PROFINITE RIGIDITY, FIBERING, AND THE FIGURE-EIGHT KNOT

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Abstract. We establish results concerning the profinite completions of 3-manifold groups. In particular, we prove that the complement of the figure-eight knot $\mathbb{S}^3 \smallsetminus \mathcal{K}$ is distinguished from all other compact 3-manifolds by the set of finite quotients of its fundamental group. In addition, we show that if M is a compact 3-manifold with $b_1(M) = 1$, and $\pi_1(M)$ has the same finite quotients as a free-by-cyclic group $F_r \rtimes \mathbb{Z}$, then M has non-empty boundary, fibres over the circle with compact fibre, and $\pi_1(M) \cong F_r \rtimes_\psi \mathbb{Z}$ for some $\psi \in \mathrm{Out}(F_r)$.

1. Introduction

We are interested in the extent to which the set of finite quotients of a 3-manifold group $\Gamma = \pi_1(M)$ determines Γ and M. In this article we shall prove that fibred 3-manifolds that have non-empty boundary and first betti number 1 can be distinguished from all other compact 3-manifolds by examining the finite quotients of $\pi_1(M)$. A case where our methods are particularly well suited is when M is the complement of the figure-eight knot. Throughout, we denote this knot by $\mathcal{K}$ and write $\Pi = \pi_1(\mathbb{S}^3 \smallsetminus \mathcal{K})$.

The manifold $\mathbb{S}^3 \smallsetminus \mathcal{K}$ and the group Π hold a special place in the interplay of 3-manifold topology and hyperbolic geometry. $\mathcal{K}$ became the first example of a *hyperbolic knot* when Riley [29] constructed a discrete faithful representation of Π into $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, where ω is a cube root of unity. Subsequently, Thurston [34] showed that the hyperbolic structure on $\mathbb{S}^3 \smallsetminus \mathcal{K}$ could be obtained by gluing two copies of the regular ideal tetrahedron from $\mathbb{H}^3$. The insights gained from understanding the complement of the figure-eight knot provided crucial underpinning for many of Thurston's great discoveries concerning the geometry and topology of 3-manifolds, such as the nature of geometric structures on manifolds obtained by Dehn surgery, and the question of when a surface bundle admits a hyperbolic structure. In our exploration of the finite quotients of 3-manifold groups, the figure-eight knot stands out once again as a special example.

Throughout this article, we allow manifolds to have non-empty boundary, but we assume that any spherical boundary components have been removed by capping-off with a 3-ball. We write $\mathcal{C}(G)$ for the set of isomorphism classes of the finite quotients of a group G.

Theorem A. *Let M be a compact connected 3-manifold. If $\mathcal{C}(\pi_1(M)) = \mathcal{C}(\Pi)$, then M is homeomorphic to $\mathbb{S}^3 \smallsetminus \mathcal{K}$.*

Corollary 1.1. *Let $J \subset \mathbb{S}^3$ be a knot and let $\Lambda = \pi_1(S^3 \smallsetminus J)$. If $\mathcal{C}(\Lambda) = \mathcal{C}(\Pi)$, then J is isotopic to $\mathcal{K}$.*

It is useful to arrange the finite quotients of a group Γ into a directed system, and to replace $\mathcal{C}(\Gamma)$ by the *profinite completion* $\widehat{\Gamma}$: the normal subgroups of finite index $N \lhd \Gamma$ are ordered by reverse inclusion and $\widehat{\Gamma} := \varprojlim \Gamma/N$. A standard argument shows that for finitely generated groups, $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ if and only if $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ (see [12]). Henceforth, we shall express our results in terms of $\widehat{\Gamma}$ rather than $\mathcal{C}(\Gamma)$.

Key words and phrases. 3-manifold groups, profinite completion, figure-eight knot.

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$\mathbb{S}^3 \smallsetminus \mathcal{K}$ has the structure of a once-punctured torus bundle over the circle (see §3), and hence Π is an extension of a free group of rank 2 by $\mathbb{Z}$. We shall use Theorem A to frame a broader investigation into 3-manifolds whose fundamental groups have the same profinite completion as a free-by-cyclic group. To that end, we fix some notation.

We write F_r to denote the free group of rank $r \geq 2$ and $b_1(X)$ to denote the first betti number $\mathrm{rk}\,\mathrm{H}^1(X, \mathbb{Z})$, where X is a space or a finitely generated group. Note that for knot complements $b_1(X) = 1$. Stallings' Fibering Theorem [31] shows that a compact 3-manifold with non-empty boundary fibres over the circle, with compact fibre, if and only if $\pi_1 N$ is of the form $F_r \rtimes \mathbb{Z}$.

In Section 3 we shall elucidate the nature of compact 3-manifolds whose fundamental groups have the same profinite completion as a group of the form $F_r \rtimes \mathbb{Z}$. In particular we shall prove:

Theorem B. *Let M be a compact connected 3-manifold and let $\Gamma = F_r \rtimes_\phi \mathbb{Z}$. If $b_1(M) = 1$ and $\widehat{\Gamma} \cong \widehat{\pi_1(M)}$, then M has non-empty boundary, fibres over the circle with compact fibre, and $\pi_1(M) \cong F_r \rtimes_\psi \mathbb{Z}$ for some $\psi \in \mathrm{Out}(F_r)$.*

Corollary 1.2. *Let M and N be compact connected 3-manifolds with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. If M has non-empty incompressible boundary and fibres over the circle, and $b_1(M) = 1$, then N has non-empty incompressible boundary and fibres over the circle, and $b_1(N) = 1$.*

One can remove the hypothesis $b_1(M) = 1$ from Theorem B at the expense of demanding more from the isomorphism $\widehat{\pi_1 M} \to \widehat{F_r \rtimes \mathbb{Z}}$: it is enough to require that $\pi_1 M$ has cyclic image under the composition

$$\pi_1 M \to \widehat{\pi_1 M} \to \widehat{F_r \rtimes \mathbb{Z}} \to \widehat{\mathbb{Z}},$$

or else that the isomorphism $\widehat{\pi_1 M} \to \widehat{F_r \rtimes \mathbb{Z}}$ is regular in the sense of [6]; see remark 5.2. Corollary 1.2 can be strengthened in the same manner.

We shall pay particular attention to the case $r = 2$. In that setting we prove that for each Γ with $b_1(\Gamma) = 1$, there are only finitely many possibilities for M, and either they are all hyperbolic or else none of them are. (We refer the reader to Theorem D at the end of the paper for a compilation of related results.) These considerations apply to the complement of the figure-eight knot, because it is a once-punctured torus bundle with holonomy

$$\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

whence $\Pi \cong F_2 \rtimes_\phi \mathbb{Z}$. In this case, a refinement of the argument which shows that there are only finitely many possibilities for M shows that in fact there is a unique possibility. Similar arguments show that the trefoil knot complement and the Gieseking manifold are uniquely defined by the finite quotients of their fundamental groups (see §3 and §6 for details).

The most difficult step in the proof of Theorem B is achieved by appealing to the cohomological criterion provided by the following theorem.

Theorem C. *Let M be a compact, irreducible, 3-manifold with non-empty boundary which is a non-empty union of incompressible tori and Klein bottles. Let $f : \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then, either* ker f *is finitely generated and free (and M is fibred) or else the closure of* ker f *in $\widehat{\pi_1 M}$ has cohomological dimension at least* 2*.*

Our proof of this theorem relies on the breakthroughs of Agol [1] and Wise [36] concerning cubical structures on 3-manifolds and the subsequent work of Przytycki-Wise [24] on the separability of embedded surface groups, as refined by Liu [20]. We isolate from this body of work a technical result of independent interest: *if M is a closed 3-manifold and $S \subset M$ is a closed embedded incompressible surface, then the closure of $\pi_1 S$ in $\widehat{\pi_1 M}$ is isomorphic to $\widehat{\pi_1 S}$.* The surfaces to which we apply this result are obtained using a construction of Freedman and Freedman [13]. Our argument also exploits the calculation of L_2-betti numbers by Lott and Lück [21], and the Cyclic Surgery Theorem [10]. Results concerning the goodness and cohomological dimension of profinite completions also play an important role in the proof of Theorem B.

