Experimental Mathematics

ISSN: 1058-6458 (Print) 1944-950X (Online) Journal homepage: http://www.tandfonline.com/loi/uexm20

Principal Congruence Links: Class Number Greater than 1

M. D. Baker & A. W. Reid

To cite this article: M. D. Baker & A. W. Reid (2017): Principal Congruence Links: Class Number Greater than 1, Experimental Mathematics, DOI: 10.1080/10586458.2017.1289867

To link to this article: http://dx.doi.org/10.1080/10586458.2017.1289867

Published online: 20 Mar 2017.

EXPERIMENTAL MATHEMATICS
http://dx.doi.org/10.1080/10586458.2017.1289867

Taylor & Francis
Taylor & Francis Group

Principal Congruence Links: Class Number Greater than 1

M. D. Baker[a] and A. W. Reid[b]

[a]IRMAR, Université de Rennes 1, Rennes, France; [b]Department of Mathematics, University of Texas, Austin, TX, USA

ABSTRACT
In a previous article, we started an enumeration of the finitely many levels for which a principal congruence manifold can be a link complement in S^3. In this article we give a complete enumeration of all the principal congruence link complements in S^3, together with their levels in the case when the class number of $\mathbf{Q}(\sqrt{-d})$ is greater than 1.

KEYWORDS
Bianchi group; link complement; congruence subgroup

2000 AMS SUBJECT CLASSIFICATION
Primary 57M25; Secondary 20H10; 57M50

1. Introduction

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbf{Q}(\sqrt{-d})$, and let Q_d denote the Bianchi orbifold $\mathbf{H}^3/\mathrm{PSL}(2, O_d)$.

As is well known Q_d is a finite volume hyperbolic orbifold with h_d cusps, where h_d is the class number of $\mathbf{Q}(\sqrt{-d})$ (see [Maclachlan and Reid 03] Chapters 8 and 9 for example). A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [Maclachlan and Reid 03] Chapter 8 for more on this).

An important class of arithmetic 3-manifolds consists of the *congruence* manifolds. Recall that a subgroup $\Gamma < \mathrm{PSL}(2, O_d)$ is called a *congruence subgroup* if there exists an ideal $I \subset O_d$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2, O_d) \to \mathrm{PSL}(2, O_d/I)\},$$

where $\mathrm{PSL}(2, O_d/I) = \mathrm{SL}(2, O_d/I)/\{\pm Id\}$. The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ. For convenience, if $n \in \mathbf{Z}$, we will denote the principal O_d-ideal $< n >$ simply by n. A manifold $M = \mathbf{H}^3/\Gamma$ is called *congruence* (resp. *principal congruence*) if $\Gamma > \Gamma(I)$ (resp. $\Gamma = \Gamma(I)$) for some ideal I.

Since the 1980s there has been considerable interest in the question of which Bianchi orbifolds are commensurable to, or stronger still, covered by a link complement $S^3 \setminus L$. This culminated in the solution of the Cuspidal Cohomology Problem for $\mathrm{PSL}(2, O_d)$ (see [Vogtmann 85]) which showed that Q_d can have a cover homeomorphic to a link complement in S^3 only if

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

Furthermore, in [Baker 01] it was shown that for every d in this list, Q_d is indeed covered by such a link complement.

Given this finite list of d, and the negative solution to the Congruence Subgroup Property for the Bianchi groups [Serre 70], one can refine these questions further, and ask for the levels for which there are congruence (resp. principal congruence) manifolds that are homeomorphic to a link complement in S^3. In a previous article [Baker and Reid 14], we started an enumeration of the finitely many levels for which a principal congruence manifold can be homeomorphic to a link complement in S^3 (see [Baker and Reid 14] or Section 4.1 for the proof of finiteness). In particular, for $h_d = 1$, the main result of that article gave 9 new examples of such principal congruence link groups bringing the known total to 18. One consequence of the results in [Baker and Reid 14] is that when $h_d = 1$ (i.e., $d = 1, 2, 3, 7, 11, 19$), Q_d is covered by a principal congruence link complement (see [Baker and Reid 14]).

In this article, we give a complete enumeration of all the principal congruence link complements in S^3, together with their levels in the case when the class number of $\mathbf{Q}(\sqrt{-d})$ is greater than 1. Our main result is the following:

Theorem 1.1. *Suppose that $h_d > 1$. Then the following list of 16 pairs (d, I) describes all principal congruence*

CONTACT A. W. Reid areid@math.utexas.edu Department of Mathematics, University of Texas Austin, Austin, TX 78712, USA; M. D. Baker mark.baker@univ-rennes1.fr IRMAR, Université de Rennes 1, 35042 Rennes Cedex, France.

subgroups $\Gamma(I) < \mathrm{PSL}(2, O_d)$ *such that* $\mathbf{H}^3/\Gamma(I)$ *is a link complement in* S^3.

1. $d = 5$: $I =< 3, (1 \pm \sqrt{-5}) >$.
2. $d = 15$: $I =< 2, (1 \pm \sqrt{-15})/2 >$,
   $I =< 3, (3 \pm \sqrt{-15})/2 >$,
   $I =< 4, (1 \pm \sqrt{-15})/2 >$,
   $I =< 5, (5 \pm \sqrt{-15})/2 >$,
   $I =< 6, (-3 \pm \sqrt{-15})/2 >$.
3. $d = 23$: $I =< 2, (1 \pm \sqrt{-23})/2 >$,
   $I =< 3, (1 \pm \sqrt{-23})/2 >$,
   $I =< 4, (-3 \pm \sqrt{-23})/2 >$.
4. $d = 31$: $I =< 2, (1 \pm \sqrt{-31})/2 >$,
   $I =< 4, (1 \pm \sqrt{-31})/2 >$,
   $I =< 5, (3 \pm \sqrt{-31})/2 >$.
5. $d = 47$: $I =< 2, (1 \pm \sqrt{-47})/2 >$,
   $I =< 3, (1 \pm \sqrt{-47})/2 >$,
   $I =< 4, (1 \pm \sqrt{-47})/2 >$.
6. $d = 71$: $I =< 2, (1 \pm \sqrt{-71})/2 >$.

Note that the cases $d = 15, I =< 2, (1 \pm \sqrt{-15})/2 >$, and $d = 23, I =< 2, (1 \pm \sqrt{-23})/2 >$ were treated in [Baker 92] and explicit link diagrams were given. To our knowledge, the other 14 examples are new (although we do not have link diagrams for any of these 14 examples).

Comparing Theorem 1.1 with the list of allowable d's given by the solution of The Cuspidal Cohomology Problem gives the following.

Corollary 1.2. *When* $d = 6$ *or 39 there are no principal congruence link complements.*

Since [Baker and Reid 14] and the completion of this article, we have been able to go further in the class number one case, and analyze some of the levels that remained to be decided (when $h_d = 1$) as to whether the corresponding principal congruence manifold is homeomorphic to a link complement in S^3. Given this (details will appear in a future article), at present there remain only two possible cases to decide upon namely:

$$d = 7, I =< (1 + 3\sqrt{-7})/2 > \text{ and}$$
$$d = 11, I =< (5 + \sqrt{-11})/2 > .$$

As a counterpoint to Corollary 1.2, we are able to prove the existence of *congruence* link complements when $d = 6$ or 39 (see Section 5), and so this, together with Theorem 1.1 and [Baker and Reid 14] proves the following theorem.

Theorem 1.3. *For all* $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$, *there exists a congruence link complement.*

As pointed out in [Baker and Reid 14], it remains an open question as to whether there are infinitely many congruence link complements.

We close the Introduction by outlining the plan of the article. In Section 2 we recall material from [Baker and Reid 14] as well as some other preliminary setup. In Section 3 we show that the principal congruence groups given in Theorem 1.1 are indeed link groups, and in Section 4, we eliminate the other (finitely many) possibilities. Finally, Section 5 contains the proof that there exist (non-principal) congruence link complements when $d = 6$ or 39.

2. Preliminaries and outline of proofs

In Sections 2.1–2.4 we gather facts and background to be used; in Section 2.5 we outline the proof of Theorem 1.1.

2.1. Indices of principal congruence subgroups

We begin by recalling the orders of the groups $\mathrm{PSL}(2, \mathcal{R})$ where $\mathcal{R}$ is a finite ring of the form O_d/I, with $I \subset O_d$ an ideal (see [Dixon 58]). For such an ideal I we have a decomposition into powers of prime ideals. Assuming that $I = \mathcal{P}_1^{a_1} \dots \mathcal{P}_r^{a_r}$, we have

$$|\mathrm{PSL}(2, O_d/I)| = \begin{cases} 6, & \text{when } N(I) = 2, \\ N(I)^3 \prod_{\mathcal{P}|I} \left(1 - \dfrac{1}{N(\mathcal{P})^2}\right), & \text{when } I =< 2 >, \\ \dfrac{N(I)^3}{2} \prod_{\mathcal{P}|I} \left(1 - \dfrac{1}{N(\mathcal{P})^2}\right), & \text{otherwise} \end{cases}$$

where $N(I) = |O_d/I|$ denotes the norm of the ideal I. Since $\mathrm{PSL}(2, O_d)/\Gamma(I) \cong \mathrm{PSL}(2, O_d/I)$, this gives a formula for the index of $\Gamma(I)$ in $\mathrm{PSL}(2, O_d)$.

2.2. Presentations

The proof of Theorem 1.1 makes use of presentations for the Bianchi groups in the cases $d \in \{5, 6, 15, 23, 31, 39, 47, 71\}$. Note that the class numbers for the relevant quadratic imaginary number fields are 2 (when $d = 5, 6, 15$), 3 (when $d = 23, 31$), 4 (when $d = 39$), 5 (when $d = 47$), and 7 (when $d = 71$).

The presentations for $d = 5, 6, 15$ are from [Swan 71] while the rest were done by Page using a suite of computer packages he recently developed (see [Page 15]) to study arithmetic Kleinian groups. We maintain the notation of [Swan 71] for $d = 5, 6, 15$ and for the others we use the notation of the presentations provided to us by Page.

$$\mathrm{PSL}(2, O_5) =< a, t, u, b, c \mid a^2 = b^2 = (ta)^3 = (ab)^2$$
$$= (aubu^{-1})^2 = acatc^{-1}t^{-1} = ubu^{-1}cbtc^{-1}t^{-1} = 1,$$
$$[t, u] = 1 >,$$

$$
\begin{aligned}
\mathrm{PSL}(2, O_6) =&< a, t, u, b, c \,|\, a^2 = b^2 = (ta)^3 = (atb)^3 \\
&= (atubu^{-1})^3 = t^{-1}ctubu^{-1}c^{-1}b^{-1} = 1, \\
&[t, u] = [a, c] = 1 >, \\
\mathrm{PSL}(2, O_{15}) =&< a, t, u, c \,|\, a^2 = (ta)^3 \\
&= ucuatu^{-1}c^{-1}u^{-1}a^{-1}t^{-1} = 1, [t, u] = [a, c] = 1 >, \\
\mathrm{PSL}(2, O_{23}) =&< g_1, g_2, g_3, g_4, g_5 \,|\, g_3^3 = (g_3g_2)^2 \\
&= g_5g_2^{-1}g_3^{-1}g_5^{-1}g_1^{-1}g_2^{-1}g_3^{-1}g_1 = g_4^{-1}g_5g_3g_2g_5^{-1}g_2g_4g_3 \\
&= 1, [g_1, g_2] = [g_4, g_5] = 1 >, \\
\mathrm{PSL}(2, O_{31}) =&< g_1, g_2, g_3, g_4, g_5 \,|\, g_2^3 = (g_2g_1^{-1})^2 \\
&= g_4g_1^{-1}g_3^{-1}g_2g_3g_4^{-1}g_2g_4g_3^{-1}g_1^{-1}g_2g_3g_4^{-1}g_2 \\
&= g_5g_3^{-1}g_2g_3g_4^{-1}g_2g_1^{-1}g_5^{-1}g_2^{-1}g_4g_3^{-1}g_2^{-1}g_3g_1 \\
&= g_2g_3g_4^{-1}g_2g_1^{-1}g_4g_3^{-1}g_2g_3g_4^{-1}g_1g_2^{-1}g_4g_3^{-1} = 1, \\
&[g_1, g_3] = [g_4, g_5] = 1 >, \\
\mathrm{PSL}(2, O_{39}) =&< g_1, g_2, g_3, g_4, g_5, g_6, g_7 \,|\, g_3^3 = \\
&(g_3g_5)^2 = (g_1^{-1}g_3^{-1})^2 = (g_5^{-1}g_1)^3 \\
&= (g_7g_5^{-1}g_7^{-1}g_1)^3 = g_5^{-1}g_1g_6^{-1}g_4^{-1}g_5g_4g_1^{-1}g_6 \\
&= g_4^{-1}g_5g_4g_2^{-1}g_7g_5^{-1}g_7^{-1}g_2 \\
&= g_6g_1^{-1}g_5g_6^{-1}g_4^{-1}g_5g_4g_1^{-1}g_4^{-1}g_5g_4g_1^{-1} = 1, \\
&[g_2, g_1] = [g_3^{-1}, g_7^{-1}] = [g_4, g_6] = 1 >, \\
\mathrm{PSL}(2, O_{47}) =&< g_1, g_2, g_3, g_4, g_5, g_6, g_7 \,|\, g_1^3 \\
&= (g_2^{-1}g_1)^2 = g_2^{-1}g_1g_6g_1^{-1}g_2g_6^{-1} \\
&= g_6g_2^{-1}g_4^{-1}g_5g_3^{-1}g_6^{-1}g_4g_2g_3g_5^{-1} \\
&= g_7^{-1}g_2^{-1}g_5^{-1}g_4g_1g_4^{-1}g_2g_7g_4g_1^{-1}g_4^{-1}g_5 \\
&= g_3g_5^{-1}g_4g_1g_4^{-1}g_2g_5g_3^{-1}g_2^{-1}g_4^{-1}g_1^{-1}g_4 \\
&= g_5^{-1}g_4g_1g_4^{-1}g_7^{-1}g_2^{-1}g_4g_1^{-1}g_4^{-1}g_5g_3^{-1}g_2g_3g_7 = 1, \\
&[g_5, g_7] = [g_3, g_2] = 1 >, \\
\mathrm{PSL}(2, O_{71}) =&< g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9 \,|\, g_8^3 \\
&= (g_8g_7^{-1})^2 = g_1^{-1}g_3g_7g_3^{-1}g_1g_7^{-1} = g_6g_3g_6^{-1}g_7g_9^{-1}g_3^{-1}g_9g_7^{-1} \\
&= g_7^{-1}g_6g_3g_6^{-1}g_5^{-1}g_2g_7g_5g_6g_3^{-1}g_6^{-1}g_2^{-1} \\
&= g_8g_7^{-1}g_1g_5g_6g_3^{-1}g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_5^{-1}g_1^{-1} \\
&= g_4^{-1}g_7^{-1}g_5^{-1}g_2g_1^{-1}g_3g_7g_9g_4g_1g_7^{-1}g_2^{-1}g_5g_7g_9^{-1}g_3^{-1} \\
&= g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_7g_9g_6g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_9^{-1} \\
&g_7^{-1}g_3^{-1}g_1 = g_2g_6g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_7g_8^{-1}g_5^{-1}g_2^{-1} \\
&g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_7g_8^{-1}g_1g_5g_6g_3^{-1}g_6^{-1} = 1, [g_8^{-1}, g_4] = 1 > .
\end{aligned}
$$

