

COVERING SPACES OF ARITHMETIC 3-ORBIFOLDS

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1. INTRODUCTION

This paper investigates properties of finite sheeted covering spaces of arithmetic hyperbolic 3-orbifolds (see §2). The main motivation is a central unresolved question in the theory of closed hyperbolic 3-manifolds; namely whether a closed hyperbolic 3-manifold is virtually Haken. Various strengthenings of this have also been widely studied. Of specific interest to us is the question of whether the fundamental group of a given hyperbolic 3-manifold M is *large*; that is to say, some finite index subgroup of $\pi_1(M)$ admits a surjective homomorphism onto a non-abelian free group. This implies that M is virtually Haken, and indeed that M has *infinite virtual first Betti number* (see §2.4 for a definition). Of course, a weaker formulation is to only ask whether the virtual first Betti number of a closed hyperbolic 3-manifold M is positive. This has been verified in many cases, see [8] for some recent work on this. However, in general, passage from positive virtual first Betti number to infinite virtual first Betti number is difficult, as is passage from infinite virtual first Betti number to large. This paper makes some progress on the latter in certain settings.

The background for our work is recent work of the first author (see for example [16] and [18]). This suggests that the questions addressed above for hyperbolic 3-manifolds that are commensurable with an orbifold may be more amenable to study. One of the aims of this paper is to address these questions for arithmetic hyperbolic 3-manifolds and in particular, provide further evidence for a positive solution to the largeness question. It is already known that many arithmetic hyperbolic 3-manifolds have infinite virtual first Betti number, mainly through the application of the theory of automorphic forms (see [4], [13], [19], [26] and [34]). For convenience, we shall refer to these collectively as *arithmetic methods*. Some geometric methods are also known using the existence of a totally geodesic surface, and largeness is known there (see [20] and [24]). However the question of largeness remains unknown in general for arithmetic hyperbolic 3-manifolds, in particular even for those for which positive virtual first Betti number is known by arithmetic methods.

Our main results are the following, the first of which explains why arithmetic manifolds are particularly well-suited to the above questions in the context of orbifolds.

Theorem 1.1. *Let $M = \mathbb{H}^3/\Gamma$ be an arithmetic hyperbolic 3-manifold. Then M is commensurable with an arithmetic hyperbolic 3-orbifold with non-empty singular locus. More precisely, Γ is commensurable with an arithmetic Kleinian group Γ_0 containing an element of order 2.*

In fact, more can be said about elements of order 2 in the commensurability class of Γ .

Theorem 1.2. *Let $M = \mathbb{H}^3/\Gamma$ be as above. Then Γ is commensurable with an arithmetic Kleinian group Γ_0 containing a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Theorem 1.1 is easily seen to be false in the setting of non-arithmetic manifolds. For example, by Margulis's result [32], in the non-arithmetic case there is a unique maximal element in the commensurability class of the group. It is easy to construct examples whereby this maximal element has no non-trivial elements of finite order (see for example [35]).

We apply these results, together with results in §3, that are in the spirit of the [16] and [18] to obtain the following results for arithmetic hyperbolic 3-manifolds.

Theorem 1.3. *Let M be an arithmetic hyperbolic 3-manifold for which the virtual first Betti number is at least 4. Then $\pi_1(M)$ is large.*

In [1] Borel shows that arithmetic manifolds (not necessarily hyperbolic) having a congruence subgroup with positive first Betti number, have infinite virtual first Betti number (see §2.4). Thus we have.

Corollary 1.4. *Let M be an arithmetic hyperbolic 3-manifold for which arithmetic methods apply to produce a cover with positive first Betti number. Then $\pi_1(M)$ is large.*

In particular this applies to all known examples of arithmetic hyperbolic 3-manifolds that have covers with positive first Betti number (we discuss some specific examples of this in §6).

As further evidence for studying orbifolds, and in particular arithmetic ones,

we also show:

Theorem 1.5. *Let M be an arithmetic hyperbolic 3-manifold commensurable with an orbifold $O = \mathbb{H}^3/\Gamma$ such that either;*

- (i) Γ contains A_4 , S_4 or A_5 or;
- (ii) Γ is derived from a quaternion algebra and contains a finite dihedral group.

Then $\pi_1(M)$ is large.

The proof of Theorem 1.3 involves establishing linear growth in mod p homology for some prime p (see §4). Some of our other main results concern this phenomena. For example, we prove the following result.

Theorem 1.6. *Let O be a 3-orbifold (with possibly empty singular locus) commensurable with a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Then O has a tower of finite-sheeted covers $\{O_i\}$ that has linear growth of mod 2 homology.*

This result is also proved in [16], using the Golod-Shafarevich inequality and the theory of p -adic Lie groups. Our proof uses only properties of hyperbolic 3-orbifolds, and as such can be considered “elementary”. A consequence of this is that we can give a new proof of the following result (see §10). This was originally proved by Lubotzky in [23], and again the proof used the Golod-Shafarevich inequality, and the theory of p -adic Lie groups.

Theorem 1.7. *No arithmetic Kleinian group has the congruence subgroup property.*

We also discuss Property (τ) in connection with orbifolds. Property (τ) is an important group-theoretic concept, introduced by Lubotzky and Zimmer [28]. It has many applications to diverse areas of mathematics, including hyperbolic 3-manifold theory (see [14] for more details). It is conjectured that if M is a closed orientable 3-manifold with infinite fundamental group, then $\pi_1(M)$ does not have Property (τ) . As is well-known, having virtually positive first Betti number implies this, but beyond this, little is known by way establishing a group does not have (τ) . Another of our main results for arithmetic Kleinian groups is.

Theorem 1.8. *Suppose that for every compact orientable 3-manifold M with*

infinite fundamental group, $\pi_1(M)$ fails to have Property (τ) . Then any arithmetic Kleinian group is large.

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2. ARITHMETIC HYPERBOLIC 3-ORBIFOLDS

2.1 We begin by recalling some facts about arithmetic Kleinian groups that will be needed (see [31] for further details).

Arithmetic Kleinian groups are obtained as follows. Let k be a number field having exactly one complex place, and B a quaternion algebra over k which ramifies at all real places of k . Let $\rho : B \rightarrow M(2, \mathbb{C})$ be an embedding, \mathcal{O} an order of B , and \mathcal{O}^1 the elements of norm one in \mathcal{O} . Then $P\rho(\mathcal{O}^1) < \mathrm{PSL}(2, \mathbb{C})$ is a finite co-volume Kleinian group, which is co-compact if and only if B is not isomorphic to $M(2, \mathbb{Q}(\sqrt{-d}))$, where d is a square free positive integer. An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ commensurable with a group of the type $P\rho(\mathcal{O}^1)$. We call $Q = \mathbb{H}^3/\Gamma$ *arithmetic* if Γ is arithmetic.

Notation: We shall denote $P\rho(\mathcal{O}^1)$ by $\Gamma_{\mathcal{O}}^1$ and the set of finite places of k that ramify the quaternion algebra B by $\mathrm{Ram}_f(B)$.

An arithmetic Kleinian group Γ is called *derived from a quaternion algebra* if $\Gamma < \Gamma_{\mathcal{O}}^1$. For convenience we state the following result that is deduced from the characterisation theorem for arithmetic Kleinian groups (see [31] Corollary 8.3.5). For a finitely generated group G we denote by $G^{(2)}$ the subgroup of G generated by the squares of elements in G .

Theorem 2.1. *Let Γ be a finite co-volume Kleinian group. Then Γ is arithmetic if and only if the group $\Gamma^{(2)}$ is derived from a quaternion algebra.*

One final fact about arithmetic Kleinian groups that we will make use of is the following. If Γ is derived from a quaternion algebra B defined over k , then

$$A_0\Gamma = \left\{ \sum a_i \gamma_i \ : \ a_i \in k, \gamma_i \in \Gamma \right\},$$

is a quaternion algebra over k (see [31] Chapter 3) and is isomorphic to B (see [31] Chapter 8). In what follows we shall just identify the two.

Remark: For convenience, we have blurred the distinction between an element $a \in \mathrm{PSL}(2, \mathbb{C})$ and a matrix $A \in \mathrm{SL}(2, \mathbb{C})$ that projects to a under the homomorphism $\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$.

2.2 Here we prove Theorems 1.1 and 1.2 (Theorem 1.1 is implicit in [10]).

We begin with a lemma.

Lemma 2.2. *Let Γ be derived from a quaternion algebra B , defined over the number field k . Let R_k denote the ring of integers of k , a and b a pair of non-commuting elements of Γ , and let $\mathcal{O} = R_k[1, a, b, a.b]$. Then \mathcal{O} is an order of B .*

Proof. To show that \mathcal{O} is an order we proceed as follows. First, since a and b do not commute, it is easy to see that $\{1, a, b, ab\}$ spans B over k . Thus $R_k[1, a, b, ab]$ contains a k -basis of B , is finitely generated and contains R_k . Also note that since Γ is derived from a quaternion algebra the elements $a, b, a.b$ and R_k -combinations of these words are integral in the algebra. To complete the proof, it suffices to prove that all products of the basis elements can be expressed as R_k -combinations of the basis elements. This follows from the Cayley-Hamilton theorem as well as some other trace identities that we include below.

$$a + a^{-1} = \mathrm{tr}(a)1,$$

$$a^2 = \mathrm{tr}(a)a - 1,$$

$$a^2b = \mathrm{tr}(a)ab - b,$$

$$aba = -\mathrm{tr}(b)1 + \mathrm{tr}(ab)a + b,$$

$$b^{-1}a^{-1} = \mathrm{tr}(b)a^{-1} - ba^{-1},$$

$$ba + ab = (\mathrm{tr}(ab) - \mathrm{tr}(a)\mathrm{tr}(b))1 + \mathrm{tr}(b)a + \mathrm{tr}(a)b.$$

In particular note that the first identity, with a replaced by ab throughout, implies that $b^{-1}a^{-1} \in \mathcal{O}$ and the last identity then implies that $a^{-1}b^{-1} \in \mathcal{O}$. \square

Remark. The discriminant of the order \mathcal{O} in Lemma 2.2 can be easily computed

and is the ideal $\langle \text{tr}[a, b] - 2 \rangle$.

Define the *normalizer* of \mathcal{O} in B by:

$$N(\mathcal{O}) = \{x \in B^* \mid x\mathcal{O}x^{-1} = \mathcal{O}\}.$$

The image, $\Gamma(\mathcal{O})$ of $N(\mathcal{O})$ in $\text{PGL}(2, \mathbb{C})$ (which is isomorphic to $\text{PSL}(2, \mathbb{C})$), is an arithmetic Kleinian group. To see this we argue as follows.

Note first that, for every $x \in \mathcal{O}^1$, $x\mathcal{O}x^{-1} = \mathcal{O}$ because \mathcal{O} is a ring. Hence $N(\mathcal{O})$ contains \mathcal{O}^1 . Furthermore, any element of $N(\mathcal{O})$ normalizes \mathcal{O}^1 (since conjugation preserves the norm). Therefore $\Gamma(\mathcal{O})$ is a subgroup of the normalizer of $\Gamma_{\mathcal{O}}^1$ in $\text{PGL}(2, \mathbb{C})$. Since $\Gamma_{\mathcal{O}}^1$ has finite co-volume, it is well-known that its normalizer is also discrete and finite co-volume. Hence, $\Gamma(\mathcal{O})$ is discrete. It also has finite co-volume, since it contains $\Gamma_{\mathcal{O}}^1$. We summarize this discussion in the following.

Corollary 2.3. *Let B and \mathcal{O} be as above. Then $\Gamma(\mathcal{O})$ is an arithmetic Kleinian group commensurable with $\Gamma_{\mathcal{O}}^1$.*

Remark. Corollary 2.3 holds more generally. Namely, if \mathcal{O} is any order of a quaternion algebra B (as in §2.1), then $N(\mathcal{O})$ always gives rise to an arithmetic Kleinian group $\Gamma(\mathcal{O})$ (see [31] Chapter 6). We have included the above proof for completeness, and since it is straightforward in this case.

Theorem 1.1 will follow immediately from the next proposition and Theorem 2.1. This will require some notation.

Let a and b be elements of $\text{SL}(2, \mathbb{C})$ without a common fixed point. Then, as noticed by Jørgenson [11], $ab - ba$ is an element of $\text{GL}(2, \mathbb{C})$ which has trace 0 and whose image in $\text{PGL}(2, \mathbb{C})$ is of order two and conjugates a to a^{-1} and b to b^{-1} . Denote this involution by $\tau_{a,b}$.