Our results here, and similar results in [6], are the first to give credence to the possibility, raised in [2] and [27], that Kleinian groups of finite co-volume might be distinguished from each other and from all other 3-manifold groups by their profinite completions, i.e. that they are *profinitely rigid* in the sense of [27]. In contrast, the fundamental groups of 3-manifolds modelled on the geometries Sol and $\mathbb{H}^2 \times \mathbb{R}$ are not profinitely rigid in general; see [14], [19] and Remark 3.7. There are also lattices in higher-dimensional semi-simple Lie groups that are not determined by their profinite completions (see [3]). We refer the reader to [27] §10 for a wider discussion of profinite rigidity for 3-manifold groups, and note that the recent work of Wilton and Zalesskii [35] provides an important advance in this area.

Theorem D prompts the question: *to what extent are free-by-cyclic groups profinitely rigid?* We shall return to this question in a future article (cf. Remark 3.7). Note that one cannot hope to distinguish a free-by-cyclic group from an arbitrary finitely presented, residually finite group by means of its finite quotients without first resolving the deep question of whether finitely generated free groups themselves can be distinguished.

Remark: During the writing of this paper we became aware that M. Boileau and S. Friedl were working on similar problems [6]. Some of their results overlap with ours, but the methods of proof are very different.

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2. Preliminaries concerning profinite groups

In this section we gather the results about profinite groups that we shall need.

2.1. **Basics.** Let Γ be a finitely generated group. We have

$$\widehat{\Gamma} = \varprojlim \Gamma/N$$

where the inverse limit is taken over the normal subgroups of finite index $N \lhd \Gamma$ ordered by reverse inclusion. $\widehat{\Gamma}$ is a compact topological group.

The natural homomorphism $i : \Gamma \to \widehat{\Gamma}$ is injective if and only if Γ is residually finite. The image of i is dense regardless of whether Γ is residually finite or not, so the restriction to Γ of any continuous epimomorphism from $\widehat{\Gamma}$ to a finite group is onto. A deep theorem of Nikolov and Segal [23] implies that if Γ is finitely generated then *every* homomorphism from $\widehat{\Gamma}$ to a finite group is continuous.

For every finite group Q, the restriction $f \mapsto f \circ i$, gives a bijection from the set of epimorphisms $\widehat{\Gamma} \twoheadrightarrow Q$ to the set of epimorphisms $\Gamma \twoheadrightarrow Q$. From consideration of the finite abelian quotients, the following useful lemma is easily deduced.

Lemma 2.1. *Let Γ_1 and Γ_2 be finitely generated groups. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $H_1(\Gamma_1, \mathbb{Z}) \cong H_1(\Gamma_2, \mathbb{Z})$.*

Notation and terminology. Given a subset X of a profinite group G, we write $\overline{X}$ to denote the closure of X in G.

Let Γ be a finitely generated, residually finite group and let Δ be a subgroup. The inclusion $\Delta \hookrightarrow \Gamma$ induces a continuous homomorphism $\widehat{\Delta} \to \widehat{\Gamma}$ whose image is $\overline{\Delta}$. The map $\widehat{\Delta} \to \widehat{\Gamma}$ is injective if and only if $\overline{\Delta} \cong \widehat{\Delta}$; and in this circumstance one says that Γ *induces the full profinite topology* *on* Δ. For finitely generated groups, this is equivalent to the statement that for every subgroup of finite index $I < \Delta$ there is a subgroup of finite index $S < \Gamma$ such that $S \cap \Delta \subset I$.

Note that if $\Delta < \Gamma$ is of finite index, then Γ induces the full profinite topology on Δ.

There are two other situations of interest to us where Γ induces the full profinite topology on a subgroup. First, suppose that Γ is a group and H a subgroup of Γ, then Γ is called H-separable if for every $g \in G \smallsetminus H$, there is a subgroup K of finite index in Γ such that $H \subset K$ but $g \notin K$; equivalently, the intersection of all finite index subgroups in Γ containing H is precisely H. The group Γ is called *LERF* (or *subgroup separable*) if it is H-separable for every finitely-generated subgroup H, or equivalently, if every finitely generated subgroup is a closed subset in the profinite topology.

It is important to note that even if a finitely generated subgroup $H < \Gamma$ is separable, it need not be the case that Γ induces the full profinite topology on H. However, LERF does guarantee this: each subgroup of finite index $H_0 < H$ is finitely generated, hence closed in Γ, so we can write H as a union of cosets $h_1H_0 \cup \cdots \cup h_nH_0$ and find subgroups of finite index $K_i < \Gamma$ such that $H_0 \subset K_i$, but $h_i \notin K_i$; the intersection of the K_i is then a subgroup of finite index in Γ with $K \cap H = H_0$.

The second situation that is important for us is the following.

Lemma 2.2. *If N is finitely generated, then any semidirect product of the form $\Gamma = N \rtimes Q$ induces the full profinite topology on N.*

Proof. The characteristic subgroups $C < N$ of finite index are a fundamental system of neighbourhoods of 1 defining the profinite topology on N, so it is enough to construct a subgroup of finite index $S < \Gamma$ such that $S \cap N = C$. As C is characteristic in N, it is invariant under the action of Q, so the action $\Psi : Q \to \mathrm{Aut}(N)$ implicit in the semidirect product descends to an action $Q \to \mathrm{Aut}(N/C)$, with image Q_C say. Define S to be the kernel of $\Gamma \to (N/C) \rtimes Q_C$. $\square$

2.2. **On the cohomology of profinite groups.** We recall some facts about the cohomology of profinite groups that we shall use. We refer the reader to [28, Chapter 6] and [30] for details.

Let G be a profinite group, M a discrete G-module (i.e. an abelian group M equipped with the discrete topology on which G acts continuously) and let $C^n(G, M)$ be the set of all continuous maps $G^n \to M$. When equipped with the usual coboundary operator $d : C^n(G, M) \to C^{n+1}(G, M)$, this defines a chain complex $C^*(G, M)$ whose cohomology groups $H^q(G; M)$ are the *continuous cohomology groups* of G with coefficients in M.

Now let Γ be a finitely generated group. Following Serre [30], we say that a group Γ is *good* if for all $q \geq 0$ and for every finite Γ-module M, the homomorphism of cohomology groups

$$H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$.

Returning to the general setting again, let G be a profinite group, the *p-cohomological dimension* of G is the least integer n such that for every discrete G-module M and for every $q > n$, the p-primary component of $H^q(G; M)$ is zero. This is denoted by $\mathrm{cd}_p(G)$. The cohomological dimension of G is defined as the supremum of $\mathrm{cd}_p(G)$ over all primes p, and this is denoted by $\mathrm{cd}(G)$.

We also retain the standard notation $\mathrm{cd}(\Gamma)$ for the cohomological dimension (over $\mathbb{Z}$) of a discrete group Γ.

Lemma 2.3. *If the discrete group Γ is good, then $\mathrm{cd}(\widehat{\Gamma}) \leq \mathrm{cd}(\Gamma)$. If, in addition, Γ is the fundamental group of a closed aspherical manifold, then $\mathrm{cd}(\widehat{\Gamma}) = \mathrm{cd}(\Gamma)$.*

Proof. If $\mathrm{cd}(\Gamma) \leq n$ then $H^q(\Gamma, M) = 0$ for every Γ-module M and every $q > n$. By goodness, this vanishing transfers to the profinite setting in the context of finite modules.

If Γ is the fundamental group of a closed aspherical d-manifold M, then $\mathrm{cd}(\Gamma) = d$. And by Poincaré duality, $H^d(\Gamma, \mathbb{F}_2) \neq 0$, whence $H^d(\widehat{\Gamma}, \mathbb{F}_2) \neq 0$. $\square$

We also recall the following result (see [30, Chapter 1 §3.3]).

Proposition 2.4. *Let p be a prime, let G be a profinite group, and H a closed subgroup of G. Then $\mathrm{cd}_p(H) \leq \mathrm{cd}_p(G)$.*

Fundamental groups of closed surfaces are good and (excluding $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$) have cohomological dimension 2. Free groups are good and have cohomological dimension 1.