Throughout the article we use ω_d as follows:

$$
\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1+\sqrt{-15}}{2}, \frac{1+\sqrt{-23}}{2}, \frac{1+\sqrt{-31}}{2}, \frac{1+\sqrt{-39}}{2}, \frac{1+\sqrt{-47}}{2}, \frac{1+\sqrt{-71}}{2}.
$$

When $d = 5, 6, 15$, $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & \omega_d \\ 0 & 1 \end{smallmatrix}\right)$ (with the obvious abuse of notation between SL and PSL). Matrix representatives for the other generators used are given in Appendix A.

2.3. Cusps

Let $\Gamma \le \mathrm{PSL}(2, O_d)$ be a finite index subgroup. For convenience in what follows, we assume that $d \neq 1, 3$. Then

- A *cusp*, $[c]$, of Γ is a Γ-orbit of points in $\mathbf{P}^1(\mathbf{Q}(\sqrt{-d}))$.
- A *peripheral subgroup* of Γ for $[c]$ is a maximal parabolic subgroup, $P_x < \Gamma$, fixing $x \in [c]$. Note that if $y \in [c]$, then P_x and P_y are conjugate; hence a peripheral subgroup for $[c]$ is determined up to conjugacy.
- A *set of peripheral subgroups for* Γ is the choice of one peripheral subgroup for each cusp of Γ.

We will use the term *cusp* to mean $[c]$, a choice of point x in $[c]$, as well as the end of $\mathbf{H}^3/\Gamma$ corresponding to $[c]$. Which one is meant should be clear from the context. Note that since $d \neq 1, 3$, each peripheral subgroup is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

2.4. Cusp representatives

As noted in the Introduction, Q_d has h_d cusps (or equivalently $\mathrm{PSL}(2, O_d)$ has h_d conjugacy classes of peripheral subgroups). In the setting of this article, $h_d > 1$ and so the orbifold Q_d will have more than one cusp.

In the cases of $d = 5, 6$, and 15, the peripheral subgroup of $\mathrm{PSL}(2, O_d)$ that fixes ∞ is given by $< t, u >$ (in terms of the generators given in Section 2.2). Below we give a set of peripheral subgroups for each of the Bianchi groups in Section 2.2. We describe a choice of peripheral subgroup and cusp by (x, P_x) (following the notation described in Section 2.3).

$$
\mathbf{d = 5}: (\infty, < t, u >), \left(\frac{1-\sqrt{-5}}{2}, < tb, tu^{-1}ct^{-1} >\right).
$$

$$
\mathbf{d = 6}: (\infty, < t, u >), \left(\frac{-\sqrt{-6}}{2}, < tb, cu >\right).
$$

$$
\mathbf{d = 15}: (\infty, < t, u >), \left(\frac{1+\sqrt{-15}}{4}, < uca, c^{-1}au^{-1}c^{-1}u^{-1}ta >\right).
$$

$$\mathbf{d=23}:(\infty, < g_1, g_2 >),\ \left(\frac{1-\sqrt{-23}}{4}, < g_4, g_5 >\right),$$

$$\left(\frac{-1-\sqrt{-23}}{4}, < g_4g_3g_2, g_2^{-1}g_5g_3g_2 >\right).$$

$$\mathbf{d=31}:(\infty, < g_1, g_3 >),\ \left(\frac{1-\sqrt{-31}}{4}, < g_4, g_5 >\right),$$

$$\left(\frac{-1-\sqrt{-31}}{4}, < g_1g_5, g_3^{-1}g_2g_3g_4^{-1}g_2g_5 >\right).$$

$$\mathbf{d=39}:(\infty, < g_1, g_2 >),\ \left(\frac{1-\sqrt{-39}}{4}, < g_4, g_6 >\right),$$

$$\left(\frac{1-\sqrt{-39}}{5}, < g_5^{-1}g_6, g_4g_1^{-1}g_6 >\right),$$

$$\left(\frac{3-\sqrt{-39}}{6}, < g_5, g_4g_2^{-1}g_7 >\right).$$

$$\mathbf{d=47}:(\infty, < g_2, g_3 >),\ \left(\frac{1-\sqrt{-47}}{4}, < g_5, g_7 >\right),$$

$$\left(\frac{3-\sqrt{-47}}{4}, < g_2g_7, g_4g_1^{-1}g_4^{-1}g_5 >\right),$$

$$\left(\frac{1-\sqrt{-47}}{6}, < g_6g_2^{-1}g_4^{-1}, g_5g_3^{-1}g_2^{-1}g_4^{-1} >\right),$$

$$\left(\frac{1+\sqrt{-47}}{6}, < g_6^{-1}g_1^{-1}g_4, g_3g_5^{-1}g_4g_1 >\right).$$

$$\mathbf{d=71}:(\infty, < g_7, g_1^{-1}g_3 >),\ \left(\frac{1-\sqrt{-71}}{4},\right.$$

$$\left. < g_2, g_6g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_7g_8^{-1}g_5^{-1} > \right),$$

$$\left(\frac{1+\sqrt{-71}}{6}, < g_3, g_6^{-1}g_7g_9^{-1} >\right),$$

$$\left(\frac{-1-\sqrt{-71}}{4}, < g_7g_2, g_6g_3g_6^{-1}g_5^{-1}g_7^{-1} >\right),$$

$$\left(\frac{-1+\sqrt{-71}}{6}, < g_7g_9g_6, g_3^{-1}g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1} >\right),$$

$$\left(\frac{3+\sqrt{-71}}{8}, < g_3g_9g_4, g_4^{-1}g_7^{-1}g_5^{-1}g_2g_7g_1^{-1} >\right),$$

$$\left(\frac{3-\sqrt{-71}}{8}, < p_1, p_2 >\right)$$

where $p_1 = g_4g_1g_7^{-1}g_2^{-1}g_5g_7g_9^{-1}g_3^{-1}g_8^{-1}g_4^{-1}$ and $p_2 = g_6g_3^{-1}g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_6^{-1}g_9^{-1}g_8^{-1}g_4^{-1}$.

We remark that finding these peripheral subgroups and expressing them in terms of the given generators was a highly nontrivial exercise. One can check that the h_d cusps correspond to different elements of the ideal class group of $\mathbf{Q}(\sqrt{-d})$, hence they are inequivalent. Also, the generators of each of the above peripheral subgroups commute (using the relations in the corresponding Bianchi group, or by direct matrix calculation using the generators given in Appendix A), and these generators correspond to primitive parabolic matrices with the correct fixed point.

2.5. *Outline of the proof of Theorem 1.1*

We conclude this section with an outline of the proof of Theorem 1.1. The methods used are those in [Baker and Reid 14] adapted to the case $h_d > 1$.

We first note that there are only finitely many groups $\Gamma(I)$ that can be link groups. Indeed, as mentioned above, the cuspidal cohomology of $\mathrm{PSL}(2, O_d)$ reduces consideration to the eight Bianchi groups given in Section 2.2. Furthermore, as explained in Section 4.1, if $\Gamma(I)$ is a link group then $N(I) < 39$.

To establish the 16 principal congruence link groups in Theorem 1.1, and eliminate the remaining $\Gamma(I)$, we use the following properties of hyperbolic link complements, which we state for the case of principal congruence groups.

1. If $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$, then $\Gamma(I)$ is generated by parabolic elements.
2. $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$ if and only if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1.

We briefly comment on these two items. The first is well known, following from the fact that a link group is generated by meridians, and such elements are represented by parabolic elements under the faithful discrete representation. For the second, if $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$, then for each component L_i of L, there is a curve x_i (a meridian curve for L_i) so that Dehn filling each L_i along x_i results in S^3. Thus, trivializing the corresponding parabolic elements $[x_i]$ in $\Gamma(I)$ gives the trivial group. Conversely, given Perelman's resolution of the Geometrization Conjecture, if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1, then $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3.

Given this, our method is as follows:

Step 1: *Check whether $\Gamma(I)$ is generated by parabolic elements.*

Let $\Gamma(I) < \mathrm{PSL}(2, O_d)$, and let P_i be the peripheral subgroup of $\mathrm{PSL}(2, O_d)$ fixing the cusp c_i for $i = 1, \ldots, h_d$ as given in Section 2.4. Now, comparing with the discussion in [Baker and Reid 14], $P_i(I) = \mathrm{P}_i \cap \Gamma(I)$ is the peripheral subgroup of $\Gamma(I)$ fixing c_i.

Let $N_d(I)$ denote the normal closure in $\mathrm{PSL}(2, O_d)$ of $\{P_1(I), \ldots, P_{h_d}(I)\}$. Note that $N_d(I) < \Gamma(I)$ since $\Gamma(I)$ is a normal subgroup of $\mathrm{PSL}(2, O_d)$. It is clear that $\Gamma(I)$ is generated by parabolic elements if and only if $N_d(I) = \Gamma(I)$. We use Magma [Bosma 97] to test whether $\Gamma(I) = N_d(I)$. In Section 3 we show that the groups in Theorem 1.1 are generated by parabolics, while Section 4 is devoted to showing that the remaining $\Gamma(I)$ are not and hence cannot be link groups.

Step 2: *Find parabolic elements in $\Gamma(I)$, one for each cusp, so that trivializing these elements trivializes the group.*

First, we obtain a set of peripheral subgroups for $\Gamma(I)$ as follows. Let $\{P_1, \ldots, P_{h_d}\}$ be a set of peripheral subgroups for $\mathrm{PSL}(2, O_d)$. Then, as in Step 1, set $P_i(I) = P_i \cap \Gamma(I)$. Note that $S = \{P_1(I), \ldots, P_{h_d}(I)\}$ is a partial set of peripheral subgroups for $\Gamma(I)$. One obtains a full set of peripheral subgroups for $\Gamma(I)$ by adding certain conjugates of the $P_i(I)$ to the partial set S as explained in Section 3.

Next, given a (full) set of peripheral subgroups for $\Gamma(I)$, we choose one parabolic from each of these peripheral subgroups and use Magma to check that trivializing these elements trivializes $\Gamma(I)$. This choice of parabolics involves trial and error. However, in all cases these parabolics are linear combinations of small powers of the generators of the $P_i(I)$ and their conjugates.

3. Proof of Theorem 1.1: Determining the principal congruence link groups

We now prove, as described in Section 2.5, that the groups given in Theorem 1.1 do correspond to link groups. We use the notation (d, I) to indicate the Bianchi group and level given in Theorem 1.1. The ideals involved are of norm 2 (5 groups), 3 (4 groups), 4 (4 groups), 5 (2 groups), and 6 (1 group). Note that if $\Gamma(I)$ is determined to be a link group in S^3, then $\Gamma(\bar{I})$ is also. Therefore, in what follows we simply refer to one of the complex conjugate pair.

3.1. Normal closures

Doing Step 1 above, we check that $\Gamma(I)$ is generated by parabolic elements. Since $N_d(I) < \Gamma(I)$, it suffices to show that $N_d(I)$ and $\Gamma(I)$ have the same index in $\mathrm{PSL}(2, O_d)$. The index of $\Gamma(I)$ is given by the formula in Section 2.1 while the index of $N_d(I)$ is calculated using Magma.

