Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds

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1. Introduction

Let M be a closed, orientable Riemannian manifold of negative curvature. The *rational length spectrum* $\mathbb{Q}L(M)$ of M is the set of all rational multiples of lengths of closed geodesics of M. The commensurability class of M is the set of all manifolds M' for which M and M' have a common finite unramified cover. Our main result is:

Theorem 1.1. *If M is an arithmetic hyperbolic 3-manifold, then the rational length spectrum and the commensurability class of M determine one another.*

This sharpens [10], where it was shown that the complex length spectrum of M determines its commensurability class.

Suppose M' is an arithmetic hyperbolic 3-manifold which is not commensurable to M. Theorem 1.1 implies $\mathbb{Q}L(M) \neq \mathbb{Q}L(M')$, though by Example 2.1 below it is possible that one of $\mathbb{Q}L(M')$ or $\mathbb{Q}L(M)$ contains the other. By the length formulas recalled in §2.1 and §2.2, each element of $\mathbb{Q}L(M) \cup \mathbb{Q}L(M')$ is a rational multiple of the logarithm of a real algebraic number. As noted by Prasad and Rapinchuk in [9], the Gelfond Schneider Theorem [1] implies that a ratio of such logarithms is transcendental if it is irrational. Thus if $\ell \in \mathbb{Q}L(M) - \mathbb{Q}L(M')$ then ℓ/ℓ' is transcendental for all non-zero $\ell' \in \mathbb{Q}L(M')$.

Recently Prasad and Rapinchuk have shown in [9] that if M is an arithmetic hyperbolic manifold of even dimension, then $\mathbb{Q}L(M)$ and the commensurability class of M determine one another. In addition, they have shown that this is not always true for arithmetic hyperbolic 5-manifolds. However, they have announced a proof that for all locally symmetric spaces associated to a specified absolutely simple Lie group, there are only finitely many commensurability classes of arithmetic lattices giving rise to a given rational length spectrum.

It is known (see [4] pp. 415–417) that for closed hyperbolic manifolds, the spectrum of the Laplace-Beltrami operator action on $L^2(M)$, counting multiplicities, determines the set of lengths of closed geodesics on M (without counting multiplicities). Hence Theorem 1.1 implies:

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Corollary 1.1. *The spectrum of the Laplacian of an arithmetic hyperbolic 3-manifold M determines the commensurability class of M.*

This result was claimed but not proved in [10] where the corresponding result was proved for arithmetic hyperbolic surfaces. There have been many constructions over the years of manifolds with the same Laplace-Beltrami spectrum which are not isometric; see [7], [12], [13], [5], [11] and [3]. Apart from [5] the methods of these papers all provide commensurable manifolds.

We now describe the organization of this paper. Some preliminary results concerning arithmetic Kleinian groups are recalled in §2. Suppose that $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a torsion-free arithmetic Kleinian group associated to an arithmetic hyperbolic three-manifold M. The invariant trace field of Γ is the number field k_Γ generated over $\mathbb{Q}$ by squares of traces of pre-images of elements of Γ in $\mathrm{SL}_2(\mathbb{C})$. It is clear that the commensurability class of M determines $\mathbb{Q}L(M)$. The first step in proving the converse is to show in Theorem 6.1(a) that k_Γ is determined by $\mathbb{Q}L(M)$. We then determine the commensurability class of M from $\mathbb{Q}L(M)$ following ideas similar to those in [10] (see Theorem 6.1(b)).

The main technical work in the proof of Theorem 1.1 is number theoretic. We give in §3 - §5 a detailed analysis of the Galois theory of number fields k having one complex place and of the quadratic extensions of k which embed in a fixed quaternion division algebra over k. One by-product is the following result:

Theorem 1.2. *Suppose that k and k' are number fields having exactly one complex place and the same Galois closure over $\mathbb{Q}$. Then after replacing k' by an isomorphic field, either $k = k'$, or k and k' are quadratic non-isomorphic extensions of a common totally real subfield k^+. In the latter case, the zeta functions $\zeta_k(s)$ and $\zeta_{k'}(s)$ are not equal.*

Since number fields with the same zeta function have the same Galois closure over $\mathbb{Q}$, this implies:

Corollary 1.2. *If k is a number field having one exactly one complex place, then k is determined up to isomorphism by its zeta function.*

This Corollary contrasts with the fact that that there are many examples of number fields which are not determined up to isomorphism by their zeta functions (see [8] and [2]).

2. Preliminaries

In this section we recall some facts about arithmetic Kleinian groups $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$; see [6] for details.

2.1. Length spectra and eigenvalues

Let Γ be a torsion free discrete finite covolume Kleinian group, so that $M = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold. For $\gamma \in \Gamma$, let λ be an eigenvalue of a pre-image of γ in $\mathrm{SL}_2(\mathbb{C})$ for which $|\lambda| > 1$. Then λ is well-defined up to multiplication by ± 1, and we will refer to $\lambda = \lambda(\gamma)$ as an eigenvalue of γ. The axis of γ in $\mathbf{H}^3$ projects to a closed geodesic $c(\gamma)$ in M which depends only on the conjugacy class of γ in Γ. This defines a bijection between the conjugacy classes of hyperbolic elements of Γ and the set of closed geodesics of $\mathbf{H}^3/\Gamma$. The length of $c(\gamma)$ is $\mathrm{l}(\gamma) = 2\ln|\lambda| = \ln|\lambda\overline{\lambda}|$ where $\lambda\overline{\lambda}$ is algebraic over $\mathbb{Q}$.

2.2. Arithmetic Kleinian groups

Let k be a number field with one complex place, and fix a non-real embedding $\rho_k : k \to \mathbb{C}$. Let B/k be a quaternion algebra which is ramified at all real places of k, and let $\rho_B : B \to \mathrm{Mat}_2(\mathbb{C})$ be an embedding extending the embedding ρ_k. Let O_k be the integers of k, and let $\mathcal{O}$ be an O_k-order of B. Define $\mathcal{O}^1$ to be the multiplicative group of elements of $\mathcal{O}$ of reduced norm 1 to k. Then $\rho_B(\mathcal{O}^1)$ is a subgroup of $\mathrm{SL}(2,\mathbb{C})$ whose projection $\overline{\rho}_B(\mathcal{O}^1)$ to $\mathrm{PSL}(2,\mathbb{C})$ is discrete and of finite covolume. A Kleinian group Γ is called arithmetic if it is commensurable with a group of the form $\overline{\rho}_B(\mathcal{O}^1)$ for some k, B, ρ_B and $\mathcal{O}$ of the above kind. If Γ is a subgroup of some $\overline{\rho}_B(\mathcal{O}^1)$, then Γ is called *derived from a quaternion algebra.* It can be shown (see [6, Theorem 8.3.1 and Cor. 8.3.6]) that a Kleinian group Γ of finite covolume is arithmetic if and only if the group $\Gamma^{(2)}$ generated by the squares of elements of Γ is derived from a quaternion algebra, and in this case

$$k = \mathbb{Q}(\{\mathrm{tr}(\gamma^2) : \gamma \in \Gamma\}) = \mathbb{Q}(\{\mathrm{tr}(\eta) : \eta \in \overline{\rho}_B(\mathcal{O}^1)\}). \tag{2.1}$$

The orbifold $M = \mathbf{H}^3/\Gamma$ is a manifold if and only if Γ has no elliptic elements, and this orbifold is compact if and only if B is a division algebra. Our analysis of the commensurability class of M hinges on the following fact (c.f. [6, Thm. 8.4.1]).

Theorem 2.1. *The commensurability class of M determines, and is determined by, the isomorphism class of B as a $\mathbb{Q}$-algebra.*

2.3. Invariant trace fields and quaternion division algebras

In this section we will suppose that k and B satisfy the conditions in §2.2 and that B is a division algebra. We fix an embedding of B into $\mathrm{Mat}_2(\mathbb{C})$, which fixes an embedding of k into $\mathbb{C}$. The following facts are proved in [6, Chapter 12].

