Reflection Groups on the Octave Hyperbolic Plane

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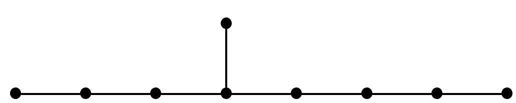
Abstract.
For two different integral forms K of the exceptional Jordan algebra we show that $\operatorname{Aut} K$ is generated by octave reflections. These provide 'geometric' examples of discrete reflection groups acting with finite covolume on the octave (or Cayley) hyperbolic plane $\mathbb{O}H^2$, the exceptional rank one symmetric space. (The isometry group of the plane is the exceptional Lie group $F_{4(-20)}$.) Our groups are defined in terms of Coxeter's discrete subring $\mathcal{K}$ of the nonassociative division algebra $\mathbb{O}$ and we interpret them as the symmetry groups of "Lorentzian lattices" over $\mathcal{K}$. We also show that the reflection group of the "hyperbolic cell" over $\mathcal{K}$ is the rotation subgroup of a particular *real* reflection group acting on $H^8 \cong \mathbb{O}H^1$. Part of our approach is the treatment of the Jordan algebra of matrices that are Hermitian with respect to any real symmetric matrix.

1 Introduction

The octave hyperbolic plane $\mathbb{O}H^2$ is the exceptional rank one symmetric space; it is very similar to the more familiar real (and complex and quaternionic) hyperbolic spaces. There is a natural notion of a hyperplane in $\mathbb{O}H^2$, and of a reflection in such a hyperplane (the mirror of the reflection). In this paper we explain this and construct two discrete groups of isometries of $\mathbb{O}H^2$ that are generated by reflections. One can also define the 'octave hyperbolic line' $\mathbb{O}H^1$, and it turns out to be isometric with the real hyperbolic space H^8. In this setting, a hyperplane is just a point and the octave reflection therein is just central inversion. We will construct a discrete group of isometries of $\mathbb{O}H^1$ generated by octave reflections and show that it is the rotation subgroup of a certain group of isometries of H^8 generated by *real* reflections. This real reflection group is easy to describe; it has a simplex for its fundamental domain, with Coxeter diagram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We construct all three of our groups by defining them arithmetically and then finding sufficiently many reflections to generate them. It turns out that in each case one can easily write down a large set of reflections preserving the relevant algebraic structure, and that the mirrors of these reflections are arranged in a pattern governed by the combinatorics of the E_7 and E_8 root lattices. We use facts about the geometry of these lattices to prove that these "known" reflections actually generate the entire group. To the author's knowledge this is the first geometric construction of discrete groups of isometries of $\mathbb{O}H^2$.

To discuss $\mathbb{O}H^2$ in any depth one must introduce various spaces of 3×3 matrices with entries in the nonassociative field $\mathbb{O}$ of octaves, such as the space J_I of Hermitian matrices; such matrices should informally regarded as "Hermitian forms" on the nonexistent vector space $\mathbb{O}^3$. One can

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define the determinant of such a "Hermitian form" in terms of a certain cubic form. The group $H_I^{\mathcal{K}}$ of linear transformations of J_I that preserve the determinant form is isomorphic to the exceptional Lie group $E_{6(-26)}$. This representation of H_I is very closely analogous to the action of $PGL_3\mathbb{C}$ on the space of 3×3 complex Hermitian matrices given for $g \in GL_3\mathbb{C}$ by $\Phi \mapsto g\Phi g^*$. (One can replace $\mathbb{C}$ here with any associative ring.) One should think of H_I as "$PGL_3\mathbb{O}$", and the stabilizer in H_I of some matrix Φ as the unitary group preserving the "Hermitian form on $\mathbb{O}^3$" with inner product matrix Φ. For appropriate choice of Φ this stabilizer is isomorphic to $G = F_{4(-20)} = \operatorname{Aut}\mathbb{O}H^2$, and one can use this to construct $\mathbb{O}H^2$.

To construct discrete subgroups of G one can simply take Φ to be integral and consider the stabilizer of Φ in the subgroup $H_I^{\mathcal{K}}$ of H_I that preserves the set of integral Hermitian matrices. By an integral matrix we mean one whose entries lie the natural discrete subring $\mathcal{K}$ of $\mathbb{O}$ discovered by Coxeter [9]. In light of the above interpretation of G and H_I, we may think of this discrete subgroup as being essentially the isometry group of the "lattice" over $\mathcal{K}$ with inner product matrix Φ. Our main result is that if Φ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \tag{1.1}$$

then the discrete group is generated by octave reflections. We note that the first of these groups has been studied by Gross [13] in a different guise (see section 7). If one uses 2×2 matrices over $\mathbb{O}$ then one finds that the analogous discrete subgroup defined by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is also generated by octave reflections, and that is has index 2 in the real reflection group with Coxeter diagram given above. We note in passing that the finite simple group ${}^3D_4(2)$ described in the ATLAS [7] as an octave reflection group in the compact form of F_4 has been realized by Elkies and Gross [10] as the stabilizer in $H_I^{\mathcal{K}}$ of a certain positive definite matrix with entries in $\mathcal{K}$ and unit determinant.

Our construction of reflection groups in $F_{4(-20)}$ reveals the source of the ideas for this investigation. In [1] we constructed a large number of discrete reflection groups acting with finite covolume on complex and quaternionic hyperbolic spaces $\mathbb{C}H^n$ and $\mathbb{H}H^n$. These groups were defined as the automorphism groups of certain Lorentzian lattices over discrete subrings of $\mathbb{C}$ and $\mathbb{H}$. (A Lorentzian lattice is a free module equipped with a Hermitian form of signature $- + + \cdots +$. See [1] for details.) This work was in turn inspired by the work of Vinberg [18] [19], Vinberg and Kaplinskaja [20], Conway [6], and Borcherds [4] on real hyperbolic reflection groups defined in terms of Lorentzian lattices over $\mathbb{Z}$.

The basic idea of [1] was that the symmetry group of a Lorentzian lattice sometimes contains a collection of reflections whose mirrors are arranged in the pattern of some positive-definite lattice. If this lattice is "good enough" (e.g., has small covering radius) then these reflections generate a group with finite covolume. We use the same idea here, although all of the arguments of [1] must be changed because $\mathbb{O}$ and $\mathcal{K}$ are not associative and so modules and lattices over them do not make sense. We are pleased that the geometric ideas continue to work in the nonassociative case even though all of the formalism must be rewritten. In this paper, the relevant positive-definite "lattice" over $\mathcal{K}$ is just $\mathcal{K}$ itself, which is a scaled copy of the E_8 root lattice.

Above, we described $F_{4(-20)}$ as the stabilizer in $E_{6(-26)}$ of any of various Hermitian matrices. However, for computations it is more convenient to describe the group as the automorphism group of a Jordan algebra J, and to describe $\mathbb{O}H^2$ in terms of the idempotent elements of J. For any 3×3 nondegenerate real symmetric matrix Φ we define J_Φ to be the set of 3×3 octave matrices that are Hermitian with respect to Φ. This vector space is closed under the Jordan multiplication $X * Y = (XY + YX)/2$, and the automorphism group of J_Φ turns out to be a real form of F_4. In particular, if Φ is indefinite then the group is $F_{4(-20)} = \operatorname{Aut}\mathbb{O}H^2$. For most of the paper we will work with J_Ψ and various integral forms thereof, where

$$\Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is a convenient setting for computations and for the construction of the reflection groups. A little bit more effort is then required to identify the groups as the stabilizers in $H_I^{\mathcal{K}}$ of the Hermitian forms (1.1). This extra work essentially consists of the introduction of the determinant forms on J_I and J_Ψ and an identification of these cubic forms with each other.

Existing treatments of $\mathbb{O}H^2$ are either elegant and abstract but not suited for computation (see [16]) or else are defined in terms of the Jordan algebra $J_{\mathrm{diag}[-1,+1,+1]}$ (see [15]). To prove our results we need to introduce and name various transformations of J_Ψ and make other detailed constructions. In light of this, our treatment of J is almost entirely self-contained. In section 2 we describe $\mathbb{O}$ and list some identities useful for computing therein. We then heuristically describe in section 3 the geometry of $\mathbb{O}H^2$, to guide the reader through the algebraic thicket of section 4, which formally defines the Jordan algebras and $\mathbb{O}H^2$, provides useful coordinates for them, and establishes key properties of their symmetry groups. Section 5 is the heart of the paper and constructs the advertised reflection groups. The proof that the groups are the entire symmetry groups of two integral forms of J requires the rather involved theorem 5.3, but we indicate how to obtain the weaker result that they have finite covolume without using this theorem. In section 6 we describe the determinant form on J, a cubic form which makes the inclusion $F_{4(-20)} \subseteq E_{6(-26)} \cong \operatorname{Aut} \mathbb{O}P^2$ visible. Finally in section 7 we use the determinant form to realize our reflection groups as the stabilizers of the integral Hermitian forms (1.1).

2 The Nonassociative Field $\mathbb{O}$

The algebra $\mathbb{O}$ of octaves is the algebra over $\mathbb{R}$ with basis $e_\infty = 1, e_0, \ldots, e_6$ with $e_a^2 = -1$ for $a \neq \infty$, any two elements lying in an associative algebra, and $e_a e_{a+1} = e_{a+3}$ for each $a \neq \infty$, with indices read modulo 7. The algebra of octaves is noncommutative and nonassociative, but otherwise a division algebra. For computational purposes it is useful to know that for distinct $a, b, c \neq \infty$ we have $e_a e_b = -e_b e_a$ and $e_a(e_b e_c) = \pm(e_a e_b) e_c$, with associativity holding just if

$$\{a, b, c\} = \{d, d+1, d+3\} \pmod 7 \text{ for some } d = 0, \ldots, 6.$$

Sometimes it is convenient to write i, j, k and ℓ for e_0, e_1, e_3 and e_2, respectively. Then i generates a copy of $\mathbb{C}$ and i, j, k generate a copy of the quaternions $\mathbb{H}$. The real part $\operatorname{Re} x$ of $x = \sum x_a e_a$ is x_∞, and the conjugate $\bar{x}$ of x is $\bar{x} = -x + 2\operatorname{Re} x$; we have $\operatorname{Re} x = \operatorname{Re} \bar{x}$. We say that x is imaginary if $\operatorname{Re} x = 0$ and we write $\operatorname{Im} \mathbb{O}$ for the set of imaginary octaves. The norm $|x|^2$ of x is defined to be $x\bar{x} = \sum x_a^2$, and x is called a unit if it has norm one. The identities $\overline{xy} = \bar{y}\bar{x}$ and $|xy|^2 = |x|^2|y|^2$ hold universally.

Since $\mathbb{O}$ is nonassociative, computations with it are sometimes complicated. Five identities we will use are

$$z(xy)z = (zx)(yz), \tag{2.1}$$

$$\operatorname{Re}((xy)z) = \operatorname{Re}(x(yz)) = \operatorname{Re}((yz)x) \tag{2.2}$$

$$(\mu x \bar{\mu})(\mu y) = \mu(xy), \tag{2.3}$$

$$(x\mu)(\bar{\mu} y \mu) = (xy)\mu, \qquad \text{and} \tag{2.4}$$

$$xy + yx = (x\bar{\mu})(\mu y) + (y\bar{\mu})(\mu x) \tag{2.5}$$

which hold for all $x, y, z \in \mathbb{O}$ and all imaginary units μ. We write $\mathrm{Re}(xyz)$ for any of the three expressions in (2.2). Identities such as these are easily proven if one takes advantage of the automorphism group of $\mathbb{O}$. This group is the compact form of the exceptional Lie group G_2, and the stabilizer of a subalgebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ acts transitively on the units orthogonal to the algebra. (See [12] for a proof.) For example, we prove (2.3). After applying an automorphism of $\mathbb{O}$ we may take $\mu = i$, $x = x_0 + \alpha j$ and $y = y_0 + \beta\ell$ with $\alpha, \beta \in \mathbb{R}$, $x_0 \in \mathbb{C}$ and $y_0 \in \mathbb{H}$. After expanding both sides of (2.3) we find that almost all terms cancel by associativity in $\mathbb{H}$. All that remains is to check

$$(ij\bar{\imath})(i\ell) = i(j\ell),$$

which one does using the definition of multiplication. Note that (2.4) may be obtained from (2.3) by taking conjugates.