Proposition 2.4. *Let Γ be derived from a quaternion algebra and $a, b \in \Gamma$ such that $H = \langle a, b \rangle$ is a non-elementary subgroup of Γ . Then $\tau_{a,b}$ is contained in an arithmetic Kleinian group commensurable with Γ .*

Proof. Since Γ is derived from a quaternion algebra, there exists an order \mathcal{D} (as in §2.1) such that $\Gamma < \Gamma_{\mathcal{D}}^1$. Let $\mathcal{O} = R_{k\Gamma}[1, a, b, a.b]$ be as in Lemma 2.2. Note that $\mathcal{O} \subset \mathcal{D}$. By Corollary 2.3, $\Gamma(\mathcal{O})$ is an arithmetic Kleinian group that is

commensurable with $\Gamma_{\mathcal{O}}^1$. This in turn is commensurable with $\Gamma_{\mathcal{D}}^1$, and hence Γ . Finally, the involution $\tau_{a,b} \in \Gamma(\mathcal{O})$. To see this note that $\tau_{a,b}(ab) = a^{-1}b^{-1}$ which is an element of \mathcal{O} by Lemma 2.2. \square

Proof of Theorem 1.2. The extension of the argument to prove Theorem 1.2 is made as follows. Γ will continue to be derived from a quaternion algebra B and we choose a and b loxodromic elements such that their axes, A_a and A_b respectively, are disjoint.

By construction, the involution $\tau_{a,b}$ rotates around the geodesic $\gamma_{a,b}$ that is the common perpendicular between A_a and A_b . We now claim that there is an involution $\tau_{\alpha,\beta}$ that acts by rotating around A_a . To prove the claim, first observe that since $\Gamma(\mathcal{O})$ (as in the proof of Theorem 1.1) has finite co-volume, there is a loxodromic element in $\Gamma(\mathcal{O})$ that has $\gamma_{a,b}$ as an axis. Since Γ is commensurable with $\Gamma(\mathcal{O})$ there is a loxodromic element $\alpha \in \Gamma$ that has $\gamma_{a,b}$ as an axis. Note that since α and $\tau_{a,b}$ share an axis, they commute. In addition, there is a loxodromic element $\beta \in \Gamma$ that has the geodesic $a\gamma_{a,b}$ as an axis. Hence, A_a is the common perpendicular of the axes $\gamma_{a,b}$ and $a\gamma_{a,b}$, and as before we can construct an involution $\tau_{\alpha,\beta}$ that acts as claimed (and commutes with a).

Note that $\tau_{a,b}$ and $\tau_{\alpha,\beta}$ are involutions and commute. Hence, the group $V = \langle \tau_{a,b}, \tau_{\alpha,\beta} \rangle$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It remains to show that V is a subgroup of an arithmetic Kleinian group commensurable with Γ . To see this let \mathcal{L} be the order associated to the group $\langle a, \alpha \rangle$ as in Lemma 2.2. The action of these involutions on a and α is given by:

$$\begin{aligned} \tau_{a,b}a\tau_{a,b} &= a^{-1}, & \tau_{a,b}\alpha\tau_{a,b} &= \alpha, \\ \tau_{\alpha,\beta}a\tau_{\alpha,\beta} &= a, & \tau_{\alpha,\beta}\alpha\tau_{\alpha,\beta} &= \alpha^{-1}. \end{aligned}$$

It follows from this that $V < \Gamma(\mathcal{L})$ and Corollary 2.3 completes the proof. \square

2.3 In this subsection we discuss implications on the Hilbert symbol of the invariant quaternion algebra associated to a Kleinian group of finite co-volume given the presence of A_4 , S_4 or A_5 subgroup and certain dihedral subgroups. In the case of S_4 and A_5 , since both of these contain A_4 , we will restrict consideration to this group.

Definition. Let G be a finite subgroup of an arithmetic Kleinian group. We shall call G *derived from a quaternion algebra* if G is contained in some group $\Gamma_{\mathcal{O}}^1$ as above.

Theorem 2.5 *Suppose that Γ is an arithmetic Kleinian group commensurable with a Kleinian group containing A_4 or a Kleinian group containing a finite dihedral group derived from a quaternion algebra. Let k and B denote the invariant trace-field and quaternion algebra of Γ . Then if $\nu \in \text{Ram}_f B$ and ν divides the rational prime p , then k_ν contains no quadratic extension of \mathbb{Q}_p .*

Proof. Note first that if Γ is commensurable with a group Γ_1 containing A_4 , then since $A_4 = A_4^{(2)}$ it follows that $A_4 < \Gamma_1^{(2)}$ and so any A_4 is derived from a quaternion algebra. Furthermore, A_4 contains a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is the dihedral group of order 4. Thus we can assume that we are in the case that Γ is a Kleinian group derived from a quaternion algebra and contains a dihedral group D_n of order $2n$.

We can assume that Γ is cocompact, otherwise, B is a matrix algebra and is unramified at all places of k . Let $x, y \in \Gamma$ generate the dihedral subgroup, with

$$D_n = \langle x, y \mid x^2 = y^2 = (xy)^n = 1 \rangle.$$

Note that since x and y do not have a common fixed point on the sphere at infinity, it follows that a Hilbert symbol for B can be computed using the basis $\{1, x, y, xy\}$. From [31] Theorem 3.6.1 we deduce that a Hilbert symbol is given by

$$B \cong \left(\frac{-4, 4 \cos^2 2\pi/n - 4}{k} \right) \cong \left(\frac{-1, 4 \cos^2 2\pi/n - 4}{k} \right)$$

We need the following information about the term $\tau_n = 4 \cos^2 2\pi/n - 4$ (cf. [30] Lemma 4.4).

Lemma 2.6. *If n is odd, or even and greater than 4, then τ_n has norm p or is a unit, depending on whether n is a power of a prime p or not.*

In the case $n = 4$, $\tau_n = -4$, and so the Hilbert symbol becomes $\left(\frac{-1, -4}{k} \right) \cong$

$$\left(\frac{-1, -1}{k}\right).$$

Given this, and the lemma, we gain some preliminary control on $\text{Ram}_f B$. For, if $\nu \in \text{Ram}_f B$ then from above we deduce that ν divides 2 or at most one other rational prime p (see [31] Theorem 2.6.6). Furthermore, the order $\mathcal{O} = R_k[1, x, y, xy]$ (recall Lemma 2.2) can be shown to have discriminant $d(\mathcal{O}) = \langle \tau_n \rangle$ (see the Remark following Lemma 2.2). Now, the discriminant of a maximal order of B , which equals the product of finite places ramifying B , divides $d(\mathcal{O}) = \langle \tau_n \rangle$ (see [31] Theorem 6.3.4). Thus it follows that if p is odd, then ν cannot divide 2. Given these remarks we can now argue as follows.

Case 1: Assume n is not a prime power, and so τ_n is a unit. Hence $d(\mathcal{O})$ is the trivial ideal, and so it follows from the discussion above that \mathcal{O} is maximal. Hence B is unramified at all finite places, and the theorem is proved in this case.

Case 2: Assume $n = p^t$ is a prime power, and $n \neq 4$. From the remarks preceding Case 1, it follows that $\text{Ram}_f B = \emptyset$ or consists of a unique place dividing p . In particular if $p \neq 2$, it cannot contain places dividing 2. Assume by way of contradiction, that $\text{Ram}_f B$ contains a place ν such that k_ν contains a quadratic extension ℓ of \mathbb{Q}_p .

Now $\mathbb{Q}(\cos 2\pi/p^t)$ is a subfield of k , and so B can be described as follows.

$$B \cong \left(\frac{-1, \tau_{p^t}}{\mathbb{Q}(\cos 2\pi/p^t)}\right) \otimes_{\mathbb{Q}(\cos 2\pi/p^t)} k.$$

Assume first that $[k : \mathbb{Q}]$ has even degree. Hence the quaternion algebra B is ramified at all real places of k (an even number). Hence if $\text{Ram}_f B$ is non-empty it consists of an even number of finite places. However, recall from above that the order has discriminant $d(\mathcal{O})$ which is either the trivial ideal or a prime ideal of norm p . As this discriminant divides that of a maximal order, and the cardinality of $\text{Ram}_f B$ is even, it follows that the discriminant of a maximal order, and hence B , is the trivial ideal. Hence we are done in this case.

Now assume that $[k : \mathbb{Q}]$ is odd, and so $[\mathbb{Q}(\cos 2\pi/p^t) : \mathbb{Q}]$ is odd. By the theory of ramification of primes in the maximal real subfield of a cyclotomic field, there is a unique $\mathbb{Q}(\cos 2\pi/p^t)$ -prime ω dividing p and this has norm p . Further-

more, since $\mathbb{Q}(\cos 2\pi/p^t)$ is assumed to have odd degree, the theory of ramification in number field extensions [33] implies that $\mathbb{Q}(\cos 2\pi/p^t)_\omega$ also will have odd degree, and so must be disjoint from the field ℓ . Hence k_ν contains a subfield L that is the compositum of $\mathbb{Q}(\cos 2\pi/p^t)$ and ℓ . L has degree 2 over $\mathbb{Q}(\cos 2\pi/p^t)$ and so applying [31] Theorem 2.6.5, L must split the algebra $\left(\frac{-1, \tau_{p^t}}{\mathbb{Q}(\cos 2\pi/p^t)_\omega}\right)$. However, then we have from above,

$$B_\nu = B \otimes_k k_\nu \cong \left(\frac{-1, \tau_{p^t}}{\mathbb{Q}(\cos 2\pi/p^t)_\omega}\right) \otimes_{\mathbb{Q}(\cos 2\pi/p^t)_\omega} L \otimes_L k_\nu \cong M(2, k_\nu)$$

which is a contradiction, since ν ramifies.

Case 3: Finally, we must deal with the case $n = 4$. This is similar to the case above. In this case, if $\nu \in \text{Ram}_f(B)$ then ν is dyadic, and so k_ν is a finite extension of \mathbb{Q}_2 . Suppose there is a quadratic extension ℓ of \mathbb{Q}_2 that is contained in k_ν . Now B is ramified at ν , and so $B_\nu = B \otimes_k k_\nu$ is isomorphic to the unique division algebra over k_ν . As above, it is easy to see that the following tensor products hold:

$$\begin{aligned} B_\nu = B \otimes_k k_\nu &= \left(\frac{-1, -1}{k_\nu}\right) \cong \left(\frac{-1, -1}{\mathbb{Q}_2}\right) \otimes_{\mathbb{Q}_2} k_\nu. \\ &\cong \left(\frac{-1, -1}{\mathbb{Q}_2}\right) \otimes_{\mathbb{Q}_2} \ell \otimes_\ell k_\nu \end{aligned}$$

However, arguing as above, a quadratic extension of the center of a quaternion algebra over a local field (in particular ℓ over \mathbb{Q}_2) splits the unique division algebra over \mathbb{Q}_2 . Hence,

$$\left(\frac{-1, -1}{\mathbb{Q}_2}\right) \otimes_{\mathbb{Q}_2} \ell \cong M(2, \ell).$$

$$B_\nu = B \otimes_k k_\nu \cong M(2, \ell) \otimes_\ell k_\nu \cong M(2, k_\nu),$$

Again, this contradicts the assumption that B_ν is a division algebra. \square

2.4 In this subsection we gather together some notions and results pertaining to congruence covers of arithmetic hyperbolic 3-orbifolds. If Γ is an arithmetic Kleinian group, there is a distinguished class of subgroups in Γ , known as the *congruence subgroups*. These are defined as follows. Notation as in §2.1.

Let \mathcal{O} be a maximal order of B , and let I be any proper 2-sided integral ideal of B contained in \mathcal{O} ; ie I is a complete R_k -lattice in B such that

$$\mathcal{O} = \{x \in B \mid xI \subset I\} = \{x \in B \mid Ix \subset I\}.$$

As noted in [31] Chapter 6.1, any proper 2-sided integral ideal of B contained in \mathcal{O} is an ideal of \mathcal{O} in the usual non-commutative ring sense. In particular \mathcal{O}/I is a finite ring.