We now do this explicitly for the group $(d, I) = (15, <2, \omega_{15}>)$. Recall from Section 2.4 that the peripheral subgroups for $\mathrm{PSL}(2, O_{15})$ are given by $P_1 = <t, u>$, $P_2 = <uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)>$. In terms of the matrices for the presentation of $\mathrm{PSL}(2, O_{15})$, we have $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $u = \left(\begin{smallmatrix} 1 & \omega_{15} \\ 0 & 1 \end{smallmatrix}\right)$, $uca = \left(\begin{smallmatrix} 1+2\omega_{15} & 4-\omega_{15} \\ 4 & 1-2\omega_{15} \end{smallmatrix}\right)$, and $(c^{-1}au^{-1}c^{-1}u^{-1}ta) = \left(\begin{smallmatrix} -3+4\omega_{15} & 8-\omega_{15} \\ 7+\omega_{15} & 1-4\omega_{15} \end{smallmatrix}\right)$. Thus we obtain the peripheral subgroups $P_i(I) = P_i \cap \Gamma(I)$ by reducing these matrices modulo I which gives $P_1(I) = <t^2, u>$, $P_2(I) = <uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2>$.

Now from Section 2.1 we have $[\mathrm{PSL}(2, O_{15}) : \Gamma(I)] = 6$ and Magma gives $[\mathrm{PSL}(2, O_{15}) : N_d(I)] = 6$, hence $\Gamma(I) = N_{15}(I)$ so that $\Gamma(I)$ is generated by parabolics. In our Magma routines below, $G = \mathrm{PSL}(2, O_d)$, $H = <P_1(I), \ldots, P_{h_d}(I)>$, and $N = N_d(I) = << H >>$ (the normal closure of H).

```
G<a,c,t,u>:=Group<a,c,t,u|a^2,
(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*
t^-1, (t,u),(a,c)>;
H:=sub<G|t^2,u,(c^-1*a*u^-1*c^-1*
u^-1*t*a)^2,u*c*a>;
N:=NormalClosure(G,H);

print Index(G,N);
6
\\
```

The remaining 15 groups are done in exactly the same way. The set of peripheral subgroups for each $\Gamma(I)$ can be read off from H in its Magma routine.

3.2. Testing for link groups

Next, we implement Step 2: Find parabolic elements from $\Gamma(I)$, one for each cusp of $\Gamma(I)$, and use Magma to show that trivializing these elements trivializes $\Gamma(I)$. We start by obtaining a set of peripheral subgroups for $\Gamma(I)$ and then choose one parabolic from each subgroup. The method depends on the norm of the ideal I.

3.2.1. Cases (d,I), N(I) = 2

Here the Bianchi groups correspond to $d = 15, 23, 31, 47, 71$. Since $N(I) = 2$, we have $\mathrm{PSL}(2, O_d/I) \cong S_3$. In order to find a set of peripheral subgroups for $\Gamma(I)$, we use the following sequence of regular covers:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{3} \mathbf{H}^3/\Gamma_1 \xrightarrow{2} Q_d$$

where $\Gamma_1 = <\Gamma(I), \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)>$.

Since each peripheral subgroup P_i of $\mathrm{PSL}(2, O_d)$ maps to a subgroup of order 2 in $\mathrm{PSL}(2, O_d/I)$, it follows that each cusp of Q_d is covered by 3 cusps of $\mathbf{H}^3/\Gamma(I)$ each with covering degree 2. Now $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/3\mathbf{Z} < \mathrm{PSL}(2, O_d/I)$, hence each cusp of Q_d is covered by 1 cusp of $\mathbf{H}^3/\Gamma_1$ with covering degree 2. Thus a set of peripheral subgroups for Γ_1 is given by $\{P_1(I), \ldots, P_{h_d}(I)\}$. Conjugating this set by the elements $\{Id, \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)\}$ gives a set of peripheral subgroups for $\Gamma(I)$.

In what follows we give the Magma routines in the cases of $d = 15, 71$ and one of the prime ideals of norm 2 (since as remarked upon above the same will hold for the complex conjugate ideal). The remaining values of $d = 23, 31, 47$ and a prime ideal of norm 2 are handled similarly. Magma routines for these (and other calculations done in this section) are available from the authors upon request.

The case $(15, < 2, \omega_{15} >)$
Let $I =< 2, \omega_{15} >$. We analyze this $\Gamma(I)$; the other four cases involving $N(I) = 2$ are dealt with in exactly the same way.

In Section 3.1, we computed the peripheral subgroups $P_1(I)$ and $P_2(I)$ and showed that $\Gamma(I)$ was generated by parabolics. This is given again in the following Magma routine.

Since Q_{15} has 2 cusps, the manifold $\mathbf{H}^3/\Gamma(I)$ has 6 cusps. Recall that $P_1(I) =< t^2, u >$, $P_2(I) = < uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2 >$. Since $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = ta$, conjugating $P_1(I)$ and $P_2(I)$ by the elements $\{Id, ta, (ta)^2\}$ gives a set of six peripheral subgroups for $\Gamma(I)$. Now, we choose one element from each of these six peripheral subgroups:

$$\{t^2, (ta)u(ta)^{-1}, (ta)^2u(ta)^{-2}, uca, (ta)uca(ta)^{-1}, (ta)^2(c^{-1}au^{-1}c^{-1}u^{-1}ta)^2(ta)^{-2}\}$$

In the Magma routine, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these six parabolics, and Magma calculates that $Q =< 1 >$ which shows that $\Gamma(I)$ is trivialized by setting these six elements equal to 1. Thus $\Gamma(< 2, \omega_{15} >)$ is indeed a 6-component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|a^2,
(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*
t^-1, (t,u),(a,c)>;
H:=sub<G|t^2,u,(c^-1*a*u^-1*c^-1*
u^-1*t*a)^2,u*c*a>;
N:=NormalClosure(G,H);

print Index(G,N);
6
\\
Q:=quo<N|t^2,(t*a)*u*(t*a)^-1,
(t*a)^2*u*(t*a)^-2,
u*c*a,(t*a)*u*c*a*(t*a)^-1,
(t*a)^2*(c^-1*a*u^-1*c^-1*
u^-1*t*a)^2*(t*a)^-2>;

print Order(Q);
1
\\
```

The case $(71, < 2, \omega_{71} >)$
Let $I =< 2, \omega_{71} >$. The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is a 21-component link complement in S^3. Note that Q_{71} has 7 cusps and that $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = g_8{}^{-1}$. The seven peripheral subgroups $P_i(I)$ can be read off from H and Q is $\Gamma(I)$ modulo the 21 parabolic elements to be trivialized.

```
G<g1,g2,g3,g4,g5,g6,g7,g8,
g9>:=Group<g1,g2,g3,g4,g5,g6,g7,g8,
g9|g8^3,(g8^-1,g4), (g8*g7^-1)^2,
g1^-1*g3*g7*g3^-1*g1*g7^-1,g6*g3*
g6^-1*g7*g9^-1*g3^-1*g9*g7^-1,g7^-1*
g6*g3*g6^-1*g5^-1*g2*g7*g5*g6*g3^-1*
g6^-1*g2^-1,g8*g7^-1*g1*g5*g6*g3^-1*
g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*
g5^-1*g1^-1,g4^-1*g7^-1*g5^-1*g2*
g1^-1*g3*g7*g9*g4*g1*g7^-1*g2^-1*g5*
g7*g9^-1*g3^-1,g5*g8*g7^-1*g5^-1*
g1^-1*g7*g9*g6*g1*g5*g8*g7^-1*g5^-1*
g1^-1*g3*g6^-1*g9^-1*g7^-1*g3^-1*g1,
g2*g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*
g6^-1*g7*g8^-1*g5^-1*g2^-1*g5*g7*
g8^-1*g5^-1*g1^-1*g7*g8^-1*g1*g5*g6*
g3^-1*g6^-1>;

H:=sub<G|g7^2, g7*g1^-1*g3, g2^2, g6*
g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*
g7*g8^-1*g5^-1,g3^2, g3*g6^-1*g7*
g9^-1, g7*g2,(g6*g3*g6^-1*g5^-1*
g7^-1)^2, (g7*g9*g6)^2,g3^-1*g1*g5*
g8*g7^-1*g5^-1*g1^-1, (g3*g9*g4)^2,
g4^-1*g7^-1*g5^-1*g2*g7*g1^-1,
(g4*g1*g7^-1*g2^-1*g5*g7*g9^-1*g3^-1*
g8^-1*g4^-1)^2,g6*g3^-1*g1*g5*g8*
g7^-1*g5^-1*g1^-1*g6^-1*g9^-1*g8^-1*
g4^-1>;

N:=NormalClosure(G,H);
print Index(G,N);
6
\\
Q:=quo<N|g7*g1^-1*g3, g8*g7^2*g8^-1,
g8^-1*g7^2*g8,g6*g1*g5*g7*g8^-1*
g5^-1*g1^-1*g3*g6^-1*g7*g8^-1*g5^-1,
g8*(g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*
g3*g6^-1*g7*g8^-1*g5^-1)*g8^-1,g8^2*
(g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*
g6^-1*g7*g8^-1*g5^-1)*g8^-2,g3*g6^-1*
g7*g9^-1, g8*(g3*g6^-1*g7*g9^-1)*
g8^-1,g8^2*(g3*g6^-1*g7*g9^-1)*
g8^-2,g7*g2,g8*(g7*g2)*g8^-1,g8^2*
```

```
(g7*g2)*g8^-2,g3^-1*g1*g5*g8*g7^-1*
g5^-1*g1^-1,g8*(g3^-1*g1*g5*g8*g7^-1*
g5^-1*g1^-1)*g8^-1,g8^2*(g3^-1*g1*g5*
g8*g7^-1*g5^-1*g1^-1)*g8^-2,g4^-1*
g7^-1*g5^-1*g2*g7*g1^-1,g8*
(g4^-1*g7^-1*g5^-1*g2*g7*g1^-1)*g8^-1,
g8^2*(g4^-1*g7^-1*g5^-1*g2*g7*g1^-1)*
g8^-2,g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*
g1^-1*g6^-1*g9^-1*g8^-1*g4^-1,g8*
(g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*
g6^-1*g9^-1*g8^-1*g4^-1)*g8^-1,g8^2*
(g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*
g6^-1*g9^-1*g8^-1*g4^-1)*g8^-2>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

3.2.2. Cases (d,I), N(I) = 3

Here the Bianchi groups involved correspond to $d = 5, 15, 23, 47$. Since $N(I) = 3$, we have $\mathrm{PSL}(2, O_d/I) \cong A_4$.

In order to find a set of peripheral subgroups for $\Gamma(I)$, we use the following sequence of regular covers:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{4} \mathbf{H}^3/\Gamma_1 \xrightarrow{3} Q_d,$$

where $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} < A_4$.

Since each peripheral subgroup P_i of $\mathrm{PSL}(2, O_d)$ maps to a subgroup of order 3 in $\mathrm{PSL}(2, O_d/I)$, it follows that each cusp of Q_d is covered by 4 cusps of $\mathbf{H}^3/\Gamma(I)$ each with covering degree 3. Also, each cusp of Q_d is covered by 1 cusp of $\mathbf{H}^3/\Gamma_1$ with covering degree 3. Thus a set of peripheral subgroups for $\mathbf{H}^3/\Gamma_1$ is given by $\{P_1(I), \dots, P_{h_d}(I)\}$. Conjugating this set by four elements of Γ_1 that correspond to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ gives a set of peripheral subgroups for $\Gamma(I)$.

The case $(15, < 3, 1 + \omega_{15} >)$

Let $I =< 3, 1 + \omega_{15} >$. We do this case in detail; the remaining three cases are done in the same way. Since Q_{15} has 2 cusps, $\mathbf{H}^3/\Gamma(I)$ has 8 cusps.

Reducing the matrix presentations of the P_i for Q_{15} given in Section 3.1 by the ideal $< 3, 1 + \omega_{15} >$, we obtain $P_1(I) =< t^3, tu >$, $P_2(I) =< (uca)^3, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. Setting $\Gamma_1 =< \Gamma(I), a, h >$, for $h = ata(at)^{-1}$, Magma checks that $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ so that conjugating $P_1(I)$ and $P_2(I)$ by $\{Id, a, h, ah\}$ gives a set of eight peripheral subgroups for $\Gamma(I)$. The following set contains one element from each of these eight peripheral subgroups:

$$\{t^{-2}u, atua^{-1}, ht^{-2}uh^{-1}, (ha)tu(ha)^{-1}, (c^{-1}au^{-1}c^{-1}u^{-1}ta)(uca)^3, a(c^{-1}au^{-1}c^{-1}u^{-1}ta)a^{-1}, h(c^{-1}au^{-1}c^{-1}u^{-1}ta)h^{-1}, ha(c^{-1}au^{-1}c^{-1}u^{-1}ta)(ha)^{-1}\}$$

As above, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these eight parabolics, and Magma calculates that $Q =< 1 >$; hence $\Gamma(I)$ is trivialized by setting these eight elements equal to 1. Thus $\Gamma(< 3, 1 + \omega_{15} >)$ is an 8-component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),
(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*
u^-1*a*t^-1>;
H:=sub<G|t^3,t*u,
(c^-1*a*u^-1*c^-1*u^-1*t*a),(u*c*a)^3>;
N:=NormalClosure(G,H);

print Index(G,N);
12
\\

h:=a*t*a*t^-1*a;
A:=sub<G|N,a,h>;
print Index(A,N);
4
\\

Q:=quo<N| t^-2*u, a*t*u*a, h*t^-2*u*
h^-1, h*a*t*u*a*h^-1,(c^-1*a*u^-1*
c^-1*u^-1*t*a)*(u*c*a)^3,a*(c^-1*a*
u^-1*c^-1*u^-1*t*a)*a,h*(c^-1*a*u^-1*
c^-1*u^-1*t*a)*h^-1,h*a*(c^-1*a*u^-1*
c^-1*u^-1*t*a)*a*h^-1>;

print Order(Q);
1
\\
```

The case $(5, < 3, 1 + \omega_5 >)$

Let $I =< 3, 1 + \omega_5 >$. The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is an 8-component link complement in S^3. Note that Q_5 has 2 cusps, the 2 peripheral subgroups $P_i(I)$ can be read off from H, and Q is $\Gamma(I)$ modulo the 8 parabolic elements to be trivialized.

