as a element of G , has order a multiple of 20 (or infinite order), which is certainly compatible with the spider relation $S^{10} = 1$ in the bimonster. Indeed, 20 is a notably small multiple of 10. Basak also found similar compatibilities using other words.

He has also made the striking discovery that the 16 triflections may be extended to a set of 26 in exactly the same way as Conway's 16 bimonster generators. Namely, Conway observed that the map from the Coxeter group of Y_{555} to the bimonster extends to a map from the Coxeter group of a larger graph, the incidence graph of the 13 points and 13 lines of the projective plane over \mathbf{F}_3 . The symmetries of this finite projective plane, including the dualities exchanging points and lines, extend to automorphisms of the bimonster. Basak found a set of 26 roots of L , the reflections in which commute or braid according to this graph; indeed all inner products are 0 except for $\langle p|l \rangle = \theta$ when p is the root corresponding to a point of $\mathbf{P}^2(\mathbf{F}_3)$ and l is the root corresponding to a line containing it. It is not really surprising that the 16 roots of Y_{555} fit in $\mathbf{C}^{1,13}$, but it *is* surprising that 26 vectors with specified inner products just happen to fit in a 14-dimensional space, and just happen to span a very natural lattice. The mirrors in B^{13} corresponding to the 13 points (resp. lines) of $\mathbf{P}^2(\mathbf{F}_3)$ are mutually orthogonal and meet at a single point, say P (resp. L). The midpoint of the segment joining P and L has stabilizer $\mathrm{GL}_3(\mathbf{F}_3):2$ in $P\Gamma$, realizing every automorphism of $\mathbf{P}^2(\mathbf{F}_3)$ including dualities exchanging points and lines. Presumably there is a map from the Artin group of the 26-node graph to G , but I have not investigated this.

A final coincidence is that one of the maximal subgroups of the monster has structure $3^{1+12} \cdot 2 \cdot \mathrm{Suz} : 2$, where Suz denotes Suzuki's sporadic finite simple group, while $P\Gamma$ contains a subgroup K with structure $(\mathrm{Im} \mathcal{E}) \cdot \Lambda_{12}^{\mathcal{E}} : (6 \cdot \mathrm{Suz})$. Here, $\mathrm{Im} \mathcal{E} \cong \mathbf{Z}$ and $\Lambda_{12}^{\mathcal{E}}$ denote the additive groups of the imaginary part of \mathcal{E} and of the complex Leech lattice. The extension defining the Heisenberg group $(\mathrm{Im} \mathcal{E}) \cdot \Lambda_{12}^{\mathcal{E}}$ is given by the imaginary part of the inner product on $\Lambda_{12}^{\mathcal{E}}$, and $6 \cdot \mathrm{Suz}$ is $\mathrm{Aut} \Lambda_{12}^{\mathcal{E}}$. Identifying the scalar ω of $6 \cdot \mathrm{Suz}$ with a generator of $\mathrm{Im} \mathcal{E}$ reduces K to $3^{1+12} \cdot 2 \cdot \mathrm{Suz}$. There are two ways to do this, corresponding to the two generators of $\mathrm{Im} \mathcal{E}$; this mimics the construction of $3^{1+12} \cdot 2 \cdot \mathrm{Suz} : 2 \subseteq M$ as a quotient of $3^{1+12} : 6 \cdot \mathrm{Suz} : 2$ in [15, secs. 3.3 and 5.2].

K is a very natural subgroup of $P\Gamma$, namely the stabilizer of the null vector $(0; 0, 1)$ in the realization of L as $\Lambda_{12}^{\mathcal{E}} \oplus \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$. Since there are no mirrors in B^{13} passing through $(0; 0, 1)$, K is a subgroup of G , not

just a subquotient of G . Therefore we have a natural subgroup of G , which modulo a natural relation has index 2 in a maximal subgroup of the monster.

3. REMARKS

We close with some remarks that seem relevant but are not in the main line of ideas.