Define

$$\mathcal{O}^1(I) = \{\alpha \in \mathcal{O}^1 : \alpha - 1 \in I\}.$$

The corresponding *principal congruence subgroup* of $\Gamma_{\mathcal{O}}^1$ is

$$\Gamma(\mathcal{O}(I)) = \text{P}\rho(\mathcal{O}^1(I)).$$

If Γ is an arithmetic Kleinian group then a subgroup $\Delta < \Gamma$ is a *congruence subgroup* of Γ if it contains some principal congruence subgroup $\Gamma(\mathcal{O}(I))$ as above.

Before stating the result about the first Betti number of congruence subgroups we require, we need some notation.

Notation. If X is a group, space or orbifold, we will denote by $b_1(X)$ the rank of $H_1(X; \mathbb{Z}) \otimes \mathbb{Q}$ and set

$$vb_1(X) = \sup\{b_1(\tilde{X}) : \tilde{X} \text{ is a finite index subgroup or finite cover of } X\}.$$

Theorem 2.7. (Borel [1]) *Suppose Γ is an arithmetic Kleinian group containing a subgroup $\Gamma(\mathcal{O}(I))$ with $b_1(\Gamma(\mathcal{O}(I))) > 0$. Then $vb_1(\Gamma) = \infty$.*

3. THE HOMOLOGY OF 3-ORBIFOLDS

Definition. Let O be a compact orientable 3-orbifold. Let $\text{sing}(O)$ be its singular locus, and let $|O|$ denote its underlying 3-manifold. Let $\text{sing}^0(O)$ and $\text{sing}^-(O)$ denote the components of the singular locus with, respectively, zero and negative Euler characteristic. For any prime p , let $\text{sing}_p(O)$ denote the union of the arcs and circles in $\text{sing}(O)$ with singularity order that is a multiple of p . Let $\text{sing}_p^0(O)$ and $\text{sing}_p^-(O)$ denote those components of $\text{sing}_p(O)$ with zero and negative Euler characteristic.

When O is closed, $\text{sing}(O)$ is a disjoint union of simple closed curves and trivalent graphs, and hence $\text{sing}(O) = \text{sing}^0(O) \cup \text{sing}^-(O)$. However, it need not be the case that $\text{sing}_p(O) = \text{sing}_p^0(O) \cup \text{sing}_p^-(O)$.

Terminology. If p is a prime, let \mathbb{F}_p denote the field of order p . If X is a group, space or orbifold, let $d_p(X)$ be the dimension of $H_1(X; \mathbb{F}_p)$.

The following lower bound on homology will be a crucial tool that we use throughout the rest of this paper.

Proposition 3.1. *Let O be a compact orientable 3-orbifold, and let p be a prime. Then $d_p(O) \geq b_1(\text{sing}_p(O))$.*

Proof. Let M denote the 3-manifold obtained from O by removing an open regular neighbourhood of its singular locus. Let $\{\mu_1, \dots, \mu_r\}$ be a collection of meridian curves, one encircling each arc or circle of the singular locus. Let n_i be the singularity order of the arc or circle that μ_i encircles. Then

$$\pi_1(O) = \pi_1(M) / \langle\langle \mu_1^{n_1}, \dots, \mu_r^{n_r} \rangle\rangle.$$

Hence,

$$H_1(O; \mathbb{F}_p) = H_1(M; \mathbb{F}_p) / \langle\langle \mu_1^{n_1}, \dots, \mu_r^{n_r} \rangle\rangle.$$

Now, when n_i is coprime to p , quotienting $H_1(M; \mathbb{F}_p)$ by $\mu_i^{n_i}$ is the same as quotienting by μ_i . And when n_i is a multiple of p , then quotienting $H_1(M; \mathbb{F}_p)$ by $\mu_i^{n_i}$ has no effect. Thus, if we let M' be the 3-manifold obtained from $|O|$ by removing an open regular neighbourhood of $\text{sing}_p(O)$, then $d_p(O) = d_p(M')$. Now, it is a well known consequence of Poincaré duality that, for the compact orientable 3-manifold M' , $d_p(M') \geq \frac{1}{2}d_p(\partial M') \geq b_1(\text{sing}_p(O))$, as required. \square

4. LINEAR GROWTH OF HOMOLOGY

Definition. Let X be a group, space or orbifold and let p be a prime. Then a collection $\{X_i\}$ of finite index subgroups or finite-sheeted covers of X with index or degree $[X : X_i]$ is said to have *linear growth of mod p homology* if

$$\inf_i d_p(X_i) / [X : X_i] > 0.$$

In this section we prove Theorem 1.6, which we restate below for convenience.

Theorem 4.1. *Let O be a 3-orbifold (with possibly empty singular locus) commensurable with a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Then O has a tower of finite-sheeted covers $\{O_i\}$ that has linear growth of mod 2 homology.*

This is a consequence of the following more general theorem (Theorem 1.1 in [16]).

Theorem 4.2. *Let O be a compact orientable 3-orbifold with non-empty singular locus and a finite-volume hyperbolic structure. Let p be a prime that divides the order of an element of $\pi_1(O)$. Then O has a tower of finite-sheeted covers $\{O_i\}$ that has linear growth of mod p homology.*

However, the proof of Theorem 4.2 required some results about p -adic analytic groups and it also used the Golod-Shafarevich inequality. In this section, we will provide a much simpler proof of the weaker Theorem 4.1.

Note first that we can assume that O itself is closed, orientable and hyperbolic and contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. For, we know that O is commensurable with some such orbifold O' . Let O'' be a common cover of O and O' . We may assume that O'' is a regular cover of O' . Suppose that we could prove Theorem 4.1 for O' , providing a sequence of covers $\{O_i\}$ with linear growth of mod 2 homology. Then the covering spaces of O' corresponding to $\pi_1(O'') \cap \pi_1(O_i)$ also have linear growth of mod 2 homology, by the following elementary result, which appears as Lemma 5.3 in [16].

Lemma 4.3. *Let $\{G_i\}$ be a sequence of finite index subgroups of a finitely generated group G , and let H be a finite index subnormal subgroup of G . If $\{G_i\}$ has linear growth of mod p homology for some prime p , then $\{G_i \cap H\}$ does also.*

So, let us suppose that O is closed, orientable and hyperbolic and contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. We will prove Theorem 4.1 by finding a tower of finite covers O_i such that

$$\inf_i b_1(\text{sing}_2^-(O_i))/[O : O_i] > 0.$$

By Proposition 3.1, $d_2(O_i)$ is at least $b_1(\text{sing}_2(O_i))$, which is, of course, at least $b_1(\text{sing}_2^-(O_i))$. Thus, $\{O_i\}$ will indeed have linear growth of mod 2 homology.

The first step is to find a finite cover \tilde{O} of O such that $\text{sing}_2^-(\tilde{O})$ is non-empty.

Proposition 4.4. *Let O be a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Then O is finitely covered by a 3-orbifold \tilde{O} such that every arc and circle of $\text{sing}(\tilde{O})$ has order 2, and which contains at least one singular vertex. In particular, $\text{sing}_2^-(\tilde{O})$ is non-empty.*

Proof. Since O is hyperbolic, Selberg's lemma implies that it has a finite-sheeted regular cover that is a manifold M . Let \tilde{O} be the cover of O corresponding to the subgroup $\pi_1(M)(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ of $\pi_1(O)$. Then M regularly covers \tilde{O} with covering group $\pi_1(\tilde{O})/\pi_1(M) = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})/(\pi_1(M) \cap (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, any arc or circle of the singular locus of \tilde{O} has order 2. Hence, $\text{sing}_2(\tilde{O}) = \text{sing}(\tilde{O})$. Since O is closed, $\text{sing}(O)$ consists of simple closed curves and trivalent graphs. Now, \tilde{O} is hyperbolic and so is obtained as the quotient of \mathbb{H}^3 by the action of $\pi_1(\tilde{O})$. The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subgroup of $\pi_1(\tilde{O})$ is realised by a finite subgroup of $\text{Isom}(\mathbb{H}^3)$, which must have a common fixed point in \mathbb{H}^3 . The image of this point in \tilde{O} is a singular vertex. Hence, some component of $\text{sing}_2(\tilde{O})$ therefore has negative Euler characteristic. \square

The next step is to pass to a finite cover O_1 such that $b_1(\text{sing}_2^-(O_1))$ is arbitrarily large. Note that, for any finite cover O_1 of \tilde{O} , $\text{sing}(O_1) = \text{sing}_2(O_1)$. Thus, $\text{sing}_2(O_1)$ consists of trivalent graphs and simple closed curves. So,

$$b_1(\text{sing}_2^-(O_1)) \geq |V(\text{sing}_2(O_1))|/2 + 1,$$

where $V(\text{sing}_2(O_1))$ is the vertices of the singular set. Thus, we will establish a lower bound on $b_1(\text{sing}_2^-(O_1))$ by finding a lower bound on the number of singular vertices of O_1 .

Theorem 4.5. *Let O be a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Then, for any integer N , O has a finite-sheeted cover O_1 such that each arc and circle of $\text{sing}(O_1)$ has order 2, and which contains at least N singular vertices. Hence, $b_1(\text{sing}_2^-(O_1)) \geq N/2 + 1$.*

The following is a key step in the proof of this theorem, and is of independent interest.

Theorem 4.6. *Let O be a compact orientable hyperbolic 3-orbifold (with possibly empty singular locus), and let n be a positive integer. Then for infinitely many n -tuples of distinct primes (p_1, \dots, p_n) , $\pi_1(O)$ admits a surjective homomorphism*

ϕ onto $\prod_{i=1}^n \mathrm{PSL}(2, p_i)$. Furthermore, if $\pi_i: \prod_{i=1}^n \mathrm{PSL}(2, p_i) \rightarrow \mathrm{PSL}(2, p_i)$ is projection onto the i^{th} factor, then we may ensure that $\ker(\pi_i \phi)$ is torsion-free, for each i .

Proof. Let $O = \mathbb{H}^3/\Gamma$. It is shown in [22] that for infinitely many rational primes p there are (reduction) homomorphisms $\phi_p: \Gamma \rightarrow \mathrm{PSL}(2, p)$. It is well-known that by avoiding a finite set of primes we can assume that the kernels are torsion-free (see Lemma 6.5.6 of [31] for example). Also, by definition of these homomorphisms, for all non-trivial elements $g \in \Gamma$, $\phi_p(g) \neq 1$ for all but a finite number of primes. Let J be the set of rational primes p given by the above construction. It also follows from the argument in [22] that, for any finite index subgroup of Γ , the restriction of ϕ_p to that subgroup is a surjection onto $\mathrm{PSL}(2, p)$, for all but finitely many primes p in J .

The homomorphism ϕ required by the proposition will be

$$(\phi_{p_1} \times \dots \times \phi_{p_n}): \Gamma \rightarrow \mathrm{PSL}(2, p_1) \times \dots \times \mathrm{PSL}(2, p_n)$$

for suitable primes p_1, \dots, p_n in J .