Corollary 2.5. *If F is a non-abelian free group and Σ is the fundamental group of a closed surface other than $\mathbb{S}^2$, then $\widehat{F}$ does not contain $\widehat{\Sigma}$.*

If one has a short exact sequence $1 \to N \to \Gamma \to Q \to 1$ where both N and Q are good, and $H^q(N; M)$ is finite for every finite Γ-module M then Γ is good (see [30, Chapter 1 §2.6]). Hence we have:

Lemma 2.6. *If $F \neq 1$ is finitely generated and free, then every group of the form $F \rtimes \mathbb{Z}$ is good and $\mathrm{cd}(\widehat{F \rtimes \mathbb{Z}}) = 2$.*

2.3. **L^2-betti numbers.** The standard reference for this material is Lück's treatise [22]. L^2-betti numbers of groups are defined analytically, but Lück's Approximation Theorem provides a purely algebraic surrogate definition of the first L^2-betti number for finitely presented, residually finite groups: if (N_i) is a sequence of finite-index normal subgroups in Γ with $\cap_i N_i = 1$, then $b_1^{(2)}(\Gamma) = \lim_i b_1(N_i)/[\Gamma : N_i]$. The fundamental group of every compact 3-manifold X is residually finite (see for example [4]), so we can use the surrogate definition $b_1^{(2)}(X) = b_1^{(2)}(\pi_1(X))$.

We shall require the following result [7, Theorem 3.2].

Proposition 2.7. *Let Λ and Γ be finitely presented residually finite groups and suppose that Λ is a dense subgroup of $\widehat{\Gamma}$. Then $b_1^{(2)}(\Gamma) \leq b_1^{(2)}(\Lambda)$.*

In particular, if $\widehat{\Lambda} \cong \widehat{\Gamma}$ then $b_1^{(2)}(\Gamma) = b_1^{(2)}(\Lambda)$.

The following special case of a result of Gaboriau [15] will also be useful.

Proposition 2.8. *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups. If Γ is finitely presented, N finitely generated and Q is infinite, then $b_1^{(2)}(\Gamma) = 0$.*

3. Groups of the form $F \rtimes \mathbb{Z}$

In this section we explore the extent to which free-by-cyclic groups are determined by their profinite completions, paying particular attention to the figure-eight knot. We remind the reader of our notation: $b_1(G)$ is the rank of $H^1(G, \mathbb{Z})$ and F_r is a free group of rank r.

Let Γ be a finitely generated group. If $b_1(\Gamma) = 1$, then there is a unique normal subgroup $N \subset \Gamma$ such that $\Gamma/N \cong \mathbb{Z}$ and the kernel of the induced map $\widehat{\Gamma} \to \widehat{\mathbb{Z}}$ is the closure $\overline{N}$ of N. We saw in Lemma 2.2 that if N is finitely generated then $\overline{N} \cong \widehat{N}$, in which case we have a short exact sequence:

$$(\dagger) \qquad 1 \to \widehat{N} \to \widehat{\Gamma} \to \widehat{\mathbb{Z}} \to 1.$$

Lemma 3.1. *Let $\Gamma_1 = N_1 \rtimes \mathbb{Z}$ and $\Gamma_2 = N_2 \rtimes \mathbb{Z}$, with N_1 and N_2 finitely generated. If $b_1(\Gamma_1) = 1$ and $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $\widehat{N}_1 \cong \widehat{N}_2$.*

Proof. For finitely generated groups, $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ implies $H_1(\Gamma_1, \mathbb{Z}) \cong H_1(\Gamma_2, \mathbb{Z})$. Thus $b_1(\Gamma_2) = 1$. We fix an identification $\widehat{\Gamma}_1 = \widehat{\Gamma}_2$. Then N_1 and N_2 are dense in the kernel of the canonical epimorphism $\widehat{\Gamma}_1 \to \widehat{\mathbb{Z}}$ described in ($\dagger$). And by Lemma 2.2, this kernel is isomorphic to $\widehat{N}_i$ for $i = 1, 2$. $\square$

3.1. **The groups $\Gamma_\phi = F_r \rtimes_\phi \mathbb{Z}$.** The action of $\mathrm{Aut}(F_2)$ on $H_1(F_2, \mathbb{Z})$ gives an epimorphim $\mathrm{Aut}(F_2) \to \mathrm{GL}(2, \mathbb{Z})$ whose kernel is the group of inner automorphisms. The isomorphism type of Γ_ϕ depends only on the conjugacy class of the image of ϕ in $\mathrm{Out}(F_2) = \mathrm{GL}(2, \mathbb{Z})$, and we often regard ϕ as an element of $\mathrm{GL}(2, \mathbb{Z})$. We remind the reader that finite-order elements of $\mathrm{GL}(2, \mathbb{Z})$ are termed *elliptic*, infinite-order elements with an eigenvalue of absolute value bigger than 1 are *hyperbolic*, and the other infinite-order elements are *parabolic*. Note that an element of infinite order in $\mathrm{SL}(2, \mathbb{Z})$ is hyperbolic if and only if $|\mathrm{tr}\,(\phi)| > 2$.

Every automorphism ϕ of F_2 sends $[a, b]$ to a conjugate of $[a, b]^{\pm 1}$ and hence can be realised as an automorphism of the once-punctured torus (orientation-preserving or reversing according to whether $\det \phi = \pm 1$).

3.2. **Features distinguished by $\widehat{\Gamma}_\phi$.**

Proposition 3.2. *Suppose $\widehat{\Gamma}_{\phi_1} \cong \widehat{\Gamma}_{\phi_2}$. If ϕ_1 is hyperbolic then ϕ_2 is hyperbolic and has the same eigenvalues as ϕ_1 (equivalently, $\det \phi_1 = \det \phi_2$ and $\mathrm{tr}\, \phi_1 = \mathrm{tr}\, \phi_2$).*

We break the proof of this proposition into three lemmas. First, for the assertion about $\det \phi$, we use the short exact sequence (†).

Lemma 3.3. *If $b_1(\Gamma_{\phi_1}) = 1$ and $\widehat{\Gamma}_{\phi_1} \cong \widehat{\Gamma}_{\phi_2}$, then $\det \phi_1 = \det \phi_2$.*

Proof. We use the shorthand $\Gamma_i = \Gamma_{\phi_i}$, fix an identification $\widehat{\Gamma}_1 = \widehat{\Gamma}_2$, and write N for the kernel of the canonical map $\widehat{\Gamma}_i \to \widehat{\mathbb{Z}}$. The canonical map $\Gamma_1 \to \Gamma_1/[\Gamma_1, \Gamma_1]\Gamma_1^3$ induces an epimorphism $N \to (\mathbb{Z}/3)^2$, whose restriction to Γ_2 has kernel $[\Gamma_2, \Gamma_2]\Gamma_2^3$. The action of $\widehat{\Gamma}_i$ by conjugation on N induces a map $\widehat{\mathbb{Z}} \to \mathrm{GL}(2, \mathbb{Z}/3)$, with cyclic image generated by the reduction of ϕ_i for $i = 1, 2$. Thus $\det \phi_1 = \det \phi_2$ is determined by whether the image of $\widehat{\mathbb{Z}}$ lies in $\mathrm{SL}(2, \mathbb{Z}/3)$ or not. □

Lemma 3.4. *If $b_1(\Gamma_{\phi_1}) = 1$ and $\widehat{\Gamma}_{\phi_1} \cong \widehat{\Gamma}_{\phi_2}$, then $\widehat{\Gamma}_{\phi_1^r} \cong \widehat{\Gamma}_{\phi_2^r}$ for all $r \neq 0$.*

Proof. $\Gamma_{\phi^r} \cong \Gamma_{\phi^{-r}}$, so assume $r > 0$. We again consider the canonical map from $\widehat{\Gamma}_1 = \widehat{\Gamma}_2$ to $\widehat{\mathbb{Z}}$. Let N_r be the kernel of the composition of this map and $\widehat{\mathbb{Z}} \to \mathbb{Z}/r$. Then N_r is the closure of $F_2 \rtimes_{\phi_i} r\mathbb{Z} < \Gamma_{\phi_i}$ for $i = 1, 2$. Thus $\widehat{\Gamma}_{\phi_1^r} \cong N_r \cong \widehat{\Gamma}_{\phi_2^r}$. (Here we have used the fact that the profinite topology on any finitely generated group induces the full profinite topology on any subgroup of finite index.) □

Lemma 3.5.