Theorem 2.2. *Suppose that Γ is derived from B and that γ is a hyperbolic element of Γ with eigenvalue $\lambda = \lambda(\gamma)$.*

- *i. The field $k(\lambda)$ generated by λ over k is a quadratic extension field of k which embeds into B. If λ is real, then λ has degree 2 over the field $k \cap \mathbb{R}$.*
- *ii. Let L be a quadratic extension of k. Then L embeds in B/k if and only if $L = k(\lambda(\gamma'))$ for some hyperbolic $\gamma' \in \Gamma$. This will be true if and only if no place of k which splits in L is ramified in B.*
- *iii. Let B_1 and B_2 be quaternion algebras over number fields k_1 and k_2. A field isomorphism $\tau : k_1 \to k_2$ extends to an isomorphism $B_1 \to B_2$ of $\mathbb{Q}$-algebras if and only if $\tau(R_1) = R_2$ when R_i is the set of places of B_i which ramify over k_i.*
- *iv. Let $\eta : k(\lambda) \to \mathbb{C}$ be an embedding. Then $\eta(k) \subset \mathbb{R}$ if and only if $|\eta(\lambda)| = 1$, and $\{\lambda, 1/\lambda, \overline{\lambda}, 1/\overline{\lambda}\}$ is the set of conjugates of λ off the unit circle.*

Lemma 2.1. *Let Γ be as in Theorem 2.2. If λ is not real then $k = \mathbb{Q}(\lambda + 1/\lambda)$ and $[\mathbb{Q}(\lambda) : k] = 2$. If λ is real then $k^+ = \mathbb{Q}(\lambda + 1/\lambda)$ is the maximal totally real subfield of k, $[k : k^+] = 2$ and $\mathbb{Q}(\lambda)$ is a degree 2 extension of k^+.*

Proof. Since Γ is derived from a quaternion algebra, $\mathrm{tr}(\gamma) = \lambda + 1/\lambda \in k$ by (2.1). Suppose that $\mathbb{Q}(\lambda + 1/\lambda)$ is a proper subfield of k. Since k has one complex place, all proper subfields of k must be totally real, so $\lambda + 1/\lambda$ is totally real. Because γ is hyperbolic, $|\lambda| \neq 1$, so $\lambda + 1/\lambda \in \mathbb{R}$ implies $\lambda \in \mathbb{R}$. Hence if λ is not real then $k = \mathbb{Q}(\lambda + 1/\lambda)$, and then $[\mathbb{Q}(\lambda) : k] = 2$

by Theorem 2.2(i). For the rest of the proof we suppose that $\lambda \in \mathbb{R}$. Then $F = \mathbb{Q}(\lambda + 1/\lambda)$ is a proper subfield of k, so $\lambda + 1/\lambda$ is totally real. Suppose that $[k : F] \neq 2$. Since k has just two non-real embeddings, the embedding $F \subset \mathbb{R}$ determined by the non-real embedding $k \subset \mathbb{C}$ we have fixed can be extended to an embedding $\eta : k(\lambda) \hookrightarrow \mathbb{C}$ such that $\eta(k) \subset \mathbb{R}$. Theorem 2.2(iv) now implies implies $2 < |\lambda + 1/\lambda| = |\eta(\lambda + 1/\lambda)| = |\eta(\lambda) + \eta(\lambda)^{-1}| \leq 2$ so the contradiction shows $[k : F] = 2$. The last sentence of the lemma now follows from this, Theorem 2.2(i) and the fact that k is not totally real.

We finish this section by showing how Theorem 1.1 can be used to provide proper inclusion of rational length sets.

Example 2.1. Let B and k be as in §2.2, and let B' be a quaternion algebra over k which is not isomorphic to B but which ramifies over every place of k where B ramifies. Let M_1 (resp. M_2) be the manifold defined by a Kleinian group Γ_1 (resp. Γ_2) without elliptic elements which is derived from B (resp. B'). Then by Theorem 2.1, M_1 and M_2 are not commensurable. By Theorem 2.2, if γ is a hyperbolic element of Γ_2 then $L = k(\lambda(\gamma))$ embeds into B over k, where $\lambda(\gamma)$ is a unit of O_L having norm 1 to k. Since O_L embeds into some maximal order $\mathcal{O}$ of B, we conclude that there is a hyperbolic element $\gamma' \in \overline{\rho}_B(\mathcal{O}^1)$ such that $\lambda(\gamma) = \lambda(\gamma')$. A positive integral power of γ' lies in a conjugate of Γ_1, so we conclude from the length formulas of §2.1 that $\mathbb{Q}L(M_2) \subset \mathbb{Q}L(M_1)$. Note that Theorem 1.1 will imply that because M_1 and M_2 are not commensurable, $\mathbb{Q}L(M_1)$ must properly contain $\mathbb{Q}L(M_2)$.

3. Number theoretic results

Let k be a number field, which at the outset we do not assume has one complex place. We will regard k as a subfield of $\mathbb{C}$ via a fixed non-real embedding $\rho_k : k \to \mathbb{C}$. Let k^{cl} be the Galois closure of k over $\mathbb{Q}$ in $\mathbb{C}$. Define $G = \mathrm{Gal}(k^{\mathrm{cl}}/\mathbb{Q})$. Let $n = [k : \mathbb{Q}]$, and let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be the embeddings of k into $\mathbb{C}$. Then $\sigma_i(k) \subset k^{\mathrm{cl}}$ for all i. We fix a left action of G on Σ by letting $\sigma \in G$ send $\sigma_i \in \Sigma$ to $\sigma \circ \sigma_i$. This fixes an embedding of G into the symmetric group $S_n = \mathrm{Perm}(\Sigma)$. Let $c \in G$ be the restriction of complex conjugation on $\mathbb{C}$ to k^{cl}. Let $\mathcal{C}$ be the conjugacy class of c in G.

3.1. Counting archimedean places

Theorem 3.1. *Suppose that H is a subgroup of G. Let $k' = (k^{\mathrm{cl}})^H$, and define $n' = [k' : \mathbb{Q}] = [G : H]$. The numbers $r_1(k')$ and $r_2(k')$ of real and complex places of k' are given by*

$$r_1(k') = \frac{\#(\mathcal{C} \cap H)}{\#\mathcal{C}} \cdot n' \quad \text{and} \quad r_2(k') = n' \left(1 - \frac{\#(\mathcal{C} \cap H)}{\#\mathcal{C}}\right)/2. \tag{3.1}$$

Proof. There is a bijection between the set G/H of left cosets gH of H in G and the emdeddings of $\gamma : k' \to \mathbb{C}$ of k' into $\mathbb{C}$ which sends gH to the restriction of g to k'. An embedding γ is real if and only if it is fixed by complex conjugation. This is equivalent to $cgH = gH$, which is the same as $g^{-1}cg \in H$. Let $Z_G(c)$ be the centralizer of c in G. The map $G \to \mathcal{C}$ which sends $g \in G$ to $g^{-1}cg$ is surjective and defines a bijection between the right cosets $Z_G(c)\backslash G$ and $\mathcal{C}$. This gives

$$\#H \cdot r_1(k') = \#\{g \in G : g^{-1}cg \in H\} = \#(\mathcal{C} \cap H) \cdot \#Z_G(c) = \frac{\#(\mathcal{C} \cap H) \cdot \#G}{\#\mathcal{C}}.$$

The equalities (3.1) now follow from this and $[G : H] = n' = r_1(k') + 2r_2(k')$.

Corollary 3.1. *One has $r_2(k') = 1$ if and only if*

$$\#\mathcal{C} - \#(\mathcal{C} \cap H) = \frac{2\#\mathcal{C}}{n'}. \tag{3.2}$$

3.2. Fields with one complex place and the same Galois closure

In this section we will make the following hypothesis.

Hypothesis 3.1 *The fields k and $k' = (k^{\mathrm{cl}})^H$ have exactly one complex place, and the same Galois closure k^{cl} over $\mathbb{Q}$. After replacing k by k', if necessary, we can suppose $n' = [k' : \mathbb{Q}] = [G : H] \geq n = [k : \mathbb{Q}]$.*

We may order the set $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of complex embeddings of k in such a way that σ_1 is not real, $\sigma_2 = \overline{\sigma_1} = c \circ \sigma_1$ is the complex conjugate of σ_1, and $\sigma_3, \ldots, \sigma_n$ are real. Let $G(1)$ be the stabilizer of σ_1 under the action of $G = \mathrm{Gal}(k^{\mathrm{cl}}/\mathbb{Q})$ on Σ. We may identify k with $(k^{\mathrm{cl}})^{G(1)} \subset \mathbb{C}$ via $\sigma_1 : k \to \mathbb{C}$.

Definition 3.1. *Identifying the element σ_i of Σ with the integer i fixes an identification of $S_n = \mathrm{Perm}(\Sigma)$ with the permutations of $\{1, \ldots, n\}$. This identifies the complex conjugation $c \in G$ with the transposition $(1, 2)$. The conjugacy class $\mathcal{C}$ is thus a set of transpositions in S_n. For all subgroups Γ of G, define the conjugation graph $\mathcal{C}(\Gamma)$ of Γ to be the union over all transpositions $(i, j) \in \mathcal{C} \cap \Gamma$ of the undirected graph which has vertices i and j and an edge between these vertices.*

Proposition 3.1. *For all subgroups Γ of G, the conjugation graph $\mathcal{C}(\Gamma)$ is a finite (possibly empty) disjoint union of complete graphs. If Γ acts transitively on $\{1, \ldots, n\}$ there are two possibilities:*

i. $\mathcal{C}(\Gamma)$ is empty, or
ii. There is a divisor $\ell(\Gamma) > 1$ of n such that $\mathcal{C}(\Gamma)$ is the disjoint union of $n/\ell(\Gamma)$ complete graphs, each of which have $\ell(\Gamma)$ vertices.

Proof. For the first statement, it is enough to show that if T is a (non-empty) connected component of $\mathcal{C}(\Gamma)$, then T must be a complete graph. Let $\{t_1, \ldots, t_m\}$ be the vertices in T. Then $m \geq 2$ by the construction of $\mathcal{C}(\Gamma)$. Since T is connected, we can order the t_i so that for all $i \geq 2$, there is an integer $j(i)$ such that $1 \leq j(i) < i$ and $(t_i, t_{j(i)})$ is a transposition in $\mathcal{C} \cap \Gamma$. Then the transpositions $\{(t_i, t_{j(i)})\}_{i=1}^m$ generate $\mathrm{Perm}(t_1, \ldots, t_m)$, so T is a complete graph. The fact that (i) or (ii) of the Proposition hold if Γ acts transitively on $\{1, \ldots, n\}$ is clear from the fact that Γ then acts transitively on the connected components of $\mathcal{C}(\Gamma)$.