We will prove, by induction on n , that infinitely many n -tuples (p_1, \dots, p_n) of distinct primes in J can be found so that the homomorphism ϕ is surjective. For $n = 1$, this is simply [22]. For the inductive step, suppose that infinitely many n -tuples (p_1, \dots, p_n) have been found with this property. Let us focus on just one (p_1, \dots, p_n) . Our aim is to extend this to an $(n+1)$ -tuple $(p_1, \dots, p_n, p_{n+1})$ where again, the homomorphism is surjective. Let K be the kernel of $\phi_{p_1} \times \dots \times \phi_{p_n}$. This is a finite index subgroup of Γ . Hence, by the above properties, there is some $p_{n+1} \in J$ such that $\phi_{p_{n+1}}(K) = \mathrm{PSL}(2, p_{n+1})$. We claim that $\phi_1 \times \dots \times \phi_{p_{n+1}}$ is a surjection onto $\prod_{i=1}^{n+1} \mathrm{PSL}(2, p_i)$. Consider an $(n+1)$ -tuple (A_1, \dots, A_{n+1}) in the target. By the inductive hypothesis, there is some element $g \in \Gamma$ such that $(\phi_{p_1} \times \dots \times \phi_{p_n}(g)) = (A_1, \dots, A_n)$. Then $\phi_{p_{n+1}}(g)$ is some element X in $\mathrm{PSL}(2, p_{n+1})$. Since the restriction of $\phi_{p_{n+1}}$ to K is surjective, there is some $h \in K$ such that $\phi_{p_{n+1}}(h) = X^{-1}A_{n+1}$. Then $\phi_{p_1} \times \dots \times \phi_{p_{n+1}}$ maps gh onto (A_1, \dots, A_{n+1}) , as required. \square

Proof of Theorem 4.5. Let $O = \mathbb{H}^3/\Gamma$. By Proposition 4.4, we may assume that the order of each arc and circle of $\mathrm{sing}(O)$ is 2, and that $\mathrm{sing}(O)$ contains at least one vertex. Let $\phi: \Gamma \rightarrow \prod_{i=1}^n \mathrm{PSL}(2, p_i)$ be the homomorphism from Theorem 4.6,

let H be the image of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ under ϕ , and let (A_1, \dots, A_n) and (B_1, \dots, B_n) be the images under ϕ of the generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $\ker(\pi_i \phi)$ is torsion-free, A_i , B_i and $A_i B_i$ are non-trivial for each i . Let $O_1 = \mathbb{H}^3/\Gamma_1$ be the covering space of O corresponding to the subgroup $\phi^{-1}(H) = \Gamma_1$, and let M be the covering space corresponding to the kernel of ϕ . Since the kernel of ϕ is torsion-free, M is a manifold. Now, M is a regular cover of O_1 , with covering group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence, each arc and circle of $\text{sing}(O_1)$ has order 2. Since $\pi_1(O_1)$ contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, O_1 contains at least one $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ vertex v . In fact, we will show that it contains at least 4^{n-1} vertices. By elementary covering space theory, the group of covering transformations of the cover $O_1 \rightarrow O$ equals $N(H)/H$, where $N(H)$ is the normaliser of H in $\prod_{i=1}^n \text{PSL}(2, p_i)$. We claim that this group has order at least 4^{n-1} . To prove the claim, note that $(I, I, \dots, I, A_i, I, \dots, I)$ and $(I, I, \dots, I, B_i, I, \dots, I)$ both commute with (A_1, \dots, A_n) and (B_1, \dots, B_n) . In particular, they lie in $N(H)$. The group these elements generate has order 2^{2n} . Hence, $N(H)/H$ has order at least 4^{n-1} . No covering transformation can fix v , because the local group of each singular point of $\text{sing}(O)$ does not contain $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as a proper subgroup. Hence the orbit of v under the group of covering transformations has order at least 4^{n-1} . In particular, O_1 contains at least this many vertices. Since n was an arbitrary positive integer, the theorem is proved. \square

Lemma 4.7. *Let X be a finite trivalent graph with V vertices. Then X contains a simple closed curve with at most $2 \log_2((V + 2)/3) + 2$ edges.*

Proof. We may assume that X is connected. Give X the path metric where each edge has length 1. For any vertex v , let $R_1(v)$ be the minimal radius of a ball centred at v that is not a tree. Fix a vertex v where $R_1(v)$ has minimal value, and set $R = \lceil R_1(v) \rceil$. Then the ball of radius R around v contains a simple closed curve of length at most $2R$. We claim that

$$R \leq \log_2((V + 2)/3) + 1.$$

For any non-negative integer r , let $B(r)$ be the ball of radius r around v . So, $B(R - 1)$ is a tree. The number of vertices in this tree is equal to $3(2^{R-1} - 1) + 1$. This is a lower bound for V . So,

$$3(2^{R-1} - 1) + 1 \leq V$$

and therefore

$$R \leq \log_2((V + 2)/3) + 1.$$

□

Proof of Theorem 4.1. By Theorem 4.5, O has a finite cover O_1 such that each arc and circle of $\text{sing}(O_1)$ has order 2 and which contains at least 50 vertices. Starting with O_1 , we will construct a tower of finite covers $\{O_i\}$. Let n_i be the number of vertices of O_i . We will ensure that the following inequality holds for each i :

$$n_{i+1} \geq 2n_i - 4(\log_2((n_i + 2)/3) + 1) \quad (*)$$

Suppose that $H_1(|O_i|; \mathbb{Z}/2\mathbb{Z})$ is non-trivial. Then $|O_i|$ has a 2-fold cover $|O_{i+1}|$, with underlying orbifold O_{i+1} . Clearly, the number of vertices is doubled, and so $(*)$ holds. So, suppose that $|O_i|$ is a mod 2 homology 3-sphere. Using Lemma 4.7, pick a simple closed curve C in $\text{sing}_2(O_i)$ with length at most $2 \log_2((n_i + 2)/3) + 2$. This bounds a compact embedded surface S . By a small isotopy, we may assume that S intersects $\text{sing}_2(O_i)$ in ∂S and in a finite number of points in the interior of S . Let O_{i+1} be the 2-fold cover of O_i dual to S . Then each vertex in $\text{sing}_2(O_i) - C$ has inverse image equal to 2 vertices in O_{i+1} . Thus, $(*)$ holds.

We claim that when n_i is a sequence satisfying $(*)$ and where $n_1 \geq 50$, then

$$\inf_i n_i/2^i > 0.$$

To prove this, we will establish the following inequality, by induction on i :

$$n_i \geq 2^i \left(1 + \frac{24}{i}\right).$$

This holds for n_1 by our hypothesis that $n_1 \geq 50$. To prove the inductive step, note that

$$\begin{aligned} n_{i+1} &\geq 2n_i - 4(\log_2((n_i + 2)/3) + 1) \\ &\geq 2n_i - 4 \log_2 n_i \\ &\geq 2^{i+1} \left(1 + \frac{24}{i}\right) - 4 \left(i + \log_2 \left(1 + \frac{24}{i}\right)\right) \\ &\geq 2^{i+1} \left(1 + \frac{24}{i}\right) - 4(i + 5) \\ &\geq 2^{i+1} \left(1 + \frac{24}{i+1}\right). \end{aligned}$$

The second inequality holds because $n_i \geq 4$. The third is true because $2x - 4 \log_2 x$ is an increasing function of x when $x > 2/\log 2$. The final inequality holds because

$$\frac{24}{i} - \frac{i+5}{2^{i-1}} \geq \frac{24}{i+1} \Leftrightarrow \frac{24}{i(i+1)} \geq \frac{i+5}{2^{i-1}},$$

which certainly holds for all integers $i \geq 1$. So,

$$\frac{n_i}{2^i} \geq \left(1 + \frac{24}{i}\right)$$

which has positive infimum. Thus, $\{O_i\}$ has linear growth of mod 2 homology. \square

5. LARGENESS CRITERIA

The main result of this section is a largeness criterion for certain hyperbolic 3-orbifolds. The next theorem is the starting point for §6.

Theorem 5.1. *Let O be a 3-orbifold (with possibly empty singular locus) commensurable with a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Suppose that $vb_1(O) \geq 4$. Then $\pi_1(O)$ is large.*

The principal ingredient in the proof of this theorem is the following result (Theorem 1.2 of [15]).

Theorem 5.2. *Let G be a finitely presented group, and suppose that, for each natural number i , there is a triple $H_i \geq J_i \geq K_i$ of finite index normal subgroups of G such that*

- (i) H_i/J_i is abelian for all i ;
- (ii) $\lim_{i \rightarrow \infty} ((\log[H_i : J_i])/[G : H_i]) = \infty$;
- (iii) $\limsup_i (d(J_i/K_i)/[G : J_i]) > 0$.

Then K_i admits a surjective homomorphism onto a free non-abelian group, for infinitely many i .

Here, $d(\)$ is the minimal number of generators of a group.

We will need the following corollary.

Corollary 5.3. *Let G be a finitely presented group, and let $\phi: G \rightarrow \mathbb{Z}$ be a surjective homomorphism. Let G_i be $\phi^{-1}(i\mathbb{Z})$. Suppose that, for some prime p , $\{G_i\}$ has linear growth of mod p homology. Then G is large.*

Proof. Set $H_i = G$, set $J_i = G_i$ and let $K_i = [G_i, G_i]G_i^p$. Then it is trivial to check that the conditions of Theorem 5.2 hold. \square

A key hypothesis in Corollary 5.3 is linear growth of mod p homology. The following gives a situation where this is guaranteed to hold.

Proposition 5.4. *Let O be a compact orientable 3-orbifold, and let C be a component of $\text{sing}_p^0(O)$ for some prime p . Let $p_i: |O_i| \rightarrow |O|$ ($i \in \mathbb{N}$) be distinct finite covering spaces of $|O|$ such that the restriction of p_i to each component of $p_i^{-1}(C)$ is a homeomorphism onto C . Let O_i be the corresponding covering spaces of O . Then $\{O_i\}$ has linear growth of mod p homology.*

Proof. By Proposition 3.1, we have

$$d_p(O_i) \geq |\text{sing}_p^0(O_i)| \geq [O_i : O]. \square$$

Combining Corollary 5.3 and Proposition 5.4, we have the following.

Theorem 5.5. *Let O be a compact orientable 3-orbifold. Suppose that $\pi_1(O)$ admits a surjective homomorphism ϕ onto \mathbb{Z} , and that some component of $\text{sing}_p^0(O)$ has trivial image under ϕ , for some prime p . Then $\pi_1(O)$ is large.*

Proof. Each meridian of the singular locus of O represents a torsion element of $\pi_1(O)$. Hence its image under ϕ is trivial. Thus, ϕ factors through a homomorphism $\psi: \pi_1(|O|) \rightarrow \mathbb{Z}$. Let $|O_i|$ be the covering space of $|O|$ corresponding to $\psi^{-1}(i\mathbb{Z})$, and let O_i be the corresponding cover of O . Proposition 5.4 gives that $\{O_i\}$ has linear growth of mod p homology. Thus, by Corollary 5.3 (with $G = \pi_1(O)$ and $G_i = \pi_1(O_i)$), $\pi_1(O)$ is large. \square

Remark 5.6. Suppose that the singular locus of O contains a circle component and that $b_1(O) \geq 2$. Then such a homomorphism ϕ as in Theorem 5.5 may always be found.

Proof of Theorem 5.1. By hypothesis, O has a finite cover O' such that $b_1(O') \geq 4$. Let O'' be the hyperbolic orbifold, commensurable with O , containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Now, O' and O'' are commensurable, and hence they have a common cover O''' , say. Since O''' is hyperbolic, it has a manifold cover M . We may assume that M regularly covers O'' . Now, b_1 does not decrease under finite covers, and so $b_1(M) \geq 4$. Since $M \rightarrow O''$ is a regular cover, it

has a group of covering transformations $\pi_1(O'')/\pi_1(M)$. This group acts on the manifold M with quotient O'' . Now, $\pi_1(O'')$ contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence some singular point of O'' has local group that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The group of covering transformations must contain the local group of this vertex. Hence, $\pi_1(O'')/\pi_1(M)$ contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let h_1 and h_2 be the commuting covering transformations of M corresponding to the generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. These are involutions. Let h_3 be the composition of h_1 and h_2 , which also is an involution. For $i = 1, 2$ and 3 , let O_i be the quotient M/h_i . Since h_i has non-empty fixed point set, $\text{sing}(O_i)$ is a non-empty collection of simple closed curves with order 2.

We claim that, for at least one $i \in \{1, 2, 3\}$, $b_1(O_i) \geq 2$. Now, each h_i induces an automorphism h_{i*} of $H_1(M; \mathbb{R})$. Note that h_{i*} is diagonalisable because its minimum polynomial divides $x^2 - 1$ and so splits as a product of distinct linear factors. Thus, $H_1(M; \mathbb{R})$ decomposes as a direct sum of eigenspaces of h_{i*} . It is clear that $b_1(O_i)$ is equal to the dimension of the $+1$ eigenspace of h_{i*} (see Proposition III.10.4 in [2] for example). Suppose that this is at most 1 for $i = 1$ and 2 . Then the dimension of the -1 eigenspace is at least 3 for $i = 1$ and 2 . Hence, the intersection of these eigenspaces has dimension at least 2. This lies in the $+1$ eigenspace for h_{3*} , and so $b_1(O_3) \geq 2$, proving the claim. So, by Theorem 5.5 and Remark 5.6, $\pi_1(O_i)$ is large and hence so is $\pi_1(O)$. \square

6. LARGENESS FOR ARITHMETIC HYPERBOLIC 3-ORBIFOLDS

An easy consequence of Theorem 1.2 and Theorem 5.1 is the following.

Theorem 6.1. *Let M be an arithmetic 3-manifold. Suppose that $vb_1(M) \geq 4$. Then $\pi_1(M)$ is large.*

Proof. When M is closed this is immediate from Theorem 1.2 and Theorem 5.1. When M has non-empty boundary, the result follows from [5], which shows that any non-compact finite-volume hyperbolic 3-manifold has large fundamental group. \square

The hypothesis $vb_1(M) \geq 4$ is known to hold in various circumstances as we now discuss.