We observe that if $A_1, \ldots, A_n$ are matrices over $\mathbb{O}$, with all except at most two of them having all real entries, then the product $A_1 \cdots A_n$ is independent of the manner in which the terms are grouped by parentheses. The same result holds if all the entries of all the matrices lie in some associative subalgebra of $\mathbb{O}$.

Finally, we will use the fact that the group of transformations of $\mathbb{O}$ generated by the left multiplications by imaginary units acts transitively on the unit sphere S^7 in $\mathbb{O}$. (Proof: every orbit contains an equator S^6 of $S^7 \subseteq \mathbb{O}$, so any two orbits meet.)

3 The Geometry of $\mathbb{O}H^2$

This section is not logically necessary for those following it. Its purpose is to tell the reader in advance what some of the answers are, to serve as a guide to the rather heavy algebra of section 4. We discovered the reflection groups mostly by geometric visualization using the model described here, together with analogies with the constructions of [1]. The geometry of $\mathbb{O}H^2$ and $\mathbb{O}P^2$ can be developed entirely synthetically, along the lines of [16] and [17], but we will use the Jordan algebra approach in order to be able to use a theorem of Borel and Harish-Chandra at a crucial point in the proof of theorem 5.4. Some of the basic ideas described here are implicit in [14] and they are all developed in [11] and [12].

Even though $\mathbb{O}$ is not associative and so modules over it can't reasonably be defined, one should imagine the fictional vector space $\mathbb{O}^3$. If it were equipped with a nondegenerate Hermitian form Φ then we could consider the set J_Φ of $\mathbb{O}$-linear transformations of $\mathbb{O}^3$ that were self-adjoint with respect to Φ. Among these would lie the orthogonal projections onto subspaces of $\mathbb{O}^3$; these would be the identity and zero operators together with two continuous families corresponding to the 1- and 2-dimensional subspaces. In particular, thinking of elements of $\mathbb{O}^3$ as row vectors with matrices acting on the right we could consider $X = v^*v$ where v^* would be the Φ-adjoint of $v \in \mathbb{O}^3$. Up to a multiplicative constant this would be the projection onto the span of v. It would turn out that such transformations could be essentially characterized by the conditions that $X^2 = \lambda X$ and $\mathrm{Tr}\, X = \lambda$ where $\lambda \in \mathbb{R}$ would be the (square) norm of v. This would allow one to recover the 1-dimensional subspaces of $\mathbb{O}^3$ and the norms of vectors in them. Furthermore, if X and Y lay in J_Φ then their Jordan product $X * Y = (XY + YX)/2$ would also lie in J_Φ, so that J_Φ would be a Jordan algebra. Finally, one could recover the notion of orthogonality of subspaces of $\mathbb{O}^3$ because if X and Y were projections to two subspaces then these subspaces would be orthogonal just if $X * Y = 0$.

The reason one goes to these lengths is that $\mathbb{O}^3$ is not in any sense a vector space over $\mathbb{O}$, so the above account is purely fictional. However, the Jordan algebra J_Φ *does* exist (although we will only treat the case of real Φ). This allows us to recover much of the structure that $\mathbb{O}^3$ would have provided if it existed. In particular, if Φ has signature $-++$ then we can recover the set of "1-dimensional subspaces of $\mathbb{O}^3$ on which Φ is negative definite". By analogy with hyperbolic geometry over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ we define $\mathbb{O}H^2$ to be this set. The group $\operatorname{Aut} J_\Phi$ turns out to be $F_{4(-20)}$, which acts transitively and with compact stabilizer on $\mathbb{O}H^2$.

The geometry of $\mathbb{O}H^2$ is very similar to more conventional hyperbolic geometry. In particular, $\mathbb{O}H^2$ is an open 16-ball and it has a natural boundary $\partial\mathbb{O}H^2$ "at infinity", a sphere S^{15} which arises from the nilpotents of J_Φ (or, heuristically, from the norm 0 elements of $\mathbb{O}^3$). Furthermore, by considering idempotents of J_Φ one obtains points "outside" $\partial\mathbb{O}H^2$ which together with $\mathbb{O}H^2$ and $\partial\mathbb{O}H^2$ form the octave projective plane $\mathbb{O}P^2$. This is essentially the same as the realization of real hyperbolic space H^n as an open ball in $\mathbb{R}P^n$, as the image of the elements of $\mathbb{R}^{n+1}$ of negative norm with respect to a quadratic form of signature $-++\cdots+$. The only difference is that there is no vector space $\mathbb{O}^3$ associated with $\mathbb{O}P^2$.

We will show that $\operatorname{Aut}\mathbb{O}H^2$ acts transitively on $\partial\mathbb{O}H^2$, and the stabilizer of a null point will be very important in our work. There is an upper-half-space model for $\mathbb{O}H^2$ which is ideal for studying the stabilizer of a null point. In this model, the points of $\mathbb{O}P^2$ (except those on the line at infinity) are described by pairs (x, z) with $x, z \in \mathbb{O}$. The points of $\mathbb{O}H^2$ are the pairs with $\operatorname{Re} z > 0$, and the points of $\partial\mathbb{O}H^2$ are the pairs with $\operatorname{Re} z = 0$, together with one extra point called ∞, which lies on the line at infinity. There are "translations" stabilizing ∞, which have the form

$$(x, z) \mapsto (x + \xi, z - \operatorname{Im}(x\bar{\xi}) + \eta)$$

with $\xi \in \mathbb{O}$ and $\eta \in \operatorname{Im}\mathbb{O}$. The translations turn out to form a 15-dimensional Lie group $\mathcal{H}^{15}$ which (obviously) acts transitively on $\partial\mathbb{O}H^2 \smallsetminus \{\infty\}$. The group of translations is closely analogous to the translations of Euclidean space $\mathbb{R}^n$ which act on H^{n+1} in the usual upper-half-space model. The only substantial difference from the real case is that $\mathcal{H}^{15}$ is nonabelian, being a sort of octave version of the Heisenberg group.

We mentioned above that one can define the notion of orthogonality of "subspaces of $\mathbb{O}^3$" purely in terms of the Jordan algebra. This leads to the concept of a hyperplane in $\mathbb{O}H^2$; usually we will call a hyperplane a line. A reflection is a nontrivial transformation of $\mathbb{O}H^2$ that fixes a line pointwise. We will see that there is a unique reflection across each line; the line is called the mirror of the reflection. The map $R' : (x, z) \mapsto (-x, z)$ is an example of a reflection which fixes ∞ and there is another reflection R which exchanges ∞ with $(0, 0)$. The coordinate expression for R is complicated, but R is analogous to a real reflection of H^n whose mirror appears as a hemisphere resting on ∂H^n in the upper-half-space model. The two reflection groups we will build will be generated by conjugates of R and R' by translations satisfying appropriate integrality conditions on ξ and η. The mirrors of the conjugates of R' will be arranged in the pattern of a discrete subgroup of $\mathcal{H}^{15}$, and if this subgroup is "dense enough" then the mirrors will "cover" $\partial\mathbb{O}H^2 \smallsetminus \{\infty\}$. This will allow us to prove that the groups have finite covolume in $\operatorname{Aut}\mathbb{O}H^2$.

4 Jordan Algebras

We will use angle-brackets '$\langle\rangle$' to denote the values of various multilinear forms, and also to denote "the linear span of" or "the group generated by". One of the latter possibilities applies when there are no vertical bars '$|$' between the brackets. In this case, the meaning will be clear from the sort of objects lying between the brackets.

If Φ is a symmetric matrix in $M_3\mathbb{R}$, then we will regard it informally as a "Hermitian form on $\mathbb{O}^3$"; we will restrict our attention to invertible Φ, corresponding to nondegenerate forms. We write $M_3\mathbb{O}$ for the real vector space of 3×3 matrices with entries in $\mathbb{O}$, and denote by X^* the conjugate-transpose of a matrix X. The vector space $J_\Phi = \{X \in M_3\mathbb{O} \mid X\Phi = \Phi X^*\}$ is closed under the Jordan multiplication $X * Y = (XY + YX)/2$, and we call it the Jordan algebra associated to Φ. We say that $X, Y \in J_\Phi$ Jordan-commute if $X * Y = 0$. We informally regard elements of J_Φ as "transformations of $\mathbb{O}^3$" that are Hermitian with respect to Φ. In most treatments (e.g., [11] and

[12]), $\mathbb{O}P^2$ is described in terms of J_I, whose automorphism group is the compact form of F_4. We will study $\Phi \neq I$ because we are interested in $\mathbb{O}H^2$ and because it is useful to formulate the theory for more general Φ—this clarifies the various roles of the matrices involved. For $g \in GL_3\mathbb{R}$ we define the transformation C_g of $M_3\mathbb{O}$ given by $C_g : X \mapsto g^{-1}Xg$. A quick computation shows that C_g carries $J_{g\Phi g^*}$ to J_Φ and preserves matrix multiplication on $M_3\mathbb{O}$. We write $O(\Phi)$ for the subgroup of $GL_3\mathbb{R}$ whose elements are unitary with respect to Φ (that is, $g \in O(\Phi)$ if $g\Phi g^* = \Phi$). Scalars in $O(\Phi)$ act trivially on J_Φ, so we have $PO(\Phi) = O(\Phi)/\{\pm I\} \subseteq \operatorname{Aut} J_\Phi$. When Φ is indefinite, $PO(\Phi)$ may be identified with the subgroup $O^+(\Phi)$ of $O(\Phi)$ that preserves each of the two null cones defined in $\mathbb{R}^3$ by Φ. We will assume this identification henceforth. For any $X \in M_3\mathbb{O}$ we define $\chi(X) = \operatorname{Re}\operatorname{Tr}(X)$ and the symmetric inner product $\langle X|Y\rangle = \chi(X * Y)$. We call $\langle X|X\rangle$ the norm of X. It is easy to see that $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$ if either X or Y is real, so $GL_3\mathbb{R}$ preserves the trace form on $M_3\mathbb{O}$. Since $\langle X|Y\rangle$ is defined in terms of the trace and the multiplication, $GL_3\mathbb{R}$ also preserves norms and inner products. We will see that all of $\operatorname{Aut} J_\Phi$ preserves the restrictions of these forms to J_Φ, and that the trace of any element of J_Φ is real.

We will mainly be concerned with the indefinite form

$$\Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and we write J for J_Ψ. For any Φ, there exists $g \in GL_3\mathbb{R}$ such that $g\Phi g^* \in \{\pm I, \pm\Psi\}$, so J_Φ is isomorphic to J_I or to J_Ψ. An element of J has the form

$$X = (a, b, c, u, v, w) = \begin{pmatrix} a & w & \bar{v} \\ v & \bar{u} & b \\ \bar{w} & c & u \end{pmatrix} \tag{4.1}$$

with $a, b, c \in \mathbb{R}$ and $u, v, w \in \mathbb{O}$. It is obvious that X has real trace. Since all elements of J_I also have real trace (proof: compute, or see [11]) and the maps C_g preserve traces, the trace of any element of any J_Φ is real. Computation reveals

$$\langle X|X\rangle = a^2 + 2bc + 2\operatorname{Re}(u^2) + 4\operatorname{Re}(vw),$$

and so by polarization we obtain

$$\langle X_1|X_2\rangle = a_1a_2 + b_1c_2 + b_2c_1 + 2\operatorname{Re}(u_1u_2) + 2\operatorname{Re}(v_1w_2 + v_2w_1). \tag{4.2}$$

In order to work with the reflection groups of section 5 we will need detailed information about J and its automorphism group. We begin by giving the multiplication table for J. For each $x \in \mathbb{O}$ we define

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$U_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{x} & 0 \\ 0 & 0 & x \end{pmatrix}, \quad V_x = \begin{pmatrix} 0 & 0 & \bar{x} \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } W_x = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ \bar{x} & 0 & 0 \end{pmatrix}.$$

With respect to this spanning set, the Jordan multiplication is given in table 4.1. Note that entries denote *twice* the Jordan product. We write U (resp. V, W) for the span of the U_x (resp. V_x, W_x).

