INVARIANT TRACE-FIELDS AND QUATERNION ALGEBRAS OF POLYHEDRAL GROUPS

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1. *Introduction*

Let P be a polyhedron in $\mathbf{H}^3$ of finite volume such that the group $\Gamma(P)$ generated by reflections in the faces of P is a discrete subgroup of $\mathrm{Isom}\,\mathbf{H}^3$. Let $\Gamma^+(P)$ denote the subgroup of index 2 consisting entirely of orientation-preserving isometries so that $\Gamma^+(P)$ is a Kleinian group of finite covolume. $\Gamma^+(P)$ is called a *polyhedral group*.

As discussed in [**12**] and [**13**] for example (see §2 below), associated to a Kleinian group Γ of finite covolume is a pair $(A\Gamma, k\Gamma)$ which is an invariant of the commensurability class of Γ; $k\Gamma$ is a number field called the invariant trace-field, and $A\Gamma$ is a quaternion algebra over $k\Gamma$. It has been of some interest recently (cf. [**13**, **16**]) to identify the invariant trace-field and quaternion algebra associated to a Kleinian group Γ of finite covolume since these are closely related to the geometry and topology of $\mathbf{H}^3/\Gamma$.

In this paper we give a method for identifying these in the case of polyhedral groups avoiding trace calculations. This extends the work in [**15**] and [**11**] on arithmetic polyhedral groups. In §6 we compute the invariant trace-field and quaternion algebra of a family of polyhedral groups arising from certain triangular prisms, and in §7 we give an application of this calculation to construct closed hyperbolic 3-manifolds with 'non-integral trace'.

2. *Invariant trace-field and quaternion algebra*

Let Γ be a finitely generated non-elementary subgroup of $\mathrm{PSL}(2,\mathbf{C})$ and $\Gamma^{(2)} = \langle \gamma^2 \,|\, \gamma \in \Gamma \rangle$.

Define $\hat{\Gamma} = \mathrm{P}^{-1}(\Gamma)$ where P is the natural projection $\mathrm{P}:\mathrm{SL}(2,\mathbf{C}) \longrightarrow \mathrm{PSL}(2,\mathbf{C})$. The trace-field of Γ is $\mathbf{Q}(\mathrm{tr}\,\Gamma) = \mathbf{Q}(\mathrm{tr}\,\hat{\gamma} : \hat{\gamma} \in \hat{\Gamma})$, and is a finite extension of $\mathbf{Q}$ in the case where Γ has finite covolume ([**18**, Chapter 5] and [**9**]). The invariant trace-field $k\Gamma$ and quaternion algebra $A\Gamma$ are given by the following theorem, see [**1**, **13**, **16**].

THEOREM 2.1. *The field* $\mathbf{Q}(\mathrm{tr}\,\Gamma^{(2)}) := k\Gamma$ *is an invariant of the commensurability class of* Γ.

The algebra

$$A\Gamma = \{\textstyle\sum a_i \hat{\gamma}_i \,|\, a_i \in k\Gamma,\ \gamma_i \in \Gamma^{(2)}\}$$

(*where all sums are finite*) *is a quaternion algebra over* $k\Gamma$ *which is an invariant of the commensurability class of* Γ.

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We now relate this invariant trace-field to the minimal field of definition as discussed in [**20**]. For this we introduce some terminology, some of which is used in [**20**].

For a vector space V over $\mathbf{C}$ and a subgroup Δ of $\mathrm{GL}(V)$, Δ is *definable over a field* K if there is a K-lattice of V which is invariant under Δ; that is if, with respect to a suitable basis of V, Δ can be represented by matrices with entries in K. We state Vinberg's results only for the case where the ambient group is PGL_2. These results are concerned with a Zariski dense subgroup Γ and we note that, if Γ is a Kleinian group of finite covolume in $\mathrm{PGL}(2,\mathbf{C})$, then Γ is Zariski dense by Borel's density theorem (for example, [**21**, p. 41]). Before extending the notion of a group being definable over K, we recall the following.

If G is an algebraic subgroup of $\mathrm{GL}(n,\mathbf{C})$ and $J \subset \mathbf{C}$ is a subring, we denote by $G(J)$ the subgroup $G \cap \mathrm{GL}(n,J)$. If $K \subset \mathbf{C}$ is a field, we say an algebraic group H is a K-form of PGL_2 if H is definable over K and is isomorphic to PGL_2 over some extension of K.

DEFINITION 2.2. *K is a field of definition of Γ* if there is a K-form H of PGL_2 and an isomorphism $\rho : \mathrm{PGL}_2 \longrightarrow H$ defined over some finite extension of K such that $\rho(\Gamma) \subset H(K)$.

Recall (see [**14**] for instance) that every K-form of PGL_2 is of the form

$$\frac{A^*}{Z(A)^*} \cong \mathrm{SO}(A_0, n)$$

where A is a quaternion algebra over K and the subspace A_0 of pure quaternions is a quadratic space with quadratic map n being the restriction of the norm. In the case where $A = M_2$, then $A_0 = \mathscr{L} = s\ell(2)$, the subspace of traceless matrices, and the above isomorphism is induced by Ad, the adjoint representation of SL_2 into $\mathrm{GL}(\mathscr{L})$.

THEOREM 2.3 (Vinberg). *Let Γ be a Zariski dense subgroup of PGL_2.*

(i) $\mathrm{Ad}\,\Gamma$ *is definable over K if and only if* $\mathbf{Q}(\mathrm{tr}\,\mathrm{Ad}\,\Gamma) \subset K$.

(ii) *If $\Gamma_1 \subset \Gamma$ of finite index, the classes of fields over which Γ_1 and Γ are defined are identical.*

(iii) *If K is a field of definition of Γ, then $\mathrm{Ad}\,\Gamma$ is definable over K.*

For $\gamma \in \Gamma$, and $\hat{\gamma}$ as above, let $\hat{\gamma}$ have distinct eigenvalues λ, λ^{-1}, so that $\hat{\gamma}$ can be represented by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Then from Figure 1 it follows easily that $\mathrm{Ad}(\hat{\gamma})$ has eigenvalues $1, \lambda^2, \lambda^{-2}$. Now

$$\mathrm{tr}(\hat{\gamma})^2 = \mathrm{tr}^2(\hat{\gamma}) - 2 = (\lambda + \lambda^{-1})^2 - 2 = \mathrm{tr}\,\mathrm{Ad}(\hat{\gamma}) - 1.$$

The same formula holds if $\hat{\gamma}$ has equal eigenvalues and so γ is parabolic.