Corollary 3.2. *Since $\Gamma = G$ acts transitively on $\{1, \ldots, n\}$, and $\mathcal{C}(G)$ contains $c = (1, 2)$, we can define $\ell \geq 2$ to be the divisor $\ell(G)$ of n. The number of elements of $\mathcal{C}$ is $(n/\ell)\ell(\ell - 1)/2 = n(\ell - 1)/2$. The normal subgroup N generated by the set $\mathcal{C}$ of all complex conjugations in G is isomorphic to the direct product over the connected components of $\mathcal{C}(G)$ of the symmetric groups on the vertices in each component. Thus $N \cong (S_\ell)^{n/\ell}$.*

Proposition 3.2. *Let H be a subgroup of G as in Hypothesis 3.1, and let $\ell = \ell(G)$ be as in Corollary 3.2. Then $n = n'$ and there are the following possibilities for the conjugation graph $\mathcal{C}(H)$:*

i. *If $\ell > 2$, then $\mathcal{C}(H)$ is the disjoint union of $(n/\ell)-1$ complete graphs on ℓ vertices together with a complete graph on $\ell-1$ vertices. There is a unique integer j in the range $1 \le j \le n$ such that j is not a vertex of $\mathcal{C}(H)$, and the edges of $\mathcal{C}(H)$ are exactly the edges of $\mathcal{C}(G)$ which do not have j as a vertex.*
ii. *If $\ell = 2$, then $\mathcal{C}(H)$ is the union of $(n/\ell)-1$ complete graphs on $\ell = 2$ vertices. There are exactly two distinct integers j in the range $1 \le j \le n$ which are not vertices of $\mathcal{C}(H)$.*

Proof. Since $n' \ge n$ in Hypothesis 3.1, corollaries 3.1 and 3.2 show

$$\#\mathcal{C} - \#(\mathcal{C} \cap H) = \frac{2\#\mathcal{C}}{n'} = (\ell-1)\frac{n}{n'} \le (\ell-1). \tag{3.3}$$

Because $\ell \ge 2$, we conclude that $\#\mathcal{C}-\#(\mathcal{C}\cap H) > 0$. Hence by Proposition 3.1, $\mathcal{C}(H) \ne \mathcal{C}(G)$ is a union of complete subgraphs of $\mathcal{C}(G)$ which contains no isolated points. By (3.3) there are at most $\ell-1$ edges of $\mathcal{C}(G)$ not in $\mathcal{C}(H)$, where $\mathcal{C}(G)$ is a disjoint union of n/ℓ complete graphs on ℓ vertices. If some component T of $\mathcal{C}(G)$ contains two components T_1 and T_2 of $\mathcal{C}(H)$, we can order the T_i so that $\#V_1 \le \ell/2$ and $\#(V - V_1) \ge \#V_2 \ge 2$ when V (resp. V_i) is the set of vertices of T (resp. T_i) . This leads to at least $2 \cdot \ell/2 = \ell$ edges of $\mathcal{C}(G)$ not in $\mathcal{C}(H)$, contradicting (3.3). Hence the intersection of $\mathcal{C}(H)$ with each connected component of $\mathcal{C}(G)$ is a complete graph, so there must be a vertex j of $\mathcal{C}(G)$ which is not a vertex of $\mathcal{C}(H)$. There are $\ell-1$ edges of $\mathcal{C}(G)$ having this j as a vertex, and none of these are in $\mathcal{C}(H)$. Hence by (3.3), these are exactly the edges of $\mathcal{C}(G)$ not in $\mathcal{C}(H)$, and this leads to (i) and (ii).

Corollary 3.3. *Suppose that $\ell > 2$ in Proposition 3.2, and let j be the integer specified in part (i) of this Proposition. Then H equals the subgroup $G(j)$ of G which stabilizes j, and k' is a conjugate field to k. In particular, k and k' are isomorphic as fields. Finally, if k^+ is the maximal totally real subfield of k, then $[k : k^+] > 2$.*

Proof. The action of H on $\mathcal{C}(G)$ sends $\mathcal{C}(H)$ to itself, so this action must fix the unique vertex j not in $\mathcal{C}(H)$. Hence $H \subset G(j)$, so $H = G(j)$ because $n' = [G : H] = n = [G : G(j)]$. Since G acts transitively on $\{1, \ldots, n\}$, $G(j) = H$ is conjugate to $G(1)$, so k and k' are isomorphic. If $[k : k^+] = 2$, then $N \cap G(1)$ must have index two in N when N is the normal subgroup of G generated by all complex conjugations in G. We see from Corollary 3.2 that N contains the symmetric group on the set of ℓ vertices which form the connected component of $\mathcal{C}(G)$ which contains the vertex 1 fixed by $G(1)$. Thus $[N : N\cap G(1)] \ge [S_\ell : S_{\ell-1}] = \ell > 2$ so $[k : k^+] > 2$.

For the rest of this section we suppose $\ell = 2$ in Proposition 3.2. We label the real embeddings $\{\sigma_3, \ldots, \sigma_n\}$ of k into $\mathbb{R}$ in such a way that the conjugacy class $\mathcal{C}$ of complex conjugations in G is the set of $n/2$ commuting transpositions $\{(1,2),(3,4),(5,6),\ldots,(n-1,n)\}$. The group $N = \prod_{c' \in \mathbb{C}} \mathbb{Z}/2 \cong (\mathbb{Z}/2)^{n/2}$ generated by the elements of $\mathcal{C}$ is normal in G, and $\overline{G} = G/N$ acts on N via the permutation action of G on $\mathcal{C}$. Let $\pi : G \to \overline{G} = G/N$ be the natural quotient homomorphism.

Proposition 3.3. *When $\ell = 2$, there is a unique homomorphism $s : \overline{G} \to G$ which is a section to π such that $s(\overline{g})$ permutes the set $\{1, 3, \ldots, n-1\}$ of odd integers in $\{1, \ldots, n\}$. This makes G the semi-direct product of N and $\overline{G}$. The conjugation action of $\overline{G}$ on $\mathcal{C}$ is faithful and transitive.*

Proof. Since G permutes the elements of $\mathcal{C} = \{(1,2), \ldots, (n-1,n)\}$, there is for each $g \in G$ a unique $n \in N$ such that ng permutes the elements of $\{1, 3, \ldots, n-1\}$. The set map

$s : \overline{G} \to G$ defined by $s(Ng) = ng$ is the unique section of π for which $s(Ng)$ permutes $\{1, 3, \ldots, n-1\}$ for all g. The uniqueness of s implies s is a homomorphism. The action of $s(\overline{G})$ on $\{1, 3, \ldots, n-1\}$ is faithful, and the action of G on $\{1, 2, \ldots, n\}$ is transitive, so it follows that the action of $\overline{G}$ on $\mathcal{C}$ is faithful and transitive.

Proposition 3.4. *Suppose that $\ell = 2$, and that H is not conjugate to $G(1)$ in G. After replacing H by a conjugate by an element of G, which does not change the isomorphism class of $k' = (k^{\mathrm{cl}})^H$, we can assume that the two vertices which do not appear in $\mathcal{C}(H)$ are 1 and 2. Let $\overline{G}(1)$ be the subgroup of $\overline{G}$ which fixes the transposition $c = (1, 2)$ in $\mathcal{C}$, and let $\tilde{G}(1) = \pi^{-1}(\overline{G}(1))$.*

- *a. The group $\tilde{G}(1)$ is the direct sum of $G(1)$ and the cyclic group $\langle c\rangle$ of order 2.*
- *b. One has $s(\overline{G}(1)) \subset G(1)$, and the group $G(1)$ is the semi-direct product $N_0.s(\overline{G}(1))$.*
- *c. Let $\xi : \tilde{G}(1) \to \tilde{G}(1)/G(1) = \mathbb{Z}/2$ be the surjection resulting from (a). There is a unique character $\chi : \tilde{G}(1) \to \mathbb{Z}/2$ of order two inflated from a character of $\overline{G}(1)$ for which H the kernel of the character $\xi + \chi : \tilde{G}(1) \to \mathbb{Z}/2$ defined by $(\xi + \chi)(g) = \xi(g) + \chi(g)$.*
- *d. Conversely, if χ is the inflation to $\tilde{G}(1)$ of any order two character of $\overline{G}(1)$, and we define H to be the kernel of the sum character $\xi + \chi : \tilde{G}(1) \to \mathbb{Z}/2$, then $k' = (k^{\mathrm{cl}})^H$ has exactly one complex place and Galois closure k^{cl} over $\mathbb{Q}$, and k' is not isomorphic to k.*