```
G<a,b,c,t,u>:=Group<a,b,c,t,u|a^2,
b^2,(t*a)^3,(a*b)^2,(a*u*b*u^-1)^2,
a*c*a*t*c^-1*t^-1,u*b*u^-1*c*b*t*c^-1*
t^-1, (t,u)>;
H:=sub<G|t^3,t*u,(t*b)^3,(t*b)*
(t*u^-1*c*t^-1)>;
N:=NormalClosure(G,H);

print Index(G,N);
```

```
12
\\
A:=sub<G|N,a,b>;
print Index(A,N);
4
\\

Q:=quo<N|t*u,a*t*u*a^-1,b*t*u*b^-1,
a*b*t^3*b^-1*a^-1,(t*b)*(t*u^-1*
c*t^-1),a*(t*b)*(t*u^-1*c*t^-1)*a^-1,
b*(t*b)*(t*u^-1*c*t^-1)*b^-1,a*b*
(t*b)^-2*(t*u^-1*c*t^-1)*b^-1*a^-1>;

print Order(Q);
1
\\
```

The cases of $d = 23, 47$ and a prime ideal of norm 3 are handled similarly.

3.2.3. Cases (d,I), N(I) = 4

Here the Bianchi groups involved correspond to $d = 15, 23, 31, 47$. In each case, $I = \mathcal{P}^2$, where $\mathrm{N}(\mathcal{P}) = 2$. Hence $[\mathrm{PSL}(2, O_d) : \Gamma(I)] = 24$. Each cusp of Q_d is covered by 6 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 4. Since $\Gamma(\mathcal{P})/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $\Gamma(\mathcal{P}) =< \Gamma(I), (\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}) >$, we exploit the intermediate covers

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{2} \mathbf{H}^3/\Gamma_1 \xrightarrow{2} \mathbf{H}^3/\Gamma(\mathcal{P}),$$

where each covering has degree 2 and $\Gamma_1 = < \Gamma(I), (\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}) >$.

In the cases $d = 15, 23$, we show that $\mathbf{H}^3/\Gamma_1$ is a link complement in $\mathbf{RP}^3$, from which it follows that $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3. For the cases $d = 31, 47$ we show that $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3.

The case $(15, < 4, \omega_{15} >)$

Here $\mathcal{P} = < 2, \omega_{15} >$ so that $I =< 4, \omega_{15} >= \mathcal{P}^2$. Note also that $(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}) = at^2a^{-1}$.

As shown in Section 3.2.1, $\mathbf{H}^3/\Gamma(\mathcal{P})$ has 6 cusps, each cusp of Q_{15} being covered by 3 cusps. Note that $\mathbf{H}^3/\Gamma(I)$ has 12 cusps and the intermediate cover $\mathbf{H}^3/\Gamma_1$ has 8 cusps.

In the following Magma routine, $P_1(I) =< t^4, u >$ and $P_2(I) =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^4 >$ are read off from H, $A = \Gamma_1$, and Q equals Γ_1 modulo 8 parabolics, one from each peripheral subgroup of Γ_1. Since Magma gives $Q \cong \mathbf{Z}/2\mathbf{Z}$, it follows that $\mathbf{H}^3/\Gamma_1$ is an 8-component link in $\mathbf{RP}^3$. Finally, since the eight peripheral elements trivialized are also in $\Gamma(I)$, we have that $\mathbf{H}^3/\Gamma(I)$ is a 12-component link complement in S^3.

```
G<a,t,u,c>:=Group<a,t,u,c|a^2,
(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*
a^-1*t^-1, (t,u),(a,c)>;

H:=sub<G|t^4,u,(c^-1*a*u^-1*c^-1*
u^-1*t*a)^4,u*c*a>;
N:=NormalClosure(G,H);
print Index(G,N);
24
\\
A:=sub<G|N,a*t^2*a>;
print Index(G,A);
12
\\
Q:=quo<A|t^4,t*a*u*(t*a)^-1,(t*a)^-1*
t^4*u*t*a,u*c*a,(t*a)*u*c*a*(t*a)^-1,
(t*a)^-1*u*c*a*(t*a),t^2*(t*a)*t^4*u*
(t*a)^-1*t^-2,t^2*u*c*a*t^-2>;

print Order(Q);
2
\\
```

The case $(47, < 4, \omega_{47} >)$

Let $I =< 4, \omega_{47} >$. The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is a 30-component link complement in S^3. Note that Q_{47} has 5 cusps, and that $(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}) = g_1$, $(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}) = g_2^2$, $(\begin{smallmatrix} 1 & 0 \\ -2 & 0 \end{smallmatrix}) = h := g_1^{-1}g_2^2g_1$. The peripheral subgroups $P_i(I)$ can be read off from H, and one obtains a set of peripheral subgroups for $\Gamma(I)$ by conjugating the $P_i(I)$ first by the elements $\{Id, g_1, g_1^2\}$ and then by those of $\{Id, g_2^2, h, g_2^2h\}$. Here some care must be taken to remove 30 redundant groups, leaving a set of 30 peripheral subgroups for $\Gamma(I)$.

Q is the quotient of $\Gamma(I)$ by the normal closure of 30 parabolic elements, 1 from each of the peripheral subgroups. Magma calculates that $Q =< 1 >$, hence $\mathbf{H}^3/\Gamma(I)$ is a 30-component link complement in S^3.

```
G<g1,g2,g3,g4,g5,g6,g7>:=Group<g1,g2,
g3,g4,g5,g6,g7|g1^3,(g3,g2),
(g2^-1*g1)^2,(g5,g7),g2^-1*g1*g6*
g1^-1*g2*g6^-1,g6*g2^-1*g4^-1*g5*
g3^-1*g6^-1*g4*g2*g3*g5^-1,g7^-1*
g2^-1*g5^-1*g4*g1*g4^-1*g2*g7*g4*
g1^-1*g4^-1*g5,g3*g5^-1*g4*g1*g4^-1*
g2*g5*g3^-1*g2^-1*g4^-1*g1^-1*g4>;

H:=sub<G|g2^4,g2*g3,g5^4,g5*g7,g2*g7,
(g4*g1^-1*g4^-1*g5)^4,(g6*g2^-1*
g4^-1)^2,(g6*g2^-1*g4^-1)*(g5*g3^-1*
g2^-1*g4^-1)^2,(g3*g5^-1*g4*g1)^2,
(g6^-1*g1^-1*g4)^2*(g3*g5^-1*g4*g1)>;
```

```
N:=NormalClosure(G,H);
print Index(G,N);
24
\\
h:=g1^-1*g2^2*g1;

Q:=quo<N|g2^-3*g3, h*g2^-3*g3*h^-1,
g1*(g2*g3)*g1^-1, h*g1*(g2^-3*g3)*
g1^-1*h^-1,g1^-1*(g2^-3*g3)*g1, g2^2*
g1^-1*(g2*g3)*g1*g2^-2,g5*g7,h*g5*g7*
h^-1,g1*(g5*g7)*g1^-1,h*g1*(g5*g7)*
g1^-1*h^-1, g1^-1*(g5*g7)*g1,g2^2*
g1^-1*(g5*g7)*g1*g2^-2, g2*g7,h*g2*g7*
h^-1,g1*(g2*g7)*g1^-1,g2^2*g1*(g2*g7)*
g1^-1*g2^-2, g1^-1*(g2*g7)*g1,
h*g1^-1*(g2*g7)*g1*h^-1,(g6*g2^-1*
g4^-1)*(g5*g3^-1*g2^-1*g4^-1)^2,h*
(g6*g2^-1*g4^-1)^2*h^-1,g1*(g6*g2^-1*
g4^-1)^-1*(g5*g3^-1*g2^-1*g4^-1)^2*
g1^-1,g2^2*g1*(g6*g2^-1*g4^-1)*
(g5*g3^-1*g2^-1*g4^-1)^2*g1^-1*g2^-2,
g1^-1*(g6*g2^-1*g4^-1)^-1*(g5*g3^-1*
g2^-1*g4^-1)^2*g1,h*g1^-1*(g6*g2^-1*
g4^-1)^-1*(g5*g3^-1*g2^-1*g4^-1)^2*g1*
h^-1,(g6^-1*g1^-1*g4)^2*(g3*g5^-1*
g4*g1),g2^2*(g3*g5^-1*g4*g1)^2*g2^-2,
g1*(g3*g5^-1*g4*g1)^2*g1^-1, h*g1*
(g3*g5^-1*g4*g1)^2*g1^-1*h^-1,g1^-1*
(g6^-1*g1^-1*g4)^2*(g3*g5^-1*
g4*g1)^-1*g1,h*g1^-1*(g3*g5^-1*
g4*g1)^2*g1*h^-1>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

The cases of $d = 23, 31$ and an ideal of norm 4 which is the square of a prime ideal of norm 2 are handled similarly. Note that the principal ideal $< 2 >$ also has norm 4. However we show in Section 4 that $\Gamma(2)$ is not a link group if $h_d > 1$.

3.2.4. Cases (d,I), N(I) = 5

Here the Bianchi groups involved correspond to $d = 15, 31$. Since $N(I) = 5$, we have $\mathrm{PSL}(2, O_d/I) \cong A_5$. Since each P_i maps to a subgroup of order 5 in $\mathrm{PSL}(2, O_d/I)$, it follows that each cusp of Q_d is covered by 12 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 5. We use the following sequence of covers of Q_d to find a set of peripheral subgroups for $\Gamma(I)$:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{4} \mathbf{H}^3/\Gamma_1 \xrightarrow{3} \mathbf{H}^3/\Gamma_2 \xrightarrow{5} Q_d$$

where $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $\Gamma_2/\Gamma(I) \cong A_4$. It follows that $\mathbf{H}^3/\Gamma_2$ has h_d cusps, $\mathbf{H}^3/\Gamma_1$ has $3h_d$ cusps, and $\mathbf{H}^3/\Gamma(I)$ has $12h_d$ cusps.

If $\Gamma_1 =< \Gamma(I), a, b >$ and $\Gamma_2 =< \Gamma_1, r >$, we obtain a set of peripheral subgroups for $\mathbf{H}^3/\Gamma(I)$ by conjugating $\{P_1(I), \dots, P_{h_d}(I)\}$ first by the elements $\{Id, r, r^2\}$, and then further conjugating by $\{Id, a, b, ab\}$.

