THE CONWAY-SLOANE CALCULUS FOR 2-ADIC LATTICES

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ABSTRACT. We motivate and explain the system introduced by Conway and Sloane for working with quadratic forms over the 2-adic integers, and prove its validity. Their system is far better for actual calculations than earlier methods, and has been used for many years, but no proof has been published before now.

1. INTRODUCTION

Our goal in this paper is to explain the system that Conway and Sloane developed for working with lattices (quadratic forms) over the ring of 2-adic integers $\mathbb{Z}_2$. Algorithms were already known for determining when two lattices were isometric, and for finding a canonical form for each one. But these were clumsy. In his influential book on quadratic forms, Cassels even wrote about 2-adic integral canonical forms: "only the masochist is invited to read the rest of this section" [5, §8.4]. To this day, 2-adic lattices retain their reputation for complexity.

But the 2-adic part of a lattice over $\mathbb{Z}$ is its most important part. Many questions about $\mathbb{Z}$-lattices reduce to p-adic versions of the same questions, where p varies over the primes. For example, consider the question of whether one $\mathbb{Z}$-lattice is isometric to another. We restrict to the case of rank ≥ 3 and some fixed indefinite signature, because then it is (almost) true that an isometry exists if and only if one exists p-adically for each p. Most questions about p-adic lattices are easy for odd p, including this isomorphism problem. So all the real work takes place at $p = 2$. Other examples of questions with this same flavor are whether a lattice represents a given number, or whether one lattice admits another as a direct summand (or as a primitive sublattice). See section 2 for a little more on this larger picture.

The Conway-Sloane calculus [8, ch. 15] is much simpler than previous approaches to 2-adic lattices, for example the original papers on invariants and canonical forms by Pall [10] and Jones [9]. It is widely used

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in modern applications, for example [1][4][7][11]. Their innovation was to introduce the "oddity fusion" and "sign walking" operations, which are notationally simple and generate all equivalences. Strangely, their formal statement of results (their Theorem 10) completely avoids these operations. So it has the same unwieldy feel as the papers of Pall and Jones just mentioned. Proofs of their theorem appear in [12] and in Bartels' unpublished dissertation [3]. But the literature contains no treatment of the calculus as it is actually used. We hope to make it more accessible. What is new here are the "givers" and "receivers" of section 4, and the "signways" of section 6. In particular, we use signways to correct an error in their formulation of canonical forms.

Here is a fairly detailed overview of the calculus. Our goal is to show what it looks like and what it involves, rather than to explain it properly. For that, see the formal development beginning in section 3.

Unimodular lattices: The first step in all approaches to $\mathbb{Z}_2$-lattices is to classify the unimodular ones. Conway and Sloane indicate them by symbols like $L = 1_2^{+2}$ or 1_3^{-3} or 1_{II}^{-4}. The main number 1 says that L is unimodular. If L is even, which is to say that all norms are even, then the subscript is II. Otherwise, L is diagonalizable and the subscript is the *oddity* $o(L)$ of L, meaning the sum mod 8 of the diagonal terms in any diagonalization. Amazingly, this is an isometry invariant, although the definition is more complicated if L is non-diagonalizable or non-unimodular; see section 3. The superscript is not a signed number, but rather a sign and a separate nonnegative integer. The integer is $\dim L$. The sign is $+$ or $-$ according to whether $\det(L) \equiv \pm 1$ or ± 3 mod 8. The sign, dimension and subscript turn out to determine the isometry class of L. We prove this in theorem 5.1.

For example, 1_3^{-3} is isometric to the lattices with diagonal inner product matrices $\langle 1, -1, 3\rangle$, $\langle -1, -1, -3\rangle$ and $\langle 3, 3, -3\rangle$: each is 3-dimensional with determinant ± 3 and diagonal entries summing to 3 mod 8. Similarly, 1_2^{+2} is isometric to $\langle 1, 1\rangle$ and $\langle -3, -3\rangle$. Passing from the symbol to a representative lattice is always this easy. And the symbols also behave cleanly under direct sum: signs multiply and dimensions and subscripts add. For subscripts this means addition in $\mathbb{Z}/8$, together with the special rule $\mathrm{II} + t = t$. For example, $1_2^{+2} \oplus 1_3^{-3} \oplus 1_{\mathrm{II}}^{-4} \cong 1_5^{+9}$.

Jordan decompositions: A general $\mathbb{Z}_2$-lattice can be expressed as a direct sum, where each term is got by rescaling a unimodular lattice by a different power of 2. This is called a Jordan decomposition and the terms are called Jordan constituents. Conway and Sloane use symbols like 1_{II}^{+2}, 2_2^{-2}, 4_1^{+3} and 64_{II}^{-2} to indicate them. These lattices are got from the unimodular lattices with the same decorations, by scaling inner products by 1, 2, 4 and 64 respectively. The *scale* of each term means this scaling factor. A general $\mathbb{Z}_2$-lattice is a direct sum of such terms, for example

$$1^2_{\mathrm{II}}\, 2^{-2}_4 4^3_{-1} 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\, 128^1_{-1} 256^1_1 512^{-4}_{\mathrm{II}} \tag{1.1}$$

where we have suppressed + signs in superscripts and $\oplus$ symbols between the terms. We will use this example many times: it is complicated enough to illustrate all possible phenomena.

There are two main ways that the case of p an odd prime is simpler than the $p = 2$ case. The first is that the unimodular classification is simpler: one needs no subscripts. The second is that the Jordan decomposition is unique up to isometry. So when p is odd, understanding a p-adic lattice amounts to a writing down something like (1.1) without subscripts. Equivalences between distinct Jordan decompositions is the subtle part of 2-adic lattice theory. Conway and Sloane introduced *oddity fusion* and *sign walking* to organize these equivalences.

Oddity fusion: An example of nonuniqueness of Jordan decomposition is

$$2^{-2}_4 4^3_{-1} \cong 2^{-2}_2 4^3_1 \cong 2^{-2}_{-2} 4^3_5 \tag{1.2}$$

These are the same except for their subscripts, and in all three cases the sum of the subscripts is 3 mod 8. This illustrates a general phenomenon called oddity fusion: when the scales of a sequence of Jordan constituents are consecutive powers of 2, and the subscripts are all numerical rather than II, then those constituents "share" their subscripts. We write $[2^{-2}4^3]_3$ rather than any particular Jordan decomposition from (1.2). A collection of terms that are bracketed in this way is called a *compartment*, and the final subscript 3 is called the *compartment oddity*. Since $[2^{-2}4^3]_3$ displays less information than any of the three symbols from (1.2), it is more canonical. Most of the simplicity of the Conway-Sloane approach comes from the use of oddity fusion.

After oddity fusion, our example (1.1) becomes

$$1^2_{\mathrm{II}}\, [2^{-2}4^3]_3 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\, [128^1\, 256^1]_0 512^{-4}_{\mathrm{II}} \tag{1.3}$$

The term of scale 16 is not part of the first compartment because of the absence of a term of scale 8. It forms a compartment by itself. We call a symbol like (1.3) a 2-*adic symbol*.

Sign walking: Oddity fusion does not generate all equivalences between 2-adic Jordan decompositions. For example, (1.3). turns out to be isometric to each of

$$\underbracket{1_{\mathrm{II}}^{-2}\,[2^2}\,4^3]_{-1}16_1^1\,32_{\mathrm{II}}^2\,64_{\mathrm{II}}^{-2}\,[128^1\,256^1]_0 512_{\mathrm{II}}^{-4} \tag{1.4}$$

$$1_{\mathrm{II}}^2\,[\underbracket{2^2 4^{-3}}]_{-1}16_1^1\,32_{\mathrm{II}}^2\,64_{\mathrm{II}}^{-2}\,[128^1\,256^1]_0 512_{\mathrm{II}}^{-4} \tag{1.5}$$

$$1_{\mathrm{II}}^2\,[2^{-2}\,\underbracket{4^{-3}]_{-1}16_{-3}^{-1}}\,32_{\mathrm{II}}^2\,64_{\mathrm{II}}^{-2}\,[128^1\,256^1]_0 512_{\mathrm{II}}^{-4} \tag{1.6}$$

In each case we have negated the signs of two terms of (1.3), and changed by 4 the oddity of each compartment involved. The under-brackets indicate the terms whose signs were changed. The rules for which pairs of terms admit such a *sign walk* are subtle enough that we postpone them to section 6. But to illustrate the flexibility they provide, we show which terms can interact with each other via some chain of sign walks:

$$\underbracket{1_{\mathrm{II}}^2\,[2^{-2}4^3]_3 16_1^1\,32_{\mathrm{II}}^2}\;\underbracket{64_{\mathrm{II}}^{-2}\,[128^1}\;\underbracket{256^1]_0 512_{\mathrm{II}}^{-4}}$$

We call these groups of terms *signways*, suggesting highways along which signs can move. In the language of Conway and Sloane, the classification of 2-adic lattices amounts to the theorem that sign walking generates all equivalences between 2-adic symbols. (Theorem 6.2.)