(1) $b_1(\Gamma_\phi) = 1$ *if and only if* $1 + \det \phi \neq \mathrm{tr}\, \phi$.
(2) *If* $b_1(\Gamma_\phi) = 1$ *then* $H_1(\Gamma_\phi, \mathbb{Z}) \cong \mathbb{Z} \oplus T$, *where* $|T| = |1 + \det \phi - \mathrm{tr}\, \phi|$.

Proof. By choosing a representative $\phi_* \in \mathrm{Aut}(F_2)$, we get a presentation for Γ_ϕ,

$$\langle a, b, t \mid tat^{-1} = \phi_*(a),\ tbt^{-1} = \phi_*(b) \rangle.$$

By abelianising, we see that $H_1(\Gamma_\phi, \mathbb{Z})$ is the direct sum of $\mathbb{Z}$ (generated by the image of t) and $\mathbb{Z}^2$ modulo the image of $\phi - I$. The image of $\phi - I$ has finite index if and only if $\det(\phi - I)$ is non-zero, and a trivial calculation shows that this determinant is $1 - \mathrm{tr}\, \phi + \det \phi$. If the index is finite, then the quotient has order $|\det(\phi - I)|$. □

Corollary 3.6. *$b_1(\Gamma_{\phi^r}) = 1$ for all $r \neq 0$ if and only if ϕ is hyperbolic.*

Proof. If ϕ is hyperbolic, then ϕ^r is hyperbolic for all $r \neq 0$; in particular $\mathrm{tr}\, \phi^r \neq 0$, and $|\mathrm{tr}\, \phi^r| > 2$ if $\det \phi^r = 1$. Thus the result follows from Lemma 3.5(1), in the hyperbolic case. If ϕ is elliptic, then $\phi^r = I$ for some r, whence $b_1(\Gamma_{\phi^r}) = 3$. If ϕ is parabolic, then there exists $r > 0$ so that ϕ^r had determinant 1 and is conjugate in $\mathrm{GL}(2, \mathbb{Z})$ to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with $n > 0$. In particular, $b_1(\Gamma_{\phi^r}) = 2$. □

Proof of Proposition 3.2. Lemma 3.4 and Corollary 3.6 imply that Γ_{ϕ_2} is hyperbolic, Lemma 3.3 shows that $\det\phi_1 = \det\phi_2$, and then Lemma 3.5(2) implies that $\operatorname{tr}\phi_1 = \operatorname{tr}\phi_2$ (since $H_1(\Gamma_{\phi_1}, \mathbb{Z}) \cong H_1(\Gamma_{\phi_2}, \mathbb{Z})$). $\square$

Remark 3.7. So far, the only quotients that we have used to explore the profinite completion of Γ_ϕ are the abelian quotients of $F_2 \rtimes_{\phi^r} \mathbb{Z}$. Since these all factor through $A_{\phi^r} = \mathbb{Z}^2 \rtimes_{\phi^r} \mathbb{Z}$, we were actually extracting information about ϕ from the groups $\widehat{A}_{\phi^r}$. Up to isomorphism, such a completion $\widehat{A}_\psi$ is determined by the *local* conjugacy class of ψ, meaning $\widehat{A}_\psi \cong \widehat{A}_{\psi'}$ if and only if the image of ψ is conjugate to the image of ψ' in $\mathrm{GL}(2, \mathbb{Z}/m)$ for all integers $m > 1$. Local conjugacy does not imply that ψ and ψ' are conjugate in $\mathrm{GL}(2, \mathbb{Z})$: Stebe [32] proved that

$$\begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}$$

have this property and Funar [14] described infinitely many such pairs. The corresponding torus bundles over the circle provide pairs of Sol manifolds whose fundamental groups are not isomorphic but have the same profinite completion [14]. In a forthcoming article with Henry Wilton we shall prove that, in contrast, *punctured* torus bundles over the circle are determined up to homeomorphism by the profinite completions of their fundamental groups.

Remark 3.8. The conclusion of Proposition 3.2 could also be phrased as saying that ϕ_1 and ϕ_2 lie in the same conjugacy class in $\mathrm{GL}(2, \mathbb{Q})$. When intersected with $\mathrm{GL}(2, \mathbb{Z})$, this will break into a finite number of conjugacy classes; how many can be determined using class field theory [33].

3.3. **The figure-eight knot.** For small examples, one can calculate the conjugacy classes in $\mathrm{GL}(2, \mathbb{Z})$ with a given trace and determinant by hand. For example, up to conjugacy in $\mathrm{GL}(2, \mathbb{Z})$ the only matrix with trace 3 and determinant 1 is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This calculation yields the following consequence of Proposition 3.2 and Lemma 3.1.

We retain the notation established in the introduction: $\mathcal{K}$ is the figure-eight knot and $\Pi = \pi_1(\mathbb{S}^3 \smallsetminus \mathcal{K})$.

Proposition 3.9. *Let $\Gamma = F \rtimes_\phi \mathbb{Z}$, where F finitely generated and free. If $\widehat{\Gamma} \cong \widehat{\Pi}$, then F has rank two, $\Gamma \cong \Pi$, and ϕ is conjugate to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in any identification of $\mathrm{Out}(F)$ with $\mathrm{GL}(2, \mathbb{Z})$.*

3.4. **Uniqueness for the trefoil knot and the Gieseking manifold.** From Lemma 3.5 we know that the only monodromies $\phi \in \mathrm{GL}(2, \mathbb{Z})$ for which $H_1(\Gamma_\phi, \mathbb{Z}) \cong \mathbb{Z}$ are those for which $(\operatorname{tr}(\phi), \det\phi)$ is one of $(1, 1)$, $(1, -1)$, $(3, 1)$. We have already discussed how the last possibility determines the figure-eight knot. Each of the other possibilities also determines a unique conjugacy class in $\mathrm{GL}(2, \mathbb{Z})$, represented by

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. The punctured-torus bundle with the first monodromy is the complement of the trefoil knot, and the second monodromy produces the Gieseking manifold, which is the unique non-orientable 3-manifold whose oriented double cover is the complement of the figure-eight knot.

Note that the first matrix is elliptic while the second is hyperbolic. The following proposition records two consequences of this discussion. For item (2) we need to appeal to Lemma 3.1.

Proposition 3.10. (1) *The only groups of the form $\Lambda = F_2 \rtimes \mathbb{Z}$ with $H_1(\Lambda, \mathbb{Z}) = \mathbb{Z}$ are the fundamental groups of (i) the figure-eight knot complement, (ii) the trefoil knot, and (iii) the Gieseking manifold.*

(2) *Let Λ be one of these three groups and let F be a free group. If $\Gamma = F \rtimes \mathbb{Z}$ and $\widehat{\Gamma} \cong \widehat{\Lambda}$, then $\Gamma \cong \Lambda$.*

Example 3.11. Instead of appealing to the general considerations in this section, one can distinguish the profinite completions of the groups for the figure-eight knot and the trefoil knot directly. Indeed, setting $y^2 = 1$ in the standard presentation

$$\Pi = \langle x, y \mid yxy^{-1}xy = xyx^{-1}yx\rangle$$

one sees that Π maps onto the dihedral group D_{10}. But T, the fundamental group of the trefoil knot, cannot map onto D_{10}, because D_{10} is centreless and has no elements of order 3, whereas $T = \langle a, b \mid a^2 = b^3\rangle$ is a central extension of $\mathbb{Z}/2 * \mathbb{Z}/3$.

3.5. **Finite Ambiguity.** We do not know if all free-by-cyclic groups can be distinguished from one another by their profinite completions (cf. Remark 3.7), but the analysis in the previous subsection enables us to show that the ambiguity in the case when the free group has rank 2 is at worst finite.