(1) There are infinitely many triflections in Γ , but their mirrors form a locally finite arrangement in B^{13} because Γ is discrete. Γ acts transitively on roots, so all the triflections are conjugate (up to inversion) and $\Delta \subseteq X$ is irreducible. The only complex reflections in Γ or $P\Gamma$ are the triflections we have considered.

(2) There exist points of B^{13} that lie on no mirrors, yet have non-trivial stabilizer in $P\Gamma$. For example, the stabilizer $6 \cdot Suz$ is possible for points near the ideal point of B^{13} given by $(0; 0, 1) \in \Lambda_{12}^{\mathcal{E}} \oplus \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$. A consequence is that $X - \Delta$ has nontrivial orbifold structure even though the obvious orbifold points have been removed. This phenomenon does not occur for finite complex reflection groups or real hyperbolic Coxeter groups.

(3) Conway and Pritchard [8] used real hyperbolic geometry to study the bimonster and certain other finite groups as quotients of Coxeter groups. I don't know of any connection between their real hyperbolic geometry and complex hyperbolic 13-space B^{13} .

(4) The study of complex reflection groups as quotients of Artin groups goes back to Coxeter [9]. In our language, he studied the Artin groups for the diagrams $A_{n=1, \dots, 5}$ modulo the relations that the generators have order 3. In the first four cases one gets a finite complex reflection group acting on \mathbf{C}^n . The last case gives a group of structure $(\text{Im } \mathcal{E}) \cdot \Lambda_4^{\mathcal{E}}: \text{Aut } \Lambda_4^{\mathcal{E}}$, which acts as a complex reflection group on the 5-ball, fixing a point on the boundary. Coxeter showed that quotienting by the central $\text{Im } \mathcal{E}$ gives a complex reflection group acting cocompactly on \mathbf{C}^4 .

(5) One can build up Y_{555} from smaller diagrams by beginning with three A_4 's, which describe finite complex reflection groups, "affinizing" them by enlarging them to A_5 's (see the previous remark), and then "hyperbolizing" the result by adjoining a single extra node, joined to each of the three affinizing nodes. This is analogous to (say) extending the E_8 Coxeter group, which acts on the 7-sphere, to E_9 , which acts on Euclidean 8-space, and then to E_{10} , which acts on real hyperbolic 9-space. Essentially the same process leads to diagrams like the one in [6], which is a "hyperbolization" of $A_{11}D_7E_6$.

(6) We have avoided the issue of making sense of the alteration of orbifold structures described right after the conjecture. But this dodge is not necessary. Suppose $b \in B^{13}$, S is its stabilizer in $P\Gamma$, $R \subseteq S$ is the subgroup generated by the triflections which fix b , and U is a small ball around b . Then R is a direct product of some copies of the triflection groups associated to A_1, \dots, A_4 in remark (4). These act on U by the direct product of their triflection representations and possibly a subspace where all the factors act trivially. By Chevalley's theorem, U/R is a smooth variety. Obviously, the image of \mathcal{H} therein is a divisor D , and S/R acts on U/R , preserving D . Let U' be the cover of U/R which is universal among those having 2-fold ramification along D and no other ramification, and let R' be the deck group of U' over U/R . By [19], R' is the direct product of Coxeter groups associated to the same diagrams A_1, \dots, A_4 as for R , and U' is an open set in \mathbf{C}^{13} . Also, R' acts on U' as the product of the Coxeter groups' standard representations, again possibly with some fixed subspace. Now, an element of S/R preserves D , so it has a lift to an automorphism of U' , in fact $|R'|$ many lifts. We take S' to be the group consisting of all such lifts. Then

$$U'/S' \cong (U'/R') / (S'/R') \cong (U/R) / (S/R) \cong U/S,$$

where the first and third isomorphisms are of complex analytic orbifolds. The middle isomorphism is one of complex analytic varieties, and is a complex analytic orbifold isomorphism away from the image of $D \subseteq U/R = U'/R'$. We have equipped $U/S \subseteq B^{13}/P\Gamma$ with an orbifold structure in which the generic point of Δ has local group $\mathbf{Z}/2$ rather than $\mathbf{Z}/3$. It is easy to see that such a structure is unique.