Some equivalence relations are like mazes, where it is not clear which "moves" to make when seeking an equivalence between two objects, or perhaps only an arcane recipe for these moves is available. This is the nature of earlier classifications of 2-adic lattices. Happily, sign walking is simple. For any given 2-adic symbol, the allowed sign walks generate an elementary abelian 2-group, acting simply transitively on the 2-adic symbols that are equivalent to it. (See the proof of theorem 6.3.) One can use sign walking to define a canonical form: walk all the $-$ signs as far left as possible, canceling pairs of such signs when possible. Then all signs will be $+$ except perhaps for the first terms of the signways.

The main virtues of the Conway-Sloane notation are that (i) it allows easy passage between the notation and the lattices, (ii) it behaves well under direct sum and scaling, and duality too, (iii) no more information is displayed than necessary, and (iv) rather than being constrained to a single canonical form, one can easily pass between all possible 2-adic symbols for a particular lattice. See the extended example 6.5 for an illustration of (iv): we find all the $\mathbb{Z}_2$-lattices whose sum with $\langle 2, 2\rangle$ is isometric to (1.3).

After some (strictly) motivational background in section 2, we cover some technical preliminaries in section 3. Then section 4 defines what we call a *fine decomposition* of a 2-adic lattice and describe some moves between them. In section 5 we classify the unimodular lattices and introduce oddity fusion. In section 6 we define 2-adic symbols and prove that sign walking generates all equivalences between them. We also discuss canonical forms and how to define some numerical invariants of 2-adic lattices. The final section is devoted to the proof of theorem 4.4.

This note developed from part of a course on quadratic forms given by the first author at the University of Texas at Austin, with his lecture treatment greatly improved by the second and third authors.

2. The larger picture

This section is meant to describe how the 2-adic lattice theory fits into the larger theory of integer quadratic forms. It is not needed later in the paper.

A lattice over $\mathbb{Z}$ or the p-adic integers $\mathbb{Z}_p$ means a free module equipped with a symmetric bilinear pairing that takes values in the fraction field $\mathbb{Q}$ or $\mathbb{Q}_p$. An *isometry* from one such lattice to another means a module isomorphism that preserves inner products. In many situations one wants to understand whether two $\mathbb{Z}$-lattices are isometric. If L is a $\mathbb{Z}$-lattice, then $L \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-lattice. If L' is another $\mathbb{Z}$-lattice, then L, L' are said to lie in the same genus if they have the same signature and $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isometric for all primes p. Isometric $\mathbb{Z}$-lattices obviously lie in the same genus.

Until work of Eichler in the 1950s, it was open whether the converse held in the indefinite case in dimension ≥ 3. Eichler discovered a subtle equivalence relation, whose equivalence classes are called spinor genera. Each genus consists of finitely many spinor genera, and each spinor genus consists of finitely many isometry classes of lattices. But some mild hypotheses promote "finitely many" to "one":

Theorem 2.1 (Eichler)**.** *An indefinite spinor genus of dimension* ≥ 3 *consists of exactly one isometry class.*

Theorem 2.2. *An indefinite genus G of dimension ≥ 3 consists of exactly one spinor genus, unless there exists some prime p such that $G \otimes \mathbb{Z}_p$ is p-adically diagonalizable, with the p-power parts of the diagonal terms all being distinct. If G is integral, then this exceptional case can only occur if $p^{\binom{n}{2}} \mid \det G$.*

Note that the integer $\det G$ and the $\mathbb{Z}_p$-lattice $G \otimes \mathbb{Z}_p$ are well-defined, by the definition of a genus. See [6] or [5, Ch. 10, Thm. 1.4] for theorem 2.1. See [8, Ch. 15, Thm. 19], or the proof of the Corollary to Lemma 3.7 in [5, Ch. 10], for theorem 2.2.

Except in quite small dimension, lattices with the distinct-powers-of-p property in Theorem 2.2 do not seem to occur in nature. So these two theorems form the basis for our statement in the introduction that for indefinite lattices of dimension ≥ 3, it is "almost" true that genera coincide with isometry classes. Even if a genus (indefinite of rank ≥ 3) does have the distinct-powers-of-p property, it might still consist of a single isometry class, and one can check this. It is just no longer guaranteed.

We have explained why questions of isometries of $\mathbb{Z}$-lattices often reduce to $\mathbb{Z}_p$-lattices. For $p > 2$, a $\mathbb{Z}_p$-lattice has only one isomorphism class of Jordan decomposition. And each Jordan constituent J is characterized by its scale, dimension and sign. In this case there is no subtlety to the isometry classification. So the $p = 2$ case accounts for most of the isometry analysis. (For odd p, the sign is defined as the Legendre symbol $\left(\frac{\det J}{p}\right) = \pm 1$, always abbreviated to $\pm$. Although we did not say so in the introduction, when $p = 2$ the sign of J is the Kronecker's generalization $\left(\frac{\det J}{2}\right)$ of the Legendre symbol.)

A second common question about a $\mathbb{Z}$-lattice L is whether a given lattice M occurs a direct summand. When L is the only lattice in its genus, and the signatures of M and L are compatible, this reduces to the question of whether $M \otimes \mathbb{Z}_p$ is a summand of $L \otimes \mathbb{Z}_p$ for all primes p. For $p > 2$ this is almost trivial: $M \otimes \mathbb{Z}_p$ is a summand if and only if each constituent of M is lower-dimensional than the corresponding constituent of L, or else has the same dimension and sign. The corresponding question for $p = 2$ is more subtle—see example 6.5 for a taste of the required analysis.

A similar common question is whether M occurs as a primitive sublattice of L. Under the same conditions as in the previous paragraph, this reduces to the problem of building a suitable candidate for the orthogonal complement of $M \otimes \mathbb{Z}_p$ in $L \otimes \mathbb{Z}_p$, for each prime p. The case of odd p is no longer trivial, but still the $p = 2$ case usually dominates the analysis. See [2] for an extended calculation of this sort.

3. Preliminaries

Now we begin our formal exposition. Henceforth, an *integer* means an element of the ring $\mathbb{Z}_2$ of 2-adic integers, and we write $\mathbb{Q}_2$ for $\mathbb{Z}_2$'s fraction field. A *lattice* means a finite-dimensional free module over $\mathbb{Z}_2$ equipped with a $\mathbb{Q}_2$-valued symmetric bilinear form.

We assume known that two odd elements of $\mathbb{Z}_2$ differ by a square factor if and only if they are congruent mod 8. All lattices considered will be nondegenerate. A lattice is *integral* if all inner products are integers. An integral lattice is called *even* if all its elements have even norm (self-inner-product), and *odd* otherwise. Given some basis for a lattice, one can seek an orthogonal basis by Gram-Schmidt diagonalization. This almost works but not quite. Instead it shows that every lattice is a direct sum of 1-dimensional lattices and copies of the two lattices $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ scaled by powers of 2. (See [5, p. 117] or §4.4 of [8, Ch. 15].)

Now suppose U is a *unimodular* lattice, meaning that it is integral and the natural map from U to its *dual lattice* $U^* := \mathrm{Hom}(U, \mathbb{Z}_2)$ is an isomorphism. Equivalently, the determinant of any inner product matrix is a unit of $\mathbb{Z}_2$. The *sign* of U means the Kronecker symbol $\left(\frac{\det U}{2}\right)$. Recall that this is defined as $+1$ or -1 according to whether $\det U \equiv \pm 1$ or ± 3 mod 8. We will always abbreviate ± 1 to $\pm$. The Kronecker symbol has special properties that are important in quadratic reciprocity. But these play no role in this paper; for us it is just a way to record partial information about the congruence class of an odd number mod 8. We only refer to it as the Kronecker symbol because that name already belongs to this function.