The first situation is:

Corollary 6.2. *Let Γ be an arithmetic Kleinian group, with the property that some congruence subgroup has positive b_1 . Then Γ is large.*

Proof. By Borel's theorem (Theorem 2.7) if a congruence subgroup has positive first betti number then we have $vb_1(\Gamma) = \infty$. Hence we can deduce largeness from Theorem 6.1. \square

As a particular case of this we have

Corollary 6.3. *Let Γ be an arithmetic Kleinian group, with the property that $vb_1(\Gamma) > 0$ by arithmetic methods. Then Γ is large.*

We discuss some particular examples of this situation at the end of this section.

Our next result allows us to weaken the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $vb_1 \geq 4$ assumption.

Theorem 6.4. *Let Γ be an arithmetic Kleinian group commensurable with a Kleinian group containing A_4 , S_4 or A_5 or a finite dihedral group that is derived from a quaternion algebra. Then Γ is large.*

To prove this theorem we need to recall the following result of Clozel [4] stated in a way that is convenient for us.

Theorem 6.5. *Let Γ be an arithmetic Kleinian group, with invariant trace-field and quaternion algebra k and B respectively. Assume that for every place $\nu \in \text{Ram}_f(B)$, k_ν contains no quadratic extension of \mathbb{Q}_p where p is a rational prime and $\nu|p$. Then Γ is commensurable with a congruence subgroup with positive first Betti number.*

Proof of Theorem 6.4. We can assume that Γ is cocompact, otherwise the result follows from [5]. By Theorem 2.5 the invariant trace-field and quaternion algebra of Γ satisfies the conditions of Theorem 6.5. Hence we can apply Corollary 6.2 to complete the proof. \square

Examples of Corollary 6.3:

1. It is known that any arithmetic Kleinian group arising from a quaternion algebra B/k (as in §2.1) with $[k : k \cap \mathbb{R}] = 2$ have congruence covers with $vb_1 > 0$ (see [13], [19] or [26]). Hence these are large.

2. In [34] it is shown that if k has one complex place and $[k : \mathbb{Q}] \leq 4$ then any arithmetic Kleinian group arising from an algebra B/k satisfies the hypothesis of Corollary 6.3 and hence is large.
3. As a particular case of 2 above, let M_W denote the Weeks manifold, the smallest arithmetic hyperbolic 3-manifold. From [3], M_W has invariant trace field of degree 3. Hence $\pi_1(M_W)$ is large.
4. Let Σ denote an arithmetic integral homology. The invariant trace-field of Σ (denoted k) has even degree over \mathbb{Q} , and the invariant quaternion algebra of Σ is unramified at all finite places (see for example [31] Theorem 6.4.3). Hence Clozel's theorem (Theorem 6.5) applies to prove $vb_1(\Sigma) > 0$, and Corollary 6.2 applies to prove largeness.

As an example of an arithmetic integral homology 3-sphere one can take the 3-fold cyclic branched cover of the $(-2, 3, 7)$ -pretzel knot. The invariant trace field is $\mathbb{Q}(\theta)$ where θ has minimal polynomial $x^6 - x^5 - x^4 + 2x^3 - 2x^2 - x + 1$. This generates a field of signature $(4, 1)$ and discriminant -104483 . All of this can be checked using Snap (see [6] for a discussion of this program).

We close this section with an example of a commensurability class of arithmetic 3-orbifolds for which no method currently known applies to provide a cover with positive first Betti number.

Let $p(x) = x^5 - x^3 - 2x^2 + 1$. Then p has three real roots and one pair of complex conjugate roots. Let t be a complex root and let $k = \mathbb{Q}(t)$. Now k has one complex place and its Galois group is S_5 . There is a unique prime \mathcal{P} of norm 11^2 in k . It follows that $k_{\mathcal{P}}$ is a quadratic extension of \mathbb{Q}_{11} . Take B ramified at the real embeddings and the prime \mathcal{P} . Then it is unknown whether any arithmetic Kleinian group arising from B has a cover with positive first Betti number.

Briefly, if Γ is any group in the commensurability class, then since k has odd degree, there are no non-elementary Fuchsian subgroups (see [31] Chapter 9). The result of Clozel (see Theorem 6.5) does not apply by the condition on \mathcal{P} , and none of the papers [13], [19] or [34] apply since $[k : \mathbb{Q}] = 5$ and the Galois group is A_5 .

7. LARGENESS AND PROPERTY (τ)

We begin by recalling the definition of Property (τ) .

Definition. Let X be a finite graph, and let $V(X)$ denote its vertex set. For any subset A of $V(X)$, let ∂A denote those edges with one endpoint in A and one not in A . Define the *Cheeger constant* of X to be

$$h(X) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(X) \text{ and } 0 < |A| \leq |V(X)|/2 \right\}.$$

Let G be a group with a finite generating set S . For any subgroup G_i of G , let $X(G/G_i; S)$ be the Schreier coset graph of G/G_i with respect to S . Then G is said to have *Property (τ) with respect to a collection of finite index subgroups $\{G_i\}$* if $\inf_i h(X(G/G_i; S)) > 0$. This turns out not to depend on the choice of finite generating set S . Also, G is said to have *Property (τ)* if it has Property (τ) with respect to the collection of all subgroups of finite index in G .

Lubotzky and Sarnak have made the following conjecture.

Conjecture 7.1. (Lubotzky-Sarnak) *The fundamental group of any closed hyperbolic 3-manifold does not have Property (τ) .*

A slight variant is the following.

Conjecture 7.2. *For any closed orientable 3-manifold M with infinite fundamental group, $\pi_1(M)$ does not have Property (τ) .*

It is a fairly routine argument that if we assume the solution to the Geometrisation Conjecture, then Conjectures 7.1 and 7.2 are equivalent. We sketch this argument in an appendix.

Now it is well-known that having $vb_1 > 0$ implies the Lubotzky-Sarnak conjecture. For if M is a hyperbolic 3-manifold, and we assume that M is finitely covered by a 3-manifold \tilde{M} with $b_1(\tilde{M}) > 0$, then $\pi_1(\tilde{M})$ admits a surjective homomorphism ϕ onto \mathbb{Z} . Let G_i be $\phi^{-1}(i\mathbb{Z})$. Then it is not hard to prove that $\pi_1(M)$ does not have Property (τ) with respect to $\{G_i\}$.

The main result in this section is as follows.

Theorem 7.3. *Conjecture 7.2 implies that any lattice in $\mathrm{PSL}(2, \mathbb{C})$ that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is large.*

Theorem 7.3 has the following surprising conclusion for arithmetic Kleinian groups.

Corollary 7.4. *Conjecture 7.2 implies that any arithmetic Kleinian group is large.*

There are two parts to the proof of Theorem 7.3:

1. Let O be the orbifold quotient of \mathbb{H}^3 by the given lattice in $\mathrm{PSL}(2, \mathbb{C})$. Prove that O has a finite cover \tilde{O} where $|\tilde{O}|$ has infinite fundamental group, and where $\mathrm{sing}_2^-(\tilde{O})$ is non-empty (Theorem 8.1).
2. Prove a result (Theorem 9.1) analogous to Corollary 5.3, which states that $\pi_1(\tilde{O})$ is large, provided $\mathrm{sing}_2^-(\tilde{O})$ is non-empty and $\pi_1(|\tilde{O}|)$ does not have Property (τ) .

8. AN UNDERLYING MANIFOLD WITH INFINITE FUNDAMENTAL GROUP

The main theorem in this section is the following.

Theorem 8.1. *Let O be a closed orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Then O has a finite cover \tilde{O} such that $\mathrm{sing}_2^-(\tilde{O})$ is non-empty, and where $|\tilde{O}|$ admits an infinite tower of finite covers $\{|O_i| \rightarrow |\tilde{O}|\}$. In particular, $\pi_1(|\tilde{O}|)$ is infinite.*

The first step is to prove the following extension of Lemma 4.7.

Proposition 8.2. *Let X be a finite trivalent graph. Then X contains a connected subgraph Y such that $b_1(Y) = 2$ and the number of edges of Y is at most $6 \log_2(b_1(X) - 1) + 12$.*

Proof. We may assume that X is connected. Consider the path metric on X , where each edge has length 1. For any vertex v and non-negative integer $n \leq b_1(X)$, let $R_n(v)$ be the minimal radius of a ball centred at v that contains a subgraph Y with $b_1(Y) \geq n$. Note that $b_1(X) \geq 2$ and so $R_2(v)$ is well-defined. Fix a vertex v where $R_2(v)$ has minimal value, and set $R = \lceil R_2(v) \rceil$. We claim that

$$R \leq \log_2(b_1(X) - 1) + 2.$$

There are three cases to consider: when $R_1(v) \leq R_2(v) - 1$, when $R_1(v) = R_2(v) - 1/2$ and when $R_1(v) = R_2(v)$. Let us concentrate on the first case.

For any non-negative integer r , let $B(r)$ be the ball of radius r around v . Since $R_1(v) < R_2(v)$, $b_1(B(R_1(v))) = 1$. Thus, $B(R_1(v))$ contains a unique simple closed curve C consisting of at most $2R_1(v)$ edges. Since we are assuming $R - 1 \geq R_1(v)$, $B(R - 1)$ is obtained from C by attaching trees. The distance between any two vertices of $B(R - 1)$ is at most $2R - 2$. By definition of R , there exist two vertices in $B(R - 1)$ joined by a path p_1 of length at most 2 in $X - B(R - 1)$, where p_1 is either an arc or simple closed curve. Let p_2 be the shortest path in $B(R - 1)$ from one of these vertices to C . If the endpoints of p_1 are distinct, let p_3 be the shortest path in $B(R - 1)$ from the other vertex to $C \cup p_2$; otherwise let p_3 be the empty set. Set $Y = p_1 \cup p_2 \cup p_3 \cup C$. Then it is clear that Y is connected and $b_1(Y) = 2$. By construction, Y consists of at most $6R$ edges.

Now, C runs through at most two of the three edges adjacent to v . Thus, if one were to remove v and its adjacent edges from $B(R - 1)$, one component would consist of a based binary tree. The number of vertices in this tree, together with v , is equal to 2^{R-1} . This is a lower bound for the number V of vertices in X . But, since X is connected and trivalent, $b_1(X) = \frac{1}{2}V + 1$. So,

$$2^{R-1} \leq V = 2b_1(X) - 2$$

and therefore

$$R \leq \log_2(b_1(X) - 1) + 2.$$

This proves the claim when $R_1(v) \leq R_2(v) - 1$. In the remaining cases, the proof is similar but simpler, and so is omitted.

We have already seen that Y contains at most $6R$ edges, and so the proposition is proved. \square

Lemma 8.3. *Let O be a closed orientable 3-orbifold with non-empty singular locus. Then $\pi_1(O)$ has a finite presentation $\langle X|R \rangle$ with*

$$|R| - |X| \leq 2b_1(\text{sing}(O)) - 2.$$

Proof. Let M be the manifold obtained from O by removing an open regular neighbourhood of its singular locus. Then $\pi_1(M)$ has a presentation $\langle X|R' \rangle$ with $|R'| - |X| = \frac{1}{2}\chi(\partial M) - 1 = \chi(\text{sing}^-(O)) - 1$. We obtain $\pi_1(O)$ from $\pi_1(M)$ by adding relations that are powers of the meridians of the singular locus. For each circle component of the singular locus, there is one such meridian. For each graph component Y , the number of meridians is equal to $-3\chi(Y)$. Hence, $\pi_1(O)$ has a presentation $\langle X|R \rangle$ with

$$\begin{aligned} |R| - |X| &= -2\chi(\text{sing}^-(O)) + |\text{sing}^0(O)| - 1 \\ &= 2b_1(\text{sing}^-(O)) - 2|\text{sing}^-(O)| + 2b_1(\text{sing}^0(O)) - |\text{sing}^0(O)| - 1 \\ &\leq 2b_1(\text{sing}(O)) - 2. \end{aligned}$$

□

The final step in the proof of Theorem 8.1 is the following proposition.