(7) The truth of the conjecture would imply that the orbifold of the previous remark is the quotient of a complex manifold by an action of the bimonster. Then each component of the preimage of Δ is a smooth hypersurface fixed pointwise by an involution α in the bimonster. The centralizer of α is $\langle \alpha \rangle$ itself times a copy of the monster, so the monster acts on the fixed-point set of α . We wonder if this complex 12-manifold (or a suitable compactification of it) could serve as the 24-dimensional "monster manifold" sought by Hirzebruch et. al. [12, pp. 86–87] in their study of elliptic cohomology.

(8) We have suggested that X may be a moduli space; we do know that it contains several lower-dimensional moduli spaces. Namely, the moduli space of unordered 12-tuples in $\mathbf{C}P^1$ is the quotient of B^9 by $P\text{Aut}(\Lambda_4^\mathcal{E} \oplus \Lambda_4^\mathcal{E} \oplus H)$, the moduli space of genus 4 curves is the quotient of B^9 by $P\text{Aut}(\Lambda_4^\mathcal{E} \oplus N \oplus [-3] \oplus H)$, where N is the orthogonal complement of a root in $\Lambda_4^\mathcal{E}$, and the moduli space of cubic threefolds is the quotient

of B^{10} by $P\text{Aut}(\Lambda_4^\mathcal{E} \oplus \Lambda_4^\mathcal{E} \oplus [-3] \oplus H)$. We remark that this last lattice may be constructed from Y_{551} in the same manner as L was from Y_{555} . See [10], [22], [1], [14], [2] and [16] for more information about these ball quotients.

(9) The Coxeter group of the Y_{555} diagram appears in the monodromy of the T_{666} surface singularity $x^6 + y^6 + z^6 + \lambda xyz = 0$, where λ is a nonzero constant (see [3, sec. 3.8] and [11]), and also in Mukai's analysis [17] of the moduli space of 12 points in $(\mathbf{P}^5)^5$. I don't know any way to fit these facts into my conjectural framework. If there is a connection then Conway and Pritchard's real-hyperbolic constructions are probably also relevant; see remark (3).

(10) Simons [20] has studied a simpler version of our situation, concerning Y_{333} and a quotient of the Coxeter group with diagram the incidence graph of the 7 lines and 7 points of $\mathbf{P}^2(\mathbf{F}_2)$, which turns out to be $\text{O}_{\bar{8}}(2):2$. I found 14 tetra reflections (order 4 complex reflections) of B^7 that satisfy the commutation and braid relations given by this diagram. One can choose the roots to have norm -2 and inner products 0 or $1 \pm i$, spanning the lattice $\Lambda_2^\mathcal{G} \oplus \Lambda_2^\mathcal{G} \oplus \Lambda_2^\mathcal{G} \oplus \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$, where \mathcal{G} is the Gaussian integers $\mathbf{Z}[i]$ and $\Lambda_2^\mathcal{G}$ is the D_4 root lattice regarded as a lattice over \mathcal{G} . Because these are tetra reflections, it is easy to turn them into involutions: one simply reduces the lattice modulo $1+i$. Then the tetra reflections act on an 8-dimensional vector space over \mathbf{F}_2 equipped with a quadratic form of minus type. This provides a nice perspective on Simons' group, although it does not seem to give a new proof of his theorem. Basak has developed these ideas in a quaternionic context in [5].