Now consider a lattice got by scaling the inner product on a unimodular lattice. We say it has *type* I or II according to whether the unimodular lattice is odd or even. For example, $\langle 2\rangle$ has type I, although it is an even lattice, because it was got by scaling the odd lattice $\langle 1\rangle$. On the other hand, $\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$ has type II, because it was got by scaling the even unimodular lattice $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$.

The last invariant of a 2-adic lattice L that we need is $\mathbb{Z}/8$-valued. It is called the *oddity* of L and written $o(L)$. It is defined in [8, p. 371], and only depends on the isometry type of the quadratic vector space $L \otimes \mathbb{Q}_2$. Its definition is strange and the fact that it is an invariant is surprising. (See [8, Ch. 15, §6.1–6.2] for a proof.) But it is very easy to compute, especially for unimodular lattices, which are all one needs it for in the Conway-Sloane calculus. To compute $o(L)$, first diagonalize its inner product matrix (over $\mathbb{Q}_2$). Then add up the odd parts of the diagonal entries mod 8, and add 4 for each diagonal entry which is an *antisquare*.

An antisquare is defined as a 2-adic number of the form $2^{\mathrm{odd}}u$ where $u \equiv \pm 3$ mod 8. The (imperfect) motivation for this language is that such a number fails to be a square in $\mathbb{Q}_2$ for two separate reasons: neither the 2-part nor the odd part are squares. (The imperfection is that -2 has the same properties but does not count as an antisquare. The name comes from the corresponding construction with 2 replaced by an odd prime. In that case the corresponding property and the definition of "antisquare" are equivalent.)

For example, $\langle 1,3,3,7\rangle$ has oddity $1+3+3+7 \equiv 4 \bmod 8$. It turns out that every odd unimodular lattice can be diagonalized over $\mathbb{Z}_2$ (lemma 4.1). The resulting diagonal terms must of course be odd, so they cannot be antisquares. So the oddity is just their sum mod 8. The calculation for even unimodular lattices is even easier: the oddity is always 0. To see this, express any such lattice as a sum of copies of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$. One can diagonalize these over $\mathbb{Q}_2$, yielding $\langle 1,-1\rangle$ and $\langle 2,6\rangle$. The first has no antisquares, so $o(\langle 1,-1\rangle) = 1-1 \equiv 0$ mod 8. The second has one antisquare (namely 6), and the odd parts of its diagonal entries are 1 and 3. So $o(\langle 2,6\rangle) = 1+3+4 \equiv 0 \bmod 8$. It follows that the oddity of every even unimodular lattice vanishes.

For unimodular lattices, the dimension, sign, type and oddity turn out to be a complete set of invariants. We prove this as theorem 5.1. Conway and Sloane express the isometry class of a unimodular lattice as $1_t^{\pm n}$ where $\pm$ is the sign, n is the dimension and t is either the formal symbol $\mathrm{I\!I}$ or an integer mod 8. We write $\mathrm{I\!I}$ for even lattices, and the oddity for odd lattices. So the subscript implicitly records the type. We just saw that type $\mathrm{I\!I}$ lattices have oddity 0, so in this case there is no point to recording it.

Except for special cases, we will not use this notation until we have classified the unimodular lattices in theorem 5.1. The special cases are in dimension 1 and the type $\mathrm{I\!I}$ case in dimension 2, where the classification is easy. Because ± 1 and ± 3 are the only square classes in $\mathbb{Z}_2^\times$, the 1-dimensional unimodular lattices are $\langle 1\rangle$, $\langle -1\rangle$, $\langle 3\rangle$ and $\langle -3\rangle$. Their symbols are 1_1^{+1}, 1_{-1}^{+1}, 1_3^{-1} and 1_{-3}^{-1} respectively. Note that the subscript determines the sign; this is unique to dimension 1. For an even unimodular lattice, we mentioned above that Gram-Schmidt orthogonalization fails to diagonalize it, but does express it as a sum of copies of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$. It follows that these are the only even unimodular lattices in 2 dimensions. They are non-isometric because their determinants are different. Their symbols are $1_{\mathrm{I\!I}}^{+2}$ and $1_{\mathrm{I\!I}}^{-2}$ respectively.

If q is a power of 2 then we write $q_{\mathrm{I\!I}}^{\pm n}$ or $q_t^{\pm n}$ for the lattice got from $1_{\mathrm{I\!I}}^{\pm n}$ or $1_t^{\pm n}$ by rescaling all inner products by q. For example, $2_{\mathrm{I\!I}}^{-2}$ has inner product matrix $\left(\begin{smallmatrix}4&2\\2&4\end{smallmatrix}\right)$. The number q is called the *scale* of the symbol (or lattice). **Caution:** in the type I case, the subscript is the oddity of the unimodular lattice, not the scaled lattice. These may differ by 4 because of the antisquare term in the definition of oddity. For example, the 2-adic lattice $2_3^{-1} \cong \langle 6\rangle$ has oddity -1, not 3.

Just like for unimodular lattices, until we prove theorem 5.1 we will only use the symbols $q_t^{\pm 1}$ and $q_{\mathrm{II}}^{\pm 2}$. We will usually omit the symbol $\oplus$ from direct sums, for example writing $1_{-1}^{+1}\,1_3^{-1}\,4_{\mathrm{II}}^{+2}$ for $1_{-1}^{+1}\oplus 1_3^{-1}\oplus 4_{\mathrm{II}}^{+2}$. To lighten the notation one usually suppresses plus signs in superscripts, for example $1_{-1}^{1}\,1_3^{-1}\,4_{\mathrm{II}}^{2}$, and/or suppresses the dimensions when they are 1, for example $1_{-1}^{+}\,1_3^{-}\,4_{\mathrm{II}}^{2}$. One could suppress even more, such as leaving the subscript blank for summands of type II. But at some point abbreviations become more error-prone than helpful.

4. Fine symbols

In this section we work with a finer decomposition of a lattice than the usual Jordan decomposition. The goal is to establish that certain "moves" between such decompositions do not change the isometry class of the lattice. This will make the corresponding facts for Jordan decompositions in the next section easy to state and prove. Theorem 4.4, proven in section 7, captures the full classification of 2-adic lattices, but in a very clumsy way. The rest of this paper recasts this classification into a simpler form.

By a *fine decomposition* of a lattice L we mean a direct sum decomposition in which each summand (or *term*) is one of $q_{\pm 1}^{1}$, $q_{\pm 3}^{-1}$ or $q_{\mathrm{II}}^{\pm 2}$, with the last case only occurring if every term of that scale has type II. The name reflects the fact that there no further decomposition of the summands is possible. A fine decomposition always exists, by starting with a decomposition as a sum of $q_t^{\pm 1}$'s and $q_{\mathrm{II}}^{\pm 2}$'s and applying the next lemma repeatedly.

Lemma 4.1. *If $\varepsilon, \varepsilon'$ are signs then $1_t^{\varepsilon 1}\,1_{\mathrm{II}}^{\varepsilon' 2}$ admits an orthogonal basis.*

Proof. Write M and N for the two summands and consider the three elements of $(M/2M)\oplus(N/2N)$ that lie in neither $M/2M$ nor $N/2N$. Any lifts of them have odd norms and even inner products. Applying row and column operations to their inner product matrix leads to a diagonal matrix with odd diagonal entries. □

In order to discuss the relation between distinct fine decompositions of a given lattice, we introduce the following special language for 1-dimensional lattices only. We call q_1^{+1} and q_{-3}^{-1} "givers" and q_{-1}^{+1} and q_3^{-1} "receivers". (Type II lattices are neither givers nor receivers.) The idea is that a giver can give away two oddity and remain a legal symbol ($q_1^{+}\to q_{-1}^{+}$ or $q_{-3}^{-}\to q_3^{-}$), while a receiver can accept two oddity. We often use a subscript R or G in place of the oddity, so that 1_G^{+} and 1_G^{-} mean 1_1^{+} and 1_{-3}^{-}, while 1_R^{+} and 1_R^{-} mean 1_{-1}^{+} and 1_3^{-}. Scaling inner products by -3 negates signs and preserves giver/receiver status, while scaling them by -1 preserves signs and reverses giver/receiver status.