Proposition 3.12. *For every $\phi \in \mathrm{GL}(2, \mathbb{Z})$, there exist only finitely many conjugacy classes $[\psi]$ in $\mathrm{GL}(2, \mathbb{Z})$ such that $\widehat{\Gamma}_\phi \cong \widehat{\Gamma}_\psi$. Moreover, all such ψ are of the same type, hyperbolic, parabolic, or elliptic.*

Proof. If ϕ is hyperbolic, this follows from Proposition 3.2. If ϕ is parabolic then either it has trace 2 and $b_1(\Gamma_\phi) = 2$, or trace -2 in which case $b_1(\Gamma_\phi) = 1$. In the former case, ϕ is conjugate in $\mathrm{GL}(2, \mathbb{Z})$ to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with $n > 0$, and $H_1(\Gamma_\phi, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/n$, so n is determined by $H_1(\Gamma_\phi, \mathbb{Z})$, hence by $\widehat{\Gamma}_\phi$. In the case where the trace is -2, we have $b_1(\Gamma_\phi) = 1$ and Lemma 3.4 reduces us to the previous case.

For the elliptic case, since the possible orders are 2, 3, 4 or 6 and there are only finitely many conjugacy classes in $\mathrm{GL}(2, \mathbb{Z})$ of elements of such orders, we are reduced to distinguishing elliptics from non-elliptics by means of $\widehat{\Gamma}_\phi$. If ϕ is elliptic and $b_1(\Gamma_\phi) = 1$, then Lemma 3.4 completes the proof, because for non-elliptics $b_1(\Gamma_{\phi^r})$ is never greater than 2, whereas for elliptics it becomes 3. The only other possibility, up to conjugacy, is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which cannot be distinguished from parabolics such as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by means of $H_1(\Gamma_\phi, \mathbb{Z})$. However, these can be distinguished on passage to subgroups of finite index, since for $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_1(\Gamma_{\phi^2}) = 3$, while for parabolics $b_1(\Gamma_{\phi^2}) = 2$. □

3.6. **Passing to finite-sheeted covers.** It is easy to see that a subgroup of finite index in a free-by-cyclic group is itself free-by-cyclic. Finite extensions of $F_r \rtimes \mathbb{Z}$, even if they are torsion-free, will not be free-by-cyclic in general, but in the setting of the following lemma one can prove something in this direction. This lemma is useful when one wants to prove that a manifold is a punctured-torus bundle by studying finite-sheeted coverings of the manifold.

Lemma 3.13. *Let Λ be a torsion-free group and let $\Gamma < \Lambda$ be a subgroup of index d. Suppose that $b_1(\Lambda) = b_1(\Gamma) = 1$. If $\Gamma \cong F_2 \rtimes_\phi \mathbb{Z}$, then $\Lambda \cong F_2 \rtimes_\psi \mathbb{Z}$, with $\phi = \psi^d$.*

Proof. As $b_1(\Gamma) = b_1(\Lambda) = 1$, there are unique normal subgroups $I \subset \Gamma$ and $J \subset \Lambda$ such that $\Gamma/I \cong \Lambda/J \cong \mathbb{Z}$. The image of Γ has finite index in Λ/J, so $I = J \cap \Gamma$. But $I \cong F_2$, and F_2 is not a proper subgroup of finite index in any torsion-free group (because such a group would be free with euler characteristic dividing $-\chi(F_2) = 1$). Thus $I = J = F_2$, and the image of Γ/I in $\Lambda/J \cong \mathbb{Z}$ has index d. □

4. Profinite completions of 3-manifold groups

We shall make use of several results about the profinite completions of 3-manifold groups. We summarize these in the following theorem, and quote their origins in the proof. Note that in what follows by the statement *a compact 3-manifold M contains an incompressible Klein bottle* $\mathbf{K}$, we shall mean that the induced map $\pi_1(\mathbf{K}) \hookrightarrow \pi_1(M)$ is injective.

Theorem 4.1. *Let X be a compact connected 3-manifold. Then:*

(1) *If $\widehat{\pi_1(X)} \cong \widehat{\pi_1(M)}$, then $H_1(X,\mathbb{Z}) \cong H_1(M,\mathbb{Z})$.*
(2) *$\pi_1(X)$ is good.*
(3) *$b_1^{(2)}(X) = 0$ if and only if $\pi_1(X)$ is virtually infinite cyclic or X is aspherical (hence irreducible) and ∂X consists of a (possibly empty) disjoint union of tori and Klein bottles.*
(4) *If X is closed and Γ is either free-by-cyclic or else the fundamental group of a non-compact finite volume hyperbolic 3-manifold, then $\widehat{\pi_1(X)}$ and $\widehat{\Gamma}$ are not isomorphic.*
(5) *Suppose that X is a Seifert fibred space and that Γ is either the fundamental group of a non-compact finite volume hyperbolic 3-manifold or else of the form $F_r \rtimes_\phi \mathbb{Z}$ where $r \geq 2$ and $[\phi]$ has infinite order in* $\mathrm{Out}(F_r)$. *Then $\widehat{\pi_1(X)}$ and $\widehat{\Gamma}$ are not isomorphic.*

Proof. (1) is a special case of Lemma 2.1. Using [1] and [36], goodness of compact 3-manifold groups follows from [8] (see also [27] Theorem 7.3 and [4] §6G.24).

Part (3) follows from the calculations of Lott and Lück [21][Theorem 0.1]. Their theorem is stated only for orientable manifolds but this is not a serious problem because, by Lück approximation (which we used to define $b_1^{(2)}(X)$), if X is a non-orientable compact 3-manifold with infinite fundamental group and $Y \to X$ is its orientable double cover, then $b_1^{(2)}(Y) = 2\, b_1^{(2)}(X)$.

To prove (4) we follow the proof of [27] Theorem 8.3. Item (3) and Proposition 2.8 tell us that $b_1^{(2)}(\Gamma) = 0$, so by Proposition 2.7 we have $b_1^{(2)}(X) = 0$. Moreover, every subgroup of finite index in Γ has non-cyclic finite quotients, so the opposite implication in (3) tells us that X is aspherical. We know that $\pi_1(X)$ is good (item (2)), so since X is closed, $\mathrm{cd}(\widehat{\pi_1 N}) = 3$ by Lemma 2.3. But Γ is also good (by (2) or Lemma 2.6), and $\mathrm{cd}(\widehat{\Gamma}) = 2$, by Lemma 2.3.

To prove the last part, we argue as follows. First, since every subgroup of finite index in Γ has infinitely many finite quotients that are not solvable, we can quickly eliminate all possibilities for X apart from those with geometric structure modelled on $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{SL}}_2$. In these two remaining cases, the projection of X onto its base orbifold gives a short exact sequence of profinite groups

$$1 \to \overline{Z} \to \widehat{\pi_1 X} \to \widehat{\Lambda} \to 1,$$

where Z is the infinite cyclic centre of $\pi_1 X$ and Λ is a non-elementary discrete group of isometries of $\mathbb{H}^2$. (In fact, following the discussion in §2, we know that $\overline{Z} = \widehat{Z}$, because $\pi_1(X)$ is LERF, but we do not need this.) Note that $\overline{Z}$ is central in $\widehat{\pi_1(X)}$, because if $[z,g] \neq 1$ for some $z \in \overline{Z}$ and $g \in \widehat{\pi_1 X}$, then for some finite quotient $q : \widehat{\pi_1(X)} \to Q$ we would have $[q(z), q(g)] \neq 1$, which would contradict the centrality of Z, since $q(\overline{Z}) = q(Z)$ and $q(\widehat{\pi_1(X)}) = q(\pi_1(X))$.

It is easy to see that the fundamental group of a finite volume hyperbolic 3-manifold has trivial centre. A trivial calculation shows that $F_r \rtimes_\phi \mathbb{Z}$ also has trivial centre, except when a power of ϕ is an inner automorphism. (In the exceptional case, if t is the generator of the $\mathbb{Z}$ factor and ϕ^r is conjugation by $u \in F_r$, then $t^{-r}u$ is central.)