THEOREM 2.4. *Let Γ be a finite covolume Kleinian group. Then $k\Gamma$ is the minimal field of definition of Γ so that K is a field of definition of Γ if and only if $k\Gamma \subset K$.*

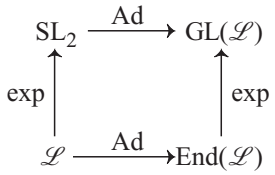

FIG. 1.

Proof. Note from above that $k\Gamma = \mathbf{Q}(\operatorname{tr}\operatorname{Ad}\Gamma)$. By Theorem 2.3(iii) and (i), it is immediate that if K is a field of definition of Γ, then $k\Gamma \subset K$. Now each element of Γ induces an automorphism of $\Gamma^{(2)}$ by conjugation and hence an automorphism of the invariant quaternion algebra $A\Gamma$. Since each automorphism of a quaternion algebra is inner, it follows that Γ embeds in the $k\Gamma$ points of $A\Gamma^*/Z(A\Gamma)^*$. Thus $k\Gamma$ is a field of definition of Γ.

3. *Gram matrices*

Now let P be a polyhedron as described in the introduction. By representing P in the Lobachevski model of $\mathbf{H}^3$, we obtain the Gram matrix $G(P)$ as follows [**19**].

Consider $V \cong \mathbf{R}^4$ equipped with a non-degenerate inner product

$$\langle \mathbf{x}, \mathbf{x} \rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

and let $C^+ = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{x} \rangle \langle 0, x_0 \rangle 0\}$. Then a model for $\mathbf{H}^3$ is PC^+ and the isometries of PC^+ are induced by the automorphisms of $(V, \langle , \rangle)$ which preserve the component C^+.

If a vector $\mathbf{v} \in V$ has positive norm, then its orthogonal complement is 3-dimensional and is hyperbolic in the sense that the restriction of the inner product is non-degenerate and indefinite. Its projection to PC^+ is a hyperbolic plane.

A polyhedron P in $\mathbf{H}^3$ will be the projection of a polyhedral angle $\mathscr{P}$ in V bounded by linear hyperplanes through the origin. For each hyperplane choose an outward pointing normal $\mathbf{e_i}$ and normalise such that $\langle \mathbf{e_i}, \mathbf{e_i} \rangle = 1$. The polyhedral angle $\mathscr{P}$ is defined by $\langle \mathbf{e_i}, \mathbf{x} \rangle \leqslant 0$. The *Gram matrix* $G(P)$ of this system of vectors is $G(P) = [a_{ij}]$ where $a_{ij} = \langle \mathbf{e_i}, \mathbf{e_j} \rangle$. If the dihedral angle between the planes H_i, H_j is α then $a_{ij} = -\cos\alpha$. More generally $a_{ij} = -(H_i, H_j)$ where (H_i, H_j) is the inversive product of the hyperbolic planes in $\mathbf{H}^3$. If P has n faces $G(P)$ is an $n \times n$ matrix, but in all cases $G(P)$ has rank 4.

If Γ is generated by reflections $\{r_i\}_{1 \leqslant i \leqslant n}$ in the faces of P, the reflection r_i in the hyperplane H_i is given by

$$r_i(\mathbf{v}) = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{e_i} \rangle \mathbf{e_i}$$

and so r_i can be represented by an $n \times n$ matrix X such that $X^t G(P) X = G(P)$.

For any

$$\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$$

define the cyclic products

$$b_{i_1 i_2 \cdots i_k} = 2^k a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} \tag{3.1}$$

and let $K(P) = \mathbf{Q}(\{b_{i_1 i_2 \cdots i_k}\})$. It is not difficult to see that it suffices to assume in the definition of $K(P)$ that the suffices $\{i_1, i_2, \ldots, i_k\}$ are distinct.

Consider the following vectors

$$\mathbf{v}_{i_1 i_2 \cdots i_k} = 2^k a_{1 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \mathbf{e}_{i_k} \tag{3.2}$$

with $\mathbf{v}_1 = 2\mathbf{e}_1$ and the suffices defined as above.

The $K(P)$-module $M(P)$ spanned by $\{\mathbf{v}_{i_1 i_2 \cdots i_k}\}$ is a Γ module. For if r_i is as above,

$$\begin{aligned} r_i(\mathbf{v}_{i_1 i_2 \cdots i_k}) &= \mathbf{v}_{i_1 i_2 \cdots i_k} - 2\langle \mathbf{v}_{i_1 i_2 \cdots i_k}, \mathbf{e}_i \rangle \mathbf{e}_i \\ &= \mathbf{v}_{i_1 i_2 \cdots i_k} - 2^{k+1} a_{1 i_1} \cdots a_{i_k i} \mathbf{e}_i. \end{aligned}$$

We also have that

$$\begin{aligned} \langle \mathbf{v}_{i_1 i_2 \cdots i_k}, \mathbf{v}_{j_1 j_2 \cdots j_l} \rangle &= 2^k a_{1 i_1} a_{i_1 i_1} \cdots a_{i_{k-1} i_k} 2^l a_{1 j_1} a_{j_1 j_2} \cdots a_{j_{l-1} j_l} a_{i_k j_l} \\ &= 1/2(2^{k+l+1} a_{1 i_1} \cdots a_{i_{k-1} i_k} a_{i_k j_l} a_{j_l j_{l-1}} \cdots a_{j_1 1}) \in K(P). \end{aligned} \tag{3.3}$$

Thus if $\mathbf{u}, \mathbf{u}' \in M(P)$ then $\langle \mathbf{u}, \mathbf{u}' \rangle \in K(P)$. Now the real dimension of M is clearly 4 and hence by the above its $K(P)$-dimension is also 4.