Proof. Any element $g \in \tilde{G}(1)$ fixes $c = (1, 2)$, so g permutes $\{1, 2\}$ and commutes with c. This leads to part (a). Since $s(\overline{G}(1))$ sends odd integers to odd integers and permutes $\{1, 2\}$ it must lie in $G(1)$. We have $H \cap N = N_0 = G(1) \cap N$ from Proposition 3.2(ii). The sequence

$$1 \longrightarrow G(1) \cap N \longrightarrow G(1) \xrightarrow{\pi} \overline{G}(1) \longrightarrow 1$$

is exact since $s(\overline{G}(1)) \subset G(1)$, and this leads to part (b). Since the action of H on $\mathcal{C}$ must fix the unique element $c = (1, 2)$ of $\mathcal{C}$ which is not in H, we have $\pi(H) \subset \overline{G}(1)$, so $H \subset \pi^{-1}(\overline{G}(1)) = \tilde{G}(1)$. Since $[G : H] = [G : G(1)]$ and $[\tilde{G}(1) : G(1)] = 2$, H must be an index two subgroup of

$$\tilde{G}(1) = \langle c\rangle \times G(1) = \langle c\rangle \times (N_0.s(\overline{G}(1))).$$

Since $H \cap N = G(1) \cap N = N_0$ has index 2 in N, and $c \notin H$, this leads to part (c). Finally, suppose we construct H and k' as in part (d). Then $n' = [G : H] = [k' : \mathbb{Q}]$ equals $n = [G : G(1)] = [k : \mathbb{Q}]$. We have $\mathcal{C} \cap H = \{(3, 4), \ldots, (n-1, n)\}$ by the definition of H as the kernel of $\xi + \chi$. So $k' = (k^{\mathrm{cl}})^H$ has exactly one complex place by Theorem 3.1. If k' were isomorphic to k, so that H is conjugate to $G(1)$, then $H = G(j)$ with $j \in \{1, 2\}$ in view of $\mathcal{C} \cap H$. Let σ be an element of $s(\overline{G}(1)) \subset G(1)$ such that $\chi(\sigma) \neq 0$ in $\mathbb{Z}/2$. Then $\xi(\sigma) = 0 \neq \xi(c)$ and σ fixes both 1 and 2 since it acts both on $\{1, 3, \ldots, n-1\}$ and $\{1, 2\}$. Hence $\xi + \chi$ is non-trivial on $\sigma \in G(1)$ and trivial on $c\sigma \notin G(2)$. This shows $H = \mathrm{Ker}(\xi + \chi)$ is not $G(1)$ or $G(2)$ so k' and k are not isomorphic. To show $(k')^{\mathrm{cl}} = k^{\mathrm{cl}}$ it will suffice to show that H contains no non-trivial normal subgroup J of G. The group $\pi(H) = \pi(G(1)) = \overline{G}(1)$ contains no non-trivial normal subgroup of $\pi(G) = \overline{G}$ since by Proposition 3.3, $\overline{G}$ is a transitive subgroup of $\mathrm{Perm}(\mathcal{C})$, and $\overline{G}(1)$ is the subgroup of $\overline{G}$ which stabilizes $c \in \mathrm{Perm}(\mathcal{C})$. It follows that $\pi(J)$ must be trivial, so $J \subset N = \mathrm{Ker}(\pi)$. However, $H \cap N = G(1) \cap N = N_0$, so J would be a non-trivial normal subgroup of G contained in $G(1)$. There is no such subgroup because G acts faithfully and transitively on $\{1, \ldots, n\}$.

Corollary 3.4. *In all cases of Proposition 3.3, the fields* $k = (k^{\mathrm{cl}})^{G(1)}$ *and* $k' = (k^{\mathrm{cl}})^H$ *are quadratic extensions of the totally real field* $k^+ = (k^{\mathrm{cl}})^{\tilde{G}(1)}$*. The Galois closure of* k^+ *over* $\mathbb{Q}$ *is* $(k^{\mathrm{cl}})^N$*. The field* k^+ *(and hence* $(k^+)^{\mathrm{cl}}$*) is determined up to isomorphism by* $(k^+)^{\mathrm{cl}}$*.*

Proof. The field k^+ is totally real because $\tilde{G}(1)$ contains $\mathcal{C}$. We have $[k : k^+] = [\tilde{G}(1) : G(1)] = 2 = [\tilde{G}(1) : H] = [k' : k^+]$. The group $\tilde{G}(1)$ contains the normal subgroup N of G, while $\tilde{G}(1)/N = \overline{G}(1)$ contains no normal subgroup of $\overline{G} = G/N$ by the argument at the end of the proof of Proposition 3.3. This means that N is the maximal normal subgroup of G contained in $\tilde{G}(1)$, so k^+ has Galois closure $(k^{\mathrm{cl}})^N$ over $\mathbb{Q}$. We have $\mathrm{Gal}((k^+)^{\mathrm{cl}}/\mathbb{Q}) = G/N = \overline{G}$, and both $G(1)$ and H have the same image $\overline{G}(1)$ in $\overline{G}$. Thus $k^+ = ((k^+)^{\mathrm{cl}})^{\overline{G}(1)}$ is determined up to isomorphism by $(k^+)^{\mathrm{cl}}$.

In view of Corollaries 3.3 and 3.4, the following result completes the proof of Theorem 1.2.

Proposition 3.5. *Suppose that* $H \neq G(1)$ *in Proposition 3.3. Then the zeta functions of* $k = (k^{\mathrm{cl}})^{G(1)}$ *and* $k' = (k^{\mathrm{cl}})^H$ *are not equal.*

Proof. By Proposition 3.4(b,c) there is a $\gamma \in s(\overline{G}(1)) \subset G(1)$ which is not in H. It will be enough to show that if $B(\gamma)$ is the conjugacy class of γ in G, then

$$\#(B(\gamma) \cap H) < \#(B(\gamma) \cap G(1)). \tag{3.4}$$

Define $\overline{B}(\pi(\gamma))$ to be the conjugacy class of $\pi(\gamma)$ in $\overline{G} = G/N$. Then π gives a surjection $\pi_B : B(\gamma) \to \overline{B}(\pi(\gamma))$. We claim that

$$\pi_B(B(\gamma) \cap H) \subset \overline{B}(\pi(\gamma)) \cap \overline{G}(1) = \pi_B(B(\gamma) \cap G(1)). \tag{3.5}$$

The first containment follows from $H \subset \tilde{G}(1)$ and $\tilde{G}(1) = \pi^{-1}(\overline{G}(1))$, and the non-trivial part of the second equality is the assertion that $\overline{B}(\pi(\gamma)) \cap \overline{G}(1) \subset \pi(B(\gamma) \cap G(1))$. Suppose that $\bar{\iota} \in \overline{G}$ and that $\bar{\iota}\pi(\gamma)\bar{\iota}^{-1} \in \overline{B}(\pi(\gamma)) \cap \overline{G}(1)$. Applying the section homomorphism $s : \overline{G} \to G$ and using the fact that $\gamma = s(\pi(\gamma))$ because $\gamma \in s(\overline{G}(1))$, we find $\iota\gamma\iota^{-1} \in s(\overline{G}(1)) \subset G(1)$ when $\iota = s(\bar{\iota})$. Thus $\iota\gamma\iota^{-1} \in B(\gamma) \cap G(1)$ satisfies $\pi_B(\iota\gamma\iota^{-1}) = \bar{\iota}\pi(\gamma)\bar{\iota}^{-1}$, so (3.5) holds.

We now claim that

$$\pi_B^{-1}(\pi_B(B(\gamma) \cap G(1))) = B(\gamma) \cap G(1) \tag{3.6}$$

where as before $\pi_B : B(\gamma) \to \overline{B}(\pi(\gamma))$ is the map induced by $\pi : G \to \overline{G}$. One containment is obvious. Suppose now that $z\gamma z^{-1}$ is an element of $\pi_B^{-1}(\pi_B(B(\gamma) \cap G(1)))$ for some $z \in G$. Since G is the semi-direct product $N.s(\overline{G})$, we can write $z = n \cdot s(g)$ for some $g \in \overline{G}$. Then $s(g)\gamma s(g)^{-1} \in s(\overline{G})$ and

$$\pi(s(g)\gamma s(g)^{-1}) \in \pi_B(B(\gamma) \cap G(1)) \subset \pi(G(1)) = \overline{G}(1).$$

Hence $\gamma' = s(g)\gamma s(g)^{-1} \in s(\overline{G}(1)) \subset G(1)$ relative to the semi-direct product description $G = N.s(\overline{G})$. Now

$$z\gamma z^{-1} = n\gamma' n^{-1} = (n\gamma' n^{-1}\gamma'^{-1})\gamma' = n(n^{-1})^{\gamma'}\gamma' \tag{3.7}$$

where $(n^{-1})^{\gamma'}$ is the image of $n^{-1} \in N$ under the conjugation action of $\gamma' \in G(1)$. Recall that

$$N = \prod_{c' \in \mathcal{C}} (\mathbb{Z}/2) \tag{3.8}$$

and that the action of G on N factors through $\overline{G} = G/N$ and is via the permutation action of $\overline{G}$ on $\mathcal{C}$. The elements of $\overline{G}(1)$ fix the element c of $\mathcal{C}$. So we conclude that for all $n \in N$, the c component of $n(n^{-1})^{\gamma'}$ relative to the description of N in (3.8) is 0. Thus $n(n^{-1})^{\gamma'}$ lies in the subgroup $N_0 \subset H \cap G(1)$. Since $\gamma' \in s(\overline{G}(1)) \subset G(1)$, we find from (3.7) that $z\gamma z^{-1} = n\gamma' n^{-1} \in G(1)$, and clearly $z\gamma z^{-1} \in B(\gamma)$. This completes the proof of (3.6).