A *fine symbol* means a sequence of symbols $q_{\mathrm{II}}^{\pm 2}$ and $q_{R\,\mathrm{or}\,G}^{\pm}$. We replace R and G by numerical subscripts whenever convenient, and regard two symbols as the same if they differ by permuting terms. Two scales are called *adjacent* if they differ by a factor of 2.

Lemma 4.2 (Sign walking)**.** *Consider a fine symbol and two terms of it that satisfy one of the following conditions:*

(0) *they have the same scale;*
(1) *they have adjacent scales and different types;*
(2) *they have adjacent scales and are both givers or both receivers;*
(3) *their scales differ by a factor of* 4 *and they both have type* I.

Consider as well the fine symbol got by negating the signs of these terms, and also changing both from givers to receivers or vice-versa in case (2)*. Then the two fine symbols represent isometric lattices.*

An alternate name for (3) might be sign jumping. Conway and Sloane informally describe it as a composition of two sign walks of type (1). For example,

$$1_1^1 2_{\mathrm{II}}^{+0} 4_1^1 \to 1_{-3}^{-1} 2_{\mathrm{II}}^{-0} 4_1^1 \to 1_{-3}^{-1} 2_{\mathrm{II}}^{+0} 4_{-3}^{-1}.$$

They also observe that this doesn't really make sense: 2_{II}^{-0} is illegal because the 0-dimensional lattice has determinant 1, hence sign $+$.

Proof. It suffices to prove the following isometries, where $\varepsilon, \varepsilon'$ are signs, X represents R or G, and X' represents R or G:

(0) $1_{\mathrm{II}}^{\varepsilon 2}\, 1_{\mathrm{II}}^{\varepsilon' 2} \cong 1_{\mathrm{II}}^{-\varepsilon 2}\, 1_{\mathrm{II}}^{-\varepsilon' 2}$ and $1_X^{\varepsilon}\, 1_{X'}^{\varepsilon'} \cong 1_X^{-\varepsilon}\, 1_{X'}^{-\varepsilon'}$
(1) $1_{\mathrm{II}}^{\varepsilon 2}\, 2_{X'}^{\varepsilon'} \cong 1_{\mathrm{II}}^{-\varepsilon 2}\, 2_{X'}^{-\varepsilon'}$ and $1_{X'}^{\varepsilon'}\, 2_{\mathrm{II}}^{\varepsilon 2} \cong 1_{X'}^{-\varepsilon'}\, 2_{\mathrm{II}}^{-\varepsilon 2}$
(2) $1_G^{\varepsilon}\, 2_G^{\varepsilon'} \cong 1_R^{-\varepsilon}\, 2_R^{-\varepsilon'}$
(3) $1_X^{\varepsilon}\, 4_{X'}^{\varepsilon'} \cong 1_X^{-\varepsilon}\, 4_{X'}^{-\varepsilon'}$

The first part of (0) is trivial except for the assertion $1_{\mathrm{II}}^{+2}\, 1_{\mathrm{II}}^{+2} \cong 1_{\mathrm{II}}^{-2}\, 1_{\mathrm{II}}^{-2}$. Choose a norm 4 vector x of the right side, that is not twice a lattice vector. Then choose y to have inner product 1 with x. The span of x and y is even of determinant $\equiv -1$ mod 8, so it is a copy of 1_{II}^{+2}. Its orthogonal complement must also be even unimodular, hence one of $1_{\mathrm{II}}^{\pm 2}$, hence 1_{II}^{+2} by considering the determinant.

The second part of (0) is best understood using numerical subscripts: we must show $1_t^{\varepsilon}\, 1_{t'}^{\varepsilon'} \cong 1_{t+4}^{-\varepsilon}\, 1_{t'+4}^{-\varepsilon'}$, i.e., $\langle t, t' \rangle \cong \langle t+4, t'+4 \rangle$. To see this, note that the left side represents $t + 4t' \equiv t + 4$ mod 8, that this is odd and therefore corresponds to some direct summand, and the determinants of the two sides are equal. Note that givers and receivers always have oddities congruent to 1 and -1 mod 4 respectively, so changing a numerical subscript by 4 doesn't alter giver/receiver status. Furthermore, the sign on 1_t^ε changes since exactly one of $t, t+4$ lies in $\{\pm 1\}$ and the other in $\{\pm 3\}$, and similarly for $1_{t'}^{\varepsilon'}$. The same argument works for (3), in the form $1_t^\varepsilon\, 4_{t'}^{\varepsilon'} \cong 1_{t+4}^{-\varepsilon}\, 4_{t'+4}^{-\varepsilon'}$.

For the first part of (1) we choose a basis for $1_{\mathrm{II}}^{\varepsilon 2}$ with inner product matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0\,\mathrm{or}\,2 \end{smallmatrix}\right)$ where the lower right corner depends on ε. Replacing the second basis vector by its sum with a generator of $2_{X'}^{\varepsilon'}$ changes the lower right corner by 2 mod 4. This toggles the 2×2 determinant between -1 and 3 mod 8. Therefore it gives an even unimodular summand of determinant -3 times that of $1_{\mathrm{II}}^{\varepsilon 2}$, hence of sign $-\varepsilon$. Since the overall determinant is an invariant, the determinant of its complement is therefore -3 times that of $2_{X'}^{\varepsilon' 2}$. So the complement is got from $2_{X'}^{\varepsilon' 2}$ by scaling by -3. We observed above that scaling by -3 negates the sign and preserves giver/receiver status, so the complement is $2_{X'}^{-\varepsilon' 2}$. The second part of (1) follows from the first by passing to dual lattices and then scaling inner products by 2. (It is easy to see that the dual lattice has the same symbol with each scale replaced by its reciprocal.)

(2) After rescaling by -3 if necessary to take $\varepsilon = +$, it suffices to prove $1_G^+\, 2_G^{\varepsilon'} \cong 1_R^-\, 2_R^{-\varepsilon'}$, i.e., $\langle 1, 2\rangle \cong \langle 3, 6\rangle$ and $\langle 1, -6\rangle \cong \langle 3, -2\rangle$. In each case one finds a vector on the left side whose norm is odd and appears on the right, and then compares determinants. □

Further equivalences between fine symbols are phrased in terms of "compartments". A *compartment* means a set of type I terms, the set of whose scales forms a sequence of consecutive powers of 2, and which is maximal with these properties. For example in $1_{\mathrm{II}}^2\, 2_G^-\, 2_R^-\, 4_G^+\, 16_R^-$, the set of scales that have type I are $\{2, 4, 16\}$. These fall into two strings of consecutive powers of 2, namely $\{2, 4\}$ and $\{16\}$. So there are two compartments, which are the sums of the terms of the corresponding scales. That is, one compartment is $2_G^-\, 2_R^-\, 4_G^+$ and the other is 16_R^-.

Lemma 4.3 (Giver permutation and conversion)**.** *Consider a fine symbol and the symbol obtained by one of the following operations. Then the lattices they represent are isometric.*

(1) *Permute the subscripts G and R within a compartment.*
(2) *Convert any four G's in a compartment to R's, or vice versa.*

Proof. Giver permutation, meaning operation (1), can be achieved by repeated use of the isomorphisms $1_G^\varepsilon\, 1_R^{\varepsilon'} \cong 1_R^\varepsilon\, 1_G^{\varepsilon'}$ and $1_G^\varepsilon\, 2_R^{\varepsilon'} \cong 1_R^\varepsilon\, 2_G^{\varepsilon'}$ (scaled up or down as necessary). To establish these we first rescale by -3 if necessary, to take $\varepsilon = +$ without loss of generality. This leaves the cases $\langle 1, -1\rangle \cong \langle -1, 1\rangle$, $\langle 1, 3\rangle \cong \langle -1, -3\rangle$, $\langle 1, -2\rangle \cong \langle -1, 2\rangle$ and $\langle 1, 6\rangle \cong \langle -1, 10\rangle$. One proves each by finding a vector on the left whose norm is odd and appears on the right, and then comparing determinants.