	A	B	C	U_y	V_y	W_y
A	$2A$	0	0	0	V_y	W_y
B		0	U_1	$2\operatorname{Re}(y) \cdot B$	0	$V_{\bar{y}}$
C			0	$2\operatorname{Re}(y) \cdot C$	$W_{\bar{y}}$	0
U_x				U_{xy+yx}	$V_{\bar{x}y}$	$W_{y\bar{x}}$
V_x					$2\operatorname{Re}(x\bar{y}) \cdot B$	$2\operatorname{Re}(xy) \cdot A + U_{\bar{y}\bar{x}}$
W_x						$2\operatorname{Re}(x\bar{y}) \cdot C$

Table 4.1 Entries denote *twice* the Jordan product in J.

We write $\operatorname{Im} U$ for the span of those U_x with $x \in \operatorname{Im} \mathbb{O}$. The nilpotent B will be very important in our analysis, and we define the height of $X \in J$ to be $\operatorname{ht}(X) = \langle X|B\rangle$. For X given by (4.1), we see by (4.2) that $\operatorname{ht}(X) = c$.

We now introduce several transformations which will turn out to be automorphisms of J. We already know that $O^+(\Psi) \subseteq \operatorname{Aut} J$. For $t \in \mathbb{R}$ and μ any imaginary unit of $\mathbb{O}$ we define the transformations

$$R : (a, b, c, u, v, w) \mapsto (a, c, b, \bar{u}, -\bar{w}, -\bar{v}) \tag{4.3}$$

$$S_\mu : (a, b, c, u, v, w) \mapsto (a, b, c, \mu u \bar{\mu}, \mu v, w \bar{\mu}) \tag{4.4}$$

$$T_{t,0} = C_{g_t} \text{ for } g_t = \begin{pmatrix} 1 & 0 & -t \\ t & 1 & -t^2/2 \\ 0 & 0 & 1 \end{pmatrix} \text{ (Note } g_t^{-1} = g_{-t}.\text{)} \tag{4.5}$$

We will soon introduce a more convenient notation for certain elements of J, which will greatly simplify these expressions. We write G for the group generated by R, the S_μ and the $T_{t,0}$. Later in this section we will see that $G = \operatorname{Aut} J$, that the S_μ generate a group isomorphic to $\operatorname{Spin}_7\mathbb{R}$, and that the $T_{t,0}$ together with their conjugates under this $\operatorname{Spin}_7\mathbb{R}$ generate a 15-dimensional nilpotent Lie group. An important element of G is the map

$$R' : (a, b, c, u, v, w) \mapsto (a, b, c, u, -v, -w), \tag{4.6}$$

which is the square of any S_μ. We will define octave reflections in section 5 and see that R' is one.

Lemma 4.1. *The transformations R, S_μ and $T_{t,0}$ are automorphisms of J and preserve the trace and norm forms. We have $\langle R, T_{t,0}\rangle = O^+(\Psi)$, and it follows that $O^+(\Psi) \subseteq G$.*

Proof: We first observe that R and $T_{t,0}$ lie in $O^+(\Psi)$: $T_{t,0}$ obviously does and $R = C_g$ for

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Verification that each S_μ is an automorphism relies on several pages of computation, using the identities (2.1)–(2.5) and the fact that $\bar{\mu} = -\mu$. It is obvious that S_μ preserves traces and since it preserves the Jordan multiplication it also preserve norms. Alternately, section 6 defines the determinant form on J and theorem 6.2 uses it to prove that $S_\mu \in \operatorname{Aut} J$. This approach reduces considerably the amount of work required but is much more circuitous.

We write elements of $\mathbb{R}^3$ as row vectors; $O^+(\Psi)$ acting on such vectors (by multiplication on the right) preserves the quadratic form Ψ. The $T_{t,0}$ are the parabolic transformations stabilizing

$(0,0,1)$ and their conjugates by R are those stabilizing $(0,1,0)$. It is obvious that these one-parameter subgroups generate $SO^+(\Psi) \cong SO^+(2,1;\mathbb{R})$. Since $R \in O^+(\Psi) \smallsetminus SO^+(\Psi)$ we see that $\langle R, T_{t,0}\rangle = O^+(\Psi)$. $\square$

Working with 3×3 matrices is tedious and there is a better notation, which closely resembles the use of "vectors in $\mathbb{O}^3$". We write elements of the real vector space $\mathbb{O}^3$ as row vectors, and define

$$\mathbb{O}^3_0 = \left\{ (x,y,z) \in \mathbb{O}^3 \;\middle|\; x,y,z \text{ all lie in some associative algebra} \right\}$$
$$\mathbb{O}^3_{00} = \left\{ (x,y,z) \in \mathbb{O}^3 \;\middle|\; y \in \mathbb{R} \right\} \subseteq \mathbb{O}^3_0.$$

Suppose Φ is given. We define the map $\pi_\Phi : \mathbb{O}^3_0 \to M_3\mathbb{O}$ by $\pi_\Phi(v) = \Phi v^* v$; we write π for π_Ψ. One checks immediately that $\pi_\Phi(\mathbb{O}^3_0) \subseteq J_\Phi$. We say that an element of J_Φ is "good" (with respect to Φ) if it is nonzero and lies in the image of π_Φ. One checks that if $(x,y,z) \in \mathbb{O}^3_0$ and $\alpha \in \mathbb{O} \smallsetminus \{0\}$ such that x,y,z and α all lie in some associative algebra then $\pi_\Phi(x,y,z) = \pi_\Phi(\alpha x, \alpha y, \alpha z)$, so any good element of J_Φ lies in the image of $\mathbb{O}^3_{00}$. It is easy to see that the span of $\pi(\mathbb{O}^3_0)$ is all of J. The elements of $\mathbb{O}^3_0$ and $\mathbb{O}^3_{00}$ have been dubbed "restricted homogeneous coordinates" by Aslaksen [3]. (Actually, Aslaksen considered a slightly different version of the special case $\Phi = I$. A very similar idea appears in [14]. The relation between these ideas is explained in [2].) For reference, we record that the ordered pair (x,z) of section 3 represents

$$\pi(x, 1, z - |x|^2/2) \in J.$$

(We will not need this identification.)

The following theorem a very useful computational tool.

Theorem 4.2. *If $v, w \in \mathbb{O}^3_0$ and all six entries of v and w lie in some associative subalgebra of $\mathbb{O}$, then for any Φ we have*
$$\langle \pi_\Phi(v) | \pi_\Phi(w) \rangle = |v \Phi w^*|^2.$$

Proof: Let α denote the single entry of $v\Phi w^*$. We write $v = (v_a)$, $w = (w_b)$ and $\Phi = (\phi_{ab})$ for $a,b = 1,2,3$. In the derivation below we have associated terms freely and used identity (2.2) and the symmetry and reality of Φ.

$$\begin{aligned}
\langle \pi_\Phi(v) | \pi_\Phi(w) \rangle &= \mathrm{Tr}(\pi_\Phi(v) * \pi_\Phi(w)) \\
&= \mathrm{Tr}(\Phi v^* v \Phi w^* w + \Phi w^* w \Phi v^* v)/2 \\
&= \mathrm{Tr}(\Phi v^* \alpha w + \Phi w^* \bar{\alpha} v)/2 \\
&= \sum_{a,b} (\phi_{ab} \bar{v}_b \alpha w_a + \phi_{ab} \bar{w}_b \bar{\alpha} v_a)/2 \\
&= \sum_{a,b} \mathrm{Re}(\phi_{ab} \bar{v}_b \alpha w_a) \\
&= \mathrm{Re}\left(\alpha \sum_{a,b} w_a \phi_{ab} \bar{v}_b \right) \\
&= \mathrm{Re}(\alpha\bar{\alpha}) = |\alpha|^2.
\end{aligned}$$

$\square$

For $\lambda \in \mathbb{R}$, we say that $X \in J_\Phi$ is λ-potent (or is a λ-potent) if $X^2 = \lambda X$. We refer to 1-, 0-, and (-1)-potents as idempotents, nilpotents and negpotents, respectively. We say that X is potent if it is λ-potent for some λ. One reason we use $\mathbb{O}_0^3$ and $\mathbb{O}_{00}^3$ is that the good elements turn out to be the most important elements of J_Φ. In particular, every good element is potent. We will see that the geometry of $\mathbb{O}H^2$ may be described in terms of the good potents of J_Φ. We define the norm $|v|_\Phi^2$ of $v \in \mathbb{O}_0^3$ to be the single entry of $v\Phi v^*$, which is automatically real. Then we have

$$\pi_\Phi(v)\pi_\Phi(v) = \Phi v^* v \Phi v^* v = \Phi v^* |v|_\Phi^2 v = |v|_\Phi^2 \pi_\Phi(v),$$

so we see that $\pi_\Phi(v)$ is $|v|_\Phi^2$-potent. (*Warning:* The norm $|v|_\Phi^2$ of $v \in \mathbb{O}_0^3$ is not the same as the norm in J_Φ of $\pi_\Phi(v)$—by theorem 4.2 the latter norm is the square of the former. This is an unavoidable inconvenience.)

We also have

$$\operatorname{Tr}(\pi_\Phi(v)) = \operatorname{Re}\operatorname{Tr}(\pi_\Phi(v)) = \operatorname{Re}\sum_{a,b} \phi_{ab}\bar{v}_b v_a = \operatorname{Re}\sum_{a,b} v_a \phi_{ab}\bar{v}_b = |v|_\Phi^2.$$

Therefore a necessary condition for $X \in J_\Phi$ to be good is that there be $\lambda \in \mathbb{R}$ such that X is λ-potent and has trace λ. In theorem 4.3 we will see that this is nearly a sufficient condition.

We can define an action of $O(\Phi)$ on $\mathbb{O}_0^3$ that is π_Φ-equivariant with its action on J_Φ. Namely, $g \in O(\Phi)$ acts by $v \mapsto vg$; observe that this preserves $\mathbb{O}_0^3$. Taking $\Phi = \Psi$ we observe that the maps $T_{t,0}$ also preserve $\mathbb{O}_{00}^3$, and we define an action of the S_μ on $\mathbb{O}_{00}^3$ by

$$S_\mu : (x, y, z) \mapsto (\mu x, y, \mu z \bar{\mu}).$$

With the aid of (2.3) and the fact that $y \in \mathbb{R}$ one can check that this action and the action of S_μ on J are π-equivariant.

An important set of transformations of $\mathbb{O}_{00}^3$ are the "translations"

$$T_{\xi,\eta} : (x, y, z) \mapsto (x + \xi y, y, z - x\bar{\xi} - |\xi|^2 y/2 + y\eta)$$

with $\xi \in \mathbb{O}$ and $\eta \in \operatorname{Im}\mathbb{O}$. Those with $\xi = 0$ are called central translations. If $\xi = t \in \mathbb{R}$ and $\eta = 0$ then this definition agrees with the action of $T_{t,0}$ on $\mathbb{O}_{00}^3$, defined in the previous paragraph by virtue of $T_{t,0} \in O(\Phi)$. All of the translations fix $(0, 0, 1)$, which is useful because $B = \pi(0, 0, 1)$ is a useful nilpotent of J. Using (2.1) and the fact that $\bar{\mu} = -\mu$ one may show

$$S_\mu \circ T_{\xi,\eta} \circ S_{\bar{\mu}} = T_{\mu\xi,\mu\eta\bar{\mu}} \tag{4.7}$$

$$T_{\xi_1,\eta_1} \circ T_{\xi_2,\eta_2} = T_{\xi_1+\xi_2,\eta_1+\eta_2+\operatorname{Im}(\xi_1\bar{\xi}_2)} \tag{4.8}$$

$$T_{\xi,\eta}^{-1} = T_{-\xi,-\eta} \tag{4.9}$$

$$[T_{\xi_1,\eta_1}, T_{\xi_2,\eta_2}] = T_{\xi_1,\eta_1}^{-1} \circ T_{\xi_2,\eta_2}^{-1} \circ T_{\xi_1,\eta_1} \circ T_{\xi_2,\eta_2} = T_{0,2\operatorname{Im}(\xi_1\bar{\xi}_2)} \quad . \tag{4.10}$$