In view of (3.5) and (3.6), we have

$$B(\gamma) \cap H \subset \coprod_{\tau \in \pi_B(B(\gamma)\cap G(1))} \pi_B^{-1}(\tau) = B(\gamma) \cap G(1). \tag{3.9}$$

where the coproduct just means the disjoint union of sets. Now note that when

$$\tau = \pi_B(\gamma) \in \pi_B(B(\gamma) \cap G(1))$$

we have $\gamma \in \pi_B^{-1}(\tau)$, but $\gamma \notin H$ by our choice of γ. Thus $\#(B(\gamma) \cap H) < \#(B(\gamma) \cap G(1))$ which completes the proof of Proposition 3.5.

Remark 3.1. The smallest possible degree over $\mathbb{Q}$ of non-isomorphic fields k and k' as in Theorem 1.2 is 6, and it is not hard to check that all minimal degree examples can be constructed in the following way. Let k^+ be a totally real non-Galois cubic extension of $\mathbb{Q}$. The Galois closure $(k^+)^{\mathrm{cl}}$ is then a totally real S_3 extension of $\mathbb{Q}$, so it contains a unique real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d > 0$ is a square free integer. Suppose that $\alpha \in k^+$ is positive at two of the real places of k^+ and negative at the other real place. Then k and k' can be taken to be isomorphic to $k^+(\sqrt{\alpha})$ and $k^+(\sqrt{d \cdot \alpha})$, respectively. A numerical example is given by letting α be the unique negative real root of $f(x) = x^3 - 4x + 1$, $k^+ = \mathbb{Q}(\alpha)$, $k = k^+(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{\alpha})$ and $k' = k^+(\sqrt{37 \cdot \alpha}) = \mathbb{Q}(\sqrt{37 \cdot \alpha})$.

4. Galois closures of fields generated by eigenvalues and logarithms of lengths.

Throughout this section we assume that Γ is an arithmetic Kleinian group derived from a quaternion algebra B/k. We view k as a subfield of $\mathbb{C}$ via a fixed a non-real embedding $\rho_k : k \to \mathbb{C}$. Let $\gamma \in \Gamma$ be a hyperbolic element with eigenvalue $\lambda = \lambda(\gamma)$, so $|\lambda| > 1$. We assume the notations of §3 concerning k. Let k^+ be the maximal totally real subfield of k.

Proposition 4.1. *If λ is real, then $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda\overline{\lambda}) = \mathbb{Q}(\lambda^2)$, so $\mathbb{Q}(\lambda)^{\mathrm{cl}} = \mathbb{Q}(\lambda\overline{\lambda})^{\mathrm{cl}}$.*

Proof. Since γ^2 has eigenvalue λ^2, we conclude from Lemma 2.1 that $k^+ \subset \mathbb{Q}(\lambda^2) \subset \mathbb{Q}(\lambda)$ and that each of $\mathbb{Q}(\lambda^2)$ and $\mathbb{Q}(\lambda)$ have degree 2 over k^+. Hence $\mathbb{Q}(\lambda^2) = \mathbb{Q}(\lambda)$.

Lemma 4.1. *Suppose that λ is not real. Then $[\mathbb{Q}(\lambda, \overline{\lambda}) : \mathbb{Q}(\lambda\overline{\lambda})] = 2$ and every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)^{\mathrm{cl}}/\mathbb{Q}(\lambda\overline{\lambda}))$ either fixes or interchanges λ and $\overline{\lambda}$.*

Proof. Since λ is not real, complex conjugation takes λ to $\overline{\lambda} \neq \lambda$ and fixes $\mathbb{Q}(\lambda\overline{\lambda})$. The Lemma now follows from the fact shown in Theorem 2.2(iv) that λ and $\overline{\lambda}$ have larger complex absolute value than any of the other conjugates of λ.

Lemma 4.2. *Suppose that $\ell = 2$ in Corollary 3.2 and that λ is not real. Then $k = \mathbb{Q}(\lambda + \lambda^{-1})$ is a degree two extension of the totally real field k^+. There are two possibilities:*

a. The field $\mathbb{Q}(\lambda) = \mathbb{Q}(\overline{\lambda})$ is quadratic over k and Galois of degree four over k^+.
b. The extensions $\mathbb{Q}(\lambda)$ and $\mathbb{Q}(\overline{\lambda})$ are distinct quadratic extensions of k. The extension $\mathbb{Q}(\lambda, \overline{\lambda})$ is a dihedral extension of degree 8 of k^+. The field $\mathbb{Q}(\lambda\overline{\lambda})$ is a non-Galois degree four extension of k^+ inside $\mathbb{Q}(\lambda, \overline{\lambda})$, and $\mathbb{Q}(\lambda\overline{\lambda}) \cap k = k^+$.

Proof. We know from Lemma 2.1 that $k = \mathbb{Q}(\lambda + \lambda^{-1})$, so $\lambda + \lambda^{-1}$ is not real. By Corollary 3.4, k is stable under complex conjugation, and $k^+ = k \cap \mathbb{R}$ is the maximal totally real subfield of k, with $[k : k^+] = 2$. By Theorem 2.2(i), $[\mathbb{Q}(\lambda) : k] = [\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1})] = 2$.

If $\mathbb{Q}(\lambda) = \mathbb{Q}(\overline{\lambda})$, complex conjugation defines an automorphism of $\mathbb{Q}(\lambda)$ over k^+ which gives a non-trivial automorphism of k. Then $[\mathbb{Q}(\lambda) : k] = [k : k^+] = 2$ implies $\mathbb{Q}(\lambda)/k^+$ is Galois of degree 4.

Now suppose $\mathbb{Q}(\lambda) \neq \mathbb{Q}(\overline{\lambda})$. Then $\mathbb{Q}(\lambda)/k^+$ is a quartic extension containing the quadratic extension k/k^+. Complex conjugation sends k to k, fixes k^+ and carries $\mathbb{Q}(\lambda)$ to $\mathbb{Q}(\overline{\lambda})$. This implies $\mathbb{Q}(\lambda, \overline{\lambda})$ is a dihedral extension of k^+ of degree 8. By Lemma 4.1, $[\mathbb{Q}(\lambda, \overline{\lambda}) : \mathbb{Q}(\lambda\overline{\lambda})] = 2$. The rest of part (b) follows from this and the fact that $\mathbb{Q}(\lambda\overline{\lambda}) = \mathbb{Q}(\lambda) \cap \mathbb{R} \supset k^+$ is fixed by complex conjugation while k is not.

Proposition 4.2. *Suppose that λ is not real, and that either $\ell > 2$ or that $\ell = 2$ and that option (b) of Lemma 4.2 holds. Then the Galois closure $\mathbb{Q}(\lambda)^{\mathrm{cl}}$ of $\mathbb{Q}(\lambda)$ over $\mathbb{Q}$ equals $\mathbb{Q}(\lambda\overline{\lambda})^{\mathrm{cl}}$.*

Proof. If $\ell = 2$, Lemma 4.2(b) implies $\mathbb{Q}(\lambda\overline{\lambda})$ is a non-Galois quartic extension of k^+ inside the dihedral degree 8 extension $\mathbb{Q}(\lambda, \overline{\lambda})$ of k^+. Hence the Galois closure of $\mathbb{Q}(\lambda\overline{\lambda})$ over k^+ is $\mathbb{Q}(\lambda, \overline{\lambda})$, and this implies that $\mathbb{Q}(\lambda\overline{\lambda})^{\mathrm{cl}} = \mathbb{Q}(\lambda)^{\mathrm{cl}}$.

The remaining case to consider is when λ is complex and $\ell > 2$. Then $\mathbb{Q}(\lambda)$ is a quadratic extension of $k = \mathbb{Q}(\lambda + \lambda^{-1})$ by Lemma 2.1. The inclusion $k^{\mathrm{cl}} \subset \mathbb{Q}(\lambda)^{\mathrm{cl}}$ gives a surjection $q : \mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\lambda)^{\mathrm{cl}}/\mathbb{Q}) \to \mathrm{Gal}(k^{\mathrm{cl}}/\mathbb{Q}) = G$. Define $\mathcal{H} = \mathrm{Gal}(\mathbb{Q}(\lambda)^{\mathrm{cl}}/\mathbb{Q}(\lambda\overline{\lambda})) \subset \mathcal{G}$. It will suffice to show that the intersection $\mathcal{J}$ of all the conjugates of $\mathcal{H}$ in $\mathcal{G}$ equals the trivial subgroup $\{e\}$.