With the restriction of the inner product, $M(P)$ is a quadratic space over $K(P)$ and so we have a representation of Γ into the orthogonal group $O(M(P))$. Let the associated quadratic map be q_P and pick an orthogonal basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ of $M(P)$. The signed determinant of $M(P)$ is the element of $K(P)^*/K(P)^{*2}$ represented by $d = q_P(\mathbf{x}_1)\, q_P(\mathbf{x}_2)\, q_P(\mathbf{x}_3)\, q_P(\mathbf{x}_4)$.

We now state the main theorem. In this we let kP and AP denote the invariant trace-field and quaternion algebra of the group $\Gamma^+(P)$.

THEOREM 3.1. (i) $[kP : kP \cap \mathbf{R}] = 2$ *and* $K(P) = kP \cap \mathbf{R}$.

(ii) $kP = K(P)(\sqrt{d})$.

(iii) $AP \cong \left(\dfrac{-q_P(\mathbf{x}_1)\, q_P(\mathbf{x}_2),\ -q_P(\mathbf{x}_1)\, (q_P(\mathbf{x}_3)}{kP} \right)$.

REMARK 3.2. This theorem is proved in [**11**] and [**15**] for the arithmetic tetrahedral groups.

REMARK 3.3. The proof of Theorem 3.1 appears in §5.

4. *Clifford algebras*

Part of our proof relies on results on the Clifford algebras of the quadratic space $M(P)$. We first recall some general information on Clifford algebras which can be found, for example, in [**6**].

Let V be a non-degenerate quadratic space over a number field k with quadratic map q and associated bilinear form B. Let us denote its group of isometries by $O(V, q)$ and recall that $O(V, q)$ is generated by reflections $\tau_{\mathbf{u}}$, defined, for $\mathbf{u}$ an anisotropic vector in V, by

$$\tau_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - \frac{2B(\mathbf{x}, \mathbf{u})}{q(\mathbf{u})} \mathbf{u}.$$

The determinant of $\tau_{\mathbf{u}}$ is -1 and each element of $\mathrm{SO}(V, q)$ is a product of an even number of reflections.

The Clifford algebra $C(V)$ for (V, q) is a k-algebra containing V whose multiplication is compatible with q in the sense that for all $\mathbf{x} \in V$, $\mathbf{x}^2 = q(\mathbf{x}).1$. The Clifford algebra is universal with this property so that for any other k-algebra D,

containing V and compatible with q, there is a unique k-algebra homomorphism $\phi: C(V) \longrightarrow D$ such that $\phi(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. If $\dim V = n$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is an orthogonal basis of V, then $\{\mathbf{x}_1^{\mathbf{e}_1}\mathbf{x}_2^{\mathbf{e}_2}\ldots\mathbf{x}_n^{\mathbf{e}_n} : \mathbf{e}_i = 0 \text{ or } 1\}$ is a basis of $C(V)$ so that $\dim C(V) = 2^n$. Clearly $C(V)$ acquires a $\mathbf{Z}_2$-grading with even and odd parts $C_0(V)$, $C_1(V)$ having dimensions 2^{n-1}. The structure of $C(V)$ depends crucially on the parity of the dimension of V.

Note that

$$q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) = (\mathbf{x}+\mathbf{y})^2 - \mathbf{x}^2 - \mathbf{y}^2 = \mathbf{xy} + \mathbf{yx} = 2B(\mathbf{x}, \mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in V$. Thus $\mathbf{x}, \mathbf{y} \in V$ are orthogonal if and only if $\mathbf{xy} = -\mathbf{yx}$ in $C(V)$. Note that, if $\mathbf{u} \in V$ is anisotropic, then it has an inverse $\mathbf{u}/q(\mathbf{u})$ in $C(V)$. Conversely, if $\mathbf{u} \in V$ has an inverse $\mathbf{v} \in C(V)$, then $q(\mathbf{u})\mathbf{v} = \mathbf{uuv} = \mathbf{u}$ and so $q(\mathbf{u}) \neq 0$. Thus $\mathbf{u} \in V$ has an inverse in $C(V)$ if and only if $\mathbf{u}$ is anisotropic. Now suppose that $\mathbf{u}$ is anisotropic. Then

$$\tau_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{xu} + \mathbf{ux}}{\mathrm{q}(\mathbf{u})}\mathbf{u} = -\mathbf{uxu}^{-1}. \tag{4.1}$$

Each isometry $\sigma \in O(V, q)$ admits a unique extension $\hat{\sigma}$ to a k-algebra automorphism of $C(V)$ by the universal property of $C(V)$. When σ is a product of reflections, this extension can be expressed explicitly using (4.1). On basis vectors

$$\hat{\sigma}(\mathbf{x}_{\mathbf{i}_1}\mathbf{x}_{\mathbf{i}_2}\ldots\mathbf{x}_{\mathbf{i}_r}) = \sigma(\mathbf{x}_{\mathbf{i}_1})\sigma(\mathbf{x}_{\mathbf{i}_2})\ldots\sigma(\mathbf{x}_{\mathbf{i}_r})$$

and $\hat{\sigma}$ preserves the grading, so by restriction determines an automorphism of $C_0(V)$. Let

$$\phi: O(V, q) \longrightarrow \mathrm{Aut}(C(V)) \tag{4.2}$$

be defined by $\phi(\sigma) = \hat{\sigma}$.

Now the opposite algebra of $C(V)$ is also an algebra in which multiplication is compatible with the quadratic map q. Since the underlying set is the same, the universal property of $C(V)$ shows that there exists a unique algebra anti-automorphism ε of $C(V)$ which fixes V. Note that $\varepsilon(\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_r) = \mathbf{u}_r\ldots\mathbf{u}_1$, ε has order 2 and preserves the grading.

If $\dim V = n$, let $\mathbf{z} = \mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n$. When n is odd, $\mathbf{z} \in C_1(V)$ and lies in the centre of $C(V)$. When n is even, $\mathbf{z} \in C_0(V)$ and $\mathbf{z}$ lies in the centre of $C_0(V)$. Note that $\mathbf{z}^2 = (-1)^{n(n-1)/2}q(\mathbf{x}_1)\ldots q(\mathbf{x}_n) = d$, which, modulo k^{*2}, is the signed determinant of (V, q).