These show that the translations form a 15-dimensional Lie group (which we call $\mathcal{H}^{15}$) which is nilpotent of class 2 and normalized by the S_μ. Its derived subgroup coincides with its center and consists of the central translations—$\mathcal{H}^{15}$ is a sort of octave version of the Heisenberg group. Because $\pi(\mathbb{O}_{00}^3)$ spans J, the π-equivariance properties of the $T_{t,0}$ and S_μ show that the action on J of any elements of $\langle T_{t,0}, S_\mu\rangle$ is completely determined by its action on $\mathbb{O}_{00}^3$. The relations above show that all of $\mathcal{H}^{15} \rtimes \langle S_\mu\rangle$ is generated by the S_μ and those $T_{t,0}$ with $t \in \mathbb{R}$, so all of $\mathcal{H}^{15} \rtimes \langle S_\mu\rangle$ maps into $G \subseteq \operatorname{Aut} J$. One reason for introducing the restricted homogeneous coordinates is that the expression for the action of $T_{\xi,\eta}$ on J is horrible for general ξ and η. We now show that the natural map from the group $\langle T_{t,0}, S_\mu\rangle$ acting on $\mathbb{O}_{00}^3$ to the group $\langle T_{t,0}, S_\mu\rangle \subseteq G$ acting on J is an isomorphism. The action of $s \in \langle S_\mu\rangle$ on $\mathbb{O}_{00}^3$ is determined by its actions on the first and third coordinates of elements of $\mathbb{O}^3_{00}$. These actions coincide with those of the image of s in G on the subspaces V and U of J. Thus $\langle S_\mu\rangle \subseteq \operatorname{Aut}\mathbb{O}^3_{00}$ acts faithfully on J. If $h \in \mathcal{H}^{15} \smallsetminus 1$ and $s \in \langle S_\mu\rangle$ then by considering the action of hs on $(0,1,0) \in \mathbb{O}^3_{00}$ we see that the corresponding action of hs on J is nontrivial. Therefore all of $\langle T_{t,0}, S_\mu\rangle \subseteq \operatorname{Aut}\mathbb{O}^3_{00}$ acts faithfully on J. This establishes an isomorphism between the group $\langle T_{t,0}, S_\mu\rangle$ acting on $\mathbb{O}^3_{00}$ and the group $\langle T_{t,0}, S_\mu\rangle$ acting on J. We will henceforth assume this identification.

The following theorem provides the justification for our assertion that the good elements of J are the most important ones, and provides the link between $\mathbb{O}^3_0$ and the structure of the Jordan algebra.

Theorem 4.3. *Suppose $X \in J\smallsetminus\{0\}$ is λ-potent and has trace λ. Then (i) under G, X is equivalent to a real matrix, and (ii) precisely one of X and $-X$ is good. Furthermore, (iii) every nilpotent of J has trace 0.*

Proof: By the π-equivariance of the actions of $O(\Psi)$ on J and $\mathbb{O}^3_0$ and those of $\langle \mathcal{H}^{15}, S_\mu\rangle$ on J and $\mathbb{O}^3_{00}$, the image under G of a good element is good. Therefore it suffices to find a transform X' of X under G such that just one of $\pm X'$ is good. Suppose $X = (a,b,c,u,v,w)$. If X Jordan-commutes with both B and C then by table 4.1 we have $X = \lambda A$ for some $\lambda \in \mathbb{R}$, obviously a real matrix. If $\lambda > 0$ then $X = \pi(\sqrt{\lambda},0,0)$ is good but $-X$ is not. If $\lambda < 0$ then the reverse applies. Since $X \neq 0$ we do not need to consider the case $\lambda = 0$.

If X fails to Jordan-commute with at least one of B and C then we may (if necessary applying $R \in G$ to exchange B with C) suppose that X fails to Jordan-commute with B. If $c = 0$ then in order for X to be λ-potent we must have $|w|^2 = \lambda c = 0$, so $w = 0$. Even if $c \neq 0$, we may suppose without loss of generality that $w = 0$, as follows. After applying suitable S_μ's to X we may take $w \in \mathbb{R}$. Computation shows that $T_{t,0}$ (for $t \in \mathbb{R}$) acts on J by

$$T_{t,0} : (a,b,c,u,v,w) \mapsto (?,?,?,?,?,w+ct)$$

where the question marks indicate irrelevant coordinates. Applying $T_{-w/c,0}$ we may take $w = 0$. Since the S_μ and $T_{t,0}$ fix B, we know that X still fails to Jordan-commute with B.

Squaring the matrix X we find that when $w = 0$ the conditions for X to be λ-potent are

$$\begin{aligned}
a(a-\lambda) &= 0 && (4.11)\\
(a+\bar{u}-\lambda)v &= 0 && (4.12)\\
vc &= 0 && (4.13)\\
u(u-\lambda)+bc &= 0 && (4.14)\\
c(2\operatorname{Re}u-\lambda) &= 0 && (4.15)\\
|v|^2 + b(2\operatorname{Re}u-\lambda) &= 0. && (4.16)
\end{aligned}$$

If $c = 0$ then by (4.14), $u = 0$ or $u = \lambda$, the latter condition being impossible by the trace condition on X. Therefore $u = 0$, which together with $w = c = 0$ shows that X Jordan-commutes with B. This contradicts our hypothesis on X, so $c = 0$ is impossible.

Since $c \neq 0$, (4.13) and (4.15) show $v = 0$ and $\operatorname{Re}u = \lambda/2$. The trace condition implies that $a = 0$, and we solve for b by finding $b = -u(u-\lambda)/c = u\bar{u}/c$. We conclude

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{u} & u\bar{u}/c \\ 0 & c & u \end{pmatrix}.$$

If $c > 0$ then $X = \pi(0, \sqrt{c}, u/\sqrt{c})$ and $-X$ is not good. If $c < 0$ then $-X = \pi(0, \sqrt{-c}, u/\sqrt{-c})$ and X is not good. This proves (ii). To prove (i), we suppose without loss of generality that $c > 0$ and simply apply $T_{0,-\operatorname{Im} u/c}$, which carries X to $\pi(0, \sqrt{c}, \operatorname{Re} u/\sqrt{c})$, a real matrix.

To prove (iii) we suppose that X is nilpotent but make no assumption about its trace. As above, we may suppose $w = 0$, so that (4.11)–(4.16) hold with $\lambda = 0$. Then (4.11) implies $a = 0$. If $bc = 0$ then by (4.14) we have $u = 0$, so $\operatorname{Tr} X = 0$. If $bc \neq 0$ then by (4.13) we find $v = 0$ and then by (4.16) we find $\operatorname{Re} u = 0$, which again shows $\operatorname{Tr} X = 0$. □

One consequence of theorem 4.3 is that if $X \in J$ is nilpotent then we know that $X = \pm\pi(v)$ for some $v \in \mathbb{O}^3_{00}$ with $|v|^2_\Psi = 0$. It is easy to enumerate all possibilities for v, and we find that either X is a multiple of $B = \pi(0, 0, 1)$ or else

$$X = h \cdot \pi(x, 1, -|x|^2/2 + z)$$

for some $h \in \mathbb{R}$, $x \in \mathbb{O}$ and $z \in \operatorname{Im} \mathbb{O}$.

Theorem 4.4.

(i) For each $\lambda \in \mathbb{R}$, G acts transitively on the good λ-potents of J.

(ii) Two orthogonal nilpotents are proportional.

(iii) The subgroup $\mathcal{H}^{15}$ of G stabilizes B and acts simply transitively on the nilpotents of any given nonzero height.

(iv) The groups G and $\operatorname{Aut} J$ coincide.

(v) The subgroup fixing B and also C is isomorphic to $\mathrm{Spin}_7\mathbb{R}$, is generated by the S_μ, has center $\{1, R'\}$, and acts on $\langle B, C, U\rangle$ as $SO(7)$.

(vi) The stabilizer of B is the semidirect product $\mathcal{H}^{15} \ltimes \mathrm{Spin}_7\mathbb{R}$.

*R*ecall that the transformation R' is defined in (4.6), as

$$R' : (a, b, c, u, v, w) \mapsto (a, b, c, u, -v, -w).$$

Proof: (i) Suppose that X and Y are good λ-potents and thus nonzero. By theorem 4.3, after applying elements of G, we may suppose that $X = \pi(v_x)$ and $Y = \pi(v_y)$ with $v_x, v_y \in \mathbb{R}^3 \smallsetminus \{0\} \subseteq \mathbb{O}^3_0$. The norm map $v \mapsto |v|^2_\Psi$ is the usual norm on $\mathbb{R}^3$ associated to the quadratic form Ψ. We know that $|v_x|^2_\Psi = |v_y|^2_\Psi = \lambda$ and that $O(\Psi) = O^+(\Psi) \times \{\pm I\}$ acts transitively on the nonzero vectors in $\mathbb{R}^3$ with any given norm. Applying an element of $O^+(\Psi) \subseteq G$ we may take $v_x = \pm v_y$ and so $X = \pi(v_x) = \pi(v_y) = Y$.

(ii),(iii) For any nilpotent X, either X or $-X$ is good (this follows from theorem 4.3). In order to have inner product $h \neq 0$ with B, we must have $X = h \cdot \pi(x, 1, -|x|^2/2 + z)$ with $x \in \mathbb{O}$, $z \in \operatorname{Im} \mathbb{O}$. For $h \neq 0$ these are obviously permuted simply transitively by $\mathcal{H}^{15}$, proving (iii). If $h = 0$ then X must have the form $\pm\pi(x, 0, z)$, but then since $0 = |(x, 0, z)|^2_\Psi = |x|^2$, we must have $x = 0$ and so X is a multiple of B. This proves (ii).

(iv) Suppose $\phi \in \operatorname{Aut} J$. By (i) and theorem 4.3 (ii),(iii), after multiplying ϕ by an element of G we may suppose $\phi(B) = \varepsilon B$, with $\varepsilon = \pm 1$. By (ii) and (iii) we may then take $\phi(C) = \varepsilon' C$ for some $\varepsilon' \in \mathbb{R}$. Since $2\phi(B) * \phi(C) = \phi(U_1)$ must be idempotent we see $\varepsilon' = 1/\varepsilon = \varepsilon$. We will eventually learn that $\varepsilon = +1$. The rest of the proof is mostly a chase through the multiplication table of J (table 4.1).

By considering $B * C$ we find that $\phi(U_1) = U_1$. We know that ϕ fixes the identity matrix I (since I is the identity element of J), so it also fixes $I - U_1 = A$. We know that ϕ fixes each of the spaces $\langle B, C, U\rangle$, $\langle A, B, \operatorname{Im} U, V\rangle$ and $\langle A, C, \operatorname{Im} U, W\rangle$, as these are the subspaces of J which Jordan-commute with A, B and C, respectively. Taking their intersection we see that ϕ stabilizes $\operatorname{Im} U$. If $x \in \operatorname{Im} \mathbb{O}$ then $x^2 \in \mathbb{R}$ and $U_x * U_x = x^2 \cdot U_1$, which shows that ϕ preserves the norm $U_x \mapsto |x|^2$ on $\operatorname{Im} U$. Next, for any $x \in \mathbb{O}$, V_x Jordan-commutes with B, so we see that $\phi(V_x) = \alpha A + \beta B + V_{x'} + U_{y'}$ for some $\alpha, \beta \in \mathbb{R}$, $x' \in \mathbb{O}$ and $y' \in \operatorname{Im} \mathbb{O}$. Consideration of $U_1 * V_x$ shows that $\alpha = \beta = y' = 0$ and therefore ϕ preserves $V \subseteq J$. Considering $V_x * V_x$ shows that ϕ preserves the norm $V_x \mapsto |x|^2$ on V. Similar reasoning proves that ϕ preserves $W \subseteq J$ and the norm $W_x \mapsto |x|^2$ thereon. We conclude that

$$\phi(a, b, c, u, v, w) = (a, \varepsilon b, \varepsilon c, \phi_1(u), \phi_2(v), \phi_3(w)),$$

with $\varepsilon = \pm 1$, each ϕ_m an orthogonal transformation, and $\phi_1(1) = 1$.

The transformation S_μ acts on $\operatorname{Im} U$ by the product of central inversion and reflection in μ. Therefore $\langle S_\mu \rangle$ (the group generated by all the S_μ) acts on $\operatorname{Im} U$ as $SO(7)$ and we may suppose that ϕ_1 is either the identity or the conjugation map. (We will see soon that the latter is impossible.)