Proposition 8.4. *Let O be a closed orientable 3-orbifold such that every arc and circle of the singular locus has singularity order 2. Let Y be a connected subgraph of $\text{sing}(O)$ that has $b_1(Y) = 2$ and at most $6\log_2(b_1(\text{sing}(O)) - 1) + 12$ edges. Let G be the fundamental group of O , and let K be the subgroup normally generated by the meridians that encircle Y . Let \tilde{O} be the covering space of O corresponding to the subgroup KG^2 . Suppose that $d_2(O) \geq 81$. Then $|\tilde{O}|$ has an infinite tower of finite covers $|\tilde{O}_i| \rightarrow |\tilde{O}|$. In particular, $\pi_1(|\tilde{O}|)$ is infinite.*

Proof. Let M_1 be the meridians of Y , one for each edge of $\text{sing}(O)$ that lies in Y . Let M_2 be those meridians of $\text{sing}(O)$ that lie in G^2 but that are not meridians of Y . Consider the group $\Gamma = \pi_1(O)/\langle\langle M_1, M_2 \rangle\rangle$. This is equal to the fundamental group of an orbifold O' with the same underlying manifold as O , and with singular set that is a subgraph of $\text{sing}(O)$. Lemma 8.3 states that $\pi_1(O')$ has a finite presentation $\langle X|R \rangle$ where the number of relations minus the number of generators is at most $2b_1(\text{sing}(O')) - 2$, which is at most $2b_1(\text{sing}(O)) - 2$. By Proposition 3.1, this is at most $2d_2(O) - 2$. Now, adding the relations in M_2 to $\pi_1(O)$ does not affect d_2 , because they lie in G^2 . Hence,

$$d_2(O) \geq d_2(\Gamma) \geq d_2(O) - |M_1| \geq d_2(O) - 6\log_2(d_2(O) - 1) - 12.$$

Therefore,

$$\begin{aligned}
& d_2(\Gamma)^2/4 - |R| + |X| - d_2(\Gamma) \\
& \geq d_2(\Gamma)^2/4 - 2d_2(O) + 2 - d_2(\Gamma) \\
& \geq (d_2(O) - 6\log_2(d_2(O) - 1) - 12)^2/4 - 3d_2(O) + 2 \\
& > 0.
\end{aligned}$$

The last inequality is a consequence of the assumption that $d_2(O) \geq 81$; it is easy to check that the given function of $d_2(O)$ is positive in this range. Hence, by the Golod-Shafarevich theorem, Γ has an infinite nested sequence of finite index subgroups and hence is infinite. The same is therefore true for Γ^2 , because it has finite index in Γ .

Let \tilde{O} be the covering space of O corresponding to KG^2 . We claim that $\pi_1(|\tilde{O}|) \cong \Gamma^2$. This will prove that $\pi_1(|\tilde{O}|)$ has an infinite nested sequence of finite index subgroups. Now, $\pi_1(|\tilde{O}|)$ is obtained from $\pi_1(\tilde{O}) = KG^2$ by quotienting each meridian of the singular locus. But the meridians in O that lift to meridians of singular components of \tilde{O} are precisely those lying in $\langle\langle M_1, M_2 \rangle\rangle$. Hence, $\pi_1(|\tilde{O}|) \cong KG^2/\langle\langle M_1, M_2 \rangle\rangle \cong \Gamma^2$. \square

Proof of Theorem 8.1. By Theorem 4.5, O is finitely covered by an orbifold O' such that each arc and circle of $\text{sing}(O')$ has order 2 and where $b_1(\text{sing}_2^-(O')) \geq 81$. By Proposition 3.1, $d_2(O') \geq 81$. Proposition 8.2 states that $\text{sing}_2^-(O')$ contains a connected subgraph Y such that $b_1(Y) = 2$ and which has at most $6\log_2(b_1(\text{sing}_2^-(O')) - 1) + 12$ edges, which is at most $6\log_2(b_1(\text{sing}_2(O')) - 1) + 12$. Let \tilde{O} be the covering space of O' corresponding to KG^2 , where $G = \pi_1(O')$ and K is the subgroup normally generated by the meridians of Y . Since $\pi_1(\tilde{O})$ contains K , the inverse image of each edge of Y in \tilde{O} is a disjoint union of copies of that edge. In particular, $\text{sing}_2^-(\tilde{O})$ is non-empty. By Proposition 8.4, $|\tilde{O}|$ has an infinite tower of finite covers. In particular, $|\tilde{O}|$ has infinite fundamental group. \square

9. THE LUBOTZKY-SARNAK CONJECTURE AND THE LARGENESS OF ORBIFOLDS

In this section, we prove the following result. Together with Theorem 8.1, this will complete the proof of Theorem 7.3.

Theorem 9.1. *Let \tilde{O} be a compact orientable 3-orbifold such that $\text{sing}_2^-(\tilde{O})$ is non-empty. Let $\{|O_i| \rightarrow |\tilde{O}|\}$ be a sequence of finite-sheeted covering spaces. Suppose that $\pi_1(|\tilde{O}|)$ does not have Property (τ) with respect to $\pi_1(|O_i|)$. Then $\pi_1(\tilde{O})$ is large.*

Proof. By passing to further finite-sheeted covers if necessary, we may assume that each $\pi_1(|O_i|)$ is a finite index normal subgroup of $\pi_1(|\tilde{O}|)$.

We are assuming that $\text{sing}_2^-(\tilde{O})$ is non-empty. Suppose that $\text{sing}_2^-(\tilde{O})$ contains a vertex with valence 1. (This may happen, for example, if this is a vertex of $\text{sing}(\tilde{O})$ with local group A_5 .) If so, remove this vertex and the adjacent edge, forming a graph Γ . Now repeat this procedure if Γ has a valence 1 vertex, and continue until every vertex of Γ has valence at least two. Remove the components of Γ which are circles. If a vertex has valence 2, amalgamate its two adjacent edges into a single edge. Repeat until every vertex of Γ has valence at least 3. Note that Γ still has negative Euler characteristic. In particular, it is non-empty. Let Γ_i be the inverse image of Γ in O_i .

Pick a 1-vertex triangulation of $|\tilde{O}|$. For convenience, we may arrange that the vertex of this triangulation is a vertex of Γ . Place a path metric on $|\tilde{O}|$ so that each edge of the triangulation has length 1 and each 3-simplex is a regular Euclidean tetrahedron.

The edges of this triangulation, when oriented, form a set S of generators for $\pi_1(|\tilde{O}|)$. We are assuming that $\pi_1(|\tilde{O}|)$ does not have Property (τ) with respect to $\{\pi_1(|O_i|)\}$. Hence, the Cheeger constants of the corresponding Schreier coset diagrams tend to zero. But each such graph X_i is just the 1-skeleton of $|O_i|$. Let A_i be a non-empty set of vertices in X_i such that $|\partial A_i|/|A_i| = h(X_i)$ and $|A_i| \leq |V(X_i)|/2$. We will use ∂A_i to construct a surface S_i that separates O_i into two pieces B_i and C_i . Place a 0-cell of S_i at the midpoint of each edge of ∂A_i . If a 2-simplex of the triangulation of $|O_i|$ intersects ∂A_i , it does so in precisely two points. Insert into this 2-simplex a geodesic joining these two points, forming a 1-cell of S_i . Then, the boundary of each 3-simplex intersects these arcs in either the empty set, or a normal curve of length three or four. In each 3-simplex of $|\tilde{O}|$ pick a representative disc spanning the 7 different curves of length 3 and 4. Use lifts of these discs to $|O_i|$ to construct the 2-cells of S_i . This therefore defines the

surface S_i . It divides O_i into two 3-orbifolds B_i and C_i , say, which contain the vertices A_i and A_i^c respectively.

Now, we may arrange that the singular set of \tilde{O} is transverse to the representative normal discs in $|\tilde{O}|$. Thus, $\text{sing}(O_i)$ is transverse to S_i . By construction, there is a uniform upper bound (independent of i) for the number of intersection points between $\text{sing}(O_i)$ and S_i in any 3-simplex of $|O_i|$. We claim, that, viewing S_i as a 2-orbifold, there is a uniform constant K_1 (independent of i) with the following property:

$$d_2(S_i) \leq K_1 |\partial A_i|. \quad (1)$$

This is because $d_2(S_i)$ is at most the sum of the number of 1-cells of S_i and its number of singular points. We have already seen that the number of singular points of S_i is bounded above by a constant times the number of 2-cells of $|S_i|$. This is bounded above by the number of 1-cells of $|S_i|$. Thus, it suffices to find a linear bound on this quantity in terms of $|\partial A_i|$. It is at most the number of 0-cells of $|S_i|$ times half the maximal valence of any 1-simplex of $|\tilde{O}|$. But there is precisely one 0-cell of $|S_i|$ in each edge of ∂A_i . This proves the claim.

We claim that there is a positive constant K_2 , independent of i , such that

$$b_1(\Gamma_i \cap B_i) \geq |V(X_i)|/8 - K_2 |\partial A_i|. \quad (2)$$

Note that $\Gamma_i \cap B_i$ is a graph in which each vertex has valence 3 or 1. The vertices with valence 1 arise at the intersection points between Γ_i and S_i . Hence, the number of such vertices is bounded above a constant times $|\partial A_i|$. Now, the Euler characteristic of $\Gamma_i \cap B_i$ is equal to the sum, over all its vertices v , of $1 - \text{val}(v)/2$, where $\text{val}(v)$ is the valence of v . We arranged that the vertex of the triangulation of $|\tilde{O}|$ was a vertex of Γ . Hence, the number of vertices of $\Gamma_i \cap B_i$ is at least $|A_i|$. This is at least $|V(X_i)|/4$ by Lemma 2.1 of [14]. So, for some constant K_2 ,

$$b_1(\Gamma_i \cap B_i) \geq -\chi(\Gamma_i \cap B_i) \geq |A_i|/2 - K_2 |\partial A_i| \geq |V(X_i)|/8 - K_2 |\partial A_i|.$$

The same inequality holds for $b_1(\Gamma_i \cap C_i)$.

By Proposition 3.1, $d_2(B_i) \geq b_1(\text{sing}_2(B_i)) \geq b_1(\Gamma_i \cap B_i)$. Combining this with inequalities (1) and (2), we see that when $h(X_i)$ is small enough,

$$d_2(B_i) \geq d_2(S_i) + 2,$$

with the corresponding inequality holding also for $d_2(C_i)$. The constant 2 here could have been replaced by any real number. Hence, the map $H^1(B_i; \mathbb{F}_2) \rightarrow H^1(S_i; \mathbb{F}_2)$ induced by inclusion has kernel with dimension at least 2. Consider the covering space O' of O_i corresponding to this kernel. This has degree at least 4. The inverse image of C_i is at least four copies of C_i . Now, in C_i , there is a properly embedded compact surface representing a non-trivial element in the kernel of $H^1(C_i; \mathbb{F}_2) \rightarrow H^1(S_i; \mathbb{F}_2)$. We may pick this surface so that it is non-separating in C_i , and so that it is disjoint from S_i . Its inverse image in O' is a non-separating surface F . Let N be the number of components of F . Then $N \geq 4$.

We claim there is a surjective homomorphism ϕ from $\pi_1(O')$ onto $*^N \mathbb{Z}/2\mathbb{Z}$, the free product of N copies of $\mathbb{Z}/2\mathbb{Z}$. The copies of $\mathbb{Z}/2\mathbb{Z}$ are indexed by the components F_1, \dots, F_N of F . Let x_i be the non-trivial element in the i^{th} copy of $\mathbb{Z}/2\mathbb{Z}$. Pick a basepoint for O' away from F . For each element g of $\pi_1(O')$, pick a representative loop ℓ . Make ℓ transverse to F via a small homotopy (keeping the endpoints of ℓ fixed). As one goes round the loop ℓ , let F_{i_1}, \dots, F_{i_r} be the components of F that one meets. Define $\phi(g)$ to be $x_{i_1} \dots x_{i_r}$. It is trivial to check that this is invariant under a homotopy of ℓ relative to its endpoints. (For example, it is a consequence of the fact that ϕ is the homomorphism induced by a collapsing map from O' onto the wedge of N copies of $\mathbb{R}P^2$.) Hence, this gives a well-defined function $\phi: \pi_1(O') \rightarrow *^N \mathbb{Z}/2\mathbb{Z}$. It is clearly a homomorphism, since concatenation of loops leads to concatenation of words. It is also surjective. This is because F is non-separating, and so, for any component F_i of F , there is a loop ℓ , based at the basepoint, which intersects F_i once and is disjoint from the remaining components of F . Hence, $\phi([\ell]) = x_i$.