The mapping

$$\phi: \mathrm{SO}(V, q) \longrightarrow \mathrm{Aut}(C_0(V))$$

has kernel $\{\pm I\} \cap \mathrm{SO}(V, q)$. For, each $\sigma \in \mathrm{SO}(V, q)$ is a product of an even number of reflections. Thus, for all $x \in C(V)$, $\hat{\sigma}(x) = vxv^{-1}$ where v is the product of an even number of vectors, so that $v \in C_0(V)$. Thus, if $\hat{\sigma}(x) = x$ for all $x \in C_0(V)$, then $v \in Z(C_0(V))$ which is spanned by $1, \mathbf{z}$. Now the anti-automorphism ε fixes $\mathbf{z}$ and so restricted to $C_0(V)$ is the usual conjugation. Now if $v = \mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_{2\mathbf{m}}$ then $n(v) = n(\mathbf{v}_1\mathbf{v}_2)\ldots n(\mathbf{v}_{2\mathbf{m}-1}\mathbf{v}_{2\mathbf{m}})$. Also $n(\mathbf{w}_1\mathbf{w}_2) = (\mathbf{w}_1\mathbf{w}_2)\varepsilon(\mathbf{w}_1\mathbf{w}_2) = \mathbf{w}_1\mathbf{w}_2{}^2\mathbf{w}_1 = q(\mathbf{w}_1)q(\mathbf{w}_2) \in k$. Thus $n(v) \in k$. Thus if $v = a\mathbf{z} + b$ then either $a = 0$ or $b = 0$. Hence either $\sigma = I$ or $\sigma = -I$.

When V has dimension 4, then $\mathbf{z}$ lies in the centre of $C_0(V)$ and $C_0(V)$ is a central simple algebra of dimension 4 over the field spanned by $1, \mathbf{z}$ over k. When k is a subfield of the reals and V has signature $(3, 1)$ then $\mathbf{z}^2 = d$ is negative so that the quaternion algebra $C_0(V)$ is defined over the non-real field $(k(\sqrt{d}))$.

Now every automorphism of $C_0(V)$ is inner and so via ϕ above we have

$$\phi : \mathrm{SO}(V,q) \longrightarrow \frac{C_0(V)^*}{Z(C_0(V))^*}. \tag{4.3}$$

5. *Proof of Theorem* 3.1

5.1.

Part (i) of Theorem 3.1 is well known and follows from a more general result.

LEMMA 5.1. *Let G be a finite covolume Kleinian group normalised by an orientation-reversing involution. Then* $[kG : kG \cap \mathbf{R}] = 2$.

Proof. Let σ be the orientation-reversing involution, which by conjugating if necessary we can assume is the extension of complex conjugation to $\mathbf{H}^3$. Pass to a subgroup G_0 of finite index in G for which $\mathbf{Q}(\mathrm{tr}\, G_0)$ coincides with kG and G_0 is normalised by σ. Since G_0 is normalised by σ it follows that if $g \in G_0$, then

$$\overline{\mathrm{tr}\, g} \in \mathbf{Q}(\mathrm{tr}\, G_0) = kG.$$

Since G_0 has finite covolume there are elements of non-real trace and it follows that complex conjugation preserves the non-real field kG. Therefore $[kG : kG \cap \mathbf{R}] = 2$.

5.2.

The next thing we shall show is that $kP \cap \mathbf{R} \subset K(P)$. To do this we apply the results of §4 to the quadratic space $M(P)$ which is defined over the real field $K(P)$ and has signature $(3,1)$, with negative signed determinant d.

Now $\Gamma^+(P)$ embeds in $\mathrm{SO}(M(P))$. Since $\Gamma^+(P)$ has trivial centre, it follows from (4.3) that it embeds, via ϕ, into

$$\frac{C_0(M(P))^*}{Z(C_0(M(P)))^*}$$

which by the results of §4 is a $K(P)(\sqrt{d})$-form. Thus $K(P)(\sqrt{d})$ is a field of definition of $\Gamma^+(P)$. Thus by Theorem 2.4, $kP \subset K(P)(\sqrt{d})$. Hence $kP \cap \mathbf{R} \subset K(P)(\sqrt{d}) \cap \mathbf{R}$ and $K(P)(\sqrt{d}) \cap \mathbf{R} = K(P)$ as required.

5.3.

For the reverse inclusion, we first note additional results proved in [**20**] and continue the notation of §2. Consider the 4-dimensional space $V = \mathbf{R}^4$ equipped with the inner product of signature $(3,1)$ as before, so that we have an embedding μ of $\Gamma(P)$ into $O(V)$. As this leaves the $K(P)$-lattice $M(P)$ invariant, then $\mu(\Gamma(P))$ is definable over $K(P)$.

THEOREM 5.2 (Vinberg [**20**]). *Let Γ be Zariski dense in $O(V)$ and generated by reflections with associated Gram matrix G. Let Δ be any subgroup of finite index in Γ. Then Δ is definable over a field K if and only if K contains the cyclic products $b_{i_1 i_2 \cdots i_k}$ of G.*

Now let us consider again $\Gamma^+(P) \subset \mathrm{PSL}(2, \mathbf{C})$. Let σ be a reflection in one of the faces of P so that the centraliser $C(\sigma)$ of σ in $\Gamma^+(P)$ will stabilise the hyperbolic plane H which is spanned by the face of P. $C(\sigma)$ has finite covolume acting on H but may contain reflections. However, it contains a non-elementary Fuchsian subgroup F of index 2 and choosing a pair of non-commuting elements in $F^{(2)}$ as generators for the quaternion algebra AP over kP (see [**10**]), we have that

$$AP \cong \left(\frac{a, b}{kP}\right)$$

with $a, b \in \ell := kP \cap \mathbf{R}$. By Lemma 5.1 we can take a basis $\{1, \alpha\}$ of kP over ℓ with $\alpha^2 \in \ell$ and $\alpha^2 < 0$.