Considering $V_x * W_y$ and using the fact that $\phi_1(\bar{x}) = \overline{\phi_1(x)}$, we find

$$\phi_1(xy) = \phi_2(x)\phi_3(y) \qquad \forall x, y \in \mathbb{O}. \tag{4.17}$$

Taking $x = y = 1$ shows that $\phi_2(1) = \overline{\phi_3(1)}$ and we write ξ for this common value. Taking $x = 1$ in (4.17) yields

$$\phi_3(y) = \bar{\xi}\phi_1(y) \tag{4.18}$$

and taking $y = 1$ in (4.17) yields

$$\phi_2(x) = \phi_1(x)\xi. \tag{4.19}$$

Plugging these into (4.17) we obtain

$$\phi_1(xy) = \phi_1(x)\xi \cdot \bar{\xi}\phi_1(y) \qquad \forall x, y \in \mathbb{O}. \tag{4.20}$$

If ϕ_1 is the conjugation map then we derive

$$\bar{y}\bar{x} = \bar{x}\xi \cdot \bar{\xi}\bar{y} \qquad \forall x, y \in \mathbb{O},$$

which is impossible since there are noncommuting x and y in some associative algebra containing ξ. Therefore $\phi_1 = I$ and so by (4.20) we find

$$xy = x\xi \cdot \bar{\xi}y \qquad \forall x, y \in \mathbb{O}.$$

Taking $z \in \mathbb{O}$ and $x = \bar{\xi}z$ we may apply the identity (2.3) to deduce $(\bar{\xi}z)y = \bar{\xi}(zy)$ for all $y, z \in \mathbb{O}$. Therefore $\bar{\xi} \in \mathbb{R}$, so $\xi = \pm 1$ and thus by (4.18) and (4.19) we have $\phi_2 = \phi_3 = \pm I$. After applying the square R' of any S_μ we may suppose that $\phi_2 = \phi_3 = I$. Finally, consideration of $B * W_x$ shows that $\phi_3 = \varepsilon\phi_2$, so $\varepsilon = +1$. Therefore ϕ is the identity, establishing (*iv*).

We have just seen that the group stabilizing each of B and C is generated by the S_μ, and that this group maps to its action $SO(7)$ on $\langle B, C, U \rangle$ with kernel equal to $\{1, R'\}$ and central in $\langle S_\mu \rangle$. Therefore $\langle S_\mu \rangle$ is isomorphic to either $\mathrm{Spin}_7\mathbb{R}$ or $SO(7) \times (\mathbb{Z}/2)$. The latter is impossible because the nontrivial central element R' is a square. This establishes (*v*), and (*vi*) follows immediately from (*iii*) and (*v*). $\square$

We now define the octave hyperbolic plane $\mathbb{O}H^2$. It is is the image in $\mathbb{R}P^{26} = PJ$ of the good negpotents of J. The "boundary" of $\mathbb{O}H^2$ is denoted $\partial\mathbb{O}H^2$ and is defined as the image of the good nilpotents. A line in $\mathbb{O}H^2$ is the set of good negpotents that Jordan-commute with some fixed good idempotent. The line associated to a given idempotent X is said to be polar to X. One may define the octave projective plane $\mathbb{O}P^2$ as the image in PJ of all the good elements of

J. Then $\partial\mathbb{O}H^2$ turns out to be the topological boundary of $\mathbb{O}H^2$ in $\mathbb{O}P^2$, and each line of $\mathbb{O}H^2$ (equivalently, each good idempotent of J) is associated with a point of $\mathbb{O}P^2 \smallsetminus \overline{\mathbb{O}H^2}$. We will not need this description of $\mathbb{O}P^2$.

Theorem 4.4 shows that G acts transitively on the points and lines of $\mathbb{O}H^2$ and 2-transitively on the points of $\partial\mathbb{O}H^2$. The remark following theorem 4.3 shows that $\partial\mathbb{O}H^2$ consists of the images of the elements $(0,0,1)$ and $(x,1,z-|x|^2/2)$ of $\mathbb{O}^3_{00}$, where $x\in\mathbb{O}$ and $z\in\operatorname{Im}\mathbb{O}$. This realizes $\partial\mathbb{O}H^2$ topologically as the one-point compactification S^{15} of $\mathbb{R}^{15}$. Similarly, $\mathbb{O}H^2$ may be identified with the image of the set $\{(x,1,z-|x|^2/2)\subseteq\mathbb{O}^3_{00}\,|\operatorname{Re}z>0\}$. This realizes $\mathbb{O}H^2$ as a 16-ball bounded by $S^{15}=\partial\mathbb{O}H^2$.

Theorem 4.5. *G is isomorphic to the group $F_{4(-20)}$, a connected simply connected simple Lie group of 52 dimensions.*

Proof: The orbit of good nilpotents in J is connected because it fibers over its image S^{15} in $\mathbb{O}H^2$ with the fibers being half-lines. The stabilizer of a nilpotent is $\mathcal{H}^{15}\ltimes\mathrm{Spin}_7\mathbb{R}$. This realizes $\operatorname{Aut}\mathbb{O}H^2$ as an iterated fibration of connected simply connected spaces and therefore it has these properties itself. We have $\dim G = 15+1+15+\dim\mathrm{Spin}_7\mathbb{R}=52$. We may also compute the homological dimension (h.d.) of G, as

$$\text{h.d.}(G)=\text{h.d.}(S^{15})+\text{h.d.}(\mathbb{R}^+)+\text{h.d.}(\mathcal{H}^{15})+\text{h.d.}(\mathrm{Spin}_7\mathbb{R})=36.$$

To show that G is semisimple and centerless it suffices to show that it has no nontrivial normal solvable subgroups. Suppose N were such a subgroup. It is easy to verify that the nilpotents of J span J and this easily implies that N acts faithfully on $\partial\mathbb{O}H^2$. As a normal subgroup of the 2-transitive group G, it acts transitively on $\partial\mathbb{O}H^2$. (This holds because G permutes the orbits of N; if there were more than one then this would contradict the 2-transitivity of G.) This realizes S^{15} as the coset space N/M for some subgroup M of N which of course is also solvable. Since solvable groups are aspherical, the long exact homotopy sequence shows that S^{15} is aspherical, which is absurd.

As a connected centerless semisimple Lie group, G is a direct product of simple Lie groups. Each factor, being normal, must act transitively on S^{15}. If M and N are distinct (thus commuting) factors then since N is transitive, the action of $g\in M$ on S^{15} is completely determined by its action on any point thereof. This implies that M acts simply-transitively on S^{15} and hence is compact. If there were more than one factor than this argument would apply to each, exhibiting G as a product of compact groups. Since G is noncompact, this is absurd. Thus there is only one factor, and G is simple.

The dimension of G and its simplicity show that it has type F_4. The dimension of any maximal compact subgroup equals h.d.(G), so we see that G is isomorphic to $F_{4(52-2\cdot 36)}=F_{4(-20)}$. □

One can see that the stabilizer G_0 of a point of $\mathbb{O}H^2$ is compact, as follows. We know $\dim G_0+\dim\mathbb{O}H^2=\dim G$, so $\dim G_0=36$. We also know that h.d.$(G_0)+$h.d.$(\mathbb{O}H^2)=$h.d.(G), so h.d.$(G_0)=36$. The compactness of G_0 follows from the equality $\dim G_0=$ h.d.(G_0). One can also show (see [16]) that $G_0\cong\mathrm{Spin}_9\mathbb{R}$. We will not need either of these facts.

5 The Reflection Groups

A reflection is a nontrivial transformation that fixes a line (its mirror) of $\mathbb{O}H^2$ pointwise. For each line L there is a unique reflection across L. One can see this by considering the line L polar to $A=\pi(1,0,0)$, which contains $B=\pi(0,0,1)$ and $C=\pi(0,1,0)$. By part (v) of theorem 4.4, the stabilizer of B and C is $\mathrm{Spin}_7\mathbb{R}$, which acts on $\langle B,C,U\rangle$ and hence on L as $SO(7)$. Therefore the central element of $\mathrm{Spin}_7\mathbb{R}$ is the only reflection across L. Explicitly, this reflection R' acts on J by

$$R':(a,b,c,u,v,w)\mapsto(a,b,c,u,-v,-w),$$

which may be written more simply on $\mathbb{O}_0^3$ as

$$R' : (x, y, z) \mapsto (-x, y, z).$$

Since G acts transitively on lines of $\mathbb{O}H^2$, the reflections form a conjugacy class in G.

Coxeter [9] discovered a natural discrete subring $\mathcal{K}$ of $\mathbb{O}$. One description [7, p. 14] of $\mathcal{K}$ is as the set of vectors $\sum x_a e_a$ with all $x_a \in \frac{1}{2}\mathbb{Z}$ such that the set of a for which $x_a \in \mathbb{Z} + \frac{1}{2}$ coincides with one of the sets

$$\{0\,1\,2\,4\}, \{0\,2\,3\,5\}, \{0\,1\,5\,6\}, \{0\,3\,4\,6\}, \{\infty\,0\,1\,3\},$$
$$\{\infty\,0\,2\,6\}, \{\infty\,0\,4\,5\}, \{\infty\,0\,1\,2\,3\,4\,5\,6\},$$

or one of their complements. We summarize here the properties of $\mathcal{K}$ that we will need.

Lemma 5.1.

(i) As a lattice, $\mathcal{K}$ is isometric to a scaled copy of the E_8 root lattice, with minimal norm 1, covering radius $1/\sqrt{2}$ and 240 units.
(ii) The elements of $\mathcal{K}$ of even norm span $\mathcal{K}$.
(iii) The deep holes of $\mathcal{K}$ nearest 0 are the halves of the elements of $\mathcal{K}$ of norm 2.
(iv) As a lattice, $\operatorname{Im}\mathcal{K}$ is a scaled copy of the E_7 root lattice, with minimal norm 1, covering radius $\sqrt{3/4}$, and 126 units.
(v) All deep holes of the E_7 lattice are equivalent under translations by elements of E_7.
(vi) The group of transformations of $\operatorname{Im}\mathcal{K}$ generated by the maps $x \mapsto \mu x \bar{\mu}$ with μ a unit of $\operatorname{Im}\mathcal{K}$ is the full rotation group of the $\mathbb{Z}$-lattice $\operatorname{Im}\mathcal{K}$.
(vii) The group of transformation of $\mathcal{K}$ generated by the maps $x \mapsto \mu x$, with μ a unit of $\operatorname{Im}\mathcal{K}$, is isomorphic to $\operatorname{Spin}_7(\mathbb{F}_2)$ and acts transitively on the elements of $\mathcal{K}$ of each norm 1 and 2.

Remarks: (vi) and (vii) identify the action of $\langle\{S_\mu : \mu \text{ is a unit of } \operatorname{Im}\mathcal{K}\}\rangle$ on $\mathbb{O}_{00}^3$ and hence on J. We will write $\operatorname{Spin}_7(2)$ for $\operatorname{Spin}_7(\mathbb{F}_2)$ and otherwise use ATLAS notation [7] for finite groups. The deep holes of a lattice L in a Euclidean space are the points of the space furthest from L; the distance from any one of these to L is called the covering radius of L.

Proof: Background information on E_7 and E_8 sufficient to prove (i), (iii), (iv), and (v) is provided in [8, ch. 4]. It is a simple exercise to prove (ii); in fact $\mathcal{K}$ is the integral span of its elements of norm 2. The map in (vi) acts on $\operatorname{Im}\mathcal{K}$ by the product of the real reflection in μ and the central involution. Therefore these maps generate the rotation subgroup of the E_7 Weyl group, which is the full rotation group of the lattice E_7 and hence of $\operatorname{Im}\mathcal{K}$. This establishes (vi).