REFERENCES

- [1] Allcock, D., The Leech lattice and complex hyperbolic reflections, *Invent. Math.* **140** (2000) 283-301.
- [2] Allcock, D., Carlson, J., and Toledo, D., The Moduli Space of Cubic Threefolds as a Ball Quotient, submitted.
- [3] Arnold, V., Gusein-Zade, S., and Varchenko, A., *Singularities of Differentiable Maps vol. II*, Birkhäuser, 1988.
- [4] Basak, T., The complex Lorentzian Leech lattice and the bimonster. *J. Algebra* **309** (2007) 32-56.
- [5] Basak, T., Reflection group of the quaternionic Lorentzian Leech lattice, *J. Algebra* **309** (2007) 57-68.
- [6] Borchers, R., Conway, J. H., Queen, L. and Sloane, N., A monster Lie algebra?, *Adv. Math.* **53** (1984) 75-79. Reprinted in Conway, J. and Sloane, N., *Sphere Packings, Lattices and Groups*, Springer-Verlag 1993, 568-571.

- [7] Conway, J. H., Norton, S. P. and Soicher, L. H., The bimonster, the group Y_{555} and the projective plane of order 3, in *Computers in Algebra* (M. C. Tangora, Ed.), pp. 27–50, Dekker, New York, 1988.
- [8] Conway, J. H. and Pritchard, A. D., Hyperbolic reflexions for the bimonster and $3Fi_{24}$, in *Groups, Combinatorics and Geometry (Durham 1990)* (M. Liebeck and J. Saxl, Eds.), London Math. Soc. Lecture Notes, vol. 165, pp. 23–45, Cambridge University Press, Cambridge 1992.
- [9] Coxeter, H. S. M., Factor groups of the braid group, in *Proceedings of the Fourth Canadian Mathematical Congress, Banff, 1957*, University of Toronto Press, Toronto 1959, pp. 95–122.
- [10] Deligne, P. and Mostow, G. D., Monodromy of hypergeometric functions and nonlattice integral monodromy, *Publ. Math. I.H.E.S.* **63** (1986) 5–89.
- [11] Gabrielov, A. M., Bifurcation, Dynkin diagrams and the modality of isolated singularities, *Functional Anal. Appl.* **8** (1974), no. 2, 94–98.
- [12] Hirzebruch, F., Berger, T. and Jung, R., *Manifolds and modular forms*, Friedr. Vieweg & Sohn, Braunschweig, 1992.
- [13] Ivanov, A. A., A geometric characterization of the Monster, in *Groups, combinatorics & geometry (Durham, 1990)*, 46–62, London Math. Soc. Lecture Note Ser., 165, Cambridge Univ. Press, Cambridge, 1992.
- [14] Kondō, S., The moduli space of curves of genus 4 and Deligne-Mostow’s complex reflection groups, *Algebraic geometry 2000, Azumino (Hotaka)*, 383–400, *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan, Tokyo, 2002.
- [15] Linton, S., Parker, R., Walsh, P. and Wilson, R., Computer construction of the monster, *J. Group Theory* **1** (1998) 307–337.
- [16] E. Looijenga and R. Swierstra, The period map for cubic threefolds, to appear.
- [17] Mukai, S., Geometric realization of T -shaped root systems and counterexamples to Hilbert’s fourteenth problem, *Algebraic transformation groups and algebraic varieties*, pp. 123–129, *Encyclopaedia Math. Sci. vol. 132*, Springer, Berlin, 2004.
- [18] Norton, S. P., Constructing the Monster, *Groups, combinatorics & geometry (Durham, 1990)*, pp. 63–76, *London Math. Soc. Lecture Note vol. 165*, Cambridge Univ. Press, Cambridge, 1992.
- [19] Orlik, P. and Solomon, L., Discriminants in the theory of reflection groups, *Nagoya Math. J.* **109** (1988) 23–45.
- [20] Simons, C., Deflating infinite Coxeter groups to finite groups, *Proceedings on Moonshine and related topics (Montréal, QC, 1999)*, 223–229, *CRM Proc. Lecture Notes* **30** Amer. Math. Soc., Providence, RI, 2001.
- [21] Swift, J., *A Modest Proposal*, pamphlet, 1729.
- [22] Thurston, W. P., Shapes of polyhedra and triangulations of the sphere, *The Epstein Birthday Schrift*, pp. 511–549 (electronic), *Geom. Topol. Monogr. vol. 1*, Geom. Topol. Publ., Coventry, 1998.

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