We know by Lemma 4.1 that every $\tilde{\gamma} \in \mathcal{H}$ either fixes each of λ and $\overline{\lambda}$ or interchanges them. If all $\tilde{\gamma} \in \mathcal{J}$ fix λ, then since $\mathcal{J}$ is normal in $\mathcal{G}$ we will see that $\mathcal{J}$ fixes all of $\mathbb{Q}(\lambda)^{\mathrm{cl}}$, so $\mathcal{J} = \{e\}$ and we are done. We may thus suppose that there is an element $\tilde{\gamma} \in \mathcal{J}$ for which $\tilde{\gamma}(\lambda) = \overline{\lambda}$ and $\tilde{\gamma}(\overline{\lambda}) = \lambda$. Then $\tilde{\gamma}(\lambda + \lambda^{-1}) = \overline{\lambda} + \overline{\lambda}^{-1}$. Since $k = \mathbb{Q}(\lambda + \lambda^{-1})$, we conclude that $\gamma = q(\tilde{\gamma}) \in G$ satisfies $\gamma\sigma_1 = \sigma_2$, where σ_1 and σ_2 are as before the non-real complex conjugate embeddings of k into $\mathbb{C}$. Since $\ell > 2$, the description of the conjugation graph $\mathcal{C}(G)$ in Proposition 3.1 and Corollary 3.2 shows that there is a $j \notin \{1, 2\}$ such that $\tau\sigma_1 = \sigma_1$ and $\tau\sigma_2 = \sigma_j$ for some $\tau \in \mathcal{G}$. Then $\tau\gamma\tau^{-1}\sigma_1 = \sigma_j$.

Let $\tilde{\tau} \in \mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\lambda)^{\mathrm{cl}}/\mathbb{Q})$ be any element for which $q(\tilde{\tau}) = \tau$. By the definition of $\mathcal{J}$ as the intersection of all the conjugates of $\mathcal{H}$ in $\mathcal{G}$, we know that $\tilde{\gamma} \in \mathcal{H}$ and $\tilde{\tau}\tilde{\gamma}\tilde{\tau}^{-1} \in \mathcal{H}$. We have $(\tilde{\tau}\tilde{\gamma}\tilde{\tau}^{-1})(\lambda + 1/\lambda) = \sigma_j(\lambda + 1/\lambda)$. On the other hand, $\tilde{\tau}\tilde{\gamma}\tilde{\tau}^{-1} \in \mathcal{H}$ and Lemma 4.1 show $(\tilde{\tau}\tilde{\gamma}\tilde{\tau}^{-1})(\lambda + 1/\lambda) \in \{\lambda + 1/\lambda, \overline{\lambda} + 1/\overline{\lambda}\}$. This would give $\sigma_j(\lambda + 1/\lambda) = \sigma_i(\lambda + 1/\lambda)$ for some $i \in \{1, 2\}$, which is impossible since $k = \mathbb{Q}(\lambda + 1/\lambda)$ and $j \notin \{1, 2\}$. The contradiction completes the proof of Proposition 4.2.

5. Cebotarev Results

We will assume the notations of the previous two sections. Let $b : \Gamma \to \mathbb{Z}^+$ be a function on hyperbolic elements of Γ and let $l_b(\gamma) = (\lambda(\gamma)\overline{\lambda(\gamma)})^{b(\gamma)}$ for $\gamma \in \Gamma$.

5.1. The intersection of Galois closures

Lemma 5.1. *The intersection $\cap_{\gamma\in\Gamma}\mathbb{Q}(l_b(\gamma))^{\mathrm{cl}}$ is equal to k^{cl} unless k is a quadratic extension of a totally real field k^+, and in the latter case this intersection equals $(k^+)^{\mathrm{cl}}$. These two alternatives correspond to $\ell > 2$ and $\ell = 2$ in the notation of Corollary 3.2.*

Proof. Suppose first that $\ell > 2$. Then the maximal totally real subfield k^+ of k has $[k : k^+] > 2$ by Corollary 3.3. On applying Lemma 2.1 to $\gamma^{b(\gamma)}$ we see that $\lambda(\gamma)^{b(\gamma)}$ is not real. Lemma 2.1 and Proposition 4.2 now show

$$\mathbb{Q}(\lambda(\gamma)^{b(\gamma)}) = k(\lambda(\gamma)) \quad \text{and} \quad \mathbb{Q}(l_b(\gamma))^{\mathrm{cl}} = \mathbb{Q}(\lambda(\gamma)^{b(\lambda)})^{\mathrm{cl}} \supset k^{\mathrm{cl}}. \tag{5.1}$$

Theorem 2.2(i) also shows that $\mathbb{Q}(\lambda(\gamma)^{b(\lambda)})$ is a quadratic extension of k, so $\mathbb{Q}(\lambda(\gamma)^{b(\lambda)})^{\mathrm{cl}}$ is an elementary abelian two-extension of k^{cl}. Hence to show that $\cap_{\gamma\in\Gamma}\mathbb{Q}(l_b(\gamma))^{\mathrm{cl}}$ is equal to k^{cl}, it will be enough to show that for each quadratic extension L of k^{cl} there is a hyperbolic element $\gamma \in \Gamma$ such that $\mathbb{Q}(\lambda(\gamma)^{b(\lambda)})^{\mathrm{cl}} \cap L = k^{\mathrm{cl}}$.

By the Cebotarev density Theorem, we can find a rational prime p which splits completely in k^{cl}, does not lie under a prime of k which ramifies in B, and for which some prime P over p in k^{cl} is inert to L. By the approximation theorem for absolute values of k, we can construct a quadratic extension F of k which is ramified at each place of k which ramifies in B, and such that each prime over p in k splits in F. By Theorem 2.2(ii) there is a hyperbolic element $\gamma \in \Gamma$ such that $k(\lambda(\gamma))$ is isomorphic to F. Then $\mathbb{Q}(\lambda(\gamma)^b) = k(\lambda(\gamma)^b) = k(\lambda(\gamma)) = F$ for all positive integers b by (5.1). Since p splits completely in F by construction, we conclude that p splits in $\mathbb{Q}(\lambda(\gamma)^{b(\lambda)})^{\mathrm{cl}} = (F)^{\mathrm{cl}}$. Since p does not split in the quadratic extension L of k^{cl}, this forces $\mathbb{Q}(\lambda(\gamma)^{b(\lambda)})^{\mathrm{cl}} \cap L = k^{\mathrm{cl}}$ as required.

Suppose now that $\ell = 2$. Then $[k : k^+] = 2$ by Corollary 3.4, and $\mathbb{Q}(l_b(\gamma)) \supset k^+$ by Lemma 4.2, so

$$(k^+)^{\mathrm{cl}} \subset \cap_{\gamma\in\Gamma}\mathbb{Q}(l_b(\gamma))^{\mathrm{cl}}. \tag{5.2}$$

Since $k^{\mathrm{cl}}/(k^+)^{\mathrm{cl}}$ is a two-extension, the right side of (5.2) is also a two-extension of $(k^+)^{\mathrm{cl}}$. Hence it will suffice to show for each quadratic extension L of $(k^+)^{\mathrm{cl}}$ it is possible to find a hyperbolic $\gamma \in \Gamma$ such that such that $\mathbb{Q}(l_b(\gamma))^{\mathrm{cl}} \cap L = (k^+)^{\mathrm{cl}}$. This can be done by a Cebotarev argument similar to the one for $\ell > 2$.

5.2. The case $\ell = 2$.

Throughout this section we will assume all the notation of the previous section and that $\ell = 2$. Thus k is a quadratic extension of a totally real field k^+.

Lemma 5.2. *There are infinitely many $\gamma \in \Gamma$ for which $\lambda = \lambda(\gamma)^{b(\gamma)}$ has the following properties.*

a. λ satisfies the conditions in option (b) of Lemma 4.2.

b. All embeddings of the field k^+ into $\mathbb{Q}(\lambda\overline{\lambda})$ over $\mathbb{Q}$ have the same image.

Proof. By the Cebotarev density theorem, we can find infinitely many primes p of $\mathbb{Q}$ which split completely in k and do not lie under any place of k ramified in B. Fix such a prime, and let q_1 and q_2 be primes of O_k over a prime q^+ of k^+ which lies over p. We can find a quadratic extension F/k which is ramified over each place of k which ramifies in B and such that q_1 is ramified in F, and q_2 splits in F. We then have $q_1 O_F = \mathcal{Q}_1^2$ and $q_2 O_F = \mathcal{Q}_2\mathcal{Q}_2'$ where $\mathcal{Q}_j$ is a prime ideal of F. By Theorem 2.2, there is an element $\gamma \in \Gamma$ such that $F = k(\lambda')$ where $\lambda' = \lambda(\gamma)$. By Theorem 2.2(i) we have $F = k(\lambda'^b)$ for all integers $b \geq 1$. Thus $F = k(\lambda)$ when $\lambda = (\lambda')^{b(\gamma)} = \lambda(\gamma)^{b(\gamma)}$. The extension F/k^+ cannot be Galois, since $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are primes of F over the same prime q^+ of k^+ which have different ramification degrees. If λ were real, then by Lemma 2.1, the extension $\mathbb{Q}(\lambda)$ would be quadratic over k^+, so $k(\lambda)$ would be Galois over k^+, which is not the case. Thus λ is not real, so either option (a) or option (b) of Lemma 4.2 holds. However, option (a) is impossible, since then $k(\lambda)$ would again again be Galois over k^+. So option (b) holds.