The case $(15, < 5, 2 + \omega_{15} >)$
The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is a 24-component link complement in S^3. Note that $\mathbf{H}^3/\Gamma(I)$ has 24 cusps. Reducing the matrix presentations for P_1 and P_2 modulo I gives $P_1(I) =< t^5, t^2u >$ and $P_2 =< (uca)^5, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. The group Γ_1 (resp. Γ_2) is given by A (resp. B). A set of peripheral subgroups for $\Gamma(I)$ is calculated as described above. Magma then calculates that quotient of $\Gamma(I)$ by the normal closure of the 24 peripheral elements given below trivializes the group.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),
(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*
u^-1*a*t^-1>;

H:=sub<G|t^5,t^2*u,(c^-1*a*u^-1*c^-1*
u^-1*t*a),(u*c*a)^5>;
N:=NormalClosure(G,H);
print Index(G,N);
60
\\
b:=(t*a*u)*a*(t*a*u)^-1;
r:=(t*c)*(t*a)*(t*c)^-1;

A:=sub<G|N,a,b>;
B:=sub<G|N,a,b,r>;
print Index(A,N);
4
\\
print Index(B,N);
12
\\
Q:=quo<N|t^-3*u, a*t^2*u*a, b*t^2*u*b,
a*b*t^2*u*b*a,r*t^2*u*r^-1,
a*r*t^2*u*r^-1*a, b*r*
t^2*u*r^-1*b, a*b*r*t^2*u*r^-1*b*a,
r^-1*t^-3*u*r, a*r^-1*t^-3*u*r*a,
b*r^-1*t^-3*u*r*b, a*b*r^-1*t^-3*u*r*
b*a,(u*c*a)^-5, a*(c^-1*a*u^-1*c^-1*
u^-1*t*a)*a*a*(u*c*a)^5*a,b*(c^-1*a*
u^-1*c^-1*u^-1*t*a)*(u*c*a)^5*b,a*b*
(c^-1*a*u^-1*c^-1*u^-1*t*a)*(u*c*a)^5*
b*a,r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*
r^-1, a*r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*
r^-1*a,b*r*(c^-1*a*u^-1*c^-1*u^-1*
t*a)*r^-1*b, a*b*r*(c^-1*a*u^-1*c^-1*
u^-1*t*a)*r^-1*b*a,r^-1*(c^-1*a*u^-1*
c^-1*u^-1*t*a)*r, a*r^-1*(c^-1*a*u^-1*
c^-1*u^-1*t*a)*r*a,b*r^-1*(c^-1*a*
u^-1*c^-1*u^-1*t*a)*r*b, a*b*r^-1*
(c^-1*a*u^-1*c^-1*u^-1*t*a)*r*b*a>;
```

```
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

The case $(31, < 5, 1+\omega_{31} >)$

Let $I =< 5, 1+\omega_{31} >$. Q_{31} has 3 cusps, hence $\mathbf{H}^3/\Gamma(I)$ has 36 cusps, and the following Magma routine shows that it is a 36-component link complement in S^3. Note that $g_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, $g_2g_1^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\Gamma_1 = A$, and $\Gamma_2 = B$.

```
G<g1,g2,g3,g4,g5>:=Group<g1,g2,g3,g4,
g5|(g1,g3),(g2)^3,(g2*g1^-1)^2,
(g5,g4),g4*g1^-1*g3^-1*g2*g3*g4^-1*g2*
g4*g3^-1*g1^-1*g2*g3*g4^-1*g2,g5*
g3^-1*g2*g3*g4^-1*g2*g1^-1*g5^-1*
g2^-1*g4*g3^-1*g2^-1*g3*g1,g2*g3*
g4^-1*g2*g1^-1*g4*g3^-1*g2*g3*g4^-1*
g1*g2^-1*g4*g3^-1>;

H:=sub<G|g1^5,g1^-1*g3, g4^5,g4*g5,
(g1*g5)^5,(g1*g5)*(g3^-1*g2*g3*g4^-1*
g2*g5)>;
N:=NormalClosure(G,H);
print Index(G,N);
60
\\
a:=g1*(g2*g1^-1)*g1^-1;
b:=g2*g5*(g2*g1^-1)*g5^-1*g2^-1;
A:=sub<G|N,a,b>;
B:=sub<G|N,a,b,g2>;
print Index(A,N);
4
\\
print Index(B,N);
12
\\

Q:=quo<N|g1^3*g3^2, b*g1^4*g3*b,
a*(g1^-1*g3)*a, b*a*(g1^-1*g3)*a*b,g2*
(g1^-1*g3)*g2^-1,b*g2*(g1^-1*g3)*
g2^-1*b,a*g2*(g1^-1*g3)*g2^-1*a,b*a*
g2*(g1^4*g3)*g2^-1*a*b, g2^-1*
(g1^-1*g3)*g2,b*g2^-1*(g1^-1*g3)*g2*b,
a*g2^-1*(g1^-1*g3)*g2*a,b*a*g2^-1*
(g1^-1*g3)*g2*a*b,g4*g5,b*g4*g5*b,
a*g4*g5*a,b*a*(g4*g5)*a*b,g2*(g4*g5)*
g2^-1,b*g2*(g4*g5)*g2^-1*b,a*g2*
(g4*g5)*g2^-1*a,b*a*g2*(g4*g5)*g2^-1*
a*b, g2^-1*(g4*g5)*g2,b*g2^-1*(g4*g5)*
g2*b, a*g2^-1*(g4*g5)*g2*a,b*a*g2^-1*
(g4*g5)*g2*a*b,(g1*g5)^-3*(g3^-1*g2*
g3*g4^-1*g2*g5)^2,b*(g1*g5)^-3*
(g3^-1*g2*g3*g4^-1*g2*g5)^2*b,
a*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*
g2*g5)^2*a,b*a*(g1*g5)^-3*(g3^-1*g2*
g3*g4^-1*g2*g5)^2*a*b,g2*(g1*g5)^-3*
(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1,
b*g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*
g2*g5)^2*g2^-1*b,a*g2*(g1*g5)^-3*
(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1*a,
b*a*g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*
g2*g5)^2*g2^-1*a*b,g2^-1*(g1*g5)^-3*
(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2,
b*g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*
g2*g5)^2*g2*b,a*g2^-1*(g1*g5)^-3*
(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2*a,
b*a*g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*
g4^-1*g2*g5)^2*g2*a*b>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

3.2.5. Case (d,I), N(I) = 6

Here the Bianchi group corresponds to $d = 15$ and $I =< 6, -2+\omega_{15} >=< 2, \omega_{15} >< 3, 1+\omega_{15} >$. Note that $N(I) = 6$ and so using Section 2.1 we deduce that $[\mathrm{PSL}(2, O_{15}) : \Gamma(I)] = 72$. Indeed, in this case we have $\mathrm{PSL}(2, O_{15}/I) \cong \mathrm{PSL}(2, \mathbf{Z}/2\mathbf{Z}) \times \mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z}) \cong S_3 \times A_4$. Now P_1 and P_2 both have order 6 in $\mathrm{PSL}(2, O_{15}/I)$, so that each cusp of Q_{15} is covered by 12 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 6. We will show that $\mathbf{H}^3/\Gamma(I)$ is a 24-component link group.

We use the following sequence of covers of Q_{15}:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{12} \mathbf{H}^3/\Gamma(\mathcal{P}) \xrightarrow{6} Q_{15}$$

where $\mathcal{P} =< 2, \omega_{15} >$. Since $\Gamma(\mathcal{P})/\Gamma(I) \cong A_4$, and $\Gamma(\mathcal{P}) =< \Gamma(I), (ta), r, s >$, we obtain a set of peripheral subgroups for $\Gamma(I)$ by first conjugating $\{P_1(I), P_2(I)\}$ by the elements $\{Id, ta, (ta)^2\}$ and then further conjugating by the elements $\{Id, r, s, rs\}$. In the following Magma routine, $A = \Gamma(\mathcal{P})$, $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = ta$, $r = t^2at^{-2}a$, and $s = at^{-2}at^2$.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),
(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*
c^-1*u^-1*a*t^-1>;
H:=sub<G|t^6,t^-2*u,(c^-1*a*u^-1*c^-1*
u^-1*t*a)^2, (u*c*a)^3>;
N:=NormalClosure(G,H);
print Index(G,N);
72
\\
r:=(t^2*a*t^-2*a);
```

```
s:=(a*t^-2*a*t^2);
A:=sub<G|N,t*a,r,s);
print Index(A,N);
12
\\
Q:=quo<N|t^-2*u,
r*t^-2*u*r^-1,
s*t^-2*u*s^-1,
r*s*t^-2*u*s^-1*r^-1,

(t*a)*t^4*u*(t*a)^-1,
r*(t*a)*t^-2*u*(t*a)^-1*r^-1,
s*(t*a)*t^-2*u*(t*a)^-1*s^-1,
r*s*(t*a)*t^-2*u*(t*a)^-1*s^-1*r^-1,

(t*a)^-1*t^-2*u*(t*a),
r*(t*a)^-1*t^-2*u*(t*a)*r^-1,
s*(t*a)^-1*t^-2*u*(t*a)*s^-1,
r*s*(t*a)^-1*t^-2*u*(t*a)*s^-1*r^-1,

(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3,
r*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*
(u*c*a)^3*r^-1,s*(c^-1*a*u^-1*c^-1*
u^-1*t*a)^2*(u*c*a)^3*s^-1,r*s*
(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*
(u*c*a)^3*s^-1*r^-1,

(t*a)*(u*c*a)^3*(t*a)^-1,
r*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)^-1*r^-1,
s*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)^-1*s^-1,
r*s*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)^-1*s^-1*r^-1,

(t*a)^-1*(u*c*a)^3*(t*a),
r*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)*r^-1,
s*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)*s^-1,
r*s*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*
t*a)^2*(u*c*a)^3*(t*a)*s^-1*r^-1>;

Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

4. Proof of Theorem 1.1: Excluding the remaining levels

In this section we eliminate (for the eight values of d in Section 2.2) the groups $\Gamma(I)$ that do not appear in Theorem 1.1 using several techniques. In Section 4.1 we establish a bound on the norm of I for which $\Gamma(I)$ can be a link complement. In Section 4.2 we use results on cuspidal cohomology to quickly eliminate certain $\Gamma(I)$. Finally, in Section 4.3, we follow the procedure described in Step 1 of Section 2.5 using Magma to show that the remaining $\Gamma(I)$ are not generated by parabolics hence are not link groups. We remind the reader that for an ideal I for which $I \neq \bar{I}$, it suffices to exclude $\Gamma(I)$ as this will automatically exclude $\Gamma(\bar{I})$.

4.1. Systole bounds

First, we remind the reader that there are only finitely many levels that can give principal congruence link complements in S^3 (see [Baker and Reid 14] Proposition 2.3 for example). We recall a proof of that here, and exhibit an explicit bound for the norm of the ideal.

From [Adams and Reid 00] we know that if $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3, its systole (i.e., the length of a shortest closed geodesic) is at most $7.35534\ldots$. Note that the argument in [Adams and Reid 00] used the 2π-Theorem of Gromov and Thurston, but using the 6-Theorem of Agol [Agol 00] and Lackenby [Lackenby 00] the argument of [Adams and Reid 00] can be redone to improve this systole bound to $7.171646\ldots$ (see also [Palaparthi 13]).

Lemma 4.1. *Suppose that $I \subset O_d$ is an ideal such that $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3. Then $N(I) < 39$.*

Proof. If $\gamma \in \Gamma(I)$ is a hyperbolic element, its complex length is $\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma)$, where $\ell_0(\gamma)$ is the translation length of γ and $\theta(\gamma)$ is the angle incurred in translating along the axis of γ by distance $\ell_0(\gamma)$. Now, as is well known $\cosh(\ell(\gamma)/2) = \pm\mathrm{tr}(\gamma)/2$, and so we get the following inequality for $\ell_0(\gamma)$: $|\mathrm{tr}(\gamma)|/2 \leq \cosh(\ell_0(\gamma)/2)$. With the systole bound given above, it follows that

$$|\mathrm{tr}(\gamma)/2| \leq \cosh(7.1717/2) \leq 18.1 \text{ and so} |\mathrm{tr}(\gamma)| < 37.$$

From Lemma 2.5 of [Baker and Reid 14] we have that if $\gamma \in \Gamma(I)$ is a hyperbolic element, then $\mathrm{tr}\,\gamma = \pm 2 \bmod I^2$. Hence $\mathrm{tr}(\gamma) \pm 2 \in I^2$, and this together with the bound on $|\mathrm{tr}(\gamma)|$ quickly gives $N(I) < 39$. □

Remark 4.2. The proof of Lemma 4.1 actually shows more: if $\Gamma(I)$ is a link group, then there exists $x \in I$ such that $|x|^2 < 39$. Since $h_d > 1$, there are ideals I of norm less than 39 for which no such element exists and hence $\Gamma(I)$ is not a link group. In particular, this eliminates $\Gamma(I)$ for the levels $I =< 23, 8+\omega_5 >$, $I =< 29, 9+\omega_6 >$, and $I = < 13, -1+2\omega_{39} >$.

4.2. Cuspidal cohomology

Given Lemma 4.1, and the Remark 4.2, we further reduce the number of groups $\Gamma(I)$ of $N(I) < 39$ that can be link groups. The case of rational integer level can be dealt with by the following result of the first author [Baker 82].

Theorem 4.3. *If $h_d > 1$, and $\Gamma(n) < \mathrm{PSL}(2, O_d)$, then $\mathbf{H}^3/\Gamma(n)$ is not homeomorphic to a link complement in S^3.*

Indeed, for certain d, it will be useful in what follows to note a stronger version of Theorem 4.3 when the primes have small norm. To state this, we recall the definition of *degree 1 cuspidal cohomology* in a form that is useful to us. Suppose that $X = \mathbf{H}^3/\Gamma$ is an orientable, non-compact, finite volume hyperbolic 3-orbifold, and U_Γ the normal subgroup of Γ generated by the parabolic elements of Γ. Then the subspace of $H_1(X, \mathbf{Q})$ which defines the degree 1 cuspidal cohomology of X (or Γ) can be identified with:

$$V_X\,(\text{ or } V_\Gamma) = (\Gamma/U_\Gamma)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Theorem 4.4. *If $h_d > 1$, then $\Gamma(n) < \mathrm{PSL}(2, O_d)$ has nontrivial degree 1 cuspidal cohomology in the following cases (using the notation introduced earlier to indicate the level and d):*

$(23, 3)$, $(23, 5)$, $(31, 2)$, $(47, 2)$, $(47, 3)$, $(71, 2)$, $(71, 3)$.

Proof. To prove this we use Zimmert sets as in the proof of Proposition 4.6 of [Baker and Reid 14] (following [Grunewald and Schwermer 81]). As in [Baker and Reid 14] for the cases stated in the theorem, the Zimmert sets all have at least two elements, and this allows one to conclude the existence of nontrivial cuspidal cohomology. □

Note that as a corollary to this we have the following.

Corollary 4.5. *Let $d \in \{23, 31, 47, 71\}$, $I \subset O_d$ an ideal, and $p = 2, 3, 5$. Suppose that (d, p) is as in Theorem 4.4 and I is divisible by $< p >$. Then $\Gamma(I)$ has nontrivial degree 1 cuspidal cohomology. In particular $\mathbf{H}^3/\Gamma(I)$ is not homeomorphic to a link complement in S^3.*

4.3. Ruling out small norm levels

We now use Magma to deal with $\Gamma(I)$ for those ideals I with $N(I) < 39$ that are not eliminated by the results in Section 4.2.

Recall from Section 2.5 that $\mathbf{H}^3/\Gamma(I)$ a link complement implies that $\Gamma(I)$ is generated by parabolics. Using the presentations for the Bianchi groups in Section 2.2, the peripheral subgroups P_i of $\mathrm{PSL}(2, O_d)$ in Section 2.4, and the matrix representatives in Appendix A, we identify the peripheral subgroups $P_i(I) = P_i \cap \Gamma(I)$ for $i = 1, \dots, h_d$. As in Section 2.5, $N_d(I)$ denotes the normal closure in $\mathrm{PSL}(2, O_d)$ of $< P_1(I), \dots, P_{h_d}(I) >$.

Consider the quotient group $B_d(I) = \mathrm{PSL}(2, O_d)/N_d(I)$. If $\Gamma(I)$ is a link group then $B_d(I)$ is a finite group with order equal to $|\mathrm{PSL}(2, O_d/I)|$. Hence if $B_d(I)$ is infinite or has order greater than $|\mathrm{PSL}(2, O_d/I)|$, then $\Gamma(I)$ cannot be a link group. Since the groups $N_d(I)$ are often not of finite index in $\mathrm{PSL}(2, O_d)$, we calculate $B_d(I)$ in the Magma routines below as

$$B_d(I) =< \mathrm{PSL}(2, O_d)|P_1(I) = \cdots = P_{h_d}(I) = 1 >$$

that is, by adding the peripheral subgroups $P_i(I)$ to the relations of $\mathrm{PSL}(2, O_d)$. We distinguish two cases:

Case 1: *$B_d(I)$ is a finite group but has order larger than $|\mathrm{PSL}(O_d/I)|$.*

Case 2: *$B_d(I)$ has a finite index subgroup with infinite abelianization.*

Remark 4.6. Note that if $B_d(I)$ is infinite or larger than $|\mathrm{PSL}(2, O_d/J)|$ for an ideal $J \subset I$, then so is $B_d(J)$ and hence $\Gamma(J)$ is not a link group.

We single out the case of $d = 6$ and discuss it in some detail in Section 4.3.1. The remaining values of d are discussed in the following subsections. All Magma routines not included are available on request.

Notation. For the reader's convenience, we recall that Magma uses the following notation to encode the abelianization of a group G. We illustrate this by way of example. If the abelianization of a group G is

$$\mathbf{Z}^4 \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/38\mathbf{Z}$$

then Magma encodes this as [2, 2, 38, 0, 0, 0, 0].

4.3.1. d = 6

We begin with some comments about the behavior of rational primes of norm < 39 in the extension $\mathbf{Q}(\sqrt{-6})/\mathbf{Q}$. The primes 2 and 3 are the only ramified primes; 5, 7, 11, 29, and 31 split; and 13, 17, 19, 23, and 37 are inert and so have norm exceeding 39. Hence, we exclude these and any ideal that they divide from further consideration. Recall also that $I =< 29, 9 + \omega_6 >$ is excluded by Remark 4.2.

Table 1 gives the ideals in O_6 whose norms are $<$ 39, which were ruled out using Magma. We give the order of $B_d(I)$ as ∞ if it contains a finite index subgroup with infinite abelianization, and as $\gg 1$ if its order is $\geq 10^6$. We also include peripheral subgroups $P_1(I)$ and $P_2(I)$.

Table 1. Levels and peripheral subgroups for $d = 6$.

Ideal	Norm	Peripheral subgroups	Order($B_d(I)$)
$< 2, \omega_6 >$	2	$< t^2, u >, < (tb)^2, cu >$	24
$< 3, \omega_6 >$	3	$< t^3, u >, < (tb)^3, cu >$	∞
$< 2 >$	4	$< t^2, u^2 >, < (tb)^2, (cu)^2 >$	∞
$< 5, 2+\omega_6 >$	5	$< t^5, t^2u >, < (tb)^5, (tb)^{-2}cu >$	$\gg 1$
$< 7, 1+\omega_6 >$	7	$< t^7, tu >, < (tb)^7, tb(cu)^{-1} >$	∞
$< 11, 4+\omega_6 >$	11	$< t^{11}, t^4u >, < (tb)^{11}, (tb)^4(cu)^{-1} >$	$\gg 1$
$< 31, 5+\omega_6 >$	31	$< t^{31}, t^5u >, < (tb)^{31}, (tb)^5(cu)^{-1} >$	$\gg 1$

The ideals of norm 8, 9, 12, 14, 15, 16, 18, 20, 21, 24, 27, 28, 30, 32, 33, 35, and 36 are sub-ideals of those in Table 1 and are eliminated by Remark 4.6.

Magma routine for $I =< 2, \omega_6 >$
Note that from Section 2.1, the order of PSL$(2, O_6/I)$ is 6.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3, (a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*b^-1,
(t,u),(a,c),t^2,u,(t*b)^2,c*u>;

print Order(B);
24
```