Since Γ has trivial centre, its image under any isomorphism $\widehat{\Gamma} \to \widehat{\pi_1(X)}$ would intersect $\overline{Z}$ trivially, and hence project to a dense subgroup of $\widehat{\Lambda}$. But using Lück approximation, it is easy to see that $b_1^{(2)}(\Lambda) \neq 0$ since Λ is either virtually free of rank at least 2 or virtually a surface group of genus at least 2. And in the light of Theorem 4.1(3) and Proposition 2.8, this would contradict Proposition 2.7. $\square$

Remark 4.2. In the setting of Theorem 4.1(5), when Γ is the fundamental group of a non-compact finite volume hyperbolic 3-manifold, one can use Proposition 6.6 of [35] together with [36] to prove the stronger statement that $\widehat{\Gamma}$ has trivial centre.

Corollary 4.3. *Let X be a compact orientable 3-manifold. If $\widehat{\pi_1(X)} \cong \widehat{\Gamma}$, where $\Gamma = F_r \rtimes \mathbb{Z}$, then X is irreducible and its boundary is a union of t incompressible tori where $1 \le t \le b_1(\Gamma)$.*

Proof. Theorem 4.1 (3), (4), (5) tells us that X is irreducible and has non-empty toral boundary. If one of the boundary tori were compressible, then by irreducibilty X would be a solid torus, which it is not since Γ has non-abelian finite quotients and $\widehat{\Gamma} = \widehat{\pi_1(X)}$.

The upper bound on the number of tori comes from the well-known "half-lives, half-dies" phenomenon described in the following standard consequence of Poincare-Lefschetz duality. □

Proposition 4.4. *Let M be a compact orientable 3-manifold with non-empty boundary. The rank of the image of*

$$H_1(\partial M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$$

is $b_1(\partial M)/2$.

It is more awkward to state the analogue of Corollary 4.3 for non-orientable manifolds M. One way around this is to note that an index-2 subgroup of $F_r \rtimes_\phi \mathbb{Z}$ is either $F_r \rtimes_{\phi^2} \mathbb{Z}$ or else is of the form $F_{2r-1} \rtimes \mathbb{Z}$, so the orientable double cover X is of the form described in Corollary 4.3 and each boundary torus in X either covers a Klein bottle in ∂X or a torus (1-to-1 or 2-to-1).

5. The Proof of Theorems B and C

The results in this section form the technical heart of the paper. In the statement of Theorem B, we assumed that $b_1(M) = 1$, so the following theorem applies in that setting. Indeed, in the light of Corollary 4.3 and the comment that follows Proposition 4.4, Theorem 5.1 completes the proof of Theorem B.

Theorem 5.1. *Let M be a compact, irreducible, 3-manifold whose boundary is a non-empty union of incompressible tori and Klein bottles. Suppose that there is an isomorphism $\widehat{\pi_1(M)} \to \widehat{F \rtimes \mathbb{Z}}$ such that $\pi_1 M$ has cyclic image under the composition*

$$\pi_1 M \to \widehat{\pi_1 M} \to \widehat{F \rtimes \mathbb{Z}} \to \widehat{\mathbb{Z}},$$

where F is finitely generated and free, and $\widehat{F \rtimes \mathbb{Z}} \to \widehat{\mathbb{Z}}$ is induced by the obvious surjection $F \rtimes \mathbb{Z} \to \mathbb{Z}$. Then, M fibres over the circle with compact fibre (equivalently, $\pi_1(M)$ is of the form $F_r \rtimes \mathbb{Z}$).

Remark 5.2. Given an arbitrary isomorphism $\widehat{\pi_1(M)} \to \widehat{F \rtimes \mathbb{Z}}$, the image of $\pi_1 M$ in $\widehat{\mathbb{Z}}$ will be a finitely generated, dense, free abelian group, but one has to contend with the fact that it might not be cyclic. In Theorem B we overcame this by requiring $b_1(M) = 1$. Boileau and Friedl [6] avoid the same problem by restricting attention to isomorphisms $\widehat{\Gamma}_1 \to \widehat{\Gamma}_2$ that induce an isomorphism on $H^1(\Gamma_i, \mathbb{Z})$. In Theorem 5.1 we remove the difficulty directly with the hypothesis on $\pi_1(M) \to \widehat{\mathbb{Z}}$.

Theorem 5.1 is proved by applying the following result to the map from $\pi_1 M$ to $\widehat{\mathbb{Z}}$. In more detail, in Theorem 5.1 we assume that we have an isomorphism $\widehat{\pi_1(M)} \to \widehat{F \rtimes \mathbb{Z}}$, which we compose with $\widehat{F \rtimes \mathbb{Z}} \to \widehat{\mathbb{Z}}$ to obtain an epimorphism of profinite groups $\Phi : \widehat{\pi_1 M} \to \widehat{\mathbb{Z}}$. Let f denote the restriction of Φ to $\pi_1 M$. The kernel of Φ is isomorphic to the free profinite group $\widehat{F}$; in particular it has cohomological dimension 1. From Proposition 2.4 it follows that the closure of ker f also has cohomological dimension 1. And we have assumed that the image of f is isomorphic to $\mathbb{Z}$, so Theorem 5.3 implies that ker f is finitely generated and free.

Theorem 5.3. *Let M be a compact, irreducible, 3-manifold with non-empty boundary which is a non-empty union of incompressible tori and Klein bottles. Let $f : \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then, either* ker f *is finitely generated and free (and M is fibred) or else the closure of* ker f *in $\widehat{\pi_1 M}$ has cohomological dimension at least* 2.

Proof. In Theorem 3 of [13], Freedman and Freedman prove that if M is a compact 3-manifold with each boundary component either a torus or a Klein bottle and $f : \pi_1(M) \to \mathbb{Z}$ has a non-trivial restriction to each boundary component, then either (1) the infinite cyclic covering $M^f \to M$ corresponding to f is homeomorphic to a compact surface times $\mathbb{R}$, or (2) the cover M^f contains a closed 2-sided embedded incompressible surface S.

In Theorem 5.3, we are assuming that the boundary components are incompressible, so if the restriction of f to some component T were trivial, then $\pi_1(T)$ would lie in the kernel of f. Let $A \cong \mathbb{Z}^2$ be a subgroup of finite index in $\pi_1(T)$. From [17], we know that $\pi_1(M)$ induces the full profinite topology on A. Therefore, by Lemma 2.3, the closure of ker f in $\widehat{\pi_1(M)}$ has cohomological dimension at least 2, as required. Thus we may assume that f satisfies the hypotheses of Theorem 3 of [13]. If alternative (1) of that theorem holds, then the kernel of f is a finitely generated free group and we are done. So we assume that alternative (2) holds and consider the closed embedded 2-sided incompressible surface S.

Since S is compact, it does not intersect its translates by suitably high powers of the basic deck transformation of M^f, so we can factor out by such a power to obtain a finite-cyclic covering $N \to M$ in which S embedded. The proof will be complete if we can prove that $\pi_1(N)$ (equivalently $\pi_1(M)$) induces the full profinite topology on $\pi_1(S)$ (since, by construction, $\pi_1(S)$ is contained in ker f).

It will be convenient in what follows to pass to a finite cover of N and S (which we continue to call N and S) so that both are orientable and S continues to be embedded. Standard separability properties ensure that if $\pi_1(M)$ induces the full profinite topology on $\pi_1(S)$, then this will get promoted to the original surface. Henceforth we assume N and S are orientable.

In more topological language, what we must prove is the following.

Proposition 5.4. *If $p : S_1 \to S$ is the finite-sheeted covering corresponding to an arbitrary finite-index subgroup of $\pi_1(S)$, then there is a finite-sheeted covering $q : N_1 \to N$ so that p is the restriction of q to a connected component of $q^{-1}(S)$.*

This will be proved in the next section, thus completing the proof of Theorem 5.3. □

5.1. **Surface separability, controlled Dehn filling and aspiral surfaces.** What we need in Proposition 5.4 is close to a recent theorem of Przytycki and Wise in [26] (see also [25] and [24]). They prove that for every closed incompressible surface S embedded in a compact 3-manifold M, the image of $\pi_1(S)$ is closed in the profinite topology of $\pi_1(M)$. But we need more (recall §2.1): all finite-index subgroups of $\pi_1(S)$ have to be closed in $\pi_1 M$ as well. In order to prove their results, Przytycki and Wise developed a technology for merging finite covers of the blocks in the JSJ decomposition of the manifold [24], [26], [25]. Liu [20] and his coauthors [11] refined this technology to establish further results, and Liu's refinement in [20] will serve us well here. But since his criterion applies only to closed manifolds, we have to perform Dehn filling on N to get us into this situation.