Because $N \geq 4$, $*^N \mathbb{Z}/2\mathbb{Z}$ contains a free non-abelian group as a finite index subgroup. The inverse image of this group in $\pi_1(O')$ also has finite index. It surjects this free non-abelian group. Hence, $\pi_1(O')$ is large, as therefore is $\pi_1(O)$. \square

Remark. If, in Theorem 9.1, we make the extra hypotheses that the covers $|O_i| \rightarrow |\tilde{O}|$ are nested, and that successive covers $|O_{i+1}| \rightarrow |O_i|$ are regular and have degree a power of 2, then this theorem would be a consequence of the following result (Theorem 1.1 of [17]), together with Lemma 9.3 below.

Theorem 9.2. *Let G be a finitely presented group, let p be a prime and suppose that $G \geq G_1 \triangleright G_2 \triangleright \dots$ is a nested sequence of finite index subgroups, such that G_{i+1} is normal in G_i and has index a power of p , for each i . Suppose that $\{G_i\}$ has linear growth of mod p homology. Then, at least one of the following must hold:*

- (i) G is large; or
- (ii) G has Property (τ) with respect to $\{G_i\}$.

Lemma 9.3. *Let \tilde{O} be a compact orientable 3-orbifold, such that $\text{sing}_2^-(\tilde{O})$ is non-empty. Let $|O_i|$ be finite covering spaces of the manifold $|\tilde{O}|$ and let O_i be the corresponding covering spaces of O . Then $\{O_i\}$ has linear growth of mod 2 homology.*

Proof. The inverse image of $\text{sing}_2^-(\tilde{O})$ under the map $O_i \rightarrow \tilde{O}$ is $\text{sing}_2^-(O_i)$. Hence, $\chi(\text{sing}_2^-(O_i)) = \chi(\text{sing}_2^-(\tilde{O}))|O_i : \tilde{O}|$. Now, $b_1(\text{sing}_2^-(O_i)) \geq |\chi(\text{sing}_2^-(O_i))|$. Thus, Proposition 3.1 implies that $\{O_i\}$ has linear growth of mod 2 homology. \square

We conclude this section by proving the following result; this can be deduced from Corollary 7.4, but we give a proof below that is simpler and more direct.

Theorem 9.4. *If $vb_1(M) > 0$ for all closed orientable 3-manifolds M with infinite fundamental group, then $\pi_1(N)$ is large for any arithmetic 3-manifold N .*

Theorem 9.4 is proved using 8.1 above together with the following.

Theorem 9.5. *Let O be a closed orientable hyperbolic 3-orbifold containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Suppose that $vb_1(|O|) > 0$. Then $\pi_1(O)$ is large.*

Proof. By Proposition 4.4, O is finitely covered by an orbifold O' such that $\text{sing}_2^-(O')$ is non-empty. We are assuming that $vb_1(|O|) > 0$. So, there is a finite cover $|O''| \rightarrow |O|$ such that $b_1(|O''|) > 0$. Consider the covering space O''' of O corresponding to $\pi_1(O') \cap \pi_1(O'')$. We claim that $O''' \rightarrow O'$ descends to a cover $|O'''| \rightarrow |O'|$ between underlying manifolds. To prove this, it suffices to show that each torsion element g in $\pi_1(O')$ lies in $\pi_1(O''')$. But g maps to a torsion element of $\pi_1(O)$ and this must lie in $\pi_1(O'')$ because $|O''| \rightarrow |O|$ is a cover. Thus, g lies in $\pi_1(O'') \cap \pi_1(O') = \pi_1(O''')$, as required. Because $|O'''| \rightarrow |O'|$ is a cover,

$\text{sing}_2^-(O''')$ is non-empty. Since $b_1(O'')$ is positive, so too is $b_1(O''')$.

Let $G = \pi_1(O''')$, and let $\phi: \pi_1(O''') \rightarrow \mathbb{Z}$ be a surjective homomorphism. Let G_i be $\phi^{-1}(i\mathbb{Z})$, and let O_i be the corresponding covering space of O''' . By Lemma 9.3, $\{\pi_1(O_i)\}$ has linear growth of mod 2 homology. So, by Corollary 5.3, $\pi_1(O''')$ is large and therefore so is $\pi_1(O)$. \square

10. THE CONGRUENCE SUBGROUP PROPERTY

In this section we show how Theorem 4.1 (proved by only the methods of 3-manifold topology and Kleinian groups) can be used to give a new proof of Lubotzky's result [23].

Theorem 10.1. *No arithmetic Kleinian group has the congruence subgroup property.*

Recall that the *congruence subgroup property* is said to hold for Γ if any finite index subgroup of Γ is a congruence subgroup. Lubotzky's original proof relied heavily on the Golod-Shafarevich inequality and the theory of p -adic analytic groups. The aim of this section to provide a more elementary proof. Following in the spirit of [25], the idea is to compare the number of subgroups of Γ of a given index with the number of congruence subgroups with that index. Therefore, for a natural number n , define $s_n(\Gamma)$ and $c_n(\Gamma)$ to be the number of subgroups of Γ (respectively, congruence subgroups of Γ) with index at most n .

Theorem 10.1 is an immediate consequence of the following two theorems, since they imply that $s_n(\Gamma) > c_n(\Gamma)$ for infinitely many n .

Theorem 10.2. *Let Γ be a lattice in $\text{PSL}(2, \mathbb{C})$ that is commensurable with a group containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then, there is a constant $k > 1$ such that*

$$s_n(\Gamma) \geq k^n$$

for infinitely many n .

Theorem 10.3. *Let Γ be an arithmetic Kleinian group. Then, there is a positive constant b such that*

$$c_n(\Gamma) \leq n^{b \log n / \log \log n},$$

for all n .

Proof of Theorem 10.2. Let O be the orbifold \mathbb{H}^3/Γ . Suppose first that O is closed. According to Theorem 4.1, O has a nested sequence of finite covers $\{O_i\}$ that have linear growth of mod 2 homology. The same conclusion holds when O is non-compact, because of [4] which guarantees that $\pi_1(O)$ has a finite index subgroup that has a free non-abelian quotient. Let λ be

$$\inf_i d_2(O_i)/[O : O_i],$$

which is therefore positive. Each homomorphism $\pi_1(O_i) \rightarrow \mathbb{Z}/2\mathbb{Z}$ gives a subgroup of $\pi_1(O_i)$ with index 1 or 2, and these subgroups are all distinct. Therefore, the number subgroups of $\pi_1(O)$ with index at most $2[O : O_i]$ is at least $2^{\lambda[O:O_i]}$. Setting $k = 2^{\lambda/2}$ proves Theorem 10.2. \square

A proof of Theorem 10.3 for arbitrary arithmetic groups was proved by Lubotzky in [25], and is also given in Section 6.1 of [27]. However, given our aim of producing a more elementary proof of Lubotzky's result in [25], we wish to avoid some of the technology that is used in [25] and [27]. We use these as our guidelines but will not reproduce all of the amendments necessary, merely commenting on salient points.

Before commencing the proof we make some comments that help simplify some of the discussion below.

First, following [25] and [27] we will work with the groups $\mathrm{SL}(2)$ rather than $\mathrm{PSL}(2)$. Now let K be an arbitrary number field with ring of integers R_K . Then $\mathrm{SL}(2, R_K)$ contains the family of congruence subgroups obtained in the usual way as:

$$\Gamma(J) = \ker(\mathrm{SL}(2, R_K) \rightarrow \mathrm{SL}(2, R_K/J)),$$

where $J \subset R_K$ is an ideal.

Connecting with the discussion in §2.4, $J.M(2, R_K)$ is a proper 2-sided integral ideal of $M(2, R_K)$ contained in $M(2, R_K)$, and the elements $\alpha \in \mathrm{SL}(2, R_K)$ such that $\alpha - 1 \in J.M(2, R_K)$ correspond precisely to the group $\Gamma(J)$. Also, given any other maximal order $\mathcal{L} \in M(2, K)$ (not $\mathrm{GL}(2, K)$ -conjugate to $M(2, R_K)$), then \mathcal{L}^1 can be conjugated to be a subgroup of a group $\mathrm{SL}(2, R_H)$ for some number field H (the Hilbert Class field of K will work, see the proof of [31] Lemma 5.2.4). Via this, the congruence subgroups $\Gamma(\mathcal{L}(I))$ of §2.4 can be described in $\mathrm{SL}(2, R_H)$

using the more traditional definition given above (see also the discussion in [31] Chapter 6.6)

For congruence subgroups of arithmetic Kleinian groups, using an embedding of B into $\mathrm{SL}(2, K)$ where K is a splitting field of B , we can use the above discussion to describe these congruence subgroups as subgroups of congruence subgroups of $\mathrm{SL}(2, R_K)$ for certain number fields K .

Proof of Theorem 10.3.

Following the above discussion, it suffices to prove our result in the following context.

Proposition 10.4. *Let K be a number field with ring of integers R_K and degree d . Then, there is a positive constant b such that*

$$c_n(\mathrm{SL}(2, R_K)) \leq n^{b \log n / \log \log n},$$

for all n .

The key result is the following “level versus index” result.

Proposition 10.5. *There is some constant c with the following property. For each congruence subgroup H of $\mathrm{SL}(2, R_K)$, $H \geq \Gamma(J)$, for some ideal $J \subset R_K$ with $N(J) \leq c[\mathrm{SL}(2, R_K) : H]$.*

We defer comment on the proof of this theorem and complete the proof of Theorem 10.3. A consequence of Proposition 10.5 is that $c_n(\mathrm{SL}(2, R_K))$ is at most the sum, $\sum s_n(\mathrm{SL}(2, R_K/J))$, where the sum is over all ideals $J \subset R_K$ with $N(J) \leq cn$. This in turn is less than the sum

$$\sum_{m=1}^{cn} s_n(\mathrm{SL}(2, R_K/mR_K)).$$

Thus, the question now reduces to a count of subgroups in the finite groups $\mathrm{SL}(2, R_K/mR_K)$. We now discuss this, following [25] and [27].

Define the *rank* of a finite group G to be

$$\mathrm{rank}(G) = \sup\{d(H) : H \leq G\}.$$

An easy argument (see [27] Lemma 1.2.2) shows the total number of subgroups of a finite group G is $\leq |G|^{\mathrm{rank}(G)}$.

The groups $\mathrm{SL}(2, R_K/mR_K)$ decompose as $\prod \mathrm{SL}(2, R_K/\mathcal{P}_j^{a_j})$, where $mR_K = \mathcal{P}_1^{a_1} \dots \mathcal{P}_t^{a_t}$ is a factorization into distinct prime (ideal) powers of the principal ideal mR_K . From this, the orders of these groups can be computed [36]. Also note that there are at most d (which recall is the degree of K) primes in K lying above a given rational prime p . Given this, an estimate of the order that suffices is m^{3d} .

To compute the rank we argue as follows. If \mathcal{P} is a K -prime dividing the rational prime p then R_K/\mathcal{P} is a finite extension of \mathbb{F}_p the field of p elements of degree at most d . Notice that if $R_{\mathcal{P}}$ denotes the \mathcal{P} -adic integers in $K_{\mathcal{P}}$ with uniformizer $\pi_{\mathcal{P}}$ then

$$\mathrm{SL}(2, R_K/\mathcal{P}^a) \cong \mathrm{SL}(2, R_{\mathcal{P}}/\pi_{\mathcal{P}}^a R_{\mathcal{P}})$$

and the latter are all homomorphic images of $\mathrm{SL}(2, R_{\mathcal{P}})$. From [7], standard aspects of uniform pro- p groups imply that these have rank (as pro- p groups) at most $3d$ (the field $K_{\mathcal{P}}$ has degree at most d over \mathbb{Q}_p). Hence we deduce:

1. Let \mathcal{P} be a K -prime. Then $\mathrm{rank}(\mathrm{SL}(2, R_K/\mathcal{P}^a)) \leq 3d$.

To pass to the rank of $\mathrm{SL}(2, R_K/mR_K)$, the argument is as in [25] and [27] with the extra care that we are working with a number field.

Thus if $m = p_1^{b_1} \dots p_l^{b_l}$ then as in [25], a simple application of the Prime Number Theorem gives $l \leq \log m / \log \log m$. As noted above, each prime p_i splits into at most d K -primes, and so arguing as in [25] we deduce using 1:

2. $\mathrm{rank}(\mathrm{SL}(2, R_K/mR_K)) \leq 3d^2 \log m / \log \log m$.