\\

Magma routine for $I =< 3, \omega_6 >$
Note that from Section 2.1, the order of PSL$(2, O_6/I)$ is 12.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3, (a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*
b^-1, (t,u),(a,c),t^3,u,(t*b)^3,
c*u>;

L:=LowIndexNormalSubgroups(G,48);
print #L;
8
```

\\

```
print AbelianQuotientInvari-
ants(L[8]`Group);
[0,0,0]
```

\\

Magma routine for $I =< 2 >$

We first point out that this case does not follow from Remark 4.6 using the ideal $< 2, \omega_6 >$, since from Section 2.1, the order of PSL$(2, O_6/I)$ is $48 > 24$. Also note that it does follow from Theorem 4.3 that $\Gamma(2)$ is not a link group. However, to deal with other levels divisible by 2 we need to use the stronger statement that $B_6(I)$ is infinite.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3, (a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*b^-1,
(t,u),(a,c),t^2,u^2,(t*b)^2,(c*u)^2>;

L:=LowIndexNormalSubgroups(B,48);
print #L;
98
```

\\

```
print AbelianQuotientInvari-
ants(L[20]`Group);
[ 0 ]
```

\\

Magma routine for $I =< 5, 2 + \omega_6 >$
Note that from Section 2.1, the order of PSL$(2, O_6/I)$ is 60.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3, (a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*b^-1,
(t,u),(a,c),t^5,t^2*u,(t*b)^5,
(t*b)^-2*(c*u)>;

print Order(B);
1966080
```

\\

Magma routine for $I =< 11, 4 + \omega_6 >$
Note that from Section 2.1, the order of PSL$(2, O_6/I)$ is 660.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3, (a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*b^-1,
(t,u),(a,c),t^11,t^4*u,(t*b)^11,
(t*b)^4*(c*u)^-1>;
L:=LowIndexNormalSubgroups(B,660);
print #L;
2
```

\\

```
print Index(B,L[2]`Group);
660
```

\\

```
print AbelianQuotientInvari-
ants(L[2]`Group);
[ 4, 120, 120, 120, 120, 120, 120,
120, 120, 120, 120 ]
```

\\

Magma routine for $I =< 31, 5 + \omega_6 >$

Note that from Section 2.1, the order of $\mathrm{PSL}(2, O_6/I)$ is 14880.

```
G<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t,u),(t*a)^3,(a,c),t^-1*c*t*u*
b*u^-1*c^-1*b^-1,(a*t*b)^3,
(a*t*u*b*u^-1)^3,t^31,t^5*u,(t*b)^31,
(t*b)^5*(c*u)^-1>;

L:=LowIndexNormalSubgroups(G,14880);
print #L;
2
\\
print Index(G,L[2]'Group);
14880
\\
M:=sub<G|(L[2]'Group),t>;
print Index(M,(L[2]'Group));
31
\\
print AbelianQuotientInvariants(M);
[ 3, 3, 3, 3, 3, 3, 68433864,
1847714328 ]
```

In the following subsections, we have followed what was done for the case of $d = 6$ above and applied the results and comments of Sections 4.1 and 4.2 to the remaining values of d. Specifically we have used Appendix B (which contains lists of prime ideals of norm less than 39 in each case), together with Lemma 4.1, Remark 4.6, Theorem 4.3, Theorem 4.4, and Corollary 4.5 to eliminate certain levels. Tables 2–6 below summarize the Magma calculations that treat all of the remaining cases to be eliminated. We do not list those ideals $J \subset I$ that are eliminated directly by the size of $B_d(I)$ (as per Remark 4.6).

4.3.2. d = 5

Note that the group $\Gamma(< 2, 1 + \omega_5 >)$ contains the element $\left(\begin{smallmatrix} \omega_5 & 2 \\ 2 & -\omega_5 \end{smallmatrix}\right)$ which has order 2, and so $\Gamma(< 2, 1 + \omega_5 >)$ cannot be a link group.

We make one comment on $I =< 29, -13 + \omega_5 >$. In this case we found, as for $I =< 31, 5 + \omega_6 >$, an intermediate subgroup with infinite abelianization as described below.

```
B<a,b,c,t,u>:=Group<a,b,c,t,u|(t,u),
a^2,b^2,(t*a)^3,(a*b)^2,
(a*u*b*u^-1)^2,a*c*a*t*c^-1*t^-1,
u*b*u^-1*c*b*t*c^-1*t^-1,
t^29,t^-13*u,(t*b)^29,(t*b)^-13*
(t*u^-1*c*t^-1)>;

L:=LowIndexNormalSubgroups(B,12180);
print #L;
2
\\
print Index(B,L[2]'Group);
12180
\\
M:=sub<B|L[2]'Group,t>;
print Index(B,M);
420
\\
AbelianQuotientInvariants(M);
[ 2, 2, 58, 58, 116, 116, 0, 0 ]
\\
```

4.3.3. d = 15, 23, 31, 39

Referring to Tables 3–6, in Table 3, set $x = c^{-1}au^{-1}c^{-1}u^{-1}ta$ to be the parabolic element commuting with uca.

In Table 4, set $p_1 = g_4g_3g_2$ and $p_2 = g_2^{-1}g_5g_3g_2$ to be the parabolic elements generating the peripheral subgroup of $\mathrm{PSL}(2, O_{23})$ fixing $\frac{-1-\sqrt{-23}}{4}$.

In Table 5, set $p = g_3^{-1}g_2g_3g_4^{-1}g_2g_5$ to be the parabolic element commuting with g_1g_5. These elements generate the peripheral subgroup of $\mathrm{PSL}(2, O_{31})$ fixing $\frac{-1-\sqrt{-31}}{4}$.

Table 6 needs no further explanation.

4.3.4. d = 47 *and* d = 71

For the remaining two Bianchi groups, we simply comment on a few cases which together with Theorems 4.3, 4.4, and Corollary 4.5 eliminate all other possible levels.

When $d = 47$, $B_{47}(I)$ is infinite (resp. $\gg 1$) for $I = < 6, \omega_{47} >$ (resp. for $I =< 7, 1 + \omega_{47} >$, $< 8, 4 + \omega_{47} >$ and $< 9, 2 + \omega_{47} >$).

When $d = 71$, $B_{71}(I)$ is infinite for $I =< 3, \omega_{71} >$ (resp. $\geq 24 \times (33)^3$ for $I =< 4, -2 + \omega_{71} >$), (resp. $\gg 1$ for $I =< 5, 1 + \omega_{71} >$).

5. Congruence links when $d = 6$ and $d = 39$

In this section we exhibit congruence link groups for $d = 6$ and $d = 39$ which proves Theorem 1.3. We begin with some notation.

For an ideal $I \subset O_d$, let $\Gamma_0(I)$ (resp. $\Gamma_1(I)$) denote the congruence subgroup obtained as the preimage of the subgroup $\{\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right) : a, x \in O_d/I\}/\{\pm Id\}$ (resp. $\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in O_d/I\}/\{\pm Id\}$) under the reduction homomorphism $\mathrm{PSL}(2, O_d) \to \mathrm{PSL}(2, O_d/I)$. Note that by definition, as long as $N(I) \geq 5$, the group $\Gamma_1(I)$ is torsion-free.

Table 2. Levels and peripheral subgroups for $d = 5$.