Note that by Lemma 4.2 there is an embedding $s_1 : k^+ \to \mathbb{Q}(\lambda\overline{\lambda})$. Suppose that there is another embedding $s_2 : k^+ \to \mathbb{Q}(\lambda\overline{\lambda})$ such that $s_1(k^+) \neq s_2(k^+)$. Regarding k^+ as a subfield of $\mathbb{Q}(\lambda\overline{\lambda})$ via s_1, the composite field $L = k^+ s_2(k^+)$ is now a totally real non-trivial extension of k^+ inside $\mathbb{Q}(\lambda\overline{\lambda})$. By option (b) of Lemma 4.2, L must be the fixed field $\mathbb{Q}(\lambda,\overline{\lambda})^{\tilde{J}}$ of the order 4 subgroup $\tilde{J}$ generated by the conjugates of $J = \mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/\mathbb{Q}(\lambda\overline{\lambda}))$ in $\mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/k^+)$. Let $\mathcal{A}$ be a prime of $\mathbb{Q}(\lambda,\overline{\lambda})$ lying over the prime $\mathcal{Q}_2$ of F. Recall that $\mathcal{Q}_2$ is unramified over k^+, since the prime q_2 of k under $\mathcal{Q}_2$ is split from k to F, and q_2 is unramified over the prime q^+ of k^+ which is unramified over $\mathbb{Q}$. However, since $\mathbb{Q}(\lambda,\overline{\lambda})$ is a Galois extension of k^+, $\mathcal{A}$ must be conjugate to a prime of $\mathbb{Q}(\lambda,\overline{\lambda})$ lying over the prime $\mathcal{Q}_1$, which is quadratically ramified over k. So it follows that $\mathcal{A}$ must be quadratically ramified over F, i.e. $\mathcal{A}^2 = \mathcal{Q}_2 O_{\mathbb{Q}(\lambda,\overline{\lambda})}$. By considering the ramification indices of primes lying below $\mathcal{A}$ in the tower of extensions $k^+ \subset F \subset \mathbb{Q}(\lambda,\overline{\lambda})$ it follows that the inertia group $I(\mathcal{A})$ of $\mathcal{A}$ in $H = \mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/k^+)$ equals $\mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/F) = \mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/\mathbb{Q}(\lambda))$. No conjugate of $I(\mathcal{A})$ lies in the group $\tilde{J}$, since $\tilde{J}$ is generated by the conjugates of J and J is a non-central group of order 2 in H which intersects $\mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/k)$ trivially. (Note that $\mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/k)$ is the Klein four subgroup generated by the conjugates of $I(\mathcal{A}) = \mathrm{Gal}(\mathbb{Q}(\lambda,\overline{\lambda})/\mathbb{Q}(\lambda))$.) Thus q^+ must ramify in the extension $L = k^+ s_2(k^+) = \mathbb{Q}(\lambda,\overline{\lambda})^{\tilde{J}}$ since no prime over q^+ in L can ramify in $\mathbb{Q}(\lambda,\overline{\lambda})$. However, we chose q^+ to be a prime over the rational prime p which splits completely in k^+. Thus p splits completely in $s_2(k^+)$ and thus also in L, which is impossible if q^+ ramifies from k^+ to L. The contradiction shows that there could not have been a second embedding $s_2 : k^+ \to \mathbb{Q}(\lambda\overline{\lambda})$ such that $s_2(k^+) \neq k^+$.

6. Proof of Theorem 1.1

Clearly the commensurability class of M determines the rational length spectrum $\mathbb{Q}L(M)$. Hence Theorem 1.1 will follow immediately from the next result and Theorem 2.1.

Theorem 6.1. *Suppose that $M_1 = \mathbf{H}^3/\Gamma_1$ and $M_2 = \mathbf{H}^3/\Gamma_2$ are arithmetic hyperbolic 3-manifolds with the same rational length spectrum. Let k_i (resp. B_i) be the invariant trace field (resp. the invariant quaternion algebra) of M_i.*

a. There is an field isomorphism $\phi : k_1 \to k_2$.
b. The isomorphism ϕ in (a) can be extended to an isomorphism $B_1 \to B_2$.

To begin the proof of Theorem 6.1, note that by (2.1), we can replace Γ_i by $\Gamma_i^{(2)}$ so as to be able to assume that Γ_i is derived from B_i. Since M_1 and M_2 have the same rational length spectrum there are functions $b_i : \Gamma_i - \{e\} \to \mathbb{Z}^+$ for $i = 1, 2$ with the following property. Suppose $i = 1, 2$ and that $j = 3 - i$ is the other element of $\{1, 2\}$. Then for each $\gamma \in \Gamma_i - \{e\}$, the product $b_i(\gamma) \cdot \mathrm{l}(\gamma)$ lies in the set $\mathcal{L}(M_j)$ of lengths of closed geodesics of M_j, where $\mathrm{l}(\gamma)$ is the length of the closed geodesic on M_i associated to γ.

Define

$$\ell_{b_i}(\gamma) = \left(\lambda(\gamma)\overline{\lambda(\gamma)}\right)^{b_i(\gamma)} = e^{b_i(\gamma)\mathrm{l}(\gamma)}$$

where $\lambda(\gamma)$ is the eigenvalue of γ. Let $S(\Gamma_i, b_i) = \{\ell_{b_i}(\gamma) : \gamma \in \Gamma_i - \{e\}\}$. Since $b_i(\gamma)\mathrm{l}(\gamma) = \mathrm{l}(\gamma') \in \mathcal{L}(M_j)$ for some $\gamma' \in \Gamma_j - \{e\}$, we conclude that

$$S(\Gamma_i, b_i) \subset S(\Gamma_j, 1_j) \tag{6.1}$$

when $1_j : \Gamma_j - \{e\} \to \mathbb{Z}^+$ is the function which takes the value 1 on all elements of $\Gamma_j - \{e\}$.

6.1. Proof of Theorem 6.1(a)

By Lemma 5.1,

$$\cap\{\mathbb{Q}(\tau)^{\mathrm{cl}} : \tau \in S(\Gamma_i, b_i)\} = (k_i')^{\mathrm{cl}} \tag{6.2}$$

where $k_i' = k_i$ except when k_i is a quadratic extension of its maximal totally real subfield k_i^+, in which case $k_i' = k_i^+$. This result is independent of b_i. So by (6.1),

$$(k_1')^{\mathrm{cl}} = (k_2')^{\mathrm{cl}} \tag{6.3}$$

It was shown in Corollaries 3.3 and 3.4 that the isomorphism class of k_i' can be determined from that of $(k_i')^{\mathrm{cl}}$. So (6.3) implies Theorem 6.1(a) if $k_i = k_i'$ for $i = 1, 2$. We thus reduce to the case in which $[k_i : k_i^+] = 2$ for at least one of $i = 1, 2$. Then (6.3) gives $[k_i : k_i^+] = 2$ and $\ell = 2$ for $i \in \{1, 2\}$.

By Lemma 5.1,

$$\cap\{\mathbb{Q}(\tau) : \tau \in S(\Gamma_i, b_i)\} = (k_i^+)^{\mathrm{cl}}.$$

The containments in (6.1) now show $(k_1)^{\mathrm{cl}} = (k_2)^{\mathrm{cl}}$. By Corollary 3.4, this forces k_1^+ and k_2^+ to be isomorphic.

In Lemma 5.2 we showed there is an element $\gamma \in \Gamma_1$ such that $\lambda = \lambda(\gamma)^{b_1(\gamma)}$ satisfies all the conditions in option (b) of Lemma 4.2 and for which all embeddings of the field k_1^+ into $\mathbb{Q}(\lambda\overline{\lambda})$ over $\mathbb{Q}$ have the same image, where $\lambda\overline{\lambda} = \ell_{b_1}(\gamma)$. Fixing one such embedding, the field $\mathbb{Q}(\ell_{b_1}(\gamma))$ is a non-Galois quartic extension of k_1^+, and the Galois closure F of $\mathbb{Q}(\ell_{b_1}(\gamma))$ over k_1^+ is a dihedral extension of k_1^+ of degree 8. Now Lemma 4.2 forces k_1 to be isomorphic to F^D where D is the unique Klein four subgroup of $\mathrm{Gal}(F/k_1^+)$ which does not contain $\mathrm{Gal}(F/\mathbb{Q}(\ell_{b_1}(\gamma)))$.

We now use the fact described above that $\ell_{b_1}(\gamma) = \ell_1(\gamma')$ for some $\gamma' \in \Gamma_2$ (see 6.1). Since we have shown $(k_1)^+$ is isomorphic to $(k_2)^+$, all embeddings of $(k_2)^+$ into $\mathbb{Q}(\ell_1(\gamma')) = \mathbb{Q}(\ell_{b_1}(\gamma))$ have the same image because of condition (b) of Lemma 5.1. This image is the same as that of $(k_1)^+$ under the embedding discussed above. Running the above arguments through now with Γ_2 replacing Γ_1, we conclude that $\ell_1(\gamma') = \ell_{b_1}(\gamma)$ implies k_2 is isomorphic to the field $F^D = k_1$.

6.2. Proof of Theorem 6.1(b)

We adopt the notations and assumptions of §6.1. By Theorem 6.1(a) we can assume that B_1 and B_2 are quaternion division algebras over the same number field k. Let R_i be the set of places of k which ramify in B_i.

Proposition 6.1. *There is an automorphism $c' : k \to k$ such that $c'(R_1) = R_2$.*

Before proving this Lemma, we note that it implies B_1 and B_2 are isomorphic as $\mathbb{Q}$-algebras by Theorem 2.2(iii), so this and Theorem 2.1 will show Theorem 6.1(b).