The central quotient of the group generated by the transformations of (vii) is identified on p. 85 of [7] as the simple group $O_7(2)$, and since the central involution is a square (the square of any S_μ), the group must be the (unique) nontrivial central extension $\operatorname{Spin}_7(2)$ of $O_7(2)$. To prove transitivity on the 240 units of $\mathcal{K}$, observe that each orbit must have at least 126 members, so any two orbits meet. Now we show transitivity on norm 2 vectors. Every norm 2 element of $\mathcal{K}$ has the form $a + b$ where a and b are orthogonal units of $\mathcal{K}$. By transitivity on units we may take $a = 1$. Then since $\operatorname{Spin}_7(2)$ contains $G_2(2) = \operatorname{Aut}\mathcal{K}$ (see [7, p. 14]), which fixes $a = 1$ and acts transitively on the units of $\operatorname{Im}\mathcal{K}$, we may take $b = i$ (say). This establishes (vii). $\square$

For n a positive integer we define K_n as the integral span in J of nA, B, C, and those U_x, $\sqrt{n}\,V_x$ and $\sqrt{n}\,W_x$ with $x \in \mathcal{K}$. It is easy to see the $\mathcal{K}$ is the integral span of the image under π of

$$\{(x, y, z) \in \mathbb{O}_0^3 | x \in \sqrt{n} \cdot \mathcal{K}, y, z \in \mathcal{K}\},$$

or of

$$\{(x, y, z) \in \mathbb{O}_{00}^3 | x \in \sqrt{n} \cdot \mathcal{K}, y \in \mathbb{Z}, z \in \mathcal{K}\}.$$

In particular, we can show that an element of G preserves K_n if it preserves either of these subsets of $\mathbb{O}_0^3$. It is easy to check that K_n is closed under Jordan multiplication and thus is an integral form of J. We write R_n for the subgroup of $\operatorname{Aut} K_n$ generated by reflections. We will see that $R_n = \operatorname{Aut} K_n$ if $n = 1$ or $n = 2$.

Recall that R' acts on $\mathbb{O}_0^3$ by $R' : (x, y, z) \mapsto (-x, y, z)$, so we see that $R' \in R_n$ for all n. The transformation R of (4.3) acts on $\mathbb{O}_0^3$ by $R : (x, y, z) \mapsto (x, -z, -y)$, fixing $\{(x, y, -y)\}$ pointwise. Since this subset of $\mathbb{O}_0^3$ corresponds to the line polar to $\pi(0, 1, 1)$, we see that R is a reflection. It is also clear that $R \in R_n$ for all n. The reflection groups we construct will be generated by conjugates by certain translations of R and R'. We first show that R_n contains a generous supply of translations.

Theorem 5.2. *For every n and every $\xi \in \mathcal{K}$, there exists $\eta \in \operatorname{Im} \mathbb{O}$ such that R_n contains the transformation $T_{\xi\sqrt{n},\eta}$.*

Proof: By (4.8) and lemma 5.1(*ii*) it suffices to prove the theorem when $|\xi|^2$ is even. Computations in $\mathbb{O}_{00}^3$ show that $T_{-\xi\sqrt{n}/2,0} \circ R' \circ T_{\xi\sqrt{n}/2,0}$ preserves K_n. (This holds even though $T_{\xi\sqrt{n}/2,0}$ might not preserve K_n.) Since it is a reflection, it lies in R_n. Therefore

$$T_{\xi\sqrt{n},0} = R' \circ T_{-\xi\sqrt{n}/2,0} \circ R' \circ T_{\xi\sqrt{n}/2,0}$$

lies in R_n. $\square$

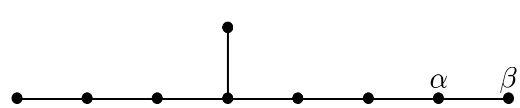
Remark: The geometric picture behind this proof is that R' and its conjugate by $T_{\xi\sqrt{n}/2,0}$ are reflections whose mirrors are parallel at infinity (that is, their mirrors do not meet in $\mathbb{O}H^2$, but both contain the point $\pi(0, 0, 1)$ of $\partial\mathbb{O}H^2$). Naturally, the product of reflections in parallel mirrors in a translation.

We now study the stabilizer in R_n of nA. Let J_- denote the subspace of J whose elements Jordan-commute with nA. We have $J_- = \langle B, C, U \rangle$, a Jordan subalgebra of J. Since R_n contains 1 and R', which are the only elements of G that fix J_- pointwise, the stabilizer in R_n of nA is completely determined by its action on J_-. In particular, it must stabilize $J_- \cap K_n$, which is independent of n and which we denote by K_-. We write R_- for the subgroup of R_n generated by all the reflections $T_{0,\eta} \circ R \circ T_{0,-\eta}$ with $\eta \in \operatorname{Im} \mathcal{K}$. (For each n, we have $\langle R, T_{0,\eta} \rangle \subseteq \operatorname{Aut} K_n$, so $R_- \subseteq R_n$.) Since R and each $T_{0,\eta}$ fixes A, we see that R_- acts on J_- and K_-.

We write $\mathbb{O}H^1$ (resp. $\partial\mathbb{O}H^1$) for the intersection of $\mathbb{O}H^2$ (resp. $\partial\mathbb{O}H^2$) with the image of J_- in $PJ = \mathbb{R}P^{26}$. One can identify $\mathbb{O}H^1$ with the real hyperbolic space H^8 and show that its stabilizer in G is $\operatorname{Spin}(8, 1)$, acting on each of J_- and H^8 as $SO(8, 1)$. (In fact this falls out of the proof of theorem 5.3 below.) Therefore R_- will acts as a group of isometries of H^8. We can describe this group explicitly:

Theorem 5.3.

(*i*) *The action of R_- on $\mathbb{O}H^1 \cong H^8$ is that of the rotation subgroup of the real hyperbolic reflection group with Coxeter diagram*

$\Delta =$ 

(*ii*) *For each $\eta \in \operatorname{Im} \mathcal{K}$ and each unit μ of $\operatorname{Im} \mathcal{K}$, the transformations $T_{0,\eta}$ and S_μ lie in R_- (and hence in R_n, for all n).*

(*iii*) *For each n, the stabilizers of B in R_n and in $\operatorname{Aut} K_n$ coincide.*

Remarks: The labels α and β on the diagram are for reference by the proof. This proof is lengthy and the reader may be satisfied with a weaker version of theorem 5.4, for which this theorem is not needed. Namely, knowing only that the stabilizer of B in R_n has finite index in its stabilizer in $\operatorname{Aut} K_n$, one can slightly modify the proof of theorem 5.4 to prove that R_n has finite index in $\operatorname{Aut} K_n$. (We indicate there how to make these modifications.) To prove this property of the stabilizers of B one need only take the translations of theorem 5.2 together with those central translations which are their commutators.

Proof: The elements of $\partial\mathbb{O}H^1$ are the images in projective space PJ of $\pi(0,0,1)$ and those $\pi(0,1,z)$ with $z \in \operatorname{Im}\mathbb{O}$. We denote these points by ∞ and z respectively, and we will describe symmetries of $\mathbb{O}H^1$ by their actions on $\operatorname{Im}\mathbb{O}\cup\{\infty\}$. We know that $\operatorname{Aut} J_-$ contains the transformations R and $T_{0,\eta}$ ($\eta \in \operatorname{Im}\mathbb{O}$). These act as follows:

$$R : z \mapsto 1/z$$
$$T_{0,\eta} : z \mapsto z + \eta.$$

These formulas also describe the action of R and $T_{0,\eta}$ on $\mathbb{O}H^1 = \{\, z \in \mathbb{O} \mid \operatorname{Re} z > 0 \,\}$. Note that the octave reflection R acts as the central inversion about $1 \in \mathbb{O}H^1$. These transformations generate the conformal group of $\partial\mathbb{O}H^1 = S^7$, so this identifies $\mathbb{O}H^1$ with H^8, as claimed above. We henceforth use the symbol η (resp. μ) exclusively to denote elements (resp. units) of $\operatorname{Im}\mathcal{K}$. By a real reflection we will mean a nontrivial transformation of H^8 that fixes a real hyperplane pointwise. This is the usual definition of reflections, and opposed to octave reflections, which act on H^8 by central inversion in points of H^8.

By definition, $R_- = \langle P_\eta \rangle$ where $P_\eta = T_{0,\eta} \circ R \circ T_{0,-\eta}$ and η varies over $\operatorname{Im}\mathcal{K}$. This group is normalized by the involution $q : z \mapsto \bar{z}$, since q commutes with R and conjugates $T_{0,\eta}$ to $T_{0,-\eta}$. The reader is cautioned that q is not the restriction to J_- of any element of $\operatorname{Aut} J$. Since q reverses orientation on $\partial\mathbb{O}H^1$, $\langle P_\eta \rangle$ is the rotation subgroup of $\langle P_\eta, q \rangle$. We write Q_η for $T_{0,\eta} \circ R \circ q \circ T_{0,-\eta}$. One can check that Q_0 is a real reflection which acts on $\partial\mathbb{O}H^1$ by inversion in the unit sphere centered at 0, so Q_η acts by inversion in the unit sphere centered at η. The next few constructions are much more easily carried out by geometric rather than symbolic computation. Figure 5.1 should assist the reader.

We know that $Q_0 \in \langle P_\eta, q \rangle$. If $|\eta|^2 = 1$ then $P_\eta \circ Q_0 \circ P_\eta$ is the (real) reflection across the P_η-translate of the mirror of Q_0—namely, the perpendicular bisector of the segment joining η and 2η. If η_1 and η_2 are two units of $\operatorname{Im}\mathcal{K}$ with mutual distance 1, then

$$(P_{\eta_1} \circ Q_0 \circ P_{\eta_1})(P_{\eta_2} \circ Q_0 \circ P_{\eta_2})(P_{\eta_1} \circ Q_0 \circ P_{\eta_1})$$

is reflection across the hyperplane of $\operatorname{Im}\mathbb{O}$ that passes through 0 and is orthogonal to $\pm(\eta_1 - \eta_2)$. These reflections generate the E_7 Weyl group, the group of all isometries of $\operatorname{Im}\mathcal{K}$ fixing 0, and so $\langle P_\eta \rangle$ contains all of the rotations of $\operatorname{Im}\mathcal{K}$ fixing 0. Since $\langle P_\eta \rangle$ is normalized by the $T_{0,\eta}$, it contains all the rotations about all of the points of $\operatorname{Im}\mathcal{K}$. These rotations generate an (infinite) group of transformations of $\operatorname{Im}\mathbb{O}$ containing all rotations and translations preserving $\operatorname{Im}\mathcal{K}$. We are now in a position to prove (*ii*). We know that R_- contains transformations acting on J_- in the same manner as do the $T_{0,\eta}$ and S_μ. Fixing μ for the moment, we see that R_- contains either S_μ or $R' \circ S_\mu$. The square of either of these is R', so $R' \in R_-$. Now, for any μ (resp. η), R_- contains either S_μ or $R' \circ S_\mu$ (resp. $T_{0,\eta}$ or $R' \circ T_{0,\eta}$) and hence contains both. This proves (*ii*).

We now finish the proof of (*i*). We know that $\langle P_\eta, q \rangle$ contains Q_0 and all the translations $z \mapsto z + \eta$, and so it contains all the Q_η. Since $q = Q_0 \circ P_0$, we see that $\langle P_\eta, q \rangle = \langle P_\eta, Q_\eta \rangle$ and so $\langle P_\eta \rangle$ is the rotation subgroup of $\langle P_\eta, Q_\eta \rangle$. We now show $\langle P_\eta \rangle \subseteq \langle Q_\eta \rangle$. If η_1 and η_2 are neighbors in $\operatorname{Im}\mathcal{K}$ then $Q_{\eta_1} \circ Q_{\eta_2} \circ Q_{\eta_1}$ is the real reflection in the perpendicular bisector of the segment joining η_1 and η_2. These real reflections generate the group of all isometries of $\operatorname{Im}\mathbb{O}$ that preserve