Ideal	Norm	Peripheral subgroups	Order$(B_d(I))$
$< 2 >$	4	$< t^2, u^2 >, < (tb)^2, (tu^{-1}ct^{-1})^2 >$	∞
$< \omega_5 >$	5	$< t^5, u >, < (tb)^5, tu^{-1}ct^{-1} >$	∞
$< 6, 1 + \omega_5 >$	6	$< t^6u, tu >, < (tb)^6, tb(tu^{-1}ct^{-1}) >$	144
$< 7, 3 + \omega_5 >$	7	$< t^7, t^3u >, < (tb)^7, (tb)^{-4}(tu^{-1}ct^{-1}) >$	$\geq 168 \times 3^6$
$< 3, 1 + \omega_5 >^2$	9	$< t^9, t^{-2}u >, < (tb)^9, (tb)^{-2}(tu^{-1}ct^{-1}) >$	$\geq 324 \times 2^{12}$
$< 3 >$	9	$< t^3, u^3 >, < (tb)^3, (tu^{-1}ct^{-1})^3 >$	∞
$< 29, -13 + \omega_5 >$	29	$< t^{29}, t^{-13}u >, < (tb)^{29}, (tb)^{-13}(tu^{-1}ct^{-1}) >$	∞

Table 3. Levels and peripheral subgroups for $d = 15$.

Ideal	Norm	Peripheral subgroups	Order$(B_d(I))$
$< 2, \omega_{15} >^3$	8	$< t^8, t^{-4}u >, < (uca)x^4, (uca)^2 >$	192×3^5
$< 3 >$	9	$< t^3, u^3 >, < (uca)^3, x^3 >$	∞
$< 10, 2 + \omega_{15} >$	10	$< t^{10}, t^2u >, < (uca)^5, x^2 >$	∞
$< 12, 4 + \omega_{15} >$	12	$< t^{12}, t^4u >, < (uca)^3, x^4 >$	∞
$< 15, -8 + \omega_{15} >$	15	$< t^{15}, t^7u >, < (uca)^{15}, x >$	∞
$< 17, 5 + \omega_{15} >$	17	$< t^{17}, t^5u >, < (uca)^{17}, (uca)^{-8}x >$	∞
$< 19, 10 + \omega_{15} >$	19	$< t^{19}, t^{10}u >, < (uca)^{19}, (uca)^4x >$	$\gg 1$
$< 31, -14 + \omega_{15} >$	31	$< t^{31}, t^{-14}u >, < (uca)^{31}, (uca)^{13}x >$	∞

Table 4. Levels and peripheral subgroups for $d = 23$.

Ideal	Norm	Peripheral subgroups	Order$(B_d(I))$
$< 2 >$	4	$< g_1^2, g_2^2 >, < g_4^2, g_5^2 >, < p_1^2, p_2^2 >$	∞
$< 6, \omega_{23} >$	6	$< g_1^6, g_1g_2 >, < g_4^3, g_4^{-1}g_5^2 >, < p_1^6, p_1^{-3}p_2 >$	∞
$< 2, \omega_{23} >^3$	8	$< g_1^8, g_1^{-1}g_2 >, < g_4^4, g_4g_5^2 >, < p_1^2p_2^{-2}, p_1^3p_2 >$	$\geq 192 \times 3^6$
$< 3, \omega_{23} >^2$	9	$< g_2^9, g_2^4g_1 >, < g_4^9, g_4^{-2}g_5 >, < p_1^9, p_1^6p_2 >$	$\geq 324 \times 2^8$
$< 13, 4 + \omega_{23} >$	13	$< g_2^{13}, g_2^5g_1 >, < g_4^{13}, g_4^{-4}g_5 >, < p_1^{13}, p_1^{-2}p_2 >$	$\gg 1$
$< 23, \omega_{23} >$	23	$< g_2^{23}, g_2^{-11}g_1 >, < g_4^{23}, g_4^{-6}g_5 >, < p_1^{23}, p_1^{-6}p_2 >$	∞

Table 5. Levels and peripheral subgroups for $d = 31$.

Ideal	Norm	Peripheral subgroups	Order$(B_d(I))$
$< 7, 2 + \omega_{31} >$	7	$< g_1^7, g_1^{-2}g_3 >, < g_4^7, g_4^{-2}g_5 >, < (g_1g_5)^7, p >$	∞
$< 2, \omega_{31} >^3$	8	$< g_1^8, g_3 >, < g_4^4, g_4g_5^2 >, < (g_1g_5)^2, (g_1g_5)^{-1}p^4 >$	∞
$< 3 >$	9	$< g_1^3, g_3^3 >, < g_4^3, g_5^3 >, < (g_1g_5)^3, p^3 >$	∞
$< 10, 1 + \omega_{31} >$	10	$< g_1^{10}, g_1^{-1}g_3 >, < g_4^{10}, g_4^{-4}g_5 >, < (g_1g_5)^{10}, (g_1g_5)p >$	$\gg 1$
$< 19, 5 + \omega_{31} >$	19	$< g_1^{19}, g_1^5g_3 >, < g_4^{19}, g_4^3g_5 >, < (g_1g_5)^{19}, (g_1g_5)^7p >$	$\gg 1$
$< -1 + 2\omega_{31} >$	31	$< g_1^{31}, g_1^{16}g_3 >, < g_4^{31}, g_4^8g_5 >, < (g_1g_5)^{31}, (g_1g_5)^5p^{-1} >$	∞

Table 6. Levels and peripheral subgroups for $d = 39$.

Ideal	Norm	Peripheral subgroups	Order $(B_d(I))$
$< 2, \omega_{39} >$	2	$< g_1^2, g_2 >, < g_4^2, g_6 >, < (g_5^{-1}g_6)^2, g_4g_1^{-1}g_6 >, < g_5^2, g_5^{-1}(g_4g_2^{-1}g_7) >$	18
$< 3, 1 + \omega_{39} >$	3	$< g_1^3, g_1g_2 >, < g_4, g_6^3 >, < (g_5^{-1}g_6)^3, (g_5^{-1}g_6)^2(g_4g_1^{-1}g_6) >, < g_5^3, g_5^{-1}(g_4g_2^{-1}g_7) >$	∞
$< 2, \omega_{39} >^2$	4	$< g_1^4, g_1^{-1}g_2 >, < g_4^4, g_6 >, < (g_5^{-1}g_6)^4, g_4g_1^{-1}g_6 >, < g_5^4, g_5^{-1}(g_4g_2^{-1}g_7) >$	72
$< 2 >$	4	$< g_1^2, g_2^2 >, < g_4^2, g_6^2 >, < (g_5^{-1}g_6)^2, (g_4g_1^{-1}g_6)^2 >, < g_5^2, (g_4g_2^{-1}g_7)^2 >$	∞
$< 5, \omega_{39} >$	5	$< g_1^5, g_2 >, < g_4^5, g_4^{-1}g_6 >, < (g_4g_1^{-1}g_6)^5, (g_4g_1^{-1}g_6)^{-2}(g_5^{-1}g_6) >, < g_5^5, g_5(g_4g_2^{-1}g_7) >$	∞
$< 2, \omega_{39} >^3$	8	$< g_1^8, g_1^2g_2 >, < g_4^4, g_6^2 >, < (g_4g_1^{-1}g_6)^2, (g_5^{-1}g_6)^4(g_4g_1^{-1}g_6) >, < g_5^4, g_5^{-2}(g_4g_2^{-2}g_7)^2 >$	∞
$< 3 >$	9	$< g_1^3, g_2^3 >, < g_4^3, g_6^3 >, < (g_5^{-1}g_6)^3, (g_4g_1^{-1}g_6)^3 >, < g_5^3, (g_4g_2^{-1}g_7)^3 >$	∞
$< 11, 3 + \omega_{39} >$	11	$< g_1^{11}, g_1^3g_2 >, < g_4^{11}, g_4^{-2}g_6 >, < (g_5^{-1}g_6)^{11}, (g_5^{-1}g_6)^3(g_4g_1^{-1}g_6) >, < g_5^{11}, g_5^4(g_4g_2^{-1}g_7) >$	$\gg 1$

In the case when $I = \mathcal{P}$ is a prime ideal of O_d lying over a split prime $p \in \mathbf{Z}$ (as will be the case of interest below), then $[\mathrm{PSL}(2, O_d) : \Gamma_0(\mathcal{P})] = p + 1$, the quotient group $\Gamma_0(\mathcal{P})/\Gamma_1(\mathcal{P})$ is cyclic of order $(p-1)/2$, and $\Gamma_1(\mathcal{P})/\Gamma(\mathcal{P}) \cong \mathbf{Z}/p\mathbf{Z}$ with $\Gamma_1(\mathcal{P}) =< \Gamma(\mathcal{P}), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) >$.

Finally, note that by definition, a peripheral subgroup of $\Gamma_0(I)$ is necessarily contained in $\Gamma_1(I)$, so that a set of peripheral subgroups for $\Gamma_0(I)$ is a partial set of peripheral subgroups for $\Gamma_1(I)$.

Proposition 5.1. $\Gamma_1(< 7, 1 + \omega_6 >)$ *and* $\Gamma_1(< 5, \omega_{39} >)$ *are congruence link subgroups of* $\mathrm{PSL}(2, O_6)$ *and* $\mathrm{PSL}(2, O_{39})$ *respectively.*

Proof. The proof follows the method in Section 3. We do $\Gamma_1(I)$ for $I =< 7, 1 + \omega_6 >$ in detail, and then comment on the case $I =< 5, \omega_{39} >$. Both Magma routines are given below.

Let $I =< 7, 1 + \omega_6 >$. From the remarks preceding the statement of Proposition 5.1, we have $[\mathrm{PSL}(2, O_6) : \Gamma_0(I)] = 8$ and $[\Gamma_0(I) : \Gamma_1(I)] = 3$. Now Q_6 has 2 cusps, and in the covering $\mathbf{H}^3/\Gamma_0(I) \longrightarrow Q_6$ each one of these 2 cusps is covered by two cusps, one of which has covering degree 1 and the other has degree 7.

Hence we deduce that $\mathbf{H}^3/\Gamma_1(I)$ has 12 cusps, and a set of peripheral subgroups for $\Gamma_1(I)$ is obtained by conjugating those of $\Gamma_0(I)$ by the elements $\{Id, x, x^{-1}\}$ where $\Gamma_0(I) =< \Gamma_1(I), x >$. A particular choice of x is $(at^{-2}a^{-1})t^{-2}(at^{-1}a^{-1})$.

Using the matrix representations for the peripheral subgroups P_1 and P_2 of $\mathrm{PSL}(2, O_6)$ given in Section 2.4, we find the following set of peripheral subgroups for $\Gamma_0(I)$:

$$(\infty, < t, u >),\ (0, a < t^7, tu > a^{-1}),$$
$$\left(\frac{-\sqrt{-6}}{2}, < (tb)^7, (tb)(cu)^{-1} >\right),$$
$$\left(\frac{-6 - \sqrt{-6}}{14}, (at^2a^{-1}) < tb, cu > (at^2a^{-1})^{-1}\right)$$

Now we obtain a set of peripheral subgroups for $\Gamma_1(I)$ by conjugating the above set for $\Gamma_0(I)$ by the elements $\{Id, x, x^{-1}\}$.

In the following Magma routine, $G = \mathrm{PSL}(2, O_6)$, K is the subgroup generated by the four peripheral subgroups of $\Gamma_0(I)$, $L[1] = \Gamma_0(I)$, and $M[1] = \Gamma_1(I)$. As before, Q is the quotient of $\Gamma_1(I)$ by the normal closure the 12 parabolic elements to be trivialized. Since Magma calculates $Q =< 1 >$, it follows that $\Gamma_1(I)$ is indeed a 12-component link group.

Note that we use the `LowIndexSubgroups` routine in Magma to obtain $\Gamma_0(I)$ from $\mathrm{PSL}(2, O_6)$ and $\Gamma_1(I)$ from $\Gamma_0(I)$. Indeed, since $N_d(I)$ is not of finite index in $\Gamma(I)$ (see Section 4.3.1), we cannot use Magma to obtain a presentation for $\Gamma(I)$ as in Section 3, and hence we cannot present $\Gamma_1(I) =< \Gamma(I), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) >$ in this way. □

```
G<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,
b^2,(t*a)^3,(a*t*b)^3,(a*t*u*
b*u^-1)^3,t^-1*c*t*u*b*u^-1*c^-1*b^-1,
(t,u),(a,c)>;

K:=sub<G|t,u,a*t^7*a,a*t*u*a,(t*b)^7,
(t*b)*(c*u)^-1,a*t^2*a*t*b*a*t^-2*a,
a*t^2*a*c*u*a*t^-2*a>;

L:=LowIndexSubgroups(G,<8,8>:
Subgroup:=K);
print #L;
1
\\
M:=LowIndexSubgroups(L[1],<3,3>:
Subgroup:=K);
print #M;
1
\\
x:=a*t^-2*a*t^2*a*t^-1*a;
A:=sub<G|M[1],x>;
A eq L[1];
true
\\
Q:=quo<M[1]|u,x*t*x^-1,x^-1*u*x,
a*t*u*a,x*a*t*u*a*x^-1,x^-1*a*t*u*a*x,
(t*b)*(c*u)^-1,x*t*b*(c*u)^-1*x^-1,
x^-1*t*b*(c*u)^-1*x,a*t^2*a*t*b*
a*t^-2*a,x*a*t^2*a*t*b*a*t^-2*a*x^-1,
x^-1*a*t^2*a*t*b*a*t^-2*a*x>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

Now let $I =< 5, \omega_{39} >$. Note that Q_{39} has 4 cusps and in the cover $\mathbf{H}^3/\Gamma_0(I) \longrightarrow Q_{39}$ each one of these 4 cusps is covered by 2 cusps, one of which has covering degree 1 and the other has degree 5. Thus we deduce that $\mathbf{H}^3/\Gamma_1(I)$ has 16 cusps, and a set of peripheral subgroups for $\Gamma_1(I)$ is obtained by conjugating one for $\Gamma_0(I)$ by the elements $\{Id, x\}$ where $\Gamma_0(I) =< \Gamma_1(I), x >$.

In the following Magma routine, $G = \mathrm{PSL}(2, O_{39})$, K gives a set of peripheral subgroups for $\Gamma_0(I)$, $L[1] = \Gamma_0(I)$, and $M[1] = \Gamma_1(I)$. Magma confirms that $\Gamma_0(I) =< \Gamma_1(I), x >$ for $x = (g_3g_1g_1^2g_3g_1)g_3^{-1}(g_3g_1g_1^2g_3g_1)$. Note that Magma returns two possibilities for $\Gamma_0(I)$: $L[1]$ and $L[2]$; however

$L[2]$ is eliminated by homology considerations. Finally, $Q =< 1 >$, so that $\Gamma_1(I)$ is a 16-component link group.