To begin the proof of Proposition 6.1, note that since the two non-real embeddings of k into $\mathbb{C}$ are taken to each other by complex conjugation, we can apply complex conjugation to the image of one of the embeddings $\rho_{B_i} : B_i \to \mathrm{Mat}_2(\mathbb{C})$ used to define Γ_i to be able to assume that the ρ_{B_i} define the same embedding $\rho : k \to \mathbb{C}$.

Lemma 6.1. *Suppose that $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are hyperbolic elements such that the lengths $\mathrm{l}(\gamma_1)$ and $\mathrm{l}(\gamma_2)$ are (non-zero) rational multiples of one another. Define $\lambda_i = \lambda(\gamma_i)$ to be the eigenvalue associated to γ_i, so that $|\lambda_i| > 1$. Then either $k(\lambda_1) = k(\lambda_2)$ or $k(\overline{\lambda_2}) = k(\lambda_1)$, and if $k(\lambda_1) \neq k(\lambda_2)$ then k is stable under complex conjugation.*

Proof. By Theorem 2.2(i), $k(\lambda_i^n) = k(\lambda_i)$ is quadratic over k for all integers $n \geq 1$. Since $\mathrm{l}(\gamma_i) = \ln|\lambda_i \overline{\lambda}_i|$ and $\mathrm{l}(\gamma_1)$ and $\mathrm{l}(\gamma_2)$ are non-zero rational multiples of one another, we can replace γ_1 and γ_2 by suitable positive powers of themselves so that the following is true. There is a real number $r > 0$ such that $\lambda_j = re^{i\theta_j}$ for some $\theta_j \in \mathbb{R}$ and $j = 1, 2$. The assumption that $k(\lambda_1) \neq k(\lambda_2)$ implies there is an automorphism $\eta \in \mathrm{Gal}(k(\lambda_1, \lambda_2)/k(\lambda_1))$ such that $\eta(\lambda_2) = 1/\lambda_2$.

Let F be the smallest Galois extension of $\mathbb{Q}$ containing k and all Galois conjugates of λ_1 and λ_2. Consider a lift τ to F of η. We have

$$\left|\frac{\tau(\overline{\lambda_2})}{\tau(\overline{\lambda_1})}\right| = |\lambda_1\lambda_2| \cdot \left|\frac{\lambda_2^{-1}\tau(\overline{\lambda_2})}{\lambda_1\tau(\overline{\lambda_1})}\right| = r^2\left|\frac{\tau(\lambda_2\overline{\lambda_2})}{\tau(\lambda_1\overline{\lambda_1})}\right| = r^2\left|\frac{\tau(r^2)}{\tau(r^2)}\right| = r^2.$$

By considering the Galois conjugates of the λ_j (see Theorem 2.2(iv)), this implies

$$|\tau(\overline{\lambda_2})| = r = 1/|\tau(\overline{\lambda_1})| \quad \text{and} \quad \tau(\overline{\lambda_1}) \in \{1/\lambda_1, 1/\overline{\lambda_1}\} \quad \text{and} \quad \tau(\overline{\lambda_2}) \in \{\lambda_2, \overline{\lambda_2}\}.$$

If $\tau(\overline{\lambda_1}) = 1/\lambda_1$ then $\tau(\lambda_1) = \lambda_1$ would imply $\overline{\lambda_1} = 1/\lambda_1$ which is impossible since λ_1 is not on the unit circle. Similarly, $\tau(\overline{\lambda_2}) \neq \lambda_2$ because $\tau(\lambda_2) = 1/\lambda_2$. Hence

$$\tau(\overline{\lambda_1}) = 1/\overline{\lambda_1} \quad \text{and} \quad \tau(\overline{\lambda_2}) = \overline{\lambda_2}.$$

Therefore

$$e^{-2i\theta_2} = \overline{\lambda_2}/\lambda_2 = \tau(\lambda_2\overline{\lambda_2}) = \tau(r^2) = \tau(\lambda_1\overline{\lambda_1}) = \lambda_1/\overline{\lambda_1} = e^{2i\theta_1}$$

so $\overline{\lambda_2}^2 = r^2e^{-2i\theta_2} = r^2e^{2i\theta_1} = \lambda_1^2$. Hence Theorem 2.2(i) shows the desired equality of fields $k(\overline{\lambda_2}) = k(\overline{\lambda_2}^2) = k(\lambda_1^2) = k(\lambda_1)$.

Suppose finally that $k(\lambda_1) \neq k(\lambda_2)$. Then $k(\lambda_1) = k(\overline{\lambda_2})$, $k(\overline{\lambda_1}) = k(\lambda_2)$ and neither λ_1 nor λ_2 can be real. By Lemma 2.1, $\mathbb{Q}(\lambda_i) = k(\lambda_i)$ is quadratic over $k = \mathbb{Q}(\lambda_i + 1/\lambda_i)$ for $i = 1, 2$. If $\overline{\lambda_2} + 1/\overline{\lambda_2} \in k = \mathbb{Q}(\lambda_2 + 1/\lambda_2)$ then k is stable under complex conjugation. Otherwise $\overline{\lambda_2} + 1/\overline{\lambda_2} \notin k$ so

$$\mathbb{Q}(\lambda_1) = k(\lambda_1) = k(\overline{\lambda_2}) = k(\overline{\lambda_2} + 1/\overline{\lambda_2}) = \mathbb{Q}(\lambda_2 + 1/\lambda_2, \overline{\lambda_2} + 1/\overline{\lambda_2})$$

is stable under complex conjugation. But then $k(\overline{\lambda_1}) = k(\lambda_2)$ and $k(\lambda_1) = \mathbb{Q}(\lambda_1)$ show

$$k(\lambda_2) = k(\overline{\lambda_1}) = \mathbb{Q}(\lambda_1, \overline{\lambda_1}) = \mathbb{Q}(\lambda_1) = k(\lambda_1)$$

contrary to hypothesis. This shows k must be stable under complex conjugation.

Proof of Proposition 6.1.

We regard k, B_1 and B_2 as subalgebras of $\mathrm{Mat}_2(\mathbb{C})$ via our fixed embedding $\rho : k \to \mathbb{C}$ and fixed extensions of this embedding to B_1 and B_2. Since $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ are length commensurable, for each $\gamma_1 \in \Gamma_1 - \{e\}$ there is an element $\gamma_2 \in \Gamma_2 - \{e\}$ for which the conclusions of Lemma 6.1 hold, and the same is true if Γ_1 and Γ_2 are interchanged.

Suppose first that for all such pairs γ_1 and γ_2 one has $k(\lambda_1) = k(\lambda_2)$ in Lemma 6.1. In view of Theorem 2.2(ii), this implies that the quadratic field extensions of k which embed into B_1 are exactly those which embed into B_2. Therefore Theorem 2.2(iii) shows B_1 and B_2 are isomorphic over k, so we can let c' be the identity isomorphism in Proposition 6.1.

For the rest of the proof we assume that there is at least one pair γ_1 and γ_2 as above such that $k(\lambda_1) = k(\overline{\lambda_2}) \neq k(\lambda_2)$. We can also assume $R_1 \neq R_2$, since otherwise the proof can be completed as before, with c' the identity isomorphism. By Lemma 6.1, complex conjugation on $\mathbb{C}$ induces an order two automorphism $c' : k \to k$. If $c'(R_1) = R_2$, then c extends to a $\mathbb{Q}$-automorphism $c' : B_1 \to B_2$ by Theorem 2.2(iii), and Proposition 6.1 follows. We therefore assume that $c'(R_1) \neq R_2$.

By exchanging B_1 and B_2 if necessary, we may suppose that $|R_2| \geq |R_1|$. Since $c'(R_1) \neq R_2 \neq R_1$, we may choose places $\mathcal{P} \in R_2 - R_1$ and $\mathcal{Q} \in R_2 - c'(R_1)$. Note that then $c'(\mathcal{Q}) \notin R_1$.

By Theorem 2.2(ii), a quadratic extension L/k embeds into B_1 if and only if no place in R_1 splits in L/k. Since $\mathcal{P}$ and $c'(\mathcal{Q})$ do not lie in R_1, we may by Theorem 2.2(ii) choose a hyperbolic element $\delta \in \Gamma_1$ with eigenvalue $\lambda(\delta)$ so that $\mathcal{P}$ and $c'(\mathcal{Q})$ both split in $k(\lambda(\delta))$. Since $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ are length commensurable, Lemma 6.1 implies that there is a $\delta' \in \Gamma_2$ with eigenvalue $\lambda(\delta')$ such that $k(\lambda(\delta')) = k(\lambda(\delta))$ or $k(\overline{\lambda(\delta')})$. If $k(\lambda(\delta')) = k(\lambda(\delta))$ then $\mathcal{P}$ splits in $k(\lambda(\delta'))$, which contradicts the fact that $k(\lambda(\delta'))$ embeds into B_2 over k and $\mathcal{P} \in R_2$ ramifies in B_2. Similarly, if $k(\lambda(\delta')) = k(\overline{\lambda(\delta)})$, then $\mathcal{Q}$ splits in $k(\lambda(\delta'))$ because $c'(\mathcal{Q})$ splits in $k(\lambda(\delta))$. This is also false since $\mathcal{Q} \in R_2$ ramifies in B_2 and $k(\lambda(\delta'))$ embeds into B_2. The contradiction completes the proof of Proposition 6.1. □

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