373K Algebra I, Homework 7

Due on Thursday March 23d–after Spring Break

Re-do Miterm 1 problems you missed or did not do from Part B

From Artin

Chapter 7 (p. 223): 5.1(c), 5.2, 5.3, 5.4, 5.6, 5.7.

Others:

1. Let $\alpha \in S_9$ denote the permutation (1 9)(2 8)(3 7)(4 6). Is α even or odd.

2. For $1 \leq r \leq n$ compute the number of r-cycles in S_n.

3. Show that an r-cycle is even if and only if r is odd.

4. Give an example of $\alpha, \beta, \gamma \in S_5$ with $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$ but $\beta\gamma \neq \gamma\beta$

5. If $\alpha \in S_n$ is an r-cycle is α^k an r-cycle?

6.(a) Show that $\alpha \in S_n$ has order 2 if and only if its cycle decomposition is a product of commuting transpositions.

(b) Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition.

7. Let p be a prime and let H_p denote the set of matrices of $\mathrm{SL}(3, \mathbf{Z}/\mathrm{p}\mathbf{Z})$ given by:

$$\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z}/p\mathbf{Z})\}.$$

(a) Prove that H_p is a subgroup of $\mathrm{SL}(3, \mathbf{Z}/\mathrm{p}\mathbf{Z})$ of order p^3 (*The mod p Heisenberg group*).

(b) Identify H_2.

(c) Prove directly (i.e. without using the class equation) that $Z(H_p)$ is non-trivial. What is it?

(d) Compute the abelianization of H_p; i.e. $H_p/[H_p, H_p]$ where $[H_p, H_p]$ is the commutator subgroup of H_p.

8. Let S be the collection all functions $f : \mathbf{R} \to \mathbf{R}$ that have derivatives of all orders. Define a binary operation $+$ on S by $(f + g)(x) = f(x) + g(x)$.

(i) Prove that $(S, +)$ is a group.

(ii) Define $\phi : (S, +) \to (\mathbf{R}, +)$ by $\phi(f) = f'(0)$. Prove that ϕ is a homomorphism. Is ϕ one-to-one?

(b) Let $C = \{f : [0, 1] \to \mathbf{R} : f \text{ is continuous}\}$, i.e. the set of continuous real-valued functions with domain $[0, 1]$. As in (a) define a binary operation $+$ on C by $(f + g)(x) = f(x) + g(x)$ which makes $(C, +)$ a group (you do not need to prove this).

Define $\sigma : (C, +) \to (\mathbf{R}, +)$ by

$$\sigma(x) = \int_0^1 f(x)dx.$$

(i) Prove that σ is a homomorphism.

(ii) Give an example of a non-zero function in the kernel of σ.