Choose a standard basis $\{1, i, j, ij\}$ of AP and define the conjugate-linear involution on AP by

$$\tau(a_0 + a_1 i + a_2 j + a_3 ij) = \overline{a_0} - \overline{a_1} i - \overline{a_2} j - \overline{a_3} ij.$$

Then exactly as in [**11**] we have the following properties:

(1) $\tau(cx + dy) = \bar{c}\tau(x) + \bar{d}\tau(y)$ for $c, d \in kP$ and $x, y \in AP$.
(2) $\tau^2 = 1$.
(3) $\tau(xy) = \tau(y)\,\tau(x)$ for $x, y \in AP$.

Furthermore if $V_\tau = \{x \in AP \mid \tau(x) = x\}$, then V_τ is a 4-dimensional ℓ-vector with basis $\{1, \alpha i, \alpha j, \alpha ij\}$ and the norm form of AP restricted to V_τ yields a quadratic form q which with respect to the above basis is given by $\mathrm{diag}\{1, -\alpha^2 a, -\alpha^2 b, \alpha^2 ab\}$. Thus over $\mathbf{R}$, $O(V_\tau) \cong O(V)$.

Now for each $y \in AP^1$, the group of elements of norm one in AP, define ϕ_y on V_τ by $\phi_y(x) = yx\tau(y)$. Using the properties of τ listed above we deduce the following.

LEMMA 5.3. *$\phi_y(x) \in V_\tau$ and $q(\phi_y(x)) = q(x)$.*

Lemma 5.3 determines a homomorphism

$$\Phi : AP^1 \longrightarrow O(V_\tau, q; \ell).$$

Following this by the isomorphism to $O(V)$, we obtain an embedding of $\Gamma^{+(2)}$ as a finite covolume subgroup. By Mostow rigidity, this embedding will be conjugate to μ. Thus this subgroup of finite index is defined over ℓ and so by Theorem 5.2, $K(P) \subset \ell$.

5.4.

We can now complete the proof of Theorem 3.1. In §5.2 it was shown that $\ell \subset K(P)$. §5.3 proves the reverse inclusion so that this together with Lemma 5.1 proves Theorem 3.1(i). Since both indices $[K(P)(\sqrt{d}) : K(P)]$, and $[kP : kP \cap \mathbf{R}]$ are 2, Theorem 3.1(ii) follows since $kP \subset K(P)(\sqrt{d})$. Finally, $\{1, \mathbf{x}_1\mathbf{x}_2, \mathbf{x}_1\mathbf{x}_3, -q_P(\mathbf{x}_1)\mathbf{x}_2\mathbf{x}_3\}$ is a standard basis of $C_0(M(P))$, and Theorem 3.1(iii) follows from the remarks above and the proof of Theorem 2.4.

We remark that the same method of proof can be used to construct the invariant trace-field and quaternion algebra of any finite covolume Kleinian group with a non-elementary Fuchsian subgroup, and which is normalised by an orientation-reversing involution.

6. *Examples*

For $q \geqslant 7$, let Γ_q denote the group generated by reflections in the triangular prism P_q shown in Figure 2, and Γ_q^+ denote the subgroup of index 2 (as in §2) (cf. [**3**]).

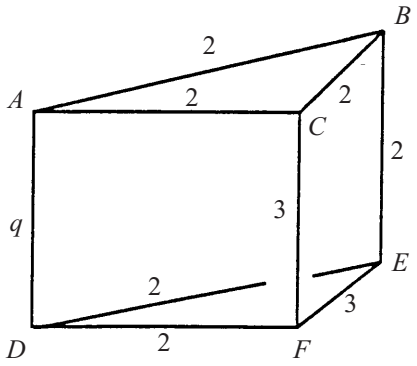

FIG. 2.

By [**19**] (see also [**15**]) the group Γ_q^+ is arithmetic exactly when $q = 7, 8, 9, 10, 14$. Using Theorem 3.1 and the method of proof we give the invariant trace-field and quaternion algebras for all Γ_q^+.

With the faces of the prism numbered as above we obtain the Gram matrix

$$G_q = \begin{pmatrix} 1 & -c & 0 & 0 & 0 \\ -c & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -\cos\pi/q \\ 0 & 0 & 0 & -\cos\pi/q & 1 \end{pmatrix}$$

where c will be the inversive product defined by the two triangular faces. Since the rank of G_q must be 4, we can determine the value of c as

$$c^2 = \frac{3\cos^2\pi/q - 2}{4\cos^2\pi/q - 3}. \tag{6.1}$$

THEOREM 6.1. *$k\Gamma_q^+ = \mathbf{Q}(\sqrt{(2+2\cos 2\pi/q)(2-6\cos 2\pi/q)})$ and*

$$A\Gamma_q^+ \cong \left(\frac{-1,-1}{k\Gamma_q^+}\right).$$

Proof. We proceed as above and compute an orthogonal basis for the quadratic space $M(P)$ over the field $K(P)$. It is not difficult to show that $K(P) = \mathbf{Q}(\cos 2\pi/q)$. With the notation of §2, we choose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{23}, \mathbf{v}_{234}$ so that $\mathbf{v}_1 = \mathbf{2e}_1, \mathbf{v}_2 = -2c\mathbf{e}_2 \cdot \mathbf{v}_{23} = 2c\mathbf{e}_3, \mathbf{v}_{234} = -2c\mathbf{e}_4$. Then

$$B_q = \begin{pmatrix} 4 & 4c^2 & 0 & 0 \\ 4c^2 & 4c^2 & 2c^2 & 0 \\ 0 & 2c^2 & 4c^2 & 2c^2 \\ 0 & 0 & 2c^2 & 4c^2 \end{pmatrix}.$$

Diagonalising gives a form equivalent over $K(P_q)$ to

$$\operatorname{diag}\{1, 4c^2, 3c^2, 4c^2(2-3c^2)/3\}.$$

Thus the signed determinant $d = 16c^6(2-3c^2)$ so that the invariant trace field is $\mathbf{Q}(\sqrt{(2+2\cos 2\pi/q)(2-6\cos 2\pi/q)})$. From the diagram of the prism given above, we see that the group Γ_q^+ contains a subgroup isomorphic to A_4. Thus the invariant quaternion algebra is given by [**5**, Lemma 9.1]:

$$\left(\frac{-1,-1}{\mathbf{Q}}\right)\otimes_{\mathbf{Q}} k\Gamma_q^+ \cong \left(\frac{-1,-1}{k\Gamma_q^+}\right).$$