We introduce some notation. Assume that N has t boundary components all of which are incompressible tori. For a choice of slopes $r_1, \dots, r_t$, denote by $N(r_1, \dots r_t)$ the manifold obtained by performing r_i-Dehn filling on the torus T_i for $i = 1, \dots, t$.

Lemma 5.5. *Let N and $p : S_1 \to S$ be as above. Then:*

(1) *There exist infinitely many collections of slopes $(r_1, \dots r_t)$ so that the Dehn fillings $N(r_1, \dots, r_t)$ of N are irreducible and the image of S in $N(r_1, \dots, r_t)$ remains incompressible (and embedded).*
(2) *There exists a finite cover $q : N' \to N(r_1, \dots, r_t)$ and an embedding $\iota : S_1 \to N'$ such that $q \circ \iota = p$.*

Proof of Proposition 5.4. Since $S \subset N(r_1, \dots, r_t)$ lies in the complement of the filling core curves of the filled tori, $S_1 \subset N'$ lies in the complement of the preimage of these curves, so by deleting them we get the desired finite cover $N_1 \to N$. □

Proof of Lemma 5.5: For part (1), incompressibility can be deduced from [10] Theorems 2.4.2 and 2.4.3, as we shall now explain.

One possibility in [10] is that N is homeomorphic to $T^2 \times I$. But we can exclude this possibility because our N contains a closed incompressible surface that is not boundary parallel whereas $T^2 \times I$ contains no such surface

We now proceed to apply [10] one torus at a time. To ensure that S remains incompressible upon r_1-Dehn filling on T_1, we simply arrange by [10] to choose (from infinitely many possibilities) a slope r_1 that has large distance (i.e. > 2) from any slope for which filling compresses S, as well as from any slope that co-bounds an annulus with some essential simple closed curve on S. Repeating this for each torus T_i in turn ensures that S remains incompressible in $N(r_1, \ldots, r_t)$.

To ensure that $N(r_1, \ldots, r_t)$ is irreducible, we make the additional stipulation (possible by [16]) that for $i = 1, \ldots, t-1$ each r_i-Dehn filling avoids the three possible slopes that result in a reducible manifold upon Dehn filling the torus T_i. (Note that [16] is stated only for manifolds with a single torus boundary component but the proof and result still apply to our context.) Now, for the final torus T_t, we simply choose r_t-Dehn filling so as to avoid the finite number of boundary slopes, and hence we avoid any essential embedded planar surface that could give rise to a reducing sphere in $N(r_1, \ldots, r_t)$.

For the proof of (2), we need to recall some of Liu's terminology [20]. Theorem 1.1 of [20] includes the statement:

A surface Σ is aspiral in the almost fibre part if and only if S is virtually essentially embedded.

Since S is embedded and incompressible in $N(r)$, it is aspiral. Assertion (2) of the lemma is equivalent to the statement that S_1 is virtually essentially embedded, so what we must argue is that S_1 is aspiral.

Let M be a closed orientable 3-manifold and Σ a closed essential surface (of genus ≥ 1) *immersed* into M. The JSJ decomposition of M induces a canonical decomposition of Σ. Following [20], the *almost fibre part* of Σ, denoted $\Phi(\Sigma)$ is the union of all its horizonal subsurfaces in the Seifert manifold pieces, together with the geometrically infinite (i.e. virtual fibre) subsurfaces in the hyperbolic pieces. Note that, by definition, for any finite covering $\Sigma_1 \to \Sigma$, the preimage of the almost fibre part $\Phi(\Sigma)$ is $\Phi(\Sigma_1)$.

To define aspiral we recall Liu's construction of the *spirality character* [20]. Given a principal $\mathbb{Q}^*$-bundle P over $\Phi(\Sigma)$, the spirality character of P, denoted $s(P)$, is the element of $H^1(\Phi(S), \mathbb{Q}^*)$ constructed as follows. For any closed loop $\alpha : [0,1] \to \Phi(\Sigma)$ based at $x_0 \in \Phi(\Sigma)$, each choice of lift for x_0 determines a lift $\tilde{\alpha} : [0,1] \to P$, and a ratio $\tilde{\alpha}(1)/\tilde{\alpha}(0) \in \mathbb{Q}^*$. This ratio depends only on $[\alpha] \in H_1(\Phi(\Sigma), \mathbb{Z})$ and so induces a homomorphism $s(P) : H_1(\Phi(\Sigma), \mathbb{Z}) \to \mathbb{Q}^*$; i.e. an element of $H^1(\Phi(\Sigma), \mathbb{Q}^*)$. The principal bundle P is called *aspiral* if $s(P)$ takes only the values ± 1 for all $[\alpha] \in H_1(\Phi(\Sigma), \mathbb{Z})$. The proof of Theorem 1.1 of [20] involves the construction of a particular principal $\mathbb{Q}^*$-bundle $\mathcal{H}$ (see Proposition 4.1 of [20]). Then, Σ is defined to be *aspiral in the almost fibre part* if the bundle $\mathcal{H}$ is aspiral.

For brevity, set $s = s(\mathcal{H})$. To see that aspirality passes to finite covers of Σ, one argues as follows. First, as above, we note that for a finite covering $\Sigma_1 \to \Sigma$, the almost fibre part $\Phi(\Sigma_1)$ is the inverse image of $\Phi(\Sigma)$. Next we observe that the naturality established in the proof of [20, Proposition 4.1] (more specifically Formula 4.5 and the paragraph following it), shows that s pulls back to the spirality character s_1 of $\Phi(\Sigma_1)$. Indeed the bundle $\mathcal{H}$ in the definition of s, which is independent of the data chosen in the construction, builds-in the data for all finite covers of Σ; see §4.2 of [20].

Thus, in our setting, S_1 is aspiral in $\Phi(S_1)$ as required. $\square$

For emphasis, we repeat the key fact that we have extracted from [26] and [20].

Theorem 5.6. *If M is a closed orientable 3-manifold and $S \subset M$ is a closed embedded incompressible surface, then $\pi_1(M)$ induces the full profinite topology on $\pi_1(S)$.*

6. Profinite rigidity for the figure-knot complement

As before, let $\mathcal{K}$ be the figure-eight knot and let $\Pi = \pi_1(\mathbb{S}^3 \smallsetminus \mathcal{K})$.

Theorem 6.1. *Let M be a compact connected 3-manifold. If $\widehat{\pi_1(M)} = \widehat{\Pi}$, then M is homeomorphic to $\mathbb{S}^3 \smallsetminus \mathcal{K}$.*

Proof. Theorem B tells us that M is a fibred manifold with fundamental group of the form $F_r \rtimes \mathbb{Z}$, and Proposition 3.9 then tells us that $r = 2$ and $\pi_1(M) \cong \Pi = F_2 \rtimes_\phi \mathbb{Z}$. The fundamental group of the fibre in M is the kernel of the unique map $\pi_1(M) \to \mathbb{Z}$, so it is free of rank 2. The only compact surface with fundamental group F_2 that supports a hyperbolic automorphism is the punctured torus, so M is the once-punctured torus bundle with holonomy ϕ, i.e. the complement of the figure-eight knot. □

Similarly, using Proposition 3.10 one can show that the complement of the trefoil knot and the Gieseking manifold are determined up to homeomorphism by their fundamental groups.

7. Related Results

We have built our narrative around the complement of the figure-eight knot, but our arguments also establish results for larger classes of manifolds. In this section we record some of these results. We write M_ϕ to denote the punctured torus bundle with monodromy $\phi \in \mathrm{GL}(2, \mathbb{Z})$.