For giver conversion, meaning operation (2), we assume first that more than one scale is present in the compartment, so we can choose terms of adjacent scales. Assuming four G's are present in the compartment, we permute a pair of them to our chosen terms, then use sign walking to convert these terms to receivers. This negates both signs. Then we permute these R's away, replacing them by the second pair of G's, and repeat the sign walking. This converts the second pair of G's to R's and restores the original signs.

For the case that only a single scale is present we first treat what will be the essential cases, namely

$$1^+_G\,1^+_G\,1^+_G\,1^+_G \cong 1^+_R\,1^+_R\,1^+_R\,1^+_R \quad\text{and}\quad 1^-_G\,1^+_G\,1^+_G\,1^+_G \cong 1^-_R\,1^+_R\,1^+_R\,1^+_R$$

That is,

$$\langle 1,1,1,1\rangle \cong \langle -1,-1,-1,-1\rangle \quad\text{and}\quad \langle -3,1,1,1\rangle \cong \langle 3,-1,-1,-1\rangle$$

In the first case we exhibit a suitable basis for the left side, namely $(2,1,1,1)$ and the images of $(-1,2,1,-1)$ under cyclic permutation of the last 3 coordinates. In the second we note that the left side is the orthogonal sum of the span of $(1,0,0,0)$ and $(0,1,1,1)$, which is a copy of $\langle -3,3\rangle$, and the span of $(0,-1,1,0)$ and $(0,0,1,-1)$, which is a copy of 1^{-2}_{II}. Since each of these is isometric to its scaling by -1, so is their direct sum.

Now we treat the general case when only a single scale is present. Suppose there are at least 4 givers. By scaling by a power of 2 it suffices to treat the unimodular case. By sign walking we may change the signs on any even number of them, so we may suppose at most one $-$ is present. (Recall that sign walking between terms of the same scale doesn't affect subscripts G or R.) By the previous paragraph we may convert four G's to R's. Then we reverse the sign walking operations to restore the original signs. □

The following theorem captures the full classification of 2-adic lattices. It is already simpler than the results in [9] and [10]. But fine symbols package information poorly, and much greater simplification is possible. We will develop this in the next two sections.

Theorem 4.4 (Equivalence of fine symbols)**.** *Two fine symbols represent isometric lattices if and only if they are related by a sequence of sign walking, giver permutation and giver conversion operations.*

Although it is natural to state the theorem here, its proof depends on Theorem 5.1. The first place we use it is to prove Theorem 6.2, so logically the proof could go anywhere in between. But in fact we have deferred it to section 7 to avoid breaking the flow of ideas.

5. Jordan symbols

In this section we define and study the Jordan decompositions of a lattice. The main point is that "oddity fusion" neatly wraps up all the giver permutation and conversion operations from the previous section. We begin by classifying the unimodular lattices:

Theorem 5.1 (Unimodular lattices)**.** *A unimodular lattice is characterized by its dimension, type, sign and oddity.*

As mentioned in section 3, the oddity is always 0 for even unimodular lattices. One checks this by diagonalizing $1_{\mathrm{II}}^{\pm 2}$ over $\mathbb{Q}_2$, obtaining $\langle 1, -1\rangle$ and $\langle 2, 6\rangle$, and computing the oddity directly.

Proof. Consider unimodular lattices U, U' with the same dimension, type, sign and oddity, and fine symbols F, F' for them. The product of the signs in F equals the sign of U, and similarly for U'. Since U and U' have the same sign, we may use sign walking to make the signs in F the same as in F'. If U, U' are even then the terms in F are now the same as in F', so $U \cong U'$. So suppose U, U' are odd.

By giver permutation, and exchanging F and F' if necessary, we may suppose that all non-matching subscripts are R in F and G in F'. And by giver conversion we may suppose that the number of non-matching subscripts is $k \leq 3$. Since changing a receiver to a giver without changing the sign increases the oddity by two, $o(U') = o(U) + 2k$. Since $o(U') \equiv o(U) \bmod 8$ we have $k = 0$. So the terms in F are the same as in F', and $U \cong U'$. $\square$

We now have license to use the notation $q_t^{\pm n}$ and $q_{\mathrm{II}}^{\pm n}$ from section 3. We say that such a symbol is *legal* if it represents a lattice. The legal symbols are

$$q_{\mathrm{II}}^{+0}$$

$$q_{\mathrm{II}}^{\pm n} \text{ with } n \text{ positive and even}$$

$$q_{\pm 1}^{+1} \text{ and } q_{\pm 3}^{-1}$$

$$q_0^{+2},\ q_{\pm 2}^{+2},\ q_4^{-2} \text{ and } q_{\pm 2}^{-2}$$

$$q_t^{\pm n} \text{ with } n > 2 \text{ and } t \equiv n \bmod 2$$

A good way to mentally organize these is to regard the conditions for dimension $\neq 1, 2$ as obvious, remember that q_4^2 and q_0^{-2} are illegal, and remember that the subscript of $q_t^{\pm 1}$ determines the sign.

The illegality of 1^2_4 and 1^{-2}_0 follows by considering all possible sums $1^{\varepsilon 1}_t 1^{\varepsilon' 1}_{t'}$. When the signs $\varepsilon, \varepsilon'$ are different, one subscript is ± 1 and the other is ± 3, so the total oddity cannot be 0. When the signs are the same, either both subscripts are in $\{\pm 1\}$ or both are in $\{\pm 3\}$, so the total oddity cannot be 4.

This calculation used the simple rules for direct sums of unimodular lattices: signs multiply and dimensions and subscripts add, subject to the special rules $\mathbb{I}\!\mathbb{I} + \mathbb{I}\!\mathbb{I} = \mathbb{I}\!\mathbb{I}$ and $\mathbb{I}\!\mathbb{I} + t = t$.

A *Jordan decomposition* of a lattice means a direct sum decomposition whose summands (called *constituents*) are unimodular lattices scaled by different powers of 2. By the *Jordan symbol* for the decomposition we mean the list of the symbols (or *terms*) $q^{\pm n}_{\mathbb{I}\!\mathbb{I}}$ and $q^{\pm n}_t$ for the summands. An example we will use in this section and the next, and mentioned already in the introduction, is

$$1^2_{\mathbb{I}\!\mathbb{I}}\, 2^{-2}_6 4^3_{-3}\, 16^1_1\, 32^2_{\mathbb{I}\!\mathbb{I}}\, 64^{-2}_{\mathbb{I}\!\mathbb{I}}\, 128^1_1\, 256^1_{-1}\, 512^{-4}_{\mathbb{I}\!\mathbb{I}} \tag{5.1}$$

It is sometimes convenient and sometimes annoying to allow trivial (0-dimensional) terms in a Jordan decomposition.

The main difficulty of 2-adic lattices is that a given lattice may have several inequivalent Jordan decompositions. The purpose of the Conway-Sloane calculus is to allow one to move easily between all possible isometry classes of Jordan decompositions. Some of the data in the Jordan symbol remains invariant under these moves. First, if one has two Jordan decompositions for the same lattice L, then each term in one has the same dimension as the term of that scale in the other. (Scaling reduces the general case to the integral case, which follows by considering the structure of the abelian group L^*/L.) Second, the type I or $\mathbb{I}\!\mathbb{I}$ of the term of any given scale is independent of the Jordan decomposition. (One can show this directly, but we won't need it until after Theorem 6.2, which implies it.) The signs and oddities of the constituents are not usually invariants of L.

We define a *compartment* of a Jordan decomposition just as we did for fine decompositions: a set of type I constituents, whose scales form a sequence of consecutive powers of 2, which is maximal with these properties. The example above has three compartments: $2^{-2}_6 4^3_{-3}$, 16^1_1 and $128^1_1\, 256^1_{-1}$. The *oddity* of a compartment means the sum of its subscripts (mod 8 as always). **Caution:** *this depends on the Jordan decomposition, and is not an isometry invariant of the underlying lattice.* See Lemma 6.1 for how it can change. Despite this non-invariance, the oddity of a compartment is still useful:

Lemma 5.2 (Oddity fusion)**.** *Consider a lattice, a Jordan symbol J for it, and the Jordan symbol J' got by reassigning all the subscripts in a compartment, in such a way that that all resulting terms are legal and the compartment's oddity remains unchanged. Then J, J' represent isometric lattices.*

Proof. By discarding the rest of J we may suppose it is a single compartment. The argument is similar to the odd case of Theorem 5.1. We refine J, J' to fine symbols F, F'. By hypothesis, the terms of J' have the same signs as those of J. It follows that for each scale, the product of the signs of F's terms of that scale is the same as the corresponding product for F'. Therefore sign walking between equal-scale terms lets us suppose that the signs in F are the same as in F'. Recall from the proof of Lemma 4.2(0) that this sort of sign walking amounts to the isomorphisms $1_t^{\varepsilon 1}\, 1_{t'}^{\varepsilon' 1} \cong 1_{t+4}^{-\varepsilon 1}\, 1_{t'+4}^{-\varepsilon' 1}$, which don't change the compartment's oddity.