7. *Application*

Let Γ be a subgroup of $\mathrm{SL}(2,\mathbf{C})$ (or $\mathrm{PSL}(2,\mathbf{C})$) whose traces consist of algebraic numbers. We say Γ has *non-integral trace* if Γ is not conjugate into ($\mathrm{SL}(2,\mathbf{A})$ (respectively $\mathrm{PSL}(2,\mathbf{A})$) where $\mathbf{A}$ is the ring of all algebraic integers in $\mathbf{C}$. A remarkable theorem of Bass is the following which forms part of the GL_2-subgroup theorem [**1**, **2**].

THEOREM 7.1. *Let Γ be a finitely generated subgroup of* $\mathrm{SL}(2,\mathbf{C})$ *with a non-integral trace. Then Γ admits a splitting as a graph of groups in the sense of* [**17**].

In the context of Kleinian groups and 3-manifolds, we have the following topological interpretation.

COROLLARY 7.2. *Let $M = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold such that Γ has a non-integral trace. Then M is Haken.*

Very few examples of closed hyperbolic 3-manifolds with non-integral trace are known. Examples of link complements in S^3 are implicit in [**13**, §5–8]. Using the examples in §6, many examples will be obtained as a consequence of the calculations which follow.

Referring to Figure 2, we note that the edge of the prism which is the axis of an elliptic element of order q is orthogonal to both triangular faces of the prism. Thus it is the unique common perpendicular to these two faces. Thus if λ is the length of that edge, then $c = \cosh\lambda$ since c is the inversive product of these two faces.

Consider a face of the prism which is a quadrilateral and contains this common perpendicular. Let y_1, y_2 denote the rotations of order 2 around the edges of this quadrilateral which abut the triangles. Then $y = y_1 y_2 \in \Gamma_q^+$ and has the geodesic containing the perpendicular as its invariant axis. Note that $y_1 y_2$ also leaves the plane spanned by the quadrilateral face invariant. Thus y is a hyperbolic element with trace $2c$. Thus $2c$ is a trace in the group Γ_q^+. Thus $\operatorname{tr} y$ is an integer if and only if $4c^2$ is an integer. Now

$$4c^2 = 3 + \frac{1}{2\cos 2\pi/q - 1}$$

and so $4c^2$ is an integer precisely when $2\cos 2\pi/q - 1$ is a unit.

7.1.

First, we want to determine when $2\cos 2\pi/q-1$ is a unit. Later we also need to know the norm of other linear combinations of $2\cos 2\pi/q$. This can be carried out as follows. Let $\Phi_q(x)$ denote the cyclotomic polynomial of degree $\phi(q)$, so that

$$\Phi_q(x) = \prod_{(r,q)=1,\,1\leqslant r<q/2}\left(x^2-2\cos\frac{2r\pi}{q}x+1\right). \tag{7.1}$$

Thus for suitable choices of x we can evaluate the norm of linear combinations of $2\cos 2\pi/q$ by evaluating the cyclotomic polynomial. In particular, for $x=\xi=e^{\pi i/3}$, x satisfies $x^2+1=x$ and so

$$|N(2\cos 2\pi/q-1)| = |\Phi_q(\xi)|.$$

LEMMA 7.3. *For $q\geqslant 7$, $u=2\cos 2\pi/q-1$ is a unit for all q except*
(i) *$q=6p^\gamma$ where p is a prime $\neq 2,3$ and $\gamma\geqslant 1$, in which case $|N(u)|=p$;*
(ii) *$q=2^\alpha\cdot 3$ where $\alpha\geqslant 2$, in which case $|N(u)|=2$;*
(iii) *$q=2\cdot 3^\beta$ where $\beta\geqslant 2$ in which case $|N(u)|=3$.*

Proof. Let $q=dm$ where $m\neq 1$. Then

$$\Phi_q(x)\left|\frac{x^{md}-1}{x^d-1}\right. = x^{d(m-1)}+x^{d(m-2)}+\ldots+x^d+1. \tag{7.2}$$

For $\eta=\xi^{\pm 1}$, one has

$$1+\eta+\ldots+\eta^{m-1} = \begin{cases} 0 & \text{if } m\equiv 0(\mathrm{mod}\,6)\\ \text{a unit} & \text{if } m\equiv\pm 1(\mathrm{mod}\,6)\\ \sqrt{3}\times\text{a unit} & \text{if } m\equiv\pm 2(\mathrm{mod}\,6)\\ 2\times\text{a unit} & \text{if } m\equiv 3(\mathrm{mod}\,6).\end{cases}$$

Also, for $\eta=\xi^{\pm 2}$, one has

$$1+\eta+\ldots+\eta^{m-1} = \begin{cases} 0 & \text{if } m\equiv 0(\mathrm{mod}\,3)\\ \text{a unit} & \text{if } m\equiv\pm 1(\mathrm{mod}\,3)\end{cases}$$

and for $\eta=\xi^3$, one has

$$1+\eta+\ldots+\eta^{m-1} = \begin{cases} 0 & \text{if } m\equiv 0(\mathrm{mod}\,2)\\ 1 & \text{if } m\equiv 1(\mathrm{mod}\,2).\end{cases}$$

Thus let us assume that $q=2^\alpha 3^\beta m_1 m_2$ where $\alpha,\beta\geqslant 0$, $(m_1,m_2)=1$ and $(m_1m_2,6)=1$.

(1) If $\alpha=\beta=0$, take $d=1$ in (7.2) to obtain that $\Phi_q(\xi)$ is a unit.

(2) If $\alpha = 0$, $\beta > 0$, take $d = 3$ to obtain that $\Phi_q(\xi)$ is a unit.
(3) If $\alpha > 0$, $\beta = 0$, take $d = 2$ to obtain that $\Phi_q(\xi)$ is a unit.
(4) If $\alpha, \beta > 0$, taking $m = m_2$ in (7.2) gives that $\Phi_q(\xi) | m_2$ and then taking $m = m_1$ gives that $\Phi_q(\xi) | m_1$.