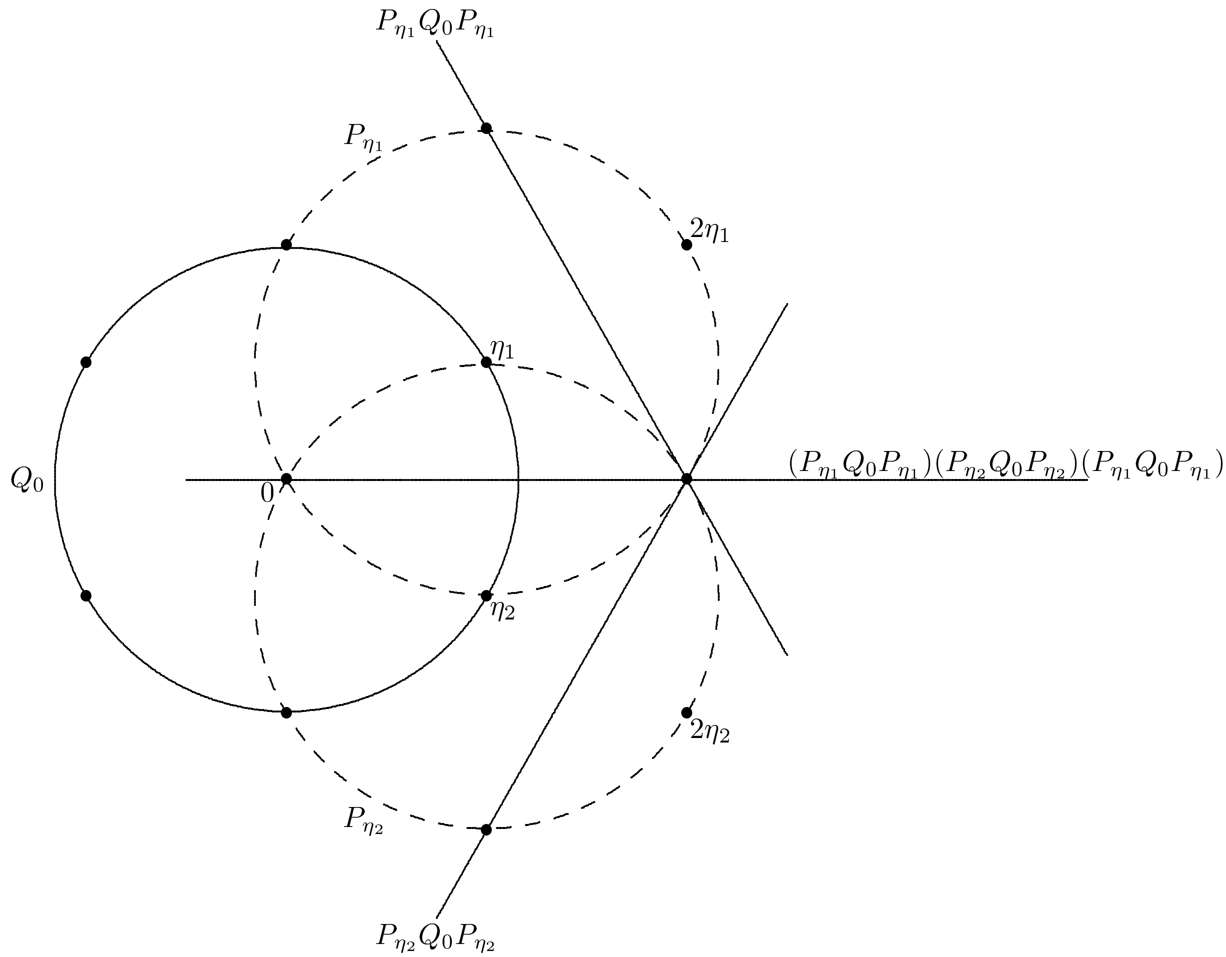


Figure 5.1. A 2-dimensional section of $\operatorname{Im}\mathbb{O}$ (see the proof of theorem 5.3). Dots indicate elements of $\operatorname{Im}\mathcal{K}$. The real reflection Q_0 acts by inversion in the solid circle. The lines are the mirrors of certain other real reflections, whose names are next to them. Each dashed circle is carried to itself by the octave reflection whose name is next to it; this reflection carries each point on the dashed circle to its antipode on the same dashed circle. The octave reflection also exchanges the center of the dashed circle with ∞.

$\operatorname{Im}\mathcal{K}$. (This is just the affine E_7 Weyl group; note that the diagram automorphism of the Coxeter group does not preserve the lattice—it exchanges it with the set of its deep holes.) In particular, it contains the map $q : z \mapsto -z$. Since $P_0 = q \circ Q_0$, we see that $P_0 \in \langle Q_\eta \rangle$. Since $\langle Q_\eta \rangle$ is normalized by all $T_{0,\eta}$, we see that $\langle P_\eta \rangle \subseteq \langle Q_\eta \rangle$. This identifies $\langle P_\eta \rangle$ as the rotation subgroup of $\langle Q_\eta \rangle$.

It remains to identify $\langle Q_\eta \rangle$. We take the nodes of Δ other than α and β to represent standard generators of the E_7 Weyl group, acting on $\operatorname{Im}\mathcal{K}$ by isometries that preserve 0. We take the node α to represent the reflection in the perpendicular bisector of the segment joining 0 and η_0, for some unit η_0 of $\mathcal{K}$. The group generated by these 8 reflections is the affine E_7 Weyl group and contains all translations of $\operatorname{Im}\mathcal{K}$. Taking the node β to represent the reflection Q_0, we see that the group generated by these 9 reflections contains all the Q_η and thus equals $\langle Q_\eta \rangle$. It is easy to see that the angles between pairs of these 9 mirrors are $\pi/3$ (resp. $\pi/2$) when the corresponding nodes of Δ are joined (resp. unjoined). Since these are integral submultiples of π, the region in H^8 bounded by the mirrors is a fundamental domain for the group generated by the 9 reflections. This identifies $\langle Q_\eta \rangle$ as the Coxeter group with diagram Δ, proving (i).

Now we prove (iii). Suppose $\phi \in \operatorname{Aut} K_n$ with $\phi(B) = B$. Then $\phi(C)$ is a good nilpotent of height 1, so $\phi(C) = \pi(x, 1, z)$ for some $x, z \in \mathbb{O}$. Since $\phi(C) \in K_n$ we must have $x \in \sqrt{n} \cdot \mathcal{K}$. After applying a translation of R_n, courtesy of theorem 5.2, we may take $x = 0$. Then since $\phi(C)$ is nilpotent and in K_n we must have $z \in \operatorname{Im} \mathcal{K}$. After applying $T_{0,-z}$, which lies in R_n by (ii), we may take $z = 0$. That is, we may suppose $\phi(C) = C$. By mimicking the proof of (iv) of theorem 5.3 and using (vi) of lemma 5.1 we see that that the simultaneous stabilizer of B and C is generated by the S_μ, which by (ii) also lie in R_n. This establishes (iii). □

Remarks: The proof of (ii) shows that R_- is a nontrivial central extension by $\mathbb{Z}/2$ of its image in $\operatorname{Aut} J_-$; the nontrivial central element is R'. One can show that Δ is the Coxeter diagram of the reflection group of the real Lorentzian lattice $E_7 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This is not surprising since this form is the negative of the restriction to the traceless elements of K_- of the norm form on J.

Here is the main result of the paper: the existence of finite covolume reflection groups acting on $\mathbb{O}H^2$.

Theorem 5.4.

(i) *Under the action of* R_1*, there is a single orbit of primitive good nilpotents of* K_1*.*

(ii) *Under the action of* R_2*, there are precisely two orbits of primitive good nilpotents of* K_2*. Elements of one orbit are characterized as such by being orthogonal to idempotents of* K_2*.*

(iii) *For* $n = 1$ *or* 2*,* R_n *coincides with* $\operatorname{Aut} K_n$ *and has finite covolume in* $G \cong F_{4(-20)}$*.*

Proof: Let $n = 1$ or 2. Suppose $X \in K_n$ is a primitive good nilpotent and has minimal height h among its images under R_n. Then either $X = B$ or $X = h \cdot \pi(x, 1, -|x|^2/2 + z)$ with $h > 0$, $x \in \mathbb{O}$ and $z \in \operatorname{Im} \mathbb{O}$. After applying a translation in R_n (of which there are plenty by theorem 5.2), we may suppose that x is at least as close to 0 as it is to any other element of $\sqrt{n} \cdot \mathcal{K}$. By theorem 5.3, R_n also contains all central translations, so we may also suppose that z is at least as close to 0 as it is to any other point of $\operatorname{Im} \mathcal{K}$. By lemma 5.1 (i), (ii), these conditions imply $|x|^2 \leq n/2$ and $|z|^2 \leq 3/4$. Observe that

$$R(X) = h \cdot (x, |x|^2/2 - z, -1)$$

has height $h(|z|^2 + |x|^4/4)$. If $n = 1$ then this is smaller than the height h of X, contrary to our hypothesis on X. Thus we have proven (i).

If $n = 2$ then either $\operatorname{ht}(R(X)) < \operatorname{ht}(X)$, contrary to our hypothesis on X, or we have $|x|^2 = 1$ and $|z|^2 = 3/4$. Since x is at least as close to 0 as to any other element of $\sqrt{2} \cdot \mathcal{K}$, we see that x is a deep hole of $\sqrt{2} \cdot \mathcal{K}$. Similarly, since z is at least as close to 0 as to any other element of $\operatorname{Im} \mathcal{K}$, we see that z is a deep hole of $\operatorname{Im} \mathcal{K}$. By lemma 5.1$(iii)$, (vii) and theorem 5.3(ii), after applying an element of $\operatorname{Spin}_7(2) \subseteq R_2$ we may take x to be any particular deep hole of $\sqrt{2} \cdot \mathcal{K}$ nearest 0, say $x = (1+i)/\sqrt{2}$. By lemma 5.1(v) and theorem 5.3(ii) we may apply some $T_{0,\eta} \in R_2$ and take z to be any particular deep hole of $\operatorname{Im} \mathcal{K}$, say $z = (i + j + k)/2$. This determines X up to the scale factor h, which is determined by the requirement that X be a primitive good element of K_2. Therefore

$$X = X_0 = \pi(1 + i, \sqrt{2}, \omega\sqrt{2}),$$

where $\omega = (-1 + i + j + k)/2$. This proves that K_2 has at most two orbits of primitive good nilpotents under R_2. It is obvious that X_0 is orthogonal to the idempotent $E = \pi(0, 1, -\bar{\omega}) \in K_2$ and one can show B is not orthogonal to any idempotent of K_2. (The key is that $A \notin K_2$.) This proves that there are exactly two orbits under R_2 and under $\operatorname{Aut} K_2$; in particular, (ii) holds.

Now suppose $\phi \in \operatorname{Aut} K_n$. By using (i) or (ii), after multiplying by an element of R_n we may suppose that $\phi(B) = B$. But then by theorem 5.3 (iii) we see that $\phi \in R_n$. This establishes the equality $R_n = \operatorname{Aut} K_n$ for $n = 1, 2$. Because $\operatorname{Aut} K_n$ is an arithmetically defined subgroup of a real semisimple Lie group, a theorem of Borel and Harish-Chandra [5] shows that it has finite covolume in G, which establishes (*iii*). □

Remarks: If we do not assume the full results of theorem 5.3, as described in the remark there, the proof above can be modified to obtain the weaker result that R_1 and R_2 have finite covolume in G. The proof above shows that if X is a primitive good nilpotent of K_1 of positive height then there is a reflection in R_1 reducing the height of X (namely, the conjugate of R by the translations used in the first paragraph of the proof). Therefore we can conclude that X is equivalent to B under R_1. Similarly, if X is a primitive good nilpotent in K_2 then after a sequence of reflections in R_2, X may be taken to have height ≤ 1. Since the stabilizer of B in $\operatorname{Aut} K_2$ acts with one orbit on the primitive good nilpotents of height 1, and by hypothesis the stabilizer of B in $R_- \subseteq R_2$ has finite index of that in $\operatorname{Aut} K_2$, we see that there are only finitely many R_2-orbits of primitive good nilpotents in K_2. The finiteness of the index of R_i in $\operatorname{Aut} K_i$ for each $i = 1, 2$ follows from this and from the assumption that B's stabilizer in R_i has finite index in its stabilizer in $\operatorname{Aut} K_i$.

6 The Determinant

In section 5 we constructed the promised octave hyperbolic reflection groups and proved that they have finite covolume. In the following section we will interpret our groups as the stabilizers of certain Hermitian forms over $\mathcal{K}$. To do this we must work in a context wider than that of our treatment so far. Namely, we must introduce the determinant form on J and its automorphism group H. By explicit construction we will show that the determinant form on J is equivalent to the one on J_I studied in [11], which will prove that $H \cong E_{6(-26)}$. The action of $E_{6(-26)}$ on J_I is similar to the action of $PGL_3\mathbb{C}$ on the space of Hermitian forms on $\mathbb{C}^3$, so this is the natural setting for discussions of the stabilizers of Hermitian forms. We will pursue this further in the next section.