Giver permutation and conversion don't change a compartment's oddity either. This is because changing a giver to a receiver, without changing the sign of that term, reduces the numerical subscript by 2. So changing one giver to a receiver, and simultaneously one receiver to a giver, leaves the compartment's oddity unchanged, as does converting between four givers and four receivers.

By giver permutation and possibly swapping F with F', we may suppose that the non-matching subscripts are R's in F and G's in F'. By giver conversion we may suppose $k \leq 3$ subscripts fail to match, and the assumed equality of oddities shows $k = 0$. So the fine symbols are the same and the lattices are isometric. □

6. 2-adic symbols

One can translate sign walking between fine symbols to the language of Jordan symbols, but it turns out to be fussier than necessary. Things become simpler once we incorporate oddity fusion into the notation as follows. The 2-*adic symbol* of a Jordan decomposition means the Jordan symbol, except that each compartment is enclosed in brackets, the enclosed terms are stripped of their subscripts, and their sum in $\mathbb{Z}/8$ (the compartment's oddity) is attached to the right bracket as a subscript. For our example (5.1) this yields

$$1_{\mathrm{II}}^2\, [2^{-2} 4^3]_3 [16^1]_1 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, [128^1\, 256^1]_0 512_{\mathrm{II}}^{-4}$$

If a compartment consists of a single term, such as $[16^1]_1$, then one usually omits the brackets:

$$1_{\mathrm{II}}^2\, [2^{-2} 4^3]_3 16_1^1\, 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, [128^1\, 256^1]_0 512_{\mathrm{II}}^{-4} \tag{6.1}$$

Lemma 5.2 shows that the isometry type of a lattice with given 2-adic symbol is well-defined.

When a compartment has total dimension ≤ 2 then its oddity is constrained by its overall sign in the same way as for an odd unimodular lattice of that dimension. For compartments of dimension 1 this is the same constraint as before. In 2 dimensions, $[1^+2^-]_0$ and $[1^-2^+]_0$ are illegal (cannot come from any fine symbol) because each term $1^+_{\dots}$ or $2^+_{\dots}$ would have ± 1 as its subscript, while each term $1^-_{\dots}$ or $2^-_{\dots}$ would have ± 3 as its subscript. There is no way to choose subscripts summing to 0. The same reasoning shows that $[1^+2^+]_4$ and $[1^-2^-]_4$ are also illegal.

Lemma 6.1 (Sign walking for 2-adic symbols)**.** *Consider the* 2*-adic symbol of a Jordan decomposition of a lattice, and two nontrivial terms of it that satisfy one of the following:*

(1) *they have adjacent scales and different types;*
(2) *they have adjacent scales and type* I*, and their compartment either has dimension* >2 *or compartment oddity* ± 2*;*
(3) *they have type* I*, their scales differ by a factor of* 4*, and the term between them is trivial.*

Then the 2*-adic symbol got by negating their signs, and changing by* 4 *the oddity of each compartment that contains at least one of the terms, represents an isometric lattice.*

As remarked after Lemma 4.2, one could also call (3) sign jumping. One can use it even if the intermediate term were nontrivial, by using two sign walks of type (2) resp. (1) if the intermediate term had type I resp. II. Our example

$$\begin{array}{rll}
 & 1^2_{\mathrm{II}}\,[2^{-2}4^3]_3 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\,[128^1\, 256^1]_0 512^{-4}_{\mathrm{II}} & \\
\text{can walk to} & \underbrace{1^{-2}_{\mathrm{II}}\,[2^2}\, 4^3]_{-1} 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\,[128^1\, 256^1]_0 512^{-4}_{\mathrm{II}} & \text{by (1),} \\
\text{or} & 1^2_{\mathrm{II}}\,[\underbrace{2^2 4^{-3}}]_{-1} 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\,[128^1\, 256^1]_0 512^{-4}_{\mathrm{II}} & \text{by (2),} \\
\text{or} & 1^2_{\mathrm{II}}\,[2^{-2}\, \underbrace{4^{-3}]_{-1} 16^{-1}_{-3}}\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\,[128^1\, 256^1]_0 512^{-4}_{\mathrm{II}} & \text{by (3),} \\
\text{or} & 1^2_{\mathrm{II}}\,[2^{-2}4^3]_3 16^1_1\, 32^2_{\mathrm{II}}\, 64^{-2}_{\mathrm{II}}\,[128^1\, \underbrace{256^{-1}]_4 512^{4}_{\mathrm{II}}} & \text{by (1),}
\end{array}$$

but no sign walk is possible between the terms of scales 128 and 256. (Underbrackets indicate the terms involved in the moves.)

Proof. Refine the Jordan decomposition to a fine decomposition F, apply the corresponding sign walk operation (1)–(3) from Lemma 4.2 to suitable terms of F, and observe the corresponding change in the Jordan symbol. In case (2) some care is required because Lemma 4.2 requires both terms of F to be givers or both to be receivers. If the compartment has dimension > 2 then we may arrange this by giver permutation (which preserves the compartment oddity and therefore doesn't change the 2-adic symbol). In dimension 2 the hypothesis

$$(\text{compartment oddity}) \equiv \pm 2$$

rules out the case that one is a giver and one a receiver, since givers and receivers have subscripts 1 and -1 mod 4. ☐

Theorem 6.2 (Equivalence of 2-adic symbols)**.** *Suppose given two lattices with Jordan decompositions. Then the lattices are isometric if and only if the* 2*-adic symbols of these decompositions are related by a sequence of the sign walk operations in Lemma 6.1.*

Proof. The previous lemma shows that sign walks preserve isometry type. So suppose the lattices are isometric. Refine the Jordan decompositions to fine decompositions, apply Theorem 4.4 to obtain a chain of intermediate fine symbols, and consider the corresponding 2-adic symbols. In the proof of Lemma 5.2 we explained why giver permutation and conversion don't change the 2-adic symbol, and that sign walking between same-scale terms also has no effect. The effects of the remaining sign walk operations are recorded in Lemma 6.1. ☐

A lattice may have more than one 2-adic symbol, but the only remaining ambiguity lies in the positions of the signs:

Theorem 6.3. *Suppose two given lattices have* 2*-adic symbols with the same scales, dimensions, types and signs. Then the lattices are isometric if and only if the symbols are equal, which amounts to having the same compartment oddities.*

Proof. If a 2-adic symbol S of a lattice L admits a sign walk affecting the signs of the terms of scales 2^i, 2^j then we write $\Delta_{i,j}(S)$ for the resulting symbol. No sign walks affect the conditions for $\Delta_{i,j}$ to act on S, since they don't change the type of any term or the oddity mod 4 of any compartment. So we may regard $\Delta_{i,j}$ as acting simultaneously on all 2-adic symbols for L. By its description in terms of negating signs and adjusting compartments' oddities, $\Delta_{i,j}$ may be regarded as an element of order 2 in the group $\{\pm 1\}^T \times (\mathbb{Z}/8)^C$ where T is the number of terms present and C is the number of compartments.