Hence using 2 and the estimate above, it follows that the total number of subgroups of $\mathrm{SL}(2, R_K/mR_K)$ is at most $m^{C \log m / \log \log m}$ where C depends on K . The count of congruence subgroups now finishes off as in [25].

We now discuss a proof of Proposition 10.6 in our context. Again, as in [25] and [27] the key assertion concerns essential subgroups of $\mathrm{SL}(2, R_K/J)$. The definition here is amended from that in [25] and [27] to work in the number field K .

Following [25] and [27] we say $H < \mathrm{SL}(2, R_K/J)$ is called *essential* if H does

not contain:

$$M(I) = \ker(\mathrm{SL}(2, R_K/J) \rightarrow \mathrm{SL}(2, R_K/I))$$

for any $I|J$ (as ideals) with $I \neq J$.

Claim. *There exists a constant $C' > 0$ (depending on K) such that for every ideal J , every essential subgroup H of $\mathrm{SL}(2, R_K/J)$ satisfies*

$$[\mathrm{SL}(2, R_K/J) : H] \geq C' N(J).$$

Proof of Claim. We follow the argument in [25] and [27] adapted to our setting. If J is a prime ideal of R_K then R_K/J is a field of order $q = p^t$ for some $t \leq d$. It is a classical result dating back to Galois (see [36] Chapter 6) that the minimal index of a proper subgroup of $\mathrm{SL}(2, R_K/J)$ is at least $q + 1$ apart from a finite number of values of q .

If $J = \mathcal{P}^a$ is the power of a prime ideal \mathcal{P} , then the argument in [25] applies in exactly the same way. The arguments in [25] appeal to Strong Approximation, but the argument for $\mathrm{SL}(2)$ can be handled directly (for example by the methods of [22]). Similarly, if $J = \mathcal{P}_1 \dots \mathcal{P}_t$ is a product of primes whose norms are powers of distinct rational primes, then the argument of [27] applies.

Now consider the case of $J = pR_K$ where p is a rational prime that splits completely in K . In this case $pR_K = \mathcal{P}_1 \dots \mathcal{P}_d$. Let $M = \mathrm{SL}(2, R_K/\mathcal{P}_1) \times \dots \times \mathrm{SL}(2, R_K/\mathcal{P}_d)$ and assume that H is an essential subgroup of M . If π_i denotes projection on the i -th factor, then we claim $\pi_i(H)$ is an essential subgroup of $\mathrm{SL}(2, R_K/\mathcal{P}_i)$; ie $\pi_i(H)$ is a proper subgroup of $\mathrm{SL}(2, R_K/\mathcal{P}_i)$.

Suppose not, and assume $i = 1$ for convenience. Let $M_1 = \mathrm{SL}(2, R_K/\mathcal{P}_1)$ and $M' = \mathrm{SL}(2, R_K/\mathcal{P}_2) \times \dots \times \mathrm{SL}(2, R_K/\mathcal{P}_d)$, so $M = M_1 \times M'$. Then $\pi_1(H) = M_1$ implies by properties of the direct product that $H \cap M_1$ is normal in M_1 . For all but a finite number of finite fields \mathbb{F} , $\mathrm{SL}(2, \mathbb{F})$ is a central extension of a finite simple group, and so $H \cap M_1 = M_1$ or is central. Both of these can be ruled out as in [27], using the essentialness of H . Hence we deduce in this case that

$$[M : H] > \prod [\mathrm{SL}(2, \mathcal{P}_i) : \pi_i(H)] > p^d,$$

which proves the claim in this case.

Now consider $J = \mathcal{P}_1^{e_1} \dots \mathcal{P}_t^{e_t}$ with some $e_i > 1$. Let $J' = \mathcal{P}_1 \dots \mathcal{P}_t$, so that $N(J') < N(J)$. Let H be an essential subgroup of $\mathrm{SL}(2, R_K/J)$, and H' denote the projection of H to $\mathrm{SL}(2, R_K/J')$. As in [27] H' is an essential subgroup of $\mathrm{SL}(2, R_K/J')$ and so the index (by above) is at least $N(J')$. Following [27] it suffices to prove that $H \cap M(J')$ has index at least $N(J)/N(J')$ in $M(J')$ (which is a product of p -groups for various primes p) and this is completed as in the last few paragraphs of [27] pp 116–117. \square

REFERENCES

1. A. BOREL, *Cohomologie de sous-groupes discrets et représentations de groupes semi-simples*, Astérisque 32-33 (1976) 73-112.
2. K. BROWN, *Cohomology of Groups*, Graduate Texts in Math. 87 (1982), Springer-Verlag.
3. T. CHINBURG, E. FRIEDMAN, K. N. JONES AND A. W. REID, *The smallest volume arithmetic hyperbolic 3-manifold*, Annali della Scuola Normale Superiore di Pisa 30 (2001) 1–40.
4. L. CLOZEL, *On the cuspidal cohomology of arithmetic subgroups of $\mathrm{SL}(2n)$ and the first betti number of arithmetic 3-manifolds*, Duke Math. J. 55 (1987) 475-486.
5. D. COOPER, D. D. LONG AND A. W. REID, *Essential closed surfaces in bounded 3-manifolds*, Journal of The American Math. Soc. 10 (1997) 553–563.
6. D. COULSEN, O. A. GOODMAN, C. D. HODGSON AND W. D. NEUMANN, *Computing arithmetic invariants of 3-manifolds*, Experimental J. Math. 9 (2000) 127–152
7. J. DIXON, M. DU SAUTOY, A. MANN, D. SEGAL, *Analytic pro- p groups*, Cambridge Studies in Advanced Mathematics, 61 (1999), Cambridge University Press.
8. N. DUNFIELD, W. THURSTON, *The virtual Haken conjecture: Experiments and examples*, Geom. Topol. 7 (2003) 399–441.

9. J. HEMPEL, *Residual finiteness for 3-manifolds*, Combinatorial group theory and topology (Alta, Utah, 1984) 379–396, Ann. of Math. Stud., 111, Princeton Univ. Press, 1987.
10. K. N. JONES AND A. W. REID, *Geodesic intersections in arithmetic hyperbolic 3-manifolds*, Duke Math. J. 89 (1997) 75–86.
11. T. JØRGENSEN, *Closed geodesics on Riemann surfaces*, Proc. Amer. Math. Soc. 72 (1978) 140–142.
12. S. KOJIMA, *Finite covers of 3-manifolds containing essential surface of Euler characteristic = 0*, Proc. Amer. Math. Soc. 101 (1987) 743–747.
13. J.-P LABESSE AND J. SCHWERMER, *On liftings and cusp cohomology of arithmetic groups*, Invent. Math. 83 (1986) 383–401.
14. M. LACKENBY, *Heegaard splittings, the virtually Haken conjecture and Property (τ)* , To appear in Invent. Math.
15. M. LACKENBY, *A characterisation of large finitely presented groups*, J. Algebra 287 (2005) 458–473.
16. M. LACKENBY, *Covering spaces of 3-orbifolds*, Preprint.
17. M. LACKENBY, *Large groups, Property (τ) and the homology growth of subgroups*, Preprint.
18. M. LACKENBY, *Some 3-manifolds and 3-orbifolds with large fundamental group*, Preprint.
19. J. S. LI AND J. J. MILLSON, *On the first betti number of a hyperbolic manifold with an arithmetic fundamental group*, Duke Math. J. 71 (1993) 365–401.
20. D. D. LONG, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. 19 (1987) 481–484.
21. D. D. LONG, G. NIBLO, *Subgroup separability and 3-manifold groups*. Math. Z. 207 (1991) 209–215.
22. D. D. LONG, A. W. REID, *Simple quotients of hyperbolic 3-manifold groups*, Proc. Amer. Math. Soc. 126 (1998) 877–880.

23. A. LUBOTZKY, *Group presentations, p -adic analytic groups and lattices in $SL(2, \mathbb{C})$* , Ann. Math. 118 (1983) 115–130.
24. A. LUBOTZKY, *Free quotients and the first Betti number of some hyperbolic manifolds*, Transform. Groups 1 (1996) 71–82.
25. A. LUBOTZKY, *Subgroup growth and congruence subgroups*, Invent. Math. 119 (1995) 267–295.
26. A. LUBOTZKY, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Ann. Math. **144** (1996) 441–452.
27. A. LUBOTZKY, D. SEGAL, *Subgroup Growth*. Progress in Mathematics, 212. Birkhäuser Verlag (2003)
28. A. LUBOTZKY, R. ZIMMER, *Variants of Kazhdan’s property for subgroups of semisimple groups*, Israel J. Math. **66** (1989) 289–299.
29. J. LUECKE, *Finite covers of 3-manifolds containing essential tori*, Trans. Amer. Math. Soc. 310 (1988) 381–391.
30. C. MACLACHLAN, G. J. MARTIN, *2-generator arithmetic Kleinian groups*, J. für die Reine und Angew. Math. **511** (1999) 95–117.
31. C. MACLACHLAN, A. W. REID, *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics, 219, Springer-Verlag (2003)
32. G. MARGULIS, *Discrete Subgroups of Semi-simple Lie Groups*, Ergebnisse der Mathematik und ihr Grenzgebiete, Springer (1991).
33. W. NARKIEWICZ, *Algebraic numbers*, Polish Scientific Publishers (1974)
34. C. S. RAJAN, *On the non-vanishing of the first Betti number of hyperbolic three manifolds*, Math. Ann. 330 (2004) 323–329.
35. A. W. REID, *Isospectrality and commensurability of arithmetic hyperbolic 2- and 3-manifolds*, Duke Math. J. 65 (1992), 215–228.
36. M. SUZUKI, *Group Theory I*, Grundlehren der math. Wissen. 247, Springer-Verlag, 1980.

The aim of this section is to establish the relationship between the following conjectures.

Conjecture 7.1. (Lubotzky-Sarnak) *The fundamental group of any closed hyperbolic 3-manifold does not have Property (τ) .*

Conjecture 7.2. *For any closed orientable 3-manifold M with infinite fundamental group, $\pi_1(M)$ does not have Property (τ) .*

Our aim is to prove the following.

Theorem A.1. *If the geometrisation conjecture is true, then Conjectures 7.1 and 7.2 are equivalent.*

Proof. Conjecture 7.2 clearly implies Conjecture 7.1 since every closed hyperbolic 3-manifold has infinite fundamental group. The aim is to prove the converse. Therefore, let M be a closed orientable 3-manifold with infinite fundamental group. Our aim is to show that either $vb_1(M) > 0$ or M is hyperbolic. Conjecture 7.1 then implies that $\pi_1(M)$ does not have Property (τ) . Assuming the geometrisation conjecture, M admits a decomposition into geometric pieces.

Suppose first that M is a connected sum $M_1 \sharp M_2$, say. By geometrisation, $\pi_1(M_i)$ is residually finite and non-trivial (see [9]). In particular, it admits a surjective homomorphism ϕ_i onto a non-trivial finite group F_i . Since $\pi_1(M)$ is isomorphic to $\pi_1(M_1) * \pi_1(M_2)$, we therefore obtain a surjective homomorphism from $\pi_1(M)$ onto $F_1 * F_2$. But, $F_1 * F_2$ has a free group as a finite index subgroup. In particular, $vb_1(F_1 * F_2) > 0$. Hence, $vb_1(M) > 0$.

So, we may assume that M is prime. Suppose that it contains an essential torus. Then it is known in this case that either M is finitely covered by a torus bundle over the circle or $\pi_1(M)$ is large. In particular, $vb_1(M) > 0$ (see [12], [29] and [21]).

Consider now the case where M is prime and atoroidal. By geometrisation, it is therefore either a Seifert fibre space or hyperbolic. In the latter case, the proof is complete. In the former case, the proof divides according to whether the base orbifold has positive, zero or negative Euler characteristic. When it is positive,

the manifold is covered either by S^3 or $S^2 \times S^1$. The former case is impossible, since $\pi_1(M)$ is infinite. In the latter case, $vb_1(M) = 1 > 0$. When the base orbifold has zero Euler characteristic, M is finitely covered by a torus-bundle over the circle. In particular, $vb_1(M) > 0$. When the base orbifold O has negative Euler characteristic, it is hyperbolic. Hence, in this case, $\pi_1(O)$ is large. But the Seifert fibration induces a surjective homomorphism from $\pi_1(M)$ onto $\pi_1(O)$, and so $\pi_1(M)$ is also large. \square

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