Theorem 7.1. *Let M_ϕ be a once-punctured torus bundle that is hyperbolic. Then there are at most finitely many compact orientable 3-manifolds $M_1, M_2, \dots, M_n$ so that $\widehat{\pi_1(M_i)} \cong \widehat{\pi_1(M_\phi)}$ and these are all hyperbolic once-punctured torus bundles M_{ϕ_i} with $\mathrm{tr}\,(\phi) = \mathrm{tr}\,(\phi_i)$ and $\det \phi = \det \phi_i$*

Proof. This follows the proof of Theorem 6.1, except that now we have the finite ambiguity provided by Proposition 3.2 rather than uniqueness. □

Theorem 7.2. *Let $K_1, K_2 \subset S^3$ be knots whose complements are hyperbolic with $\mathbb{S}^3 \smallsetminus K_i = \mathbb{H}^3/\Gamma_i$. Assume that $\mathbb{S}^3 \smallsetminus K_1$ fibres over the circle with fibre a surface of genus g. If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, then $\mathbb{S}^3 \smallsetminus K_2$ is fibred with fibre a surface of genus g.*

Proof. From Theorem B we know that $\mathbb{S}^3 \smallsetminus K_2$ is fibred and from Lemma 3.1 we know that the fibres have the same euler characteristic. That the genus is the same now follows, since in both cases the fibre is a Seifert surface and so the surface has a single boundary component, hence the same genus. □

Theorem 7.3. *Let M and N be compact orientable 3-manifolds with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. Suppose that ∂M is an incompressible torus and that $b_1(M) = 1$. If M fibres over the circle with fibre a surface of euler characteristic χ, then N fibres over the circle with fibre a surface of euler characteristic χ, and ∂N is a torus.*

Proof. This follows from Theorem B, Lemma 3.1 and Corollary 4.3. □

Theorem 4.1(5) shows that the fundamental group of a torus knot complement cannot have the same profinite completion as that of a Kleinian group of finite co-volume.

Theorem 7.4. *Let $K_1, K_2 \subset S^3$ be knots whose complements are geometric. Let $\Gamma_i = \pi_1(S^3 \smallsetminus K_i)$ and assume that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Then:*

- *K_1 is hyperbolic if and only if K_2 is hyperbolic.*
- *K_1 is fibred if and only if K_2 is fibred.*

The following is simply a summary of earlier results.

Theorem D. *Let N and M be compact connected 3-manifolds. Assume $b_1(N) = 1$ and $\pi_1(N) \cong F_r \rtimes_\phi \mathbb{Z}$. If $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$, then $b_1(M) = 1$ and*

1. *$\pi_1 M \cong F_r \rtimes_\psi \mathbb{Z}$, for some $\psi \in \mathrm{Out}(F_r)$,*
2. *M is fibred, and*
3. *∂M is either a torus or a Klein bottle.*

Moreover, when $r = 2$,

4. *for each N there are only finitely many possibilities for M,*
5. *M is hyperbolic if and only if N is hyperbolic, and*
6. *if M is hyperbolic, it is a once-punctured torus bundle.*

References

[1] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves, and J. Manning, Documenta Math. **18** (2013), 1045–1087.

[2] I. Agol, *Virtual properties of 3-manifolds*, to appear in the Proceedings of the 2014 I.C.M.

[3] M. Aka, *Arithmetic groups with isomorphic finite quotients*, J. Algebra **352**, (2012), 322–340.

[4] M. Aschenbrenner, S. Friedl and H. Wilton, *3-Manifold groups*, arXiv:1205.0202v3.

[5] G. Baumslag, *Residually finite groups with the same finite images*, Compositio Math. **29** (1974), 249–252.

[6] M. Boileau and S. Friedl, *The profinite completion of 3-manifold groups, fiberedness and the Thurston norm*, preprint.

[7] M. R. Bridson, M. Conder and A. W. Reid, *Determining Fuchsian groups by their finite quotients*, to appear Israel J. Math.

[8] W. Cavendish, *Finite-sheeted covering spaces and solenoids over 3-manifolds*, Ph.D Thesis, Princeton University (2012).

[9] M. Culler, W. Jaco and H. Rubinstein, *Incompressible surfaces in once-punctured torus bundles*, Proc. L. M. S. **45** (1982), 385–419.

[10] M. Culler, C. McA. Gordon, J. Luecke and P. B. Shalen, *Dehn surgery on knots*, Annals of Math. **125** (1987), 237–300.

[11] P. Derbez, Y. Liu, S.-C. Wang, *Chern–Simons theory, surface separability, and volumes of 3-manifolds*, arXiv:1401.0073, J. Topology (to appear).

[12] J.D. Dixon, E.W. Formanek, J.C. Poland, and L. Ribes, *Profinite completions and isomorphic finite quotients*, J. Pure Appl. Algebra **23** (1982), 227–231.

[13] B. Freedman and M. H. Freedman, *Kneser-Haken finiteness for bounded 3-manifolds, locally free groups and cyclic covers*, Topology **37** (1998), 133-147.

[14] L. Funar, *Torus bundles not distinguished by TQFT invariants*, Geometry and Topology **17** (2013), 2289–2344.

[15] D. Gaboriau, *Invariants ℓ^2 de relations d'eq́uivalence et de groupes*, Publ. Math. I.H.E.S. **95** (2002), 93–150.

[16] C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385–409.

[17] E. Hamilton, *Abelian subgroup separability of Haken 3-manifolds and closed hyperbolic n-orbifolds*, Proc. L. M. S. **83** (2001), 626–646.

[18] J. Hempel, *3-Manifolds*, Annals of Math. Studies **86**, Princeton Univ. Press (1976).

[19] J. Hempel, *Some 3-manifold groups with the same finite quotients*, arXiv:1409.3509.

[20] Y. Liu, *A characterization of virtually embedded subsurfaces in 3-manifolds*, arXiv:1406.4674, to appear Trans. Amer. Math. Soc.

[21] J. Lott and W. Lück, *L^2-topological invariants of 3-manifolds*, Invent. Math. **120** (1995), 15–60.

[22] W. Lück, *L^2-invariants of regular coverings of compact manifolds and CW-complexes*, Handbook of Geometric Topology, North-Holland, Amsterdam (2002), 735–817.

[23] N. Nikolov and D. Segal, *On finitely generated profinite groups. I. Strong completeness and uniform bounds*, Annals of Math. **165** (2007), 171–238.

[24] P. Przytycki and D. T. Wise, *Graph manifolds with boundary are virtually special*, J. Topology **7** (2014), 419–435.

[25] P. Przytycki and D. T. Wise, *Mixed 3-manifolds are virtually special*, arXiv:1205.6742.

[26] P. Przytycki and D. T. Wise, *Separability of embedded surfaces in 3-manifolds*, Compositio Math. **150** (2014), 1623–1630.

[27] A. W. Reid, *Profinite properties of discrete groups*, to appear, Proceedings of Groups St Andrews 2013.

[28] L. Ribes and P. A. Zalesskii, *Profinite Groups*, Ergeb. der Math. **40**, Springer-Verlag (2000).

[29] R. Riley, *A quadratic parabolic group*, Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.

[30] J-P. Serre, *Galois Cohomology*, Springer-Verlag (1997).

[31] J.R. Stallings, *On fibering certain 3-manifolds*, Topology of 3- manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs NJ, 1962, pp. 95–100.
[32] P. Stebe, *Conjugacy separability of groups of integer matrices*, Proc. Amer. Math. Soc. **32** (1972), 1–7.
[33] O.Taussky-Todd, *Introduction into connections between algebraic number theory and integral matrices*, Appendix to: H.Cohn, A classical invitation to algebraic numbers and class fields, Universitext, Springer-Verlag, 1978.
[34] W. P. Thurston, *The Geometry and Topology of 3-Manifolds*, Princeton University mimeographed notes, (1979).
[35] H. Wilton and P. A. Zalesskii, *Distinguishing geometries using finite quotients*, arXiv:1411.5212.
[36] D. T. Wise, *The structure of groups with a quasi-convex hierarchy*, preprint 2012.

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