```
G<g1,g2,g3,g4,g5,g6,g7>:=Group<g1,g2,
g3,g4,g5,g6,g7|g3^3,(g4,g6),(g3*g5)^2,
(g2,g1),(g1^-1*g3^-1)^2,(g3^-1,g7^-1),
(g5^-1*g1)^3,g5^-1*g1*g6^-1*g4^-1*g5*
g4*g1^-1*g6,g4^-1*g5*g4*g2^-1*g7*
g5^-1*g7^-1*g2,(g7*g5^-1*g7^-1*g1)^3,
g6*g1^-1*g5*g6^-1*g4^-1*g5*g4*g1^-1*
g4^-1*g5*g4*g1^-1>;

K:=sub<G|g1,g2,(g3*g1)*g1^5*(g3*g1),
(g3*g1)*g2*(g3*g1),g5^-1*g6,
g4*g1^-1*g6,(g3*g1)*(g4*g1^-1*g6)^5*
(g3*g1),(g3*g1)*(g4*g1^-1*g6)^-2*
(g5^-1*g6)*(g3*g1),g4^5, g4*g6^-1,
(g3*g1)*g1^2*g4*g1^-2*(g3*g1),
(g3*g1)*g1^2*g6*g1^-2*(g3*g1),g5^5,
g5*(g4*g2^-1*g7),(g3*g1)*g1*(g5)*
g1^-1*(g3*g1), (g3*g1)*g1*
(g4*g2^-1*g7)*g1^-1*(g3*g1)>;

L:=LowIndexSubgroups(G,<6,6>:
Subgroup:=K);
print #L;
2
\\
print AbelianQuotientInvari-
ants(L[1]);
[2,2,0,0,0,0,0,0,0,0]
\\
print AbelianQuotientInvari-
ants(L[2]);
[2,0,0,0,0,0,0,0,0,0,0,0]
\\
M:=LowIndexSubgroups(L[1],<2,2>:
Subgroup:=K);
print #M;
1
\\
x:=g3*g1*g1^2*g3*g1*g3^-1*
g3*g1*g1^2*g3*g1;
A:=sub<G|M[1],x>;
A eq L[1];
true
\\
Q:=quo<M|g1, x*g2*x^-1, (g3*g1)*g2*
g3*g1,x*g3*g1*g2*g3*g1*x^-1,
g4*g1^-1*g6, x*g4*g1^-1*g6*x^-1,
(g3*g1)*(g4*g1^-1*g6)^5*(g3*g1),
x*(g3*g1)*(g4*g1^-1*g6)^-2*(g5^-1*g6)*
(g3*g1)*x^-1,g4*g6^-1, x*g4*g6^-1*
x^-1, g3*g1*g1^2*g6*g1^-2*g3*g1,x*g3*
g1*g1^2*g6*g1^-2*g3*g1*x^-1, g5*g4*
g2^-1*g7, x*g5^5*x^-1,g3*g1*g1*g5*
g1^-1*g3*g1, x*g3*g1*g1*g5*g1^-1*
g3*g1*x^-1>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

Appendix A

In this appendix we gather together the matrix generators for the groups $\mathrm{PSL}(2, O_d)$ as given in Section 2.2 and used throughout. For $d = 5, 6, 15$, $t = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $u = \left(\begin{smallmatrix}1&\omega_d\\0&1\end{smallmatrix}\right)$, and $a = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Also, recall that:

$$\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1+\sqrt{-15}}{2}, \frac{1+\sqrt{-23}}{2}, \frac{1+\sqrt{-31}}{2}, \frac{1+\sqrt{-39}}{2}, \frac{1+\sqrt{-47}}{2}, \frac{1+\sqrt{-71}}{2}.$$

$$\mathbf{d=5}: a,\ b = \begin{pmatrix}-\omega_5 & 2\\ 2 & \omega_5\end{pmatrix},$$

$$c = \begin{pmatrix}-\omega_5-4 & -2\omega_5\\ 2\omega_5 & \omega_5-4\end{pmatrix},\ t,\ u.$$

$$\mathbf{d=6}: a,\ b = \begin{pmatrix}-1-\omega_6 & 2-\omega_6\\ 2 & 1+\omega_6\end{pmatrix},$$

$$c = \begin{pmatrix}5 & -2\omega_6\\ 2\omega_6 & 5\end{pmatrix},\ t,\ u.$$

$$\mathbf{d=15}: a,\ c = \begin{pmatrix}4 & 1-2\omega_{15}\\ 2\omega_{15}-1 & 4\end{pmatrix},\ t,\ u.$$

$$\mathbf{d=23}: g_1 = \begin{pmatrix}1 & -1+\omega_{23}\\ 0 & 1\end{pmatrix},\ g_2 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

$$g_3 = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix},\ g_4 = \begin{pmatrix}3+\omega_{23} & -4+\omega_{23}\\ -2+\omega_{23} & -1-\omega_{23}\end{pmatrix},$$

$$g_5 = \begin{pmatrix}5-\omega_{23} & 1+2\omega_{23}\\ 2+\omega_{23} & -3+\omega_{23}\end{pmatrix}.$$

$$\mathbf{d=31}: g_1 = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix},\ g_2 = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & \omega_{31} \\ 0 & 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} 3 & -2+2\omega_{31} \\ \omega_{31} & -5 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 3-2\omega_{31} & 7+\omega_{31} \\ 4 & -1+2\omega_{31} \end{pmatrix}.$$

$$\mathbf{d = 39}: g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} 1 & \omega_{39} \\ 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} -3-\omega_{39} & 7-2\omega_{39} \\ 2-\omega_{39} & 5+\omega_{39} \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 3-\omega_{39} & 2+\omega_{39} \\ 3 & -1+\omega_{39} \end{pmatrix},$$

$$g_6 = \begin{pmatrix} 7-\omega_{39} & 2+3\omega_{39} \\ 2+\omega_{39} & -5+\omega_{39} \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 6-\omega_{39} & -1+2\omega_{39} \\ 1-2\omega_{39} & 5+\omega_{39} \end{pmatrix}.$$

$$\mathbf{d = 47}: g_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & -1+\omega_{47} \\ 0 & 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} -2+\omega_{47} & 5 \\ -3 & 1+\omega_{47} \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 5 & -6+3\omega_{47} \\ \omega_{47} & -7 \end{pmatrix},$$

$$g_6 = \begin{pmatrix} -4+\omega_{47} & 3+\omega_{47} \\ -3-\omega_{47} & -4+\omega_{47} \end{pmatrix},$$

$$g_7 = \begin{pmatrix} -1+2\omega_{47} & 11+\omega_{47} \\ 4 & -3+2\omega_{47} \end{pmatrix}.$$

$$\mathbf{d = 71}: g_1 = \begin{pmatrix} -5 & 5-3\omega_{71} \\ -1+\omega_{71} & -10-\omega_{71} \end{pmatrix},$$

$$g_2 = \begin{pmatrix} -3+2\omega_{71} & -17-\omega_{71} \\ -4 & 1-2\omega_{71} \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 5 & -2\omega_{71} \\ 1-\omega_{71} & -7 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} -5 & 2+\omega_{71} \\ -2-\omega_{71} & -3+\omega_{71} \end{pmatrix},$$

$$g_5 = \begin{pmatrix} -6-3\omega_{71} & 13-2\omega_{71} \\ 5-\omega_{71} & 4+\omega_{71} \end{pmatrix},$$

$$g_6 = \begin{pmatrix} -1+2\omega_{71} & 12 \\ -6 & -1+2\omega_{71} \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \; g_8 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$g_9 = \begin{pmatrix} 1+\omega_{71} & -7 \\ 3 & -2+\omega_{71} \end{pmatrix}.$$

Appendix B

In this appendix we list the prime ideals of norm < 39 in O_d (as dictated by Lemma 4.1). We also include information on the splitting type of the rational prime p (i.e., whether p is ramified, inert, or split). The norm of any O_d-prime $\mathcal{P}$ dividing p is then easily computed: it is p when p is ramified or split, and p^2 when p is inert.

d = 5

Rational prime	Splitting type
2	ramified
3	split
5	ramified
7	split
23	split
29	split

d = 6

Rational prime	Splitting type
2	ramified
3	ramified
5	split
7	split
11	split
29	split
31	split

d = 15

Rational prime	Splitting type
2	split
3	ramified
5	ramified
17	split
19	split
23	split
31	split

d = 23

Rational prime	Splitting type
2	split
3	split
5	inert
13	split
23	ramified
29	split
31	split

d = 31

Rational prime	Splitting type
2	split
3	inert
5	split
7	split
19	split
31	ramified

d = 39

Rational prime	Splitting type
2	split
3	ramified
11	split
13	ramified

d = 47

Rational prime	Splitting type
2	split
3	split
7	split
17	split
37	split

d = 71

Rational prime	Splitting type
2	split
3	split
19	split
29	split
37	split

Acknowledgments

The authors are very grateful to A. Page who provided presentations for some of the Bianchi groups that were needed in our work. This work was done during visits to the University of Texas by M. D. Baker, and to Université de Rennes 1 by A. W. Reid. A. W. Reid also wishes to thank Max Planck Institüt, Bonn, for its hospitality while this work was completed. The authors are also most grateful to the referee for their careful reading of the article and useful comments that helped improve the organization of the article.

Funding

A. W. Reid is supported in part by the National Science Foundation.

References

[Adams and Reid 00] C. C. Adams and A. W. Reid. "Systoles of Hyperbolic 3-manifolds." *Math. Proc. Camb. Phil. Soc.* 128 (2000), 103–110.

[Agol 00] I. Agol. "Bounds on Exceptional Dehn Filling." *Geomet. Topol.* 4 (2000), 431–449.

[Baker 82] M. D. Baker. *Link Complements and the Homology of Arithmetic Subgroups of* $\mathrm{PSL}(2, \mathbf{C})$. I.H.E.S. preprint (1982).

[Baker 01] M. D. Baker. "Link Complements and the Bianchi Modular Groups." *Trans. A. M. S.* 353 (2001), 3229–3246.

[Baker 92] M. D. Baker. "Link Complements and Integer Rings of Class Number Greater than One." In *Topology '90*, edited by B. N. Apanasov et al., pp. 55–59. *Ohio State Univ. Math. Res. Inst. Publ. 1*. Berlin: de Gruyter, 1992.

[Baker and Reid 14] M. D. Baker and A. W. Reid. "Principal Congruence Link Complements." *Ann. Fac. Sci. Toulouse* 23 (2014), 1063–1092.

[Bosma 97] W. Bosma, J. Cannon, and C. Playoust. "The Magma Algebra System. I. The User Language." *J. Symbol. Comput.* 24 (1997), 235–265.

[Dixon 58] L. E. Dixon. *Linear Groups, with an Exposition of the Galois Field Theory*. New York: Dover Publications, 1958.

[Goerner 11] M. Goerner. "Visualizing Regular Tessellations: Principal Congruence Links and Equivariant Morphisms from Surfaces to 3-Manifold." Ph.D Thesis, U. C. Berkeley, 2011.

[Grunewald and Schwermer 81] F. Grunewald and J. Schwermer. "A Non-vanishing Theorem for the Cuspidal Cohomology of SL_2 Over Imaginary Quadratic Integers." *Math. Annalen* 258 (1981), 183–200.

[Lackenby 00] M. Lackenby. "Word Hyperbolic Dehn Surgery." *Invent. Math.* 140 (2000), 243–282.

[Maclachlan and Reid 03] C. Maclachlan and A. W. Reid. *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics, 219. New York: Springer-Verlag, 2003.

[Page 15] A. Page. "Computing Arithmetic Kleinian Groups." *Math. Comp.* 84 (2015), 2361–2390.

[Palaparthi 13] S. Palaparthi. "Closed Geodesic Lengths in Hyperbolic Link Complements in S^3." *Int. J. Pure Appl. Math.* 83 (2013), 45–53.

[Serre 70] J.-P. Serre. "Le probléme des groupes de congruence pour SL_2." *Ann. Math.* 92 (1970), 489–527.

[Swan 71] R. G. Swan. "Generators and Relations for Certain Special Linear Groups." *Adv. Math.* 6 (1971), 1–77.

[Vogtmann 85] K. Vogtmann. "Rational Homology of Bianchi Groups." *Math. Ann.* 272 (1985), 399–419.