From these we deduce that u is a unit unless $q = 2^\alpha 3^\beta p^\gamma$ where p is a prime ($\neq 2, 3$), $\alpha, \beta > 0$ and $\gamma \geqslant 0$.

(5) If $\alpha > 1$, take $m = 2$ so that $\Phi_q(\xi) | 2$.
(6) If $\beta > 1$, take $m = 3$ so that $\Phi_q(\xi) | 3$.
(7) If $\gamma \geqslant 1$, take $m = p$ so that $\Phi_q(\xi) | p$.

From these we see that u is a unit unless $q = 2^\alpha \cdot 3$, $2 \cdot 3^\beta$, $6p^\gamma$ where $\alpha, \beta \geqslant 2$, $\gamma \geqslant 1$. Now

$$\Phi_{6p^\gamma}(x) = \frac{(x^{3p^\gamma}+1)(x^{p^{\gamma-1}}+1)}{(x^{p^\gamma}+1)(x^{3p^{\gamma-1}}+1)} = \frac{x^{3p^{\gamma-1}(p-1)} - x^{3p^{\gamma-1}(p-2)} + \ldots + 1}{x^{p^{\gamma-1}(p-1)} - x^{p^{\gamma-1}(p-2)} + \ldots + 1}. \tag{7.3}$$

Directly from this $|\Phi_{6p^\gamma}(\xi)| = p$. The other two cases are similar.

COROLLARY 7.4. *$\operatorname{tr} y$ fails to be an integer precisely when $q = 6p^\alpha, 2^a 3, 2 \cdot 3^b$.*

7.2.

Here we extend the analysis of the non-integral traces further. First we state a result which is a strengthening of Theorem 7.1. The method of proof is that of [**7**], and we only sketch the relevant ideas. We need to recall the following. Let k be a number field, and $\mathscr{P}$ be a k-prime divisor of the rational prime p. We say $\mathscr{P}$ is of *degree* 1 if the norm of $\mathscr{P}$ is p.

THEOREM 7.5. *Let Γ be a finitely generated subgroup of $\mathrm{SL}(2, k)$ with non-integral traces. Furthermore, assume there is a non-integral trace t such that $\langle t \rangle = \mathscr{P}^{-a}.I$ where $\mathscr{P}$ is a k-prime of degree 1 and I is a fractional ideal for which $\mathscr{P}$ does not appear in its unique factorisation into prime ideals. Then Γ splits as a non-trivial free product with amalgamation.*

Sketch proof. Following the methods of [**1**] and [**17**], the degree 1 assumption on the prime $\mathscr{P}$ yields the existence of a faithful representation of Γ into $\mathrm{SL}(2, \mathbf{Q}_p)$ where $\mathscr{P} | p$, and so we get an action of Γ on the tree T_p of SL_2 over $\mathbf{Q}_p$. The non-integral trace assumption implies this action is non-trivial, and hence we deduce a splitting of Γ. The ideas of [**7**, §5] show that Γ is dense in $\mathrm{SL}(2, \mathbf{Q}_p)$ in the p-adic topology, and applying [**17**, Theorem 3] proves that the splitting is a free product with amalgamation.

Some of the groups $\Gamma_q^{+(2)}$ satisfy the hypothesis of Theorem 7.5 as we now show.

By Corollary 7.4 taking $q = 6p$ where p is a prime greater than 5 gives that the groups Γ_q^+ have non-integral trace. Note that the prime ideal $\mathscr{P}$ involved in the denominator is of degree 1 in $\mathbf{Q}(\cos 2\pi/q)$. Let $\mathscr{P} = (2\cos 2\pi/q - 1)$ in $\mathbf{Q}(\cos 2\pi/q)$.

We will now show that $\mathscr{P}$ splits in the extension $k\Gamma_q^+ | \mathbf{Q}(\cos 2\pi/q)$ for suitable choices of p. It will then follow from properties of primes in field extensions that the pair of primes in $k\Gamma_q^+$ above such p will be of degree 1.

Recall from Theorem 6.1 that $k\Gamma_q^+$ is generated by the element $\alpha = \sqrt{(2+2\cos 2\pi/q)(2-6\cos 2\pi/q)}$ and we first show that $\mathscr{P}$ is unramified in the extension. The discriminant of the basis 1, α of $k\Gamma_q^+$ over $\mathbf{Q}(\cos 2\pi/q)$ has norm $2^{\phi(q)}N(2+2\cos 2\pi/q)\,N(2-6\cos 2\pi/q)$. From (7.1) and (7.3), $N(2+2\cos 2\pi/q) = \Phi_q(-1) = 1$. Again from (7.1),

$$N(2-6\cos 2\pi/q) = \left(\frac{3}{x_0}\right)^{\phi(q)/2} \Phi_q(x_0)$$

where $x_0 = (1+2\sqrt{-2})/3$. Now $\Phi_q(x) | x^{3p}+1$ and so in the ring of integers in $\mathbf{Q}(\sqrt{-2})$, which is a Euclidean domain, $N(2-6\cos 2\pi/q)$, which is a rational integer, divides $(1+2\sqrt{-2})^{3p}+3^{3p}$. Now

$$(1+2\sqrt{-2})^{3p}+3^{3p} \equiv [(1+2\sqrt{-2})^3+3^3]^p \equiv [\sqrt{-2}(-1+\sqrt{-2})]^{3p} \not\equiv 0 (\mathrm{mod}\, p).$$

Thus $\mathscr{P}$ is unramified in the extension $k\Gamma | \mathbf{Q}(\cos 2\pi/q)$ and we can apply Kummer's theorem (for example, [**8**, p. 20]). Reducing the minimum polynomial of $\alpha (\mathrm{mod}\, \mathscr{P})$ gives x^2+3. Thus if we choose p such that $(-3/p) = 1$ then $\mathscr{P}$ splits in the extension. These remarks give the following corollary.