The assertion of the lemma is that if a sequence of sign walks on S restores the original signs, then it also restores the original oddities. We rephrase this in terms of the subgroup A of $\{\pm 1\}^T \times (\mathbb{Z}/8)^C$ generated by the $\Delta_{i,j}$. Namely: projecting A to the $\{\pm 1\}^T$ factor has trivial kernel. This is easy to see because the $\Delta_{i,j}$ are ordered so that they are $\Delta_{i_1,j_1},\ldots,\Delta_{i_n,j_n}$ with $i_1<j_1\leq i_2<j_2\leq\cdots\leq i_n<j_n$. The linear independence of their projections to $\{\pm1\}^T$ is obvious. $\square$

To get a canonical symbol for a lattice L one starts with any 2-adic symbol S and walks all the minus signs as far left as possible, canceling them when possible. To express this formally, we say two scales *can interact* if their terms are as in Lemma 6.1. (We noted in the previous proof that the ability of two scales to interact is independent of the particular 2-adic symbol representing L.) We define a *signway* as an equivalence class of scales, under the equivalence relation generated by interaction. The language suggests a pathway or highway along which signs can move. Signs can move (or cancel) between two adjacent scales except when both terms have type II, or when both terms have dimension 1 and together form a compartment of oddity 0 or 4. And signs can jump across a missing scale, provided both terms have type I. In our example the signways are the following:

$$\underbrace{1^2_{\mathrm{II}}\,[2^{-2}4^3]_3 16^1_1\,32^2_{\mathrm{II}}}\ \underbrace{64^{-2}_{\mathrm{II}}\,[128^1}\ \underbrace{256^1]_0 512^{-4}_{\mathrm{II}}}$$

Note that the absence of a term of scale 8 doesn't break the first signway, while signs cannot move between the terms of the "bad" compartment $[128^1\,256^1]_0$.

Each signway has a term of smallest scale, and by sign walking we may suppose that all minus signs are moved to these terms or canceled with each other. Then we say the symbol is in *canonical form*, which for our example is

$$1^{-2}_{\mathrm{II}}\,[2^2 4^3]_{-1} 16^1_1\,32^2_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^1\,256^{-1}]_4 512^4_{\mathrm{II}}$$

Theorem 6.3 implies:

Corollary 6.4 (Canonical form)**.** *Given lattices L, L' and 2-adic symbols S, S' for them in canonical form, $L\cong L'$ if and only if $S=S'$.* $\square$

Conway and Sloane's discussion of the canonical form is in terms of "trains", each of which is a union of one or more of our signways. Our example has two trains, the second consisting of the last two signways. They asserted that signs can walk up and down the length of a train, so that after walking signs leftward, there is at most one sign per train. But this is not true, as pointed out in [1]. One cannot walk the minus sign in $[128^1\,256^{-1}]_4$ leftward because there is no way to assign the subscripts in $128^-_{\pm3}\,256^+_{\pm1}$ so that the compartment has oddity 0.

Example 6.5. As an extended demonstration of sign walking, we determine the lattices M with the property that $M\oplus\langle 2,2\rangle\cong L$ where L is from (6.1). Note that $\langle 2,2\rangle = 2^2_2$. Obviously we require

$$M = \underbracket{1^{\pm 2}_{\mathrm{II}}}\,\underbracket{4^{\pm 3}_{?}16^{\pm 1}_{?}32^{\pm 2}_{\mathrm{II}}}\,\underbracket{64^{\pm 2}_{\mathrm{II}}[128^{\pm 1}}\,\underbracket{256^{\pm 1}]_{?}512^{\pm 4}_{\mathrm{II}}}$$

We have marked the signways with underbrackets. The 3rd and 4th of these become the 2nd and 3th signways of L after summing with 2^2_2. No sign walking is possible between distinct signways. So the isomorphism $M \oplus 2^2_2 \cong L$ shows that these signways in M must coincide with the corresponding signways in L. Next, the first two signways of M fuse with the 2^2_2 summand to form the first signway of L. The overall sign of this in L is $-$, so the total number of $-$ signs in the first two signways of M must be odd. By sign walking in the second signway of M, we reduce to

$$M \cong \left(1^{-2}_{\mathrm{II}}4^{3}_{t}16^{1}_{u}32^{2}_{\mathrm{II}} \;\text{ or }\; 1^{2}_{\mathrm{II}}4^{-3}_{t}16^{1}_{u}32^{2}_{\mathrm{II}}\right) \oplus 64^{-2}_{\mathrm{II}}[128^{1}256^{1}]_{0}512^{-4}_{\mathrm{II}}$$

where t and u are unknowns. Now we sum with 2^2_2 to get

$$L \cong \left(1^{-2}_{\mathrm{II}}[2^{2}4^{3}]_{2+t}16^{1}_{u}32^{2}_{\mathrm{II}} \;\text{ or }\; 1^{2}_{\mathrm{II}}[2^{2}4^{-3}]_{2+t}16^{1}_{u}32^{2}_{\mathrm{II}}\right) \oplus \cdots$$

Then we sign walk between the first two terms, or between the second and third, to make the signs match those in (6.1). That is,

$$L \cong \left(1^{2}_{\mathrm{II}}[2^{-2}4^{3}]_{6+t}16^{1}_{u}32^{2}_{\mathrm{II}} \;\text{ or }\; 1^{2}_{\mathrm{II}}[2^{-2}4^{3}]_{6+t}16^{1}_{u}32^{2}_{\mathrm{II}}\right) \oplus \cdots$$

Both this and (6.1) represent L, and the signs match, so the subscripts must too. Therefore $6+t=3$ and $u=1$. That is,

$$M \cong 1^{\pm 2}_{\mathrm{II}}4^{\mp 3}_{5}16^{1}_{1}32^{2}_{\mathrm{II}}64^{-2}_{\mathrm{II}}[128^{1}256^{1}]_{0}512^{-4}_{\mathrm{II}}$$

where one ambiguous sign is $+$ and the other is $-$. The two possibilities are distinct because their scale 1 terms have different signs and are involved in no sign walks. (More formally: canonicalization does not affect the first signway. So after canonicalization the symbols will still be different.) It follows that the isometry group of L has two orbits on summands isomorphic to $\langle 2,2\rangle$.

One can use the ideas of the proof of Theorem 6.3 to give numerical invariants for lattices, if one prefers them to a canonical form. For example, The following invariants come from Theorem 10 of [8, Ch. 15], which is proven in [12]. One records the scales, dimensions and types, the *adjusted oddity* of each compartment, and the overall sign of each signway (the product of the signs of the signway's terms). Here the adjusted oddity of a compartment means its oddity plus 4 for each $-$ sign appearing in its 1st, 3rd, 5th, ... position, with each $-$ sign after that compartment counted as occurring in the "$(k+1)$st" position, where k is the number of terms in the compartment. It is easy to check that sign walking leaves these quantities unchanged.

These invariants are clumsy because of the definition of adjusted oddity. The adjusted oddity also has the ugly feature that it depends on signs outside the signway containing the relevant compartment. This goes against the principle we used to great effect in example 6.5: distinct signways are isolated from each other.

Furthermore, these invariants are really just a complicated way of recording the canonical form while pretending not to. We will show how to construct the unique 2-adic symbol in canonical form having the same invariants as any chosen 2-adic lattice. To do this we first observe that the types of the compartments, together with the adjusted oddities (hence the compartment oddities mod 4), determine the signways. The sign of the first term of each signway is equal to the given overall sign of that signway, and the other signs are $+$. The signs then allow one to compute the compartment oddities from the adjusted oddities.

7. Equivalences between fine decompositions

In this section we give the deferred proof of Theorem 4.4: two fine symbols represent isometric lattices if and only if they are related by sign walks and giver permutation and conversion. Logically, it belongs anywhere between Theorems 5.1 and 6.2. The next two lemmas are standard; our proofs are adapted from Cassels [5, pp. 120–122].

Lemma 7.1. *Suppose L is an integral lattice, that $x, x' \in L$ have the same odd norm, and that their orthogonal complements $x^\perp, x'^\perp$ are either both odd or both even. Then $x^\perp \cong x'^\perp$.*

Proof. First, $(x-x')^2$ is even. If it is twice an odd number then the reflection in $x-x'$ is an isometry of L. This reflection exchanges x and x', so it gives an isometry between $x^\perp$ and $x'^\perp$. This argument applies in particular if $x \cdot x'$ is even. So we may restrict to the case that $x \cdot x'$ is odd and $(x-x')^2$ is divisible by 4. Next, note that $(x+x')^2$ differs from $(x-x')^2$ by $4x \cdot x' \equiv 4 \bmod 8$. So by replacing x' by $-x'$ we may suppose that $(x-x')^2 \equiv 4 \bmod 8$. This replacement is harmless because $\pm x'$ have the same orthogonal complement.