Recall that we defined $\chi(X) = \operatorname{Re}\operatorname{Tr}(X)$ for any $X \in M_3\mathbb{O}$. (Of course, we are only interested in elements of Jordan algebras J_Φ, which have real traces.) Pages 30-31 of [12] show that if $X, Y, Z \in M_3\mathbb{O}$ then $\langle X|YZ\rangle = \langle XY|Z\rangle = \langle ZX|Y\rangle$, and it follows therefrom that $\chi(X^2X) = \chi(XX^2)$ and that $\langle X|Y|Z\rangle = \langle X * Y|Z\rangle$ is a symmetric trilinear form. Since it is symmetric, it is obtained by polarization from the cubic form $\mathcal{C}$ given by

$$\mathcal{C}(X) = \langle X|X|X\rangle = \chi(X^3).$$

Following [11] we define the cubic "determinant" form by

$$\det(X) = \frac{1}{6}[2\chi(X^3) - 3\chi(X^2)\chi(X) + \chi(X)^3].$$

The name of this cubic form derives from analogies between it and the usual determinant form on (say) $M_3\mathbb{R}$. In particular, in the special cases will consider below the formulas (6.1) and (6.2) for the determinant closely resemble the usual expression. For any Φ we write H_Φ for the group of all linear transformations preserving the determinant form on the real vector space underlying J_Φ. We write H for H_Ψ.

Theorem 6.1. *For $X \in J$ given by (4.1),*

$$\det(X) = a|u|^2 + b|w|^2 + c|v|^2 - abc - 2\operatorname{Re}(uvw). \tag{6.1}$$

Proof: The computation is long and tedious but mostly straightforward. One should begin by writing $2\operatorname{Tr}(X^3) = \operatorname{Tr}(X^2X) + \operatorname{Tr}(XX^2)$ before multiplying together the matrices. This device makes the resulting sum simplify because many of its terms appear in $\mathbb{O}$-conjugate pairs. Manipulations involving no special $\mathbb{O}$ identities except obvious applications of (2.2) show

$$\det(X) = a|u|^2 + b|w|^2 + c|v|^2 - abc + [2\operatorname{Re}(\bar{u}vw) - 4\operatorname{Re}(vw)\operatorname{Re}(u)],$$

and so we examine the term in brackets. Further use of (2.2) allows the derivation

$$\begin{aligned}2\,\mathrm{Re}(\bar{u}vw) - 4\,\mathrm{Re}(vw)\,\mathrm{Re}(u) &= \mathrm{Re}[2\,\mathrm{Re}(\bar{u}vw) - 4\,\mathrm{Re}(vw)\,\mathrm{Re}(u)]\\ &= \mathrm{Re}[\bar{u}vw + \bar{w}\bar{v}u - (vw + \bar{w}\bar{v})(u + \bar{u})]\\ &= \mathrm{Re}[\bar{u}vw + \bar{w}\bar{v}u - vw\bar{u} - vwu - \bar{w}\bar{v}\bar{u} - \bar{w}\bar{v}u]\\ &= \mathrm{Re}[u\bar{w}\bar{v} + u\bar{w}\bar{v} - u\bar{w}\bar{v} - uvw - uvw - u\bar{w}\bar{v}]\\ &= -2\,\mathrm{Re}(uvw),\end{aligned}$$

which completes the proof. □

Since the action of $GL_3\mathbb{R}$ on $M_3\mathbb{O}$ preserves matrix multiplication and traces, it also preserves the determinant form. In light of this, the next theorem assures us that $G \subseteq H$; it also provides a quick alternate proof of most of theorem 4.1.

Theorem 6.2. *The transformations S_μ preserve the restriction to J of the determinant form and act as automorphisms of the Jordan algebra J.*

Proof: To show that $\det S_\mu(X) = \det X$ for $X \in J$ given by (4.1) it suffices to show

$$\mathrm{Re}(\mu u\bar{\mu} \cdot \mu v \cdot w\bar{\mu}) = \mathrm{Re}(uvw).$$

This follows from the derivation

$$\begin{aligned}\mathrm{Re}(\mu u\bar{\mu} \cdot \mu v \cdot w\bar{\mu}) &= -\,\mathrm{Re}(\mu u\bar{\mu} \cdot \mu v \cdot w\mu)\\ &= -\,\mathrm{Re}(\mu u\bar{\mu} \cdot \mu(vw)\mu)\\ &= -\,\mathrm{Re}(\mu u\bar{\mu} \cdot \mu \cdot (vw)\mu)\\ &= -\,\mathrm{Re}(\mu u\bar{\mu}\mu \cdot (vw)\mu)\\ &= -\,\mathrm{Re}(\mu u \cdot (vw) \cdot \mu)\\ &= -\,\mathrm{Re}(\mu \cdot \mu u \cdot vw)\\ &= \mathrm{Re}(uvw),\end{aligned}$$

where we have used (2.1), (2.2), and the fact that $\mu = -\bar{\mu}$. (This is in spirit the same argument as that applied to J_I in [11, §6].)

Combined with the fact that the S_μ preserve traces and norms, this proves that $S_\mu \in \mathrm{Aut}\,J$. This follows because the Jordan multiplication can be defined in terms of the trace, norm and determinant forms: Suppose $X, Y \in J$. Supposing that we know the trace, norm and determinant forms, we know the value of $\mathcal{C}$ for every element of J. Therefore we know the value of the symmetric trilinear form $\langle X|Y|Z\rangle$ obtained from $\mathcal{C}$ by polarization, for every $Z \in J$. By the definition of $\langle X|Y|Z\rangle$, we know the value of $\langle X * Y|Z\rangle$ for every Z, and since the inner product is nondegenerate this uniquely determines $X * Y$. □

There are elements of H that do not lie in G, for example if $r \in \mathbb{R} \smallsetminus \{0\}$ then H contains the map

$$F_r : (a, b, c, u, v, w) \mapsto (ar^4, br^{-2}, cr^{-2}, ur^{-2}, vr, wr).$$

We may identify H with the real Lie group $E_{6(-26)}$ as follows. In [11] Freudenthal calculated the determinant form on J_I. An element of J_I has the form

$$\begin{pmatrix} d & z & \bar{y} \\ \bar{z} & e & x \\ y & \bar{x} & f \end{pmatrix},$$

for $d, e, f \in \mathbb{R}$ and $x, y, z \in \mathbb{O}$. We denote this element of J_I by $[d, e, f, x, y, z]$. It is shown in [11] that

$$\det[d, e, f, x, y, z] = def + 2\operatorname{Re}(xyz) - d|x|^2 - e|y|^2 - f|z|^2, \tag{6.2}$$

and that the H_I is isomorphic to $E_{6(-26)}$. To identify H with this Lie group we need only observe that the map $g : J \to J_I$ defined by

$$g : (a, b, c, u, v, w) \mapsto [-a, -c, -b, -u, -v, -w]$$

identifies the determinant forms.

So far in this paper we have informally regarded elements of a Jordan algebra J_Φ as "transformations of $\mathbb{O}^3$" that are Hermitian with respect to Φ. When one considers just the determinant form on J_I and not the Jordan algebra structure, it is more appropriate to regard elements of J_I as "Hermitian forms on $\mathbb{O}^3$". One should also regard their determinants as those of Hermitian forms rather than of "linear transformations".

7 Stabilizers of Integral Hermitian Forms

We write $H_I^{\mathcal{K}}$ for the stabilizer in H_I of $J_I \cap M_3\mathcal{K}$. Given the material we have developed so far, it is quite easy to realize the groups R_n as stabilizers in $H_I^{\mathcal{K}}$ of appropriate forms over $\mathcal{K}$.

One can show that $F_{4(-20)}$ is a maximal (proper) subgroup of $E_{6(-26)}$, and since not all of H fixes the identity matrix I, we see that we may define G as the subgroup of H fixing $I \in J$. Recall that for n a positive integer we defined

$$K_n = \{(na, b, c, u, v\sqrt{n}, w\sqrt{n}) | a, b, c \in \mathbb{Z}, u, v, w \in \mathcal{K}\}.$$

From the above, we conclude that $\operatorname{Aut} K_n$ is the subgroup of H that preserves K_n as a set and fixes $nI \in K_n$.

We define the map $h_n : J \to J_I$ by $h_n(X) = -n^{-1/3} \cdot (g \circ F_r)(X)$ where $r = n^{-1/6}$ and F_r and g are as defined in section 6. Computation reveals that h_n carries K_n bijectively to $J_I \cap M_3\mathcal{K}$ and multiplies determinants by $-1/n$. This shows that $\operatorname{Aut} K_n$ is isomorphic to the subgroup of $H_I^{\mathcal{K}}$ that fixes $h_n(nI)$. Computing $h_n(nI)$ we obtain:

Theorem 7.1. *For any n, $\operatorname{Aut} K_n$ is isomorphic to the subgroup of $H_I^{\mathcal{K}}$ that preserves the matrix (or "Hermitian form")*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & n \\ 0 & n & 0 \end{pmatrix}.$$

An element $g \in PGL_3\mathbb{Z}$ acts on 3-dimensional integral quadratic forms Φ by $\Phi \mapsto g\Phi g^*$, and obviously acts on $J_I \cap M_3\mathcal{K}$ in the same way, realizing $PGL_3\mathbb{Z}$ as a subgroup of $H_I^{\mathcal{K}}$. The stabilizer in $PGL_3\mathbb{Z}$ of an integral quadratic form Φ is (the central quotient of) the automorphism group of the lattice with inner product matrix Φ. This makes it natural and satisfying to regard the groups $\operatorname{Aut} K_n$ (and in particular the reflection groups R_1 and R_2) as the symmetry groups of "Lorentzian lattices" over $\mathcal{K}$ with the "inner product matrices" given in theorem 7.1.

Gross [13] has investigated integral forms of various semisimple algebraic groups, and observed that the stabilizer in $H_I^{\mathcal{K}}$ of the matrix

$$\Psi' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

is a model over $\mathbb{Z}$ for the unique form of F_4 over $\mathbb{Q}$ which is split at each prime p and has rank 1 over $\mathbb{R}$. Since Ψ' is equivalent to $-\Psi$ under $PGL_3\mathbb{Z} \subseteq H_I^{\mathcal{K}}$ we see that this integral form of F_4 is the reflection group R_1. In particular, one of our groups has been investigated before and found to be a natural integral form of F_4.

Finally, one can consider 2-dimensional versions of our constructions. As introduced in section 5, J_- is the space of 2×2 octave matrices which are Hermitian with respect to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (We actually introduced J_- in a slightly different but equivalent way.) That is, J_- consists of the matrices $\begin{pmatrix} \bar{u} & b \\ c & u \end{pmatrix}$ with $b, c \in \mathbb{R}$ and $u \in \mathbb{O}$. There are no subtleties involved in the definition of the determinant $|u|^2 - bc$ of an element of J_-, and the group preserving just the vector space structure and determinant form on J_- is obviously the real orthogonal group $O(9,1)$. The group of Jordan algebra automorphisms is the stabilizer $O(8,1)$ therein of the identity matrix. We also defined K_- as $M_2\mathcal{K} \cap J_-$.

The determinant form on the space

$$J'_- = \left\{ \begin{pmatrix} e & x \\ \bar{x} & f \end{pmatrix} \;\middle|\; e, f \in \mathbb{R}, x \in \mathbb{O} \right\}$$

of matrices in $M_2\mathbb{O}$ that are Hermitian with respect to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is given by $ef - |x|^2$. Thus the map $J_- \to J'_-$ given by

$$\begin{pmatrix} \bar{u} & b \\ c & u \end{pmatrix} \mapsto \begin{pmatrix} b & u \\ \bar{u} & c \end{pmatrix}$$

negates determinants, carries $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and identifies K_- with $M_2\mathcal{K} \cap J'_-$. Therefore, as in the 3-dimensional case, we may interpret the group $\operatorname{Aut} K_-$ as begin essentially the automorphism group of the "Lorentzian lattice" with inner product matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

In section 5 we defined a group R_- which acted on K_-; it turned out that R_- was a central extension by $\mathbb{Z}/2$ of the rotation subgroup of a certain Coxeter group. The map $R_- \to \operatorname{Aut} K_-$ is almost an isomorphism. The kernel of the map is the central $\mathbb{Z}/2$ of R_- and the image has index two in $\operatorname{Aut} K_-$, which is the entire Coxeter group. (The real reflections of $H^8 = \mathbb{O}H^1$ arise from automorphisms of J_- that do not extend to J.) This description of $\operatorname{Aut} K_-$ follows from the fact that the central involution of K_- does not lie in $\operatorname{Aut} K_-$, together with the remark concerning the lattice $E_7 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ following the proof of theorem 5.3.

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