COROLLARY 7.6. *When $q = 6p$ where p is a prime such that $p \equiv 1 (\mathrm{mod}\, 3)$, then the group $\Gamma_q^{+(2)}$ has a non-integral trace whose denominator lies in an ideal of $k\Gamma_q^+$ of norm p.*

By Dirichlet's theorem on primes in arithmetic progression, it follows that there are infinitely many such primes. Together with Theorem 7.5 we have the following corollary.

COROLLARY 7.7. *With q as in Corollary* 7.6, *let Γ be a subgroup of finite index in $\Gamma_q^{+(2)}$. Then Γ splits as a non-trivial free product with amalgamation.*

The main interest to us here is not just the existence of this splitting but rather the passage from the group theoretical splitting to the existence of a closed incompressible surface. To prove our next theorem we will require some notation and to recall some standard facts; see [**4**, Chapter 1] for details.

Let M be a compact orientable irreducible 3-manifold, $\tilde{M}$ denote the universal cover of M and $p: \tilde{M} \longrightarrow M$ be the covering projection. Assume that $\pi_1(M)$ acts without inversions on a tree T. Let E be the set of midpoints of edges in T.

A properly embedded surface S in M is *associated to the action* if there exists a $\pi_1(M)$-equivarient map $\tilde{\phi}: \tilde{M} \longrightarrow T$ which is transverse to E, and such that $\tilde{\phi}^{-1}(E) = p^{-1}(S)$.

We have the following lemma (see [**4**, Proposition 1.3.6]).

LEMMA 7.8. *Let S be a surface associated to the action of $\pi_1(M)$ on T. Suppose that S' is obtained from S by compressing S along a disc, then S' is also associated to the action.*

THEOREM 7.9. *With q as in Corollary 7.6 and Γ a torsion-free subgroup of finite index in Γ_q there is an incompressible component of the surface in $M = \mathbf{H}^3/\Gamma$ associated to the action of Γ on T_p (as above) which is non-separating.*

Proof. With reference to Figure 2, Γ_q^+ can be generated by three elements x, y, z where x is the rotation around the edge CF, y is the hyperbolic element described earlier, which is the product of the rotations around the edges AC and DF and z is the rotation around AD. By suitably locating the prism in H^3, we obtain that x, y, z are the images of the matrices

$$X = \frac{1}{u}\begin{pmatrix} -1 & 2\cos 2\pi/q - 1 \\ -e^{2\pi i/q} & e^{2\pi i/q} \end{pmatrix}, \qquad Y = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \qquad Z = \begin{pmatrix} e^{\pi i/q} & 0 \\ 0 & e^{-\pi i/q} \end{pmatrix}$$

where $u = e^{2\pi i/q} - 1$. Now the stabiliser in Γ_q^+ of the plane H spanned by the face ABC is a $\mathbf{Z}_2$-extension of the triangle group $(2, 3, q)$. However, on passing to $\Gamma_q^{+(2)}$ the corresponding stabiliser is the Fuchsian triangle group F of signature $(3, 3, 3p)$ which is generated by the images of X and Z^2. We note that $X, Z^2 \in \mathrm{SL}(2, \mathbf{Q}(e^{2\pi i/q}))$ and that the trace field kF of F is $\mathbf{Q}(\cos 2\pi/q)$. Now x, z^2 generate a non-elementary subgroup of $\Gamma_q^{+(2)}$ so that the quaternion algebra $AF = kF[I, X, Z^2, XZ^2]$ and $A = A\Gamma_q^+ = k\Gamma[I, X, Z^2, XZ^2]$. Thus $A\Gamma \cong AF \otimes_{kF} k\Gamma$.

As before, let $\mathscr{P}$ be the ideal generated by $2\cos 2\pi/q - 1$ in kF and let $\mathscr{Q}$ be one of the ideals of $k\Gamma$ which divide $\mathscr{P}$. Since, by Theorem 6.1, A can only have finite ramification at dyadic primes

$$\begin{aligned} M_2((k\Gamma)_{\mathscr{Q}}) &\cong A \otimes_{k\Gamma} (k\Gamma)_{\mathscr{Q}} \cong AF \otimes_{kF} k\Gamma \otimes_{k\Gamma} (k\Gamma)_{\mathscr{Q}} \\ &\cong AF \otimes_{kF} (kF)_{\mathscr{P}} \cong M_2((kF)_{\mathscr{P}}). \end{aligned}$$

Now $L = \mathbf{Q}(e^{2\pi i/q})$ splits the algebra AF and, furthermore, the minimum polynomial of the generating element over kF is

$$x^2 - 2\cos 2\pi/q\, x + 1 \equiv x^2 - x + 1 (\mathrm{mod}\, \mathscr{P}).$$

Since $(-3/p) = 1$, the prime $\mathscr{P}$ splits in the extension L. Let $\mathscr{R}$ be one of the primes in L which divide $\mathscr{P}$ so that $L_{\mathscr{R}} \cong (kF)_{\mathscr{P}}$. In $M_2(L)$, since $N(e^{2\pi i/q} - 1) = \Phi_q(1) = 1$, all entries of X, Z^2 are integral. Additionally, since $2\cos 2\pi/q - 1$ can be chosen to be a uniformiser π in $(kF)_{\mathscr{P}}$, then the images of X, Z^2 locally at $\mathscr{Q}$ lie in the intersection of adjacent maximal orders of $A\Gamma \otimes_{k\Gamma} (k\Gamma)_{\mathscr{Q}}$ and so stabilise an edge of the tree T_p [**17**].

Now if Γ_0 is any torsion-free subgroup of finite index in $\Gamma_q^{+(2)}$ we therefore deduce that an incompressible surface S associated to the action given by Corollary 7.6 must contain a component that is totally geodesic since its fundamental group is a subgroup of F. Notice that the quotient H/F gives rise to an embedded non-separating 2-orbifold in the 3-orbifold $\mathbf{H}^3/\Gamma_q^{+(2)}$. Thus in the finite cover $\mathbf{H}^3/\Gamma_0$ we see an embedded totally geodesic non-separating surface. By construction this surface is associated to the action of Γ_0 on T_p as above.

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