If it happens that $(x-x') \cdot L \subseteq 2\mathbb{Z}_2$ then the reflection in $x-x'$ preserves L and we may argue as before. So suppose some $y \in L$ has odd inner product with $x-x'$. Then the inner product matrix of $x, x-x', y$ is

$$\begin{pmatrix} 1 & 0 & ? \\ 0 & 0 & 1 \\ ? & 1 & ? \end{pmatrix} \quad \bmod 2,$$

which has odd determinant. Therefore these three vectors span a unimodular summand of L, so L has a Jordan decomposition whose unimodular part L_0 contains both x and x'. Note that x's orthogonal complement in L_0 is even just if its orthogonal complement in L is, and similarly for x'. So by discarding the rest of the decomposition we may suppose $L = L_0$, without losing our hypothesis that $x^\perp, x'^\perp$ are both odd or both even. Now, $x^\perp$ is unimodular with $\det(x^\perp) = (\det L)/x^2$ and oddity $o(x^\perp) = o(L) - x^2$, and similarly for x'. Since $x^2 = x'^2$, Theorem 5.1 implies $x^\perp \cong x'^\perp$. □

Lemma 7.2. *Suppose L is an integral lattice and $U, U' \subseteq L$ are isometric even unimodular sublattices. Then $U^\perp \cong U'^\perp$.*

Proof. $U \oplus \langle 1 \rangle$ has an orthogonal basis $x_1, \ldots, x_n$ by Lemma 4.1, and we write $x'_1, \ldots, x'_n$ for the basis for $U' \oplus \langle 1 \rangle$ corresponding to it under some isometry $U \cong U'$. Apply the previous lemma n times, starting with $L \oplus \langle 1 \rangle$. (In the nth application we need the observation that the orthogonal complements of U, U' in L are both even or both odd. This holds because these orthogonal complements are even or odd according to whether L is.) □

Lemma 7.3. *Suppose L is an integral lattice and that 1^+_G is a term in some fine symbol for L. Then we may apply a sequence of sign walking and giver permutation and conversion operations to transform any other fine symbol F for L into one possessing a term 1^+_G.*

Proof. We claim first that after some of these operations we may suppose F has a term $1^+_{\cdots}$. Because L is odd, F's terms of scale 1 have the form $1^\pm_{R\,\mathrm{or}\,G}$. If F has more than one such term then we can obtain a sign $+$ by sign walking, so suppose it has only one term, of sign $-$. If there are type I terms of scale 4 then again we can use sign walking, so suppose all scale 4 terms have type $\mathrm{I\!I}$. We can do the same thing if there are any terms $2^{\pm 2}_{\mathrm{I\!I}}$. Or terms $2^{\pm 1}_{R\,\mathrm{or}\,G}$, if the compartment consisting of the scale 1 and 2 terms has at least two givers or two receivers. This holds in particular if there is more than one term of scale 2. So we have reduced to the case

$$F = 1^-_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{I\!I}}\, 8^{\cdots}_{\cdots} \cdots \quad \text{or} \quad F = 1^-_{R\,\mathrm{or}\,G}\, 2^\pm_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{I\!I}}\, 8^{\cdots}_{\cdots} \cdots$$

where in the latter case the subscripts cannot be both G's or both R's. (Here and below, the superscript and subscript dots indicate any possibilities for the number of terms at that scale, and their decorations in that position. In particular, there might be no terms of that scale. The dots at the end indicate terms of higher scale than the ones already listed.) So in the second case there is one R and one G. By giver permutation we may suppose

$$F = 1^{-}_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots \quad \text{or} \quad F = 1^{-}_{G}\, 2^{\pm}_{R}\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots$$

None of these cases occur, because these lattices don't represent 1 mod 8, contrary to the hypothesis that some fine decomposition has a term 1^+_G. This non-representation is easy to see because L is $\langle \pm 3\rangle$ or $\langle 5, -2\rangle$ or $\langle 5, 6\rangle$, plus a lattice in which all norms are divisible by 8.

So we may suppose F has a term $1^+_{\cdots}$, and must show that after further operations we may suppose it has a term 1^+_G. In particular, we may suppose that our term $1^+_{\cdots}$ is 1^+_R. If the compartment C containing it has any givers then we may use giver permutation to complete the proof. So suppose C consists of receivers. If there are 4 receivers then we may convert them to givers, reducing to the previous case. If C has two terms of different scales, neither of which is our 1^+_R term, then we may use sign walking to convert them to givers, again reducing to a known case. Only a few cases remain, none of which actually occur, by a similar argument to the previous paragraph.

Namely, after more sign walking we may take F to be

$$\big(1^+_R\, 2^\pm_R \text{ or } 1^+_R\, 2^+_R\, 2^\pm_R\big) 4^{\cdots}_{\mathrm{II}} \cdots \quad \text{or} \quad \big(1^+_R \text{ or } 1^+_R\, 1^\pm_R \text{ or } 1^+_R\, 1^+_R\, 1^\pm_R\big) 2^{\cdots}_{\mathrm{II}} \cdots$$

The first set of possibilities is

$$\Big(\langle -1, -2\rangle \text{ or } \langle -1, 6\rangle \text{ or } \langle -1, -2, -2\rangle \text{ or } \langle -1, -2, 6\rangle\Big) \oplus (\text{a lattice with all norms divisible by } 8)$$

none of which represent 1 mod 8. The second set of possibilities is

$$\Big(\langle -1\rangle \text{ or } \langle -1, -1\rangle \text{ or } \langle -1, 3\rangle \text{ or } \langle -1, -1, -1\rangle \text{ or } \langle -1, -1, 3\rangle\Big) \oplus (\text{a lattice with all norms divisible by } 4)$$

and only the last two cases represent 1 mod 8. But in these cases every vector x of norm 1 mod 8 projects to $\bar{x} := (1, 1, 1)$ in $U/2U$, where U is the summand $\langle -1, -1, -1\rangle$ or $\langle -1, -1, 3\rangle$. There are no odd-norm vectors orthogonal to x since the orthogonal complement of $\bar{x}$ in $U/2U$ consists entirely of self-orthogonal vectors. So while these lattices admit norm 1 summands, they do not admit fine decompositions with 1^+_G terms. □

Lemma 7.4. *Suppose $\varepsilon = \pm$. Then Lemma 7.3 holds with $1^{\varepsilon 2}_{\mathrm{II}}$ in place of 1^+_G.*

Proof. If F has two terms of scale 1, or a scale 2 term of type I, then we can use sign walking. The only remaining case is $F = 1^{-\varepsilon 2}_{\mathrm{II}}\, 2^{\cdots}_{\mathrm{II}}\, 4^{\cdots}_{\cdots} \cdots$. Write U for the $1^{-\varepsilon 2}_{\mathrm{II}}$ summand and note that any two elements of L

with the same image in $U/2U$ have the same norm mod 4. Direct calculation shows that the norms of the nonzero elements of $U/2U$ are $0, 0, 2$ or $2, 2, 2$ mod 4, depending on ε. Now consider the summand $U' \cong 1_{\mathrm{II}}^{\varepsilon 2}$ of L that we assumed to exist. By considering norms mod 4 we see that $U'/2U' \to U/2U$ cannot be injective, so it must have image 0 or $\mathbb{Z}/2$. Since all self-inner products in $U/2U$ vanish, we obtain the absurdity that all inner products in U' are even. $\square$

Proof of Theorem 4.4. The "if" part has already been proven in Lemmas 4.2 and 4.3, so we prove "only if". We assume the result for all lattices of lower dimension. By scaling by a power of 2 we may suppose L is integral and some inner product is odd, so each of F and F' has a nontrivial unimodular term.

First suppose L is odd, so the unimodular terms of F and F' have type I. By rescaling L by an odd number we may suppose F has a term 1_G^+. By Lemma 7.3 we may apply our moves to F' so that it also has a term 1_G^+. The orthogonal complements of the corresponding summands of L are both even (if the unimodular Jordan constituents are 1-dimensional) or both odd (otherwise). By Lemma 7.1 these orthogonal complements are isometric. They come with fine decompositions, given by the remaining terms in F, F'. By induction on dimension these fine decompositions are equivalent by our moves.

If L is even then the same argument applies, using Lemmas 7.4 and 7.2 in place of Lemmas 7.3 and 7.1